TAIWANESE JOURNAL OF MATHEMATICS
Vol. 26, No. 6, pp. 1163–1201, December 2022
DOI: 10.11650/tjm/220601

Blow-up in Coupled Solutions for a 4-dimensional Semilinear Elliptic Kuramoto–Sivashinsky System

Lilia Larbi and Nihed Trabelsi*

Abstract. The existence of singular limit solutions is established for a nonlinear elliptic Kuramoto–Sivashinsky system with exponential nonlinearity and Navier boundary conditions, by means of the nonlinear domain decomposition method.

1. Introduction and statement of the result

Generally, the real life phenomena that can be satisfactorily modeled by a single partial differential equation are very rare. So a system of coupled partial differential equations is needed to yield a suitable model. For example, the study of the nonlinear system is important for various phenomena of biology and physics. On the other hand, they present also some challenging mathematical problems which allowed several researchers as in [9,26] to be more interested about these systems, moreover to create new theories and methods to treat them.

Let Ω be a bounded open domain in $\mathbb{R}^4$. We consider the following elliptic system of Kuramoto–Sivashinsky type:

$$(1.1)\qquad \begin{cases} \frac{\partial u_1}{\partial t} + \Delta^2 u_1 - \gamma_1 \Delta u_1 + \sigma_1 u_1 - \lambda_1 |\nabla u_1|^q = \beta_1 f(x,u_1,u_2) & \text{in } \Omega, \\ \frac{\partial u_2}{\partial t} + \Delta^2 u_2 - \gamma_2 \Delta u_2 + \sigma_2 u_2 - \lambda_2 |\nabla u_2|^q = \beta_2 g(x,u_1,u_2) & \text{in } \Omega, \\ \Delta u_1 = \Delta u_2 = u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

Here γ_i, σ_i, β_i and λ_i are real constants, $i = 1,2$, $q \in [1,4]$, $f, g \in C(\mathbb{R}^4 \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and the solutions of (1.1) are sought in the space $W^{4,2}(\mathbb{R}^4) \times W^{4,2}(\mathbb{R}^4)$.

This model arises in many applications of mathematical physics, which are usually used to describe some phenomena appearing in physics, engineering and other sciences. For example, this problems occurs in the study of the static deflection of an elastic plate in a fluid, in the problem of periodic oscillations and travelling waves in a suspension bridge and as well as in the micro-electromechanical systems (see [15, 17, 22, 27]).

Received August 23, 2021; Accepted June 7, 2022.
Communicated by François Hamel.
2020 *Mathematics Subject Classification.* 35B38, 35J75, 35J60, 35J08.
Key words and phrases. critical points, singular limit solution, nonlinear elliptic equations, Green's function.
*Corresponding author.

A fundamental goal in the study of boundary value problems as (1.1) is to determine whether solutions develop a singularity. The issue of blow-up is important, since it can have bearing on the physical relevance and validity of the underlying model. However, this is notoriously difficult question for a wide range of equations such as fourth order equation like the stationary nonhomogeneous Kuramoto–Sivashinsky equation with a strong nonlinearity such as e^u:

$$\Delta^2 u - \gamma\Delta u - \lambda|\nabla u|^q = \rho^4 e^u. \tag{1.2}$$

The Kuramoto–Sivashinsky equation was independently obtained by Kuramoto and Suzuki [14] and by Sivashinsky [24] in the study of a reaction-diffusion system and flame front propagation. This equation may be also found in the study of $2D$ Kolmogorov fluid flows [25]. In [21] the authors investigated equation (1.2) with Navier boundary conditions when the parameters ρ, λ and γ tend to 0 and $q \in [1,4]$. In fact, they distinguished the cases $q \in [1,4)$ and $q = 4$ and they constructed an approximate solution for the interior problem, and then, by means of the nonlinear decomposition method, infer the existence of a singular limit solution. For $\lambda = 0$, we refer the reader to [20] where the author considers the problem, without gradient term. In the same spirit, in [2] the authors considered a biharmonic system in dimension 4 and with exponential nonlinearity, where singular sets may also intercept each other. Here, instead, we consider the full system which is the counterpart of equation (1.2), i.e., with diffusive term, represented by Δu, and convection term, $|\nabla u|^q$, and with the aim to extend the aforementioned results. Note that these new terms have significant influence on the existence of a solution, as well as on its asymptotic behavior. We restrict $q \in [1,4)$ and consider the setting in which the gradient term does not prevail on the biLaplace operator.

More precisely, our main interest is to introduce a rather efficient method to solve the Kuramoto–Sivashinsky elliptic system given as follows:

$$\begin{cases} \Delta^2 u_1 - \gamma_1\Delta u_1 - \lambda_1|\nabla u_1|^q = \rho^4 e^{\gamma u_1 + (1-\gamma)u_2} & \text{in } \Omega, \\ \Delta^2 u_2 - \gamma_2\Delta u_2 - \lambda_2|\nabla u_2|^q = \rho^4 e^{\xi u_2 + (1-\xi)u_1} & \text{in } \Omega, \\ \Delta u_1 = \Delta u_2 = u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^4$. Here $q \in [1,4)$ and ρ, γ, ξ, γ_i and λ_i, $i = 1,2$ are real parameters. We assume that $\gamma, \xi \in (0,1)$ such that $\gamma + \xi > 1$ so in the following, we have

$$\frac{1-\gamma}{\xi}, \frac{1-\xi}{\gamma} \in (0,1).$$

We are interested in the study of the existence of solutions with singular limits, i.e., a solution for which there exists a limit function having n blow-up points $(z_1, \ldots, z_n)$ where $u^\rho(z_i) \to +\infty$ as $\rho \to 0$.

1.1. A tour in the literature

Before stating our main result, summarized in Theorem 1.6 below, we focus on some known basic results in the field. The system (1.3) may be seen as generalization of the equation

$$\Delta^2 u = 6e^{4u} \quad \text{in } \mathbb{R}^4. \tag{1.4}$$

Equation (1.4) is invariant under translation, rotation, dilation in the Euclidean space and the Kelvin transform. In [16], Lin proved the following important classification result of finite-mass solutions of equation (1.4).

Theorem 1.1. [16] *Let u be a solution of* (1.4), *satisfying the finite-mass condition*

$$\int_{\mathbb{R}^4} e^{4u}\, dx < \infty, \tag{1.5}$$

and $|u(x)| = o(|x|^2)$ at ∞. Then there exists some point $x^0 \in \mathbb{R}^4$ such that u is radially symmetric about x^0 and

$$u(x) = \ln \frac{2\lambda}{1 + \lambda^2 |x - x^0|^2}.$$

This result is decisive for solving completely (1.4) under (1.5), because it reduces the problem to a simple ODE problem. In [29], Wei and Ye constructed a nonradial solution of Liouville equation (1.4) under (1.5) with the following asymptotic behavior:

$$u(x) = -\sum_{j=1}^{k} a_j (x_j - x_j^0)^2 - \alpha \ln |x| + c_0 + o(1), \quad |x| > 1 \quad \text{and} \quad \int_{\mathbb{R}^4} e^{4u(x)}\, dx = \frac{4\pi^2 \alpha}{3}$$

for each fixed $x^0 \in \mathbb{R}^4$, $1 \leq k \leq 4$, $\alpha \in (1 - k/4, 2)$ and $a_j > 0$ for $1 \leq j \leq k$.

In dimension 4, other authors were motivated by similar problems, we refer the reader to [3, 5, 10, 11]. Wei in [28] studied the behavior of solutions of the nonlinear eigenvalue problem in $\mathbb{R}^4$:

$$\begin{cases} \Delta^2 u = \lambda f(u) & \text{in } \Omega, \\ \Delta u = u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

When $f(u) = e^u$, (1.6) originates in the context of conformal geometry, by prescribing the so-called Q-curvature on 4-dimensional Riemannian manifolds. For more details and background material, we refer to [1, 8, 12, 19] for a Q-curvature problem on 2-dimensional Riemannian manifolds for a nonlinearity polynomial function. Before stating the result of [28], we will introduce some notations.

Let $G(z,z')$ defined over $\Omega\times\Omega$ be the Green function associated to the bi-laplacian operator with Navier boundary conditions, which is the solution of

$$\begin{cases}\Delta^2 G(z,z') = 64\pi^2\delta_{z=z'} & \text{in } \Omega,\\ \Delta G(z,z') = G(z,z') = 0 & \text{on } \partial\Omega\end{cases}$$

and denote by $H(z,z') := G(z,z') + 8\ln|z-z'|$ its regular part. Consider now the functional E defined on the set $\{(z_1,\dots,z_m)\in\Omega^m; z_i\neq z_j \text{ for all } 1\le i\neq j\le m\}$ by

$$E(z_1,\dots,z_m) := \sum_{j=1}^m H(z_j,z_j) + \sum_{j\neq l} G(z_j,z_l),$$

and denote by u^* the function defined on $\Omega\setminus\{z_1,\dots,z_m\}$ by

$$u^*(z) := \sum_{j=1}^m G(z,z_j).$$

We recall that a critical point is called *nondegenerate* when the Hessian matrix computed in this point is nonzero. In [28], the author proved the following result.

Theorem 1.2. [28] *Let Ω be a regular bounded convex domain in $\mathbb{R}^4$ and f be a smooth nonnegative increasing function such that*

$$e^{-u}f(u)\to 1 \quad \textit{as } u\to+\infty.$$

For u_λ solution of (1.6), *denote by $\Sigma_\lambda = \lambda\int_\Omega f(u_\lambda)\,dx$. Then there are only three possibilities:*

(i) *The $\{\Sigma_\lambda\}$ accumulate to* 0, *then $\|u_\lambda\|_{L^\infty(\Omega)}\to 0$ as $\lambda\to 0$.*

(ii) *The $\{\Sigma_\lambda\}$ accumulate to $+\infty$, then $u_\lambda(x)\to+\infty$ for all $x\in\Omega$ as $\lambda\to 0$.*

(iii) *The $\{\Sigma_\lambda\}$ accumulate to $64\pi^2 m$ for some positive integer m, then the limiting function $u^* = \lim_{\lambda\to 0} u_\lambda$ has m blow-up points, $\{z_1,\dots,z_m\}$, i.e., there exists a set $S := \{z_1,\dots,z_m\}\subset\Omega$ such that $(u_\lambda(x))_\lambda$ has a limit for $x\in\overline{\Omega}\setminus S$, while $u_{\lambda|S}\to+\infty$.*

Moreover, $(z_1,\dots,z_m)$ is a critical point of E.

In [5], the authors considered the following problem

$$\begin{cases}\Delta^2 u = \rho^4 e^u & \text{in } \Omega,\\ u = \Delta u = 0 & \text{on } \partial\Omega.\end{cases} \tag{1.7}$$

They constructed a non-minimal solution with singular limit as the parameter ρ tends to 0. Their results can be stated as follows:

Theorem 1.3. [5] *Let Ω be a regular open subset of $\mathbb{R}^4$ and $z_1, \dots, z_m \in \Omega$ be given points. Assume that $(z_1, \dots, z_m)$ is a nondegenerate critical point of E, then there exist $\rho_0 > 0$ and $(u_\rho)_{\rho \in (0,\rho_0)}$ a one parameter family of solutions of* (1.7)*, such that*

$$\lim_{\rho \to 0} u_\rho = u^* \quad \text{in } \mathcal{C}^{4,\alpha}_{\text{loc}}(\Omega \setminus \{z_1, \dots, z_m\}).$$

In dimension 2, the asymptotic behavior of the analogous problem

$$-\Delta u = \rho^2 e^u \ \text{ in } \Omega \subset \mathbb{R}^2, \quad u = 0 \ \text{ on } \partial\Omega \tag{1.8}$$

has been studied by Liouville in [18]. He gave a global representation for all solutions of (1.8) which are defined in $\mathbb{R}^2$. It is well known that as the parameter ρ tends to 0, non-minimal solutions exist and they have singular limits. In [6], Baraket and Pacard proved

Theorem 1.4. [6] *Let Ω be a regular open subset of $\mathbb{R}^2$ and $z_1, \dots, z_m \in \Omega$. Assume that $(z_1, \dots, z_m)$ is a nondegenerate critical point of the function*

$$F\colon (z_1, \dots, z_m) \in \mathbb{C}^m \mapsto \sum_j h(z_j, z_j) + \sum_{j \neq l} g(z_j, z_l),$$

then there exist $\rho_0 > 0$ and $(u_\rho)_{\rho \in (0,\rho_0)}$ a one parameter family of solutions of (1.8) *such that*

$$\lim_{\rho \to 0} u_\rho = u^* := \sum_{j=1}^m g(\,\cdot\,, z_j) \quad \text{in } \mathcal{C}^{2,\alpha}_{\text{loc}}(\Omega \setminus \{z_1, \dots, z_m\}).$$

Here g is the Green's function of $-\Delta$ in $\mathbb{R}^2$ defined as the solution of

$$\begin{cases} -\Delta g(z, z') = 8\pi \delta_{z=z'} & \text{in } \Omega, \\ g(z, z') = 0 & \text{on } \partial\Omega, \end{cases}$$

and h is its regular part defined by

$$h(z, z') := g(z, z') + 4 \ln |z - z'|.$$

Some generalizations can be found in [4, 7, 13].

In this paper, given $\varepsilon > 0$ and $\sigma_{\gamma,\lambda} = \max(\gamma, \lambda)$, $\lambda := \lambda_i$, $\gamma := \gamma_i$, $i = 1, 2$, assume that ε, γ and λ satisfy

(A1) if $0 < \varepsilon < \sigma_{\gamma,\lambda}$, then $\sigma_{\gamma,\lambda}^{1+\mu/2} \varepsilon^{-\mu} \to 0$ as $\sigma_{\gamma,\lambda} \to 0$ for any $\mu \in (1, 5 - q)$, $1 \le q < 4$.

(A2) if $0 < \varepsilon < \sigma_{\gamma,\lambda}$, then $\sigma_{\gamma,\lambda}^{1+\delta/2} \varepsilon^{-\delta} \to 0$ as $\sigma_{\gamma,\lambda} \to 0$ for any $\delta \in \big(0, \min\big\{1, 4 - q, \frac{\gamma+\xi-1}{\gamma}, \frac{\gamma+\xi-1}{\xi}\big\}\big)$, $1 \le q < 4$.

We prove the following results.

Theorem 1.5. *Let Ω be a regular bounded domain of $\mathbb{R}^4$, $\varepsilon > 0$ satisfying* (A1)–(A2) *and $z_1, \dots, z_m \in \Omega$ be given distinct points. Let moreover $p \in \{1, \dots, m\}$ and suppose that $(z_1, \dots, z_p, \dots, z_m)$ is a nondegenerate critical point of the function*

$$\mathcal{F}(z_1, \dots, z_m) = \frac{1-\xi}{2\gamma} \sum_{i=1}^{p} H(z_i, z_i) + \frac{1-\gamma}{2\xi} \sum_{j=p+1}^{m} H(z_j, z_j) + \frac{1-\gamma}{\xi} \frac{1-\xi}{\gamma} \sum_{i=1, j=p+1}^{i=p, j=m} G(z_i, z_j),$$

then there exist γ_0 and ξ_0 in $(0,1)$ such that for all $\gamma \in (\gamma_0, 1)$ and $\xi \in (\xi_0, 1)$, there exist $\rho_0 > 0$ and $\left(u_1^{\rho,\gamma_1,\lambda_1}, u_2^{\rho,\gamma_2,\lambda_2}\right)_{0<\rho\le\rho_0, 0<\gamma^i\le\gamma_0, 0<\lambda^i\le\lambda_0}$, $i = 1, 2$, a one parameter family of solutions of (1.3), *such that*

$$\lim_{\substack{\rho\to 0\\ \gamma_1\to 0\\ \lambda_1\to 0}} u_1^{\rho,\gamma_1,\lambda_1} = \frac{1}{\gamma} \sum_{i=1}^{p} G(z_i, \cdot) \quad \text{in } \mathcal{C}^{4,\alpha}_{\text{loc}}(\Omega \setminus \{z_1, \dots, z_p\}),$$

$$\lim_{\substack{\rho\to 0\\ \gamma_2\to 0\\ \lambda_2\to 0}} u_2^{\rho,\gamma_2,\lambda_2} = \frac{1}{\xi} \sum_{i=p+1}^{m} G(z_i, \cdot) \quad \text{in } \mathcal{C}^{4,\alpha}_{\text{loc}}(\Omega \setminus \{z_{p+1}, \dots, z_m\}).$$

To facilitate the presentation, we will look at the special case where we have only two singular points.

Theorem 1.6. *Let Ω be a regular bounded domain of $\mathbb{R}^4$, $\varepsilon > 0$ satisfying* (A1)–(A2) *and $z_1, z_2 \in \Omega$ be given distinct points. Suppose that (z_1, z_2) is a nondegenerate critical point of the function*

$$\mathcal{F}(z_1, z_2) = \frac{1-\xi}{2\gamma} H(z_1, z_1) + \frac{1-\gamma}{2\xi} H(z_2, z_2) + \frac{1-\gamma}{\xi} \frac{1-\xi}{\gamma} G(z_1, z_2),$$

then there exist γ_0 and ξ_0 in $(0,1)$ such that for all $\gamma \in (\gamma_0, 1)$ and $\xi \in (\xi_0, 1)$, there exist $\rho_0 > 0$ and $\left(u_1^{\rho,\gamma_1,\lambda_1}, u_2^{\rho,\gamma_2,\lambda_2}\right)_{0<\rho\le\rho_0, 0<\gamma^i\le\gamma_0, 0<\lambda^i\le\lambda_0}$, $i = 1, 2$, a one parameter family of solutions of (1.3), *such that*

$$\lim_{\substack{\rho\to 0\\ \gamma_1\to 0\\ \lambda_1\to 0}} u_1^{\rho,\gamma_1,\lambda_1} = \frac{1}{\gamma} G(z_1, \cdot) \quad \text{in } \mathcal{C}^{4,\alpha}_{\text{loc}}(\Omega \setminus \{z_1\}),$$

$$\lim_{\substack{\rho\to 0\\ \gamma_2\to 0\\ \lambda_2\to 0}} u_2^{\rho,\gamma_2,\lambda_2} = \frac{1}{\xi} G(z_2, \cdot) \quad \text{in } \mathcal{C}^{4,\alpha}_{\text{loc}}(\Omega \setminus \{z_2\}).$$

Note that Theorem 1.5 is a generalisation of Theorem 1.6. We will only prove Theorem 1.6, where we have only two singular points.

We now briefly describe the plan of the paper. In Section 2 we prove Theorem 1.6, motivated by the techniques of Baraket et al. [5]. In Subsection 2.1, we introduce and recall some weighted Hölder spaces, the linearized operators and the harmonic extensions which are crucial in the following sections and moreover we compute the first approximate solution of (1.3) in a large ball using the appropriate transformation. We also recall some known results about the biLaplace operator in weighted spaces. In Subsection 2.2, we study a nonlinear interior problem for which we prove the existence of an infinite dimensional family of solutions of (1.3) defined on a large ball and close to the first approximation of the solution. In Subsection 2.3, we prove the existence of an infinite dimensional family of solutions of the exterior problem of (1.3), i.e., far from of the singularities with arbitrary data on the edges. Finally in Subsection 2.4, with a suitable choice of these data, we gather the solutions obtained in the previous sections via a nonlinear version of the Cauchy data matching in order to find global solutions of (1.3) on the whole domain Ω.

2. Proof of Theorem 1.6

2.1. Construction of the approximate solution

We denote by ε the smallest positive parameter satisfying

$$\rho^4 = \frac{384\varepsilon^4}{(1+\varepsilon^2)^4}.$$

Let

$$u_\varepsilon(z) = 4\ln\frac{1+\varepsilon^2}{\varepsilon^2+|z|^2},$$

which is a solution of

$$\Delta^2 u = \rho^4 e^u \quad \text{in } \mathbb{R}^4. \tag{2.1}$$

Hence for all $\tau > 0$, the function

$$u_{\varepsilon,\tau}(z) = 4\ln\frac{\tau(1+\varepsilon^2)}{\varepsilon^2+|\tau z|^2} \tag{2.2}$$

is also solution of (2.1).

2.1.1. Some results on the operators to be inverted and their appropriate spaces

First we introduce some definitions and notations of the appropriate functional spaces which are weighted Hölder spaces that we will need in order to invert the linearized operator $\mathbb{L}$ defined below.

Definition 2.1. Given $k \in \mathbb{N}$, $\alpha \in (0,1)$, $\mu \in \mathbb{R}$ and $|z| = r$, the weighted Hölder space $\mathcal{C}_\mu^{k,\alpha}(\mathbb{R}^4)$ is defined as the space of all functions $w \in \mathcal{C}_{\mathrm{loc}}^{k,\alpha}(\mathbb{R}^4)$ for which the norm

$$\|w\|_{\mathcal{C}_\mu^{k,\alpha}(\mathbb{R}^4)} := \|w\|_{\mathcal{C}^{k,\alpha}(\overline{B}_1(0))} + \sup_{r\geq 1}\left((1+r^2)^{-\mu/2}\|w(r\,\cdot\,)\|_{\mathcal{C}^{k,\alpha}(\overline{B}_1(0)\setminus B_{1/2}(0))}\right)$$

is finite.

Definition 2.2. Given $\overline{r} \geq 1$, $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\mu \in \mathbb{R}$, the weighted Hölder space is defined as the space of all functions $w \in \mathcal{C}_{\mathrm{loc}}^{k,\alpha}(\mathbb{R}^4)$ for which the norm

$$\|w\|_{\mathcal{C}_\mu^{k,\alpha}(B_{\overline{r}})} := \|w\|_{\mathcal{C}^{k,\alpha}(B_1(0))} + \sup_{1\leq r\leq \overline{r}}\left(r^{-\mu}\|w(r\,\cdot\,)\|_{\mathcal{C}^{k,\alpha}(\overline{B}_1(0)\setminus B_{1/2}(0))}\right)$$

is finite.

Definition 2.3. We set $\overline{B}_1^* = \overline{B}_1 \setminus \{0\}$, given $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\mu \in \mathbb{R}$, the weighted Hölder space $\mathcal{C}_\mu^{k,\alpha}(\overline{B}_1^*)$ is defined as the space of all functions $w \in \mathcal{C}_{\mathrm{loc}}^{k,\alpha}(\overline{B}_1^*)$ for which the norm

$$\|w\|_{\mathcal{C}_\mu^{k,\alpha}(\overline{B}_1^*(0))} := \sup_{r\leq 1/2}\left(r^{-\mu}\|w(r\,\cdot\,)\|_{\mathcal{C}^{k,\alpha}(\overline{B}_2(0)\setminus B_1(0))}\right)$$

is finite.

We define the linear fourth order elliptic operator $\mathbb{L}$ by

$$\mathbb{L} := \Delta^2 - \frac{384}{(1+r^2)^4},$$

which corresponds to the linearization of (2.1) about the radial symmetric solution $u_{\varepsilon=1,\tau=1}$ defined by (2.2). When $k > 2$, we let $[\mathcal{C}_\mu^{k,\alpha}(\overline{\Omega})]_0$ to be the subspace of functions $w \in \mathcal{C}_\mu^{k,\alpha}(\overline{\Omega})$ satisfying $\Delta w = w = 0$ on $\partial\Omega$.

Proposition 2.4. [5] *All bounded solution of $\mathbb{L}w = 0$ on $\mathbb{R}^4$ are linear combinations of*

$$\phi_0(z) = 4\frac{1-|z|^2}{1+|z|^2} \quad \textit{and} \quad \phi_i(z) = \frac{8z_i}{1+|z|^2} \quad \textit{for } i = 1,\dots,4.$$

Moreover, for $\mu > 1$, $\mu \notin \mathbb{Z}$, then $\mathbb{L}\colon \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \to \mathcal{C}_{\mu-4}^{0,\alpha}(\mathbb{R}^4)$ is surjective.

In the following, we denote a right inverse of $\mathbb{L}$ by $\mathcal{G}_\mu$. Similarly, using the fact that any bounded bi-harmonic solution on $\mathbb{R}^4$ is constant, we claim

Proposition 2.5. [5] *Let $\delta > 0$, $\delta \notin \mathbb{Z}$. Then Δ^2 is surjective from $\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)$ to $\mathcal{C}_{\delta-4}^{0,\alpha}(\mathbb{R}^4)$.*

We denote by $\mathcal{K}_\delta\colon \mathcal{C}_{\delta-4}^{0,\alpha}(\mathbb{R}^4) \to \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)$ a right inverse of Δ^2 for $\delta > 0$, $\delta \notin \mathbb{Z}$.

Finally, we consider punctured domains. Given $\widetilde{z}_1 \neq \widetilde{z}_2 \in \Omega$, we define $\widetilde{\mathbf{z}} := (\widetilde{z}_1, \widetilde{z}_2)$ and $\overline{\Omega}^*(\widetilde{\mathbf{z}}) := \overline{\Omega} \setminus \{\widetilde{z}_1, \widetilde{z}_2\}$. Let $r_0 > 0$ be small such that $\overline{B}_{r_0}(\widetilde{z}_i)$ are disjoint and contained in Ω. For all $r \in (0, r_0)$, we define

$$\overline{\Omega}_r(\widetilde{\mathbf{z}}) := \overline{\Omega} \setminus \bigcup_{i=1}^{2} B_r(\widetilde{z}_i).$$

Definition 2.6. Let $k \in \mathbb{R}$, $\alpha \in (0,1)$ and $\nu \in \mathbb{R}$ the weighted Hölder space $\mathcal{C}^{k,\alpha}_{\nu}(\overline{\Omega}^*(\tilde{z}))$ is defined as the space of all functions $w \in \mathcal{C}^{k,\alpha}_{\mathrm{loc}}(\overline{\Omega}^*(\tilde{z}))$ for which the norm

$$\|w\|_{\mathcal{C}^{k,\alpha}_{\nu}(\overline{\Omega}^*(\tilde{z}))} := \|w\|_{\mathcal{C}^{k,\alpha}(\overline{\Omega}_{r_0/2}(\tilde{z}))} + \sum_{i=1}^{2} \sup_{0<r\le r_0/2} \big(r^{-\nu}\|w(\tilde{z}_i + r\,\cdot)\|_{\mathcal{C}^{k,\alpha}(\overline{B}_2(0)\setminus B_1(0))}\big)$$

is finite. Furthermore, for $k \ge 2$, we denote by $\big[\mathcal{C}^{k,\alpha}_{\nu}(\overline{\Omega}^*(\tilde{\mathbf{z}}))\big]_0$ the space of all functions $w \in \mathcal{C}^{k,\alpha}_{\nu}(\overline{\Omega}^*(\tilde{\mathbf{z}}))$ satisfying $\Delta w = w = 0$ on $\partial\Omega$.

We recall the following result.

Proposition 2.7. [5] *Let $\nu < 0$, $\nu \notin \mathbb{Z}$. Then Δ^2 is surjective from $\big[\mathcal{C}^{4,\alpha}_{\nu}(\overline{\Omega}^*(\tilde{\mathbf{z}}))\big]_0$ to $\mathcal{C}^{0,\alpha}_{\nu-4}(\overline{\Omega}^*(\tilde{\mathbf{z}}))$.*

We denote by $\widetilde{\mathcal{K}}_{\nu} \colon \mathcal{C}^{0,\alpha}_{\nu-4}(\overline{\Omega}^*(\tilde{\mathbf{z}})) \to \big[\mathcal{C}^{4,\alpha}_{\nu}(\overline{\Omega}^*(\tilde{\mathbf{z}}))\big]_0$ a right inverse of Δ^2 for $\nu < 0$, $\nu \notin \mathbb{Z}$.

Before proceeding to the different stages of the construction of the solution to the problem 1.3, we will briefly explain the purpose of the technique followed: First, before building an approximate solution for the problem, we perform a suitable transformation in order to work in a larger ball; then, we will build an exact solution of our problem inside small balls centered in singularities with arbitrary data on the edge of each small ball, in order to give a certain degree of freedom to the edge data. Next, we build an exact solutions of the problem outside the balls with always arbitrary data on the edge of the balls and finally, with a suitable choice of these data at the edges, we gather interior and exterior solutions in order to obtain a global solution on the whole domain Ω.

2.1.2. Ansatz and first estimates

For all $\sigma \ge 1$, we denote by $\xi_{\mu,\sigma} \colon \mathcal{C}^{0,\alpha}_{\mu}(\overline{B}_{\sigma}(0)) \to \mathcal{C}^{0,\alpha}_{\mu}(\mathbb{R}^4)$ the extension operator defined by

$$\begin{cases} \xi_{\mu,\sigma}(f)(z) = f(z) & \text{for } |z| \le \sigma, \\ \xi_{\mu,\sigma}(f)(z) = \chi\big(\frac{|z|}{\sigma}\big) f\big(\sigma\frac{z}{|z|}\big) & \text{for } |z| \ge \sigma. \end{cases} \tag{2.3}$$

Here χ is a cut-off function over $\mathbb{R}_+$, which is equal to 1 for $t \le 1$ and equal to 0 for $t \ge 2$.

It is easy to check that there exists a constant $\bar{c} := \bar{c}(\mu) > 0$ independent of σ, such that

$$\|\xi_{\mu,\sigma}(w)\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathbb{R}^4)} \le \bar{c}\|w\|_{\mathcal{C}^{0,\alpha}_{\mu}(\overline{B}_{\sigma}(0))}. \tag{2.4}$$

For all $\gamma_i, \lambda_i, \varepsilon, \tau > 0$ and $\gamma, \xi \in (0,1)$, we define

$$r_\varepsilon := r_{\varepsilon,\gamma_i,\lambda_i} := \max\big(\gamma_i^{1/2}, \lambda_i^{1/2}, \varepsilon^{1/2}, \varepsilon^{\frac{\gamma+\xi-1}{\gamma}}, \varepsilon^{\frac{\gamma+\xi-1}{\xi}}\big) \text{ and } R_\varepsilon := R_{\varepsilon,\gamma_i,\lambda_i} := \frac{\tau r_\varepsilon}{\varepsilon}, \quad i = 1, 2.$$

Here, we are interested in the study of the system (1.3) near $B_{r_\varepsilon}(z_1)$, namely

$$\begin{aligned} &\Delta^2 u_1 - \gamma_1 \Delta u_1 - \lambda_1 |\nabla u_1|^q = \rho^4 e^{\gamma u_1 + (1-\gamma) u_2} && \text{in } B_{r_\varepsilon}(z_1), \\ &\Delta^2 u_2 - \gamma_2 \Delta u_2 - \lambda_2 |\nabla u_2|^q = \rho^4 e^{\xi u_2 + (1-\xi) u_1} && \text{in } B_{r_\varepsilon}(z_1). \end{aligned} \tag{2.5}$$

Using the following transformation

$$v_1(z) = u_1\left(\frac{\varepsilon}{\tau} z\right) + \frac{8}{\gamma}\ln\varepsilon - \frac{4}{\gamma}\ln\frac{\tau(1+\varepsilon^2)}{2} \quad \text{and} \quad v_2(z) = u_2\left(\frac{\varepsilon}{\tau} z\right),$$

the previous system can be written as

(2.6)

$$\begin{aligned} &\Delta^2 v_1 - \gamma_1\left(\frac{\varepsilon}{\tau}\right)^2 \Delta v_1 - \lambda_1\left(\frac{\varepsilon}{\tau}\right)^{4-q} |\nabla v_1|^q = 24 e^{\gamma v_1 + (1-\gamma)v_2} && \text{in } B_{R_\varepsilon}(z_1), \\ &\Delta^2 v_2 - \gamma_2\left(\frac{\varepsilon}{\tau}\right)^2 \Delta v_2 - \lambda_2\left(\frac{\varepsilon}{\tau}\right)^{4-q} |\nabla v_2|^q = 24 \frac{2^{4(\frac{\gamma+\xi-1}{\gamma})}\varepsilon^{8(\frac{\gamma+\xi-1}{\gamma})}}{\left(\tau(1+\varepsilon^2)\right)^{4(\frac{\gamma+\xi-1}{\gamma})}} e^{\xi v_2 + (1-\xi) v_1} && \text{in } B_{R_\varepsilon}(z_1). \end{aligned}$$

Here $\tau > 0$ is a constant which will be fixed later. We denote by $\overline{u} = u_{\varepsilon=\tau=1}$, we look for a solution of (2.6) of the form

$$\begin{aligned} v_1(z) &= \frac{1}{\gamma}\overline{u}(z-z_1) - \frac{1-\gamma}{\gamma\xi} G(z,z_2) - \frac{\ln\gamma}{\gamma} + h_1^1(z), \\ v_2(z) &= \frac{1}{\xi} G(z,z_2) + h_2^1(z), \end{aligned}$$

this amounts to solve the system

$$\begin{aligned} \mathbb{L} h_1^1 &= \frac{384}{\gamma(1+r^2)^4}\left[e^{\gamma h_1^1 + (1-\gamma) h_2^1} - \gamma h_1^1 - 1\right] \\ &\quad + \gamma_1\left(\frac{\varepsilon}{\tau}\right)^2 \Delta\left(\frac{1}{\gamma}\overline{u} - \frac{1-\gamma}{\gamma\xi} G(z,z_2) + h_1^1\right) \\ &\quad + \lambda_1\left(\frac{\varepsilon}{\tau}\right)^{4-q}\left|\nabla\left(\frac{1}{\gamma}\overline{u} - \frac{1-\gamma}{\gamma\xi} G(z,z_2) + h_1^1\right)\right|^q, \\ \Delta^2 h_2^1 &= \frac{384 C_\varepsilon \varepsilon^{8(\frac{\gamma+\xi-1}{\gamma})}}{(1+r^2)^{4\frac{(1-\xi)}{\gamma}}} e^{\xi h_2^1 + \left[1 - \frac{(1-\xi)(1-\gamma)}{\gamma\xi}\right] G(z,z_2) + (1-\xi)\left(-\frac{\ln\gamma}{\gamma} + h_1^1\right)} \\ &\quad + \gamma_2\left(\frac{\varepsilon}{\tau}\right)^2 \Delta\left(\frac{1}{\xi} G(z,z_2) + h_2^1\right) + \lambda_2\left(\frac{\varepsilon}{\tau}\right)^{4-q}\left|\nabla\left(\frac{1}{\xi} G(z,z_2) + h_2^1\right)\right|^q \end{aligned} \tag{2.7}$$

in $B_{R_\varepsilon}(z_1)$, where $r = |z - z_1|$ and $C_\varepsilon = [\tau(1+\varepsilon^2)]^{4(\frac{1-\gamma-\xi}{\gamma})}$.

For $q \in [1,4)$, we fix $\mu \in (1, 5-q)$ and $\delta \in \left(0, \min\left\{1, 4-q, \frac{\gamma+\xi-1}{\gamma}, \frac{\gamma+\xi-1}{\xi}\right\}\right)$. To find a solution of (2.7), it is enough to find a fixed point (h_1^1, h_2^1) in a small ball of $\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)$ which solves the system

$$h_1^1 = \mathcal{G}_\mu \circ \xi_{\mu,R_\varepsilon} \circ \mathcal{T}_1(h_1^1, h_2^1), \quad h_2^1 = \mathcal{K}_\delta \circ \xi_{\delta,R_\varepsilon} \circ \mathcal{T}_2(h_1^1, h_2^1), \tag{2.8}$$

where

$$\mathcal{T}_1(h_1^1, h_2^1) = \frac{384}{\gamma(1+r^2)^4}\left[e^{\gamma h_1^1+(1-\gamma)h_2^1} - \gamma h_1^1 - 1\right] + \gamma_1\left(\frac{\varepsilon}{\tau}\right)^2 \Delta\left(\frac{1}{\gamma}\overline{u} - \frac{1-\gamma}{\gamma\xi}G(z,z_2) + h_1^1\right) + \lambda_1\left(\frac{\varepsilon}{\tau}\right)^{4-q}\left|\nabla\left(\frac{1}{\gamma}\overline{u} - \frac{1-\gamma}{\gamma\xi}G(z,z_2) + h_1^1\right)\right|^q,$$

$$\mathcal{T}_2(h_1^1, h_2^1) = \frac{384 C_\varepsilon \varepsilon^{8(\frac{\gamma+\xi-1}{\gamma})}}{(1+r^2)^{4\frac{(1-\xi)}{\gamma}}} e^{\xi h_2^1 + \left[1-\frac{(1-\xi)(1-\gamma)}{\gamma\xi}\right]G(z,z_2) + (1-\xi)\left(-\frac{\ln\gamma}{\gamma} + h_1^1\right)} + \gamma_2\left(\frac{\varepsilon}{\tau}\right)^2 \Delta\left(\frac{1}{\xi}G(z,z_2) + h_2^1\right) + \lambda_2\left(\frac{\varepsilon}{\tau}\right)^{4-q}\left|\nabla\left(\frac{1}{\xi}G(z,z_2) + h_2^1\right)\right|^q.$$

We denote by $\mathcal{N}$ $(= \mathcal{N}_{\varepsilon,\tau})$ and $\mathcal{M}$ $(= \mathcal{M}_{\varepsilon,\tau})$ the nonlinear operators appearing on the right-hand side of equation (2.8).

Lemma 2.8. *Given $\kappa > 0$, there exist $\varepsilon_\kappa > 0$, $c_\kappa > 0$ and $\gamma_0 \in (0,1)$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\gamma \in (\gamma_0, 1)$, $\mu \in (1, 5-q)$ and $\delta \in \left(0, \min\left\{1, 4-q, \frac{\gamma+\xi-1}{\gamma}, \frac{\gamma+\xi-1}{\xi}\right\}\right)$,*

$$\|\mathcal{N}(0,0)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2, \quad \|\mathcal{M}(0,0)\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2,$$

$$\|\mathcal{N}(h_1^1, h_2^1) - \mathcal{N}(k_1^1, k_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2 \|h_1^1 - k_1^1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} + c_\kappa(1-\gamma)\|h_2^1 - k_2^1\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}$$

and

$$\|\mathcal{M}(h_1^1, h_2^1) - \mathcal{M}(k_1^1, k_2^1)\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2 \|(h_1^1, h_2^1) - (k_1^1, k_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)},$$

where $\|(h_1^1, h_2^1) - (k_1^1, k_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} = \|h_1^1 - k_1^1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} + \|h_2^1 - k_2^1\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}$ for all $(h_1^1, h_2^1), (k_1^1, k_2^1) \in \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4) \times \mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)$ satisfying

$$\|(h_1^1, h_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2, \quad \|(k_1^1, k_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2. \tag{2.9}$$

Proof. We have

$$\begin{aligned}
\sup_{r\le R_\varepsilon} r^{4-\mu}|\mathcal{T}_1(0,0)| &\le c_\kappa\gamma_1\left(\frac{\varepsilon}{\tau}\right)^2 \sup_{r\le R_\varepsilon} r^{4-\mu}\left|\Delta\left(\frac{1}{\gamma}\overline{u} + \frac{1-\gamma}{\gamma\xi}G(z,z_2)\right)\right| \\
&\quad + c_\kappa\lambda_1\left(\frac{\varepsilon}{\tau}\right)^{4-q} \sup_{r\le R_\varepsilon} r^{4-\mu}\left|\nabla\left(\frac{1}{\gamma}\overline{u} - \frac{1-\gamma}{\gamma\xi}G(z,z_2)\right)\right|^q \\
&\le c_\kappa\gamma_1\left(\frac{\varepsilon}{\tau}\right)^2 \sup_{r\le R_\varepsilon} r^{4-\mu}\frac{2+r^2}{(1+r^2)^2} + c_\kappa\gamma_1\left(\frac{\varepsilon}{\tau}\right)^2 \sup_{r\le R_\varepsilon} r^{4-\mu}r^{-2} \\
&\quad + c_\kappa\lambda_1\left(\frac{\varepsilon}{\tau}\right)^{4-q} \sup_{r\le R_\varepsilon} r^{4-\mu}\frac{r^q}{(1+r^2)^q} + c_\kappa\lambda_1\left(\frac{\varepsilon}{\tau}\right)^{4-q} \sup_{r\le R_\varepsilon} r^{4-\mu}r^{-q} \\
&\le c_\kappa\gamma_1\left(\frac{\varepsilon}{\tau}\right)^2 \sup_{r\le R_\varepsilon} r^{2-\mu} + c_\kappa\gamma_1\left(\frac{\varepsilon}{\tau}\right)^2 \sup_{r\le R_\varepsilon} r^{2-\mu} \\
&\quad + c_\kappa\lambda_1\left(\frac{\varepsilon}{\tau}\right)^{4-q} \sup_{r\le R_\varepsilon} r^{4-\mu}\frac{r^q}{(1+r^2)^q} + c_\kappa\lambda_1\left(\frac{\varepsilon}{\tau}\right)^{4-q} \sup_{r\le R_\varepsilon} r^{4-\mu-q}.
\end{aligned}$$

Taking into account that for r very large we have $(1+r^2)^{-\beta} \sim r^{-2\beta}$, we obtain

$$\begin{aligned}\sup_{r\le R_\varepsilon} r^{4-\mu}|\mathcal{T}_1(0,0)| &\le 2\gamma_1\varepsilon^2 \sup_{r\le R_\varepsilon} r^{2-\mu} + 2\lambda_1\varepsilon^{4-q}\sup_{r\le R_\varepsilon} r^{4-\mu-q}\\ &\le c_\kappa\gamma_1\max\{\varepsilon^\mu r_\varepsilon^{2-\mu},\varepsilon^2\} + c_\kappa\lambda_1\max\{\varepsilon^\mu r_\varepsilon^{4-\mu-q},\varepsilon^{4-q}\}\\ &\le c_\kappa r_\varepsilon^2.\end{aligned}$$

Making use of Proposition 2.4 together with (2.4), for $\mu \in (1, 5-q)$, we get that there exists c_0 such that

$$\|\mathcal{N}(0,0)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2 \tag{2.10}$$

for the second estimate, we have

$$\begin{aligned}&\sup_{r\le R_\varepsilon} r^{4-\delta}|\mathcal{T}_2(0,0)|\\ &= \sup_{r\le R_\varepsilon} r^{4-\delta}\frac{384C_\varepsilon\varepsilon^{8(\frac{\gamma+\xi-1}{\gamma})}}{(1+r^2)^{4\frac{(1-\xi)}{\gamma}}} e^{\left(1-\frac{(1-\xi)(1-\gamma)}{\gamma\xi}\right)G(z,z_2)-(1-\xi)\frac{\ln\gamma}{\gamma}}\\ &\quad + \gamma_2\left(\frac{\varepsilon}{\tau}\right)^2\sup_{r\le R_\varepsilon} r^{4-\delta}\left|\Delta\left(\frac{1}{\xi}G(z,z_2)\right)\right| + \lambda_2\left(\frac{\varepsilon}{\tau}\right)^{4-q}\sup_{r\le R_\varepsilon} r^{4-\delta}\left|\nabla\left(\frac{1}{\xi}G(z,z_2)\right)\right|^q\\ &\le c_\kappa\sup_{r\le R_\varepsilon} 384C_\varepsilon\varepsilon^{8(\frac{\gamma+\xi-1}{\gamma})}S(r) + c_\kappa\gamma_2\left(\frac{\varepsilon}{\tau}\right)^2\sup_{r\le R_\varepsilon} r^{2-\delta} + c_\kappa\lambda_2\left(\frac{\varepsilon}{\tau}\right)^{4-q}\sup_{r\le R_\varepsilon} r^{4-\delta-q},\end{aligned}$$

where $S(r) = \frac{r^{4-\delta}}{(1+r^2)^{4\frac{(1-\xi)}{\gamma}}}$.

If $4-\delta+8(\xi-1)\gamma^{-1} \le 0$, then S is bounded on $\mathbb{R}_+$. If $4-\delta+8(\xi-1)\gamma^{-1} > 0$, then $\sup_{[0,r_\varepsilon/\varepsilon)} S(r) = S(r_\varepsilon/\varepsilon)$. Using the same argument as above, we get

$$\begin{aligned}&\|\mathcal{M}(0,0)\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}\\ &\le c_\kappa\left(\max\left\{\varepsilon^{8(\frac{\gamma+\xi-1}{\gamma})},\varepsilon^{4+\delta}r_\varepsilon^{4-\delta+8(\xi-1)\gamma^{-1}}\right\} + \gamma_2\varepsilon^\delta r_\varepsilon^{2-\delta} + \lambda_2\varepsilon^\delta r_\varepsilon^{4-\delta-q}\right) \le c_\kappa r_\varepsilon^2.\end{aligned} \tag{2.11}$$

Now, we recall an important result that plays a center role in our estimates see for example [23] and references therein:

Lemma 2.9. *Given x and y two reals numbers, $x > 0$, $q \ge 1$ and for all small $\eta > 0$, there exists a positive constant C_η such that*

$$\big||x+y|^q - x^q\big| \le (1+\eta)qx^{q-1}|y| + C_\eta|y|^q.$$

Recall the following conditions:

(A1) if $0 < \varepsilon < \sigma_{\gamma,\lambda}$, then $\sigma_{\gamma,\lambda}^{1+\mu/2}\varepsilon^{-\mu} \to 0$ as $\sigma_{\gamma,\lambda} \to 0$ for any $\mu \in (1, 5-q)$, $1 \le q < 4$.

(A2) if $0 < \varepsilon < \sigma_{\gamma,\lambda}$, then $\sigma_{\gamma,\lambda}^{1+\delta/2}\varepsilon^{-\delta} \to 0$ as $\sigma_{\gamma,\lambda} \to 0$ for any $\delta \in \big(0, \min\big\{1, 4-q, \frac{\gamma+\xi-1}{\gamma}, \frac{\gamma+\xi-1}{\xi}\big\}\big)$, $1 \le q < 4$.

To derive the third estimate, for h_i, k_i, $i = 1, 2$, verifying (2.9), it is necessary to consider condition (A1). Then, we have

$$
\begin{aligned}
&\sup_{r\le R_\varepsilon} r^{4-\mu}|\mathcal{T}_1(h_1^1, h_2^1) - \mathcal{T}_1(k_1^1, k_2^1)| \\
&\le c_\kappa \sup_{r\le R_\varepsilon} \frac{384 r^{4-\mu}}{\gamma(1+r^2)^4}\big|\big(e^{\gamma h_1^1 + (1-\gamma)h_2^1} - \gamma h_1^1 - 1\big) - \big(e^{\gamma k_1^1 + (1-\gamma)k_2^1} - \gamma k_1^1 - 1\big)\big| \\
&\quad + c_\kappa \gamma_1 \varepsilon^2 \sup_{r\le R_\varepsilon} r^{4-\mu}\left(\left|\Delta\left(\frac{1}{\gamma}\overline{u} - \frac{1-\gamma}{\gamma\xi}G(z,z_2) + h_1^1\right) - \Delta\left(\frac{1}{\gamma}\overline{u} - \frac{1-\gamma}{\gamma\xi}G(z,z_2) + k_1^1\right)\right|\right) \\
&\quad + c_\kappa \lambda_1 \varepsilon^{4-q} \sup_{r\le R_\varepsilon} r^{4-\mu}\left(\left|\nabla\left(\frac{1}{\gamma}\overline{u} - \frac{1-\gamma}{\gamma\xi}G(z,z_2) + h_1^1\right)\right|^q - \left|\nabla\left(\frac{1}{\gamma}\overline{u} - \frac{1-\gamma}{\gamma\xi}G(z,z_2) + k_1^1\right)\right|^q\right) \\
&\le c_\kappa \sup_{r\le R_\varepsilon} \frac{384 r^{4-\mu}}{\gamma(1+r^2)^4}\big(\gamma^2|(h_1^1)^2 - (k_1^1)^2| + (1-\gamma)|h_2^1 - k_2^1|\big) + c_\kappa\gamma_1\varepsilon^2 \sup_{r\le R_\varepsilon} r^{4-\mu}|\Delta(h_1^1 - k_1^1)| \\
&\quad + c_\kappa \lambda_1 \varepsilon^{4-q} \sup_{r\le R_\varepsilon} r^{4-\mu}\Bigg(\left|\nabla\left(\frac{1}{\gamma}\overline{u} - \frac{1-\gamma}{\gamma\xi}G(z,z_2) + k_1^1 + h_1^1 - k_1^1\right)\right|^q \\
&\qquad\qquad - \left|\nabla\left(\frac{1}{\gamma}\overline{u} - \frac{1-\gamma}{\gamma\xi}G(z,z_2) + k_1^1\right)\right|^q\Bigg).
\end{aligned}
$$

Again, making use of Lemma 2.9 and recall that a functions w in $\mathcal{C}_\mu^{k,\alpha}(\mathbb{R}^4)$ are bounded by a constant times $(1+r^2)^{\mu/2}$ and have their ℓ-th partial derivatives that are bounded by $(1+r^2)^{(\mu-\ell)/2}$, for $\ell = 1, \ldots, k+\alpha$. (a.e. $|\nabla^\ell w| \le c_\kappa r^{\mu-\ell}\|w\|_{\mathcal{C}_\mu^{k,\alpha}(\mathbb{R}^4)}$, $(1+r^2)^{(\mu-\ell)/2} \sim r^{\mu-\ell}$ for r very large) and finally recalling that

$$
\|(h_1^1, h_2^1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2, \quad \|(k_1^1, k_2^1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2,
$$

we deduce that

$$
\begin{aligned}
&\sup_{r\le R_\varepsilon} r^{4-\mu}|\mathcal{T}_1(h_1^1, h_2^1) - \mathcal{T}_1(k_1^1, k_2^1)| \\
&\le c_\kappa \sup_{r\le R_\varepsilon} r^{4-\mu}\big(\gamma^2 r^{2\mu}\big(\|h_1^1\|_{\mathcal{C}_\mu^{4,\alpha}} + \|k_1^1\|_{\mathcal{C}_\mu^{4,\alpha}}\big)\|h_1^1 - k_1^1\|_{\mathcal{C}_\mu^{4,\alpha}} + (1-\gamma)r^\delta\|h_2^1 - k_2^1\|_{\mathcal{C}_\delta^{4,\alpha}}\big) \\
&\quad + c_\kappa\gamma_1\varepsilon^2 \sup_{r\le R_\varepsilon} r^{4-\mu} r^{\mu-2}\|h_1^1 - k_1^1\|_{\mathcal{C}_\mu^{4,\alpha}} \\
&\quad + c_\kappa\lambda_1\varepsilon^{4-q} \sup_{r\le R_\varepsilon} r^{4-\mu}\left(\left|\nabla\left(\frac{1}{\gamma}\overline{u} - \frac{1-\gamma}{\gamma\xi}G(z,z_2) + k_1^1\right)\right|^{q-1} + |\nabla(h_1^1 - k_1^1)|^{q-1}\right)|\nabla(h_1^1 - k_1^1)| \\
&\le c_\kappa\big(\gamma^2\big(\|h_1^1\|_{\mathcal{C}_\mu^{4,\alpha}} + \|k_1^1\|_{\mathcal{C}_\mu^{4,\alpha}}\big)\|h_1^1 - k_1^1\|_{\mathcal{C}_\mu^{4,\alpha}} + (1-\gamma)\|h_2^1 - k_2^1\|_{\mathcal{C}_\delta^{4,\alpha}}\big) + c_\kappa\gamma_1\varepsilon^2 R_\varepsilon^2\|h_1^1 - k_1^1\|_{\mathcal{C}_\mu^{4,\alpha}} \\
&\quad + c_\kappa\lambda_1\varepsilon^{4-q} \sup_{r\le R_\varepsilon} r^{4-\mu}\big(|\nabla\overline{u}|^{q-1} + |\nabla G(z,z_2)|^{q-1} + |\nabla k_1^1|^{q-1} + |\nabla h_1^1|^{q-1}\big)|\nabla(h_1^1 - k_1^1)| \\
&\le c_\kappa\big(\gamma^2\big(\|h_1^1\|_{\mathcal{C}_\mu^{4,\alpha}} + \|k_1^1\|_{\mathcal{C}_\mu^{4,\alpha}}\big)\|h_1^1 - k_1^1\|_{\mathcal{C}_\mu^{4,\alpha}} + (1-\gamma)\|h_2^1 - k_2^1\|_{\mathcal{C}_\delta^{4,\alpha}}\big) + c_\kappa\gamma_1\varepsilon^2 R_\varepsilon^2\|h_1^1 - k_1^1\|_{\mathcal{C}_\mu^{4,\alpha}} \\
&\quad + c_\kappa\lambda_1\varepsilon^{4-q}\big(R_\varepsilon^{4-q} + R_\varepsilon^{4+\mu q-\mu-q}\big(\|k_1^1\|_{\mathcal{C}_\mu^{4,\alpha}}^{q-1} + \|h_1^1\|_{\mathcal{C}_\mu^{4,\alpha}}^{q-1}\big)\big)\|h_1^1 - k_1^1\|_{\mathcal{C}_\mu^{4,\alpha}} \\
&\le c_\kappa r_\varepsilon^2\|h_1^1 - k_1^1\|_{\mathcal{C}_\mu^{4,\alpha}} + c_\kappa(1-\gamma)\|h_2^1 - k_2^1\|_{\mathcal{C}_\delta^{4,\alpha}} + c_\kappa\gamma_1 r_\varepsilon^2\|h_1^1 - k_1^1\|_{\mathcal{C}_\mu^{4,\alpha}} \\
&\quad + c_\kappa\lambda_1 r_\varepsilon^{4-q}\big[1 + (r_\varepsilon^{2+\mu}\varepsilon^{-\mu})^{q-1}\big]\|h_1^1 - k_1^1\|_{\mathcal{C}_\mu^{4,\alpha}}.
\end{aligned}
$$

Note that in the estimates above we made use of $\nabla\overline{u}(r) \simeq r^{-1}$ and $|\nabla^i G(z,z')| \leq c|z-z'|^{-i}$ for $i \geq 1$. Moreover, the estimates

$$c_\kappa r_\varepsilon^{2+\mu}\varepsilon^{-\mu} \leq \begin{cases} c_\kappa \varepsilon^{1-\mu/2} & \text{for } \varepsilon \geq \max(\lambda,\gamma), \\ c_\kappa \lambda^{1+\mu/2}\varepsilon^{-\mu} & \text{for } \lambda > \max(\varepsilon,\gamma), \\ c_\kappa \gamma^{1+\mu/2}\varepsilon^{-\mu} & \text{for } \gamma > \max(\varepsilon,\lambda) \end{cases}$$

together with condition (A1), yield $\lambda_1 r_\varepsilon^{4-q}\big[1 + \big(r_\varepsilon^{2+\mu}\varepsilon^{-\mu}\big)^{q-1}\big] \leq r_\varepsilon^2$.

Making use of Proposition 2.4 together with (2.4) and using the condition (A1) for $\mu \in (1, 5-q)$, we conclude that

$$(2.12)\quad \|\mathcal{N}(h_1^1,h_2^1) - \mathcal{N}(k_1^1,k_2^1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa r_\varepsilon^2 \|h_1^1 - k_1^1\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} + c_\kappa(1-\gamma)\|h_2^1 - k_2^1\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)}.$$

Similarly we get the estimate for $\mathcal{M}$, then

$$\begin{aligned}
&\sup_{r\leq R_\varepsilon} r^{4-\delta}|\mathcal{T}_2(h_1^1,h_2^1) - \mathcal{T}_2(k_1^1,k_2^1)| \\
&\leq c_\kappa \sup_{r\leq R_\varepsilon} r^{4-\delta}\frac{384 C_\varepsilon \varepsilon^{8\frac{\gamma+\xi-1}{\gamma}}}{(1+r^2)^{4\frac{1-\xi}{\gamma}}} e^{\left(1-\frac{(1-\xi)(1-\gamma)}{\gamma\xi}\right)G(z,z_2)}\left|e^{\xi h_2^1 + (1-\xi)\left(-\frac{\ln\gamma}{\gamma}+h_1^1\right)} - e^{\xi k_2^1 + (1-\xi)\left(-\frac{\ln\gamma}{\gamma}+k_1^1\right)}\right| \\
&\quad + c_\kappa \gamma_2 \varepsilon^2 \sup_{r\leq R_\varepsilon} r^{4-\delta}|\Delta(h_2^1 - k_2^1)| \\
&\quad + c_\kappa \lambda_2 \varepsilon^{4-q} \sup_{r\leq R_\varepsilon} r^{4-\delta}\left(\left|\nabla\left(\frac{1}{\xi}G(z,z_2) + h_2^1\right)\right|^q - \left|\nabla\left(\frac{1}{\xi}G(z,z_2) + k_2^1\right)\right|^q\right) \\
&\leq c_\kappa 384 C_\varepsilon \varepsilon^{8\frac{\gamma+\xi-1}{\gamma}} \sup_{r\leq R_\varepsilon} \frac{r^{4-\delta}}{(1+r^2)^{4\frac{1-\xi}{\gamma}}}\left(\xi\|h_2^1 - k_2^1\|_{\mathcal{C}_\delta^{4,\alpha}} + (1-\xi)\|h_1^1 - k_1^1\|_{\mathcal{C}_\mu^{4,\alpha}}\right) \\
&\quad + c_\kappa \gamma_2 \varepsilon^2 R_\varepsilon^2 \|h_2^1 - k_2^1\|_{\mathcal{C}_\delta^{4,\alpha}} \\
&\quad + c_\kappa \lambda_2 \varepsilon^{4-q} \sup_{r\leq R_\varepsilon} r^{4-\delta}\big(|\nabla G(z,z_2)|^{q-1} + |\nabla h_2^1|^{q-1} + |\nabla k_2^1|^{q-1}\big)|\nabla(h_2^1 - k_2^1)| \\
&\leq c_\kappa 384 C_\varepsilon \varepsilon^{8\frac{\gamma+\xi-1}{\gamma}} \sup_{r\leq R_\varepsilon} \frac{r^{4-\delta}}{(1+r^2)^{4\frac{1-\xi}{\gamma}}}\left(\xi\|h_2^1 - k_2^1\|_{\mathcal{C}_\delta^{4,\alpha}} + (1-\xi)\|h_1^1 - k_1^1\|_{\mathcal{C}_\mu^{4,\alpha}}\right) \\
&\quad + c_\kappa \gamma_2 \varepsilon^2 R_\varepsilon^2 \|h_2^1 - k_2^1\|_{\mathcal{C}_\delta^{4,\alpha}} + c_\kappa \lambda_2 \varepsilon^{4-q}\big(R_\varepsilon^{4-q} + R_\varepsilon^{4+\delta q-\delta-q}\big(\|k_2^1\|_{\mathcal{C}_\delta^{4,\alpha}}^{q-1} + \|h_2^1\|_{\mathcal{C}_\delta^{4,\alpha}}^{q-1}\big)\big)\|h_2^1 - k_2^1\|_{\mathcal{C}_\delta^{4,\alpha}} \\
&\leq c_\kappa 384 C_\varepsilon \varepsilon^{8\frac{\gamma+\xi-1}{\gamma}} \sup_{r\leq R_\varepsilon} \frac{r^{4-\delta}}{(1+r^2)^{4\frac{1-\xi}{\gamma}}}\left(\xi\|h_2^1 - k_2^1\|_{\mathcal{C}_\delta^{4,\alpha}} + (1-\xi)\|h_1^1 - k_1^1\|_{\mathcal{C}_\mu^{4,\alpha}}\right) \\
&\quad + c_\kappa \gamma_2 r_\varepsilon^2 \|h_2^1 - k_2^1\|_{\mathcal{C}_\delta^{4,\alpha}} + c_\kappa \lambda_2 r_\varepsilon^{4-q}\big[1 + (r_\varepsilon^{2+\delta}\varepsilon^{-\delta})^{q-1}\big]\|h_2^1 - k_2^1\|_{\mathcal{C}_\delta^{4,\alpha}}.
\end{aligned}$$

Note that the estimates

$$c_\kappa r_\varepsilon^{2+\delta}\varepsilon^{-\delta} \leq \begin{cases} c_\kappa \varepsilon^{1-\delta/2} & \text{for } \varepsilon \geq \max(\lambda,\gamma), \\ c_\kappa \lambda^{1+\delta/2}\varepsilon^{-\delta} & \text{for } \lambda > \max(\varepsilon,\gamma), \\ c_\kappa \gamma^{1+\delta/2}\varepsilon^{-\delta} & \text{for } \gamma > \max(\varepsilon,\lambda) \end{cases}$$

together with condition (A2), yield $\lambda_2 r_\varepsilon^{4-q}\big[1 + \big(r_\varepsilon^{2+\delta}\varepsilon^{-\delta}\big)^{q-1}\big] \leq r_\varepsilon^2$.

We conclude that

$$\|\mathcal{M}(h_1^1,h_2^1)-\mathcal{M}(k_1^1,k_2^1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2 \|(h_1^1,h_2^1)-(k_1^1,k_2^1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)}. \tag{2.13}$$ □

Reducing ε_κ if necessary, we can assume that $\bar{c}_\kappa r_\varepsilon^2 \le 1/2$ for all $\varepsilon \in (0,\varepsilon_\kappa)$. There exists also $\gamma_0 \in (0,1)$ such that $c_\kappa(1-\gamma) \le 1/2$ for all $\gamma \in (\gamma_0,1)$. Therefore (2.10)–(2.13) are enough to show that

$$(h_1^1,h_2^1) \mapsto (\mathcal{N}(h_1^1,h_2^1),\mathcal{M}(h_1^1,h_2^1))$$

is a contraction from the ball

$$\left\{(h_1^1,h_2^1) \in \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4) : \|(h_1^1,h_2^1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2\right\}$$

into itself and hence a unique fixed point (h_1^1,h_2^1) exists in this set, which is a solution of (2.8). Hence we have shown the following proposition.

Proposition 2.10. *Given $\kappa>0$, there exist $\varepsilon_\kappa>0$, $c_\kappa>0$ and $\gamma_0\in(0,1)$ such that for all $\varepsilon\in(0,\varepsilon_\kappa)$, $\gamma\in(\gamma_0,1)$, there exists a unique $(h_1^1,h_2^1) := (h^1_{1,\varepsilon,\gamma_1,\lambda_1}, h^1_{2,\varepsilon,\gamma_2,\lambda_2})$ solution of* (2.8) *such that*

$$\|(h_1^1,h_2^1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2.$$

Hence

$$v_1(z) := \frac{1}{\gamma}\bar{u}(z-z_1) - \frac{1-\gamma}{\gamma\xi}G(z,z_2) - \frac{\ln\gamma}{\gamma} + h_1^1(z),$$
$$v_2(z) := \frac{1}{\xi}G(z,z_2) + h_2^1(z)$$

solve (2.5) *in* $B_{R_\varepsilon}(z_1)$.

Similarly, we get also

Proposition 2.11. *Given $\kappa>0$, there exist $\varepsilon_\kappa>0$, $c_\kappa>0$ and $\xi_0\in(0,1)$ such that for all $\varepsilon\in(0,\varepsilon_\kappa)$, $\xi\in(\xi_0,1)$, there exists a unique $(h_1^2,h_2^2) := (h^2_{1,\varepsilon,\gamma_1,\lambda_1}, h^2_{2,\varepsilon,\gamma_2,\lambda_2})$ solution of* (2.8) *such that*

$$\|(h_1^2,h_2^2)\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)\times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2.$$

Hence

$$v_1(z) := \frac{1}{\gamma}G(z,z_1) + h_1^2(z),$$
$$v_2(z) := \frac{1}{\xi}\bar{u}(z-z_2) - \frac{1-\xi}{\gamma\xi}G(z,z_1) - \frac{\ln\xi}{\xi} + h_2^2(z)$$

solve (2.5) *in* $B_{R_\varepsilon}(z_2)$.

2.1.3. Bi-harmonic extensions

Next, we will study the properties of interior and exterior bi-harmonic extensions. Given $(\varphi,\psi),(\widetilde{\varphi},\widetilde{\psi}) \in \mathcal{C}^{4,\alpha}(S^3)\times\mathcal{C}^{2,\alpha}(S^3)$, where S^3 is the three-dimensional unit sphere, we define respectively $H^{\text{int}} = H^{\text{int}}(\varphi,\psi;\cdot\,) = H^{\text{int}}_{\varphi,\psi}$ to be the solution of

$$\begin{cases} \Delta^2 H^{\text{int}} = 0 & \text{in } B_1(0), \\ H^{\text{int}} = \varphi & \text{on } \partial B_1(0), \\ \Delta H^{\text{int}} = \psi & \text{on } \partial B_1(0), \end{cases}$$

and $H^{\text{ext}} = H^{\text{ext}}(\widetilde{\varphi},\widetilde{\psi};\cdot\,) = H^{\text{ext}}_{\widetilde{\varphi},\widetilde{\psi}}$ to be the solution of

$$\begin{cases} \Delta^2 H^{\text{ext}} = 0 & \text{in } \mathbb{R}^4 \setminus B_1(0), \\ H^{\text{ext}} = \widetilde{\varphi} & \text{on } \partial B_1(0), \\ \Delta H^{\text{ext}} = \widetilde{\psi} & \text{on } \partial B_1(0), \end{cases}$$

which decays at infinity.

Definition 2.12. Given $k\in\mathbb{N}$, $\alpha\in(0,1)$ and $\nu\in\mathbb{R}$, we define the space $\mathcal{C}^{k,\alpha}_{\nu}(\mathbb{R}^4 - B_1(0))$ as the space of functions $w\in\mathcal{C}^{k,\alpha}_{\text{loc}}(\mathbb{R}^4 - B_1(0))$ for which the following norm

$$\|w\|_{\mathcal{C}^{k,\alpha}_{\nu}(\mathbb{R}^4 - B_1(0))} := \sup_{r\geq 1}\big(r^{-\nu}\|w(r\,\cdot\,)\|_{\mathcal{C}^{k,\alpha}_{\nu}(\overline{B}_2(0)-B_1(0))}\big)$$

is finite.

We denote by $e_1,\dots,e_4$ the coordinate functions on S^3.

Lemma 2.13. [3] *Assume that*

$$\int_{S^3}(8\varphi-\psi)\,dv_{S^3} = 0 \quad \textit{and} \quad \int_{S^3}(12\varphi-\psi)e_\ell\,dv_{S^3} = 0 \quad \textit{for } \ell = 1,\dots,4. \tag{2.14}$$

Then there exists $c>0$ such that

$$\|H^{\text{int}}_{\varphi,\psi}\|_{\mathcal{C}^{4,\alpha}_{2}(\overline{B}^*_1(0))} \leq c\big(\|\varphi\|_{\mathcal{C}^{4,\alpha}(S^3)} + \|\psi\|_{\mathcal{C}^{2,\alpha}(S^3)}\big).$$

Moreover there exists $c>0$ such that if

$$\int_{S^3}\widetilde{\psi}\,dv_{S^3} = 0, \tag{2.15}$$

then

$$\|H^{\text{ext}}_{\widetilde{\varphi},\widetilde{\psi}}\|_{\mathcal{C}^{4,\alpha}_{-1}(\mathbb{R}^4 - B_1(0))} \leq c\big(\|\widetilde{\varphi}\|_{\mathcal{C}^{4,\alpha}(S^3)} + \|\widetilde{\psi}\|_{\mathcal{C}^{2,\alpha}(S^3)}\big).$$

If $F \subset L^2(S^3)$ is a subspace of $L^2(S^3)$, we denote by $F^\perp$ the subspace of all elements which are orthogonal to $1, e_1, \dots, e_4$. We will need the following result.

Lemma 2.14. [3] *The mapping*

$$\begin{array}{rccc} \mathcal{P}: & \mathcal{C}^{4,\alpha}(S^3)^\perp \times \mathcal{C}^{2,\alpha}(S^3)^\perp & \longrightarrow & \mathcal{C}^{3,\alpha}(S^3)^\perp \times \mathcal{C}^{1,\alpha}(S^3)^\perp \\ & (\varphi, \psi) & \longmapsto & \big(\partial_r(H^{\text{int}}_{\varphi,\psi} - H^{\text{ext}}_{\varphi,\psi}), \partial_r(\Delta H^{\text{int}}_{\varphi,\psi} - \Delta H^{\text{ext}}_{\varphi,\psi})\big) \end{array}$$

is an isomorphism.

2.2. The nonlinear interior problem

Here, we look for a solution of the following system as in Section 2.1.2, we will just add the interior harmonic extension and the perturbation term v_i, $i = 1, 2$,

$$\Delta^2 v_1 - \gamma_1 \left(\frac{\varepsilon}{\tau}\right)^2 \Delta v_1 - \lambda_1 \left(\frac{\varepsilon}{\tau}\right)^{4-q} |\nabla v_1|^q = 24 e^{\gamma v_1 + (1-\gamma) v_2} \quad \text{in } B_{R_\varepsilon}(z_1),$$

$$\Delta^2 v_2 - \gamma_2 \left(\frac{\varepsilon}{\tau}\right)^2 \Delta v_2 - \lambda_2 \left(\frac{\varepsilon}{\tau}\right)^{4-q} |\nabla v_2|^q = 24 \frac{2^{4(\frac{\gamma+\xi-1}{\gamma})} \varepsilon^{8(\frac{\gamma+\xi-1}{\gamma})}}{\big(\tau(1+\varepsilon^2)\big)^{4(\frac{\gamma+\xi-1}{\gamma})}} e^{\xi v_2 + (1-\xi) v_1} \quad \text{in } B_{R_\varepsilon}(z_1).$$

Given $\varphi^1 := (\varphi_1^1, \varphi_2^1) \in (\mathcal{C}^{4,\alpha}(S^3))^2$ and $\psi^1 := (\psi_1^1, \psi_2^1) \in (\mathcal{C}^{2,\alpha}(S^3))^2$ such that (φ_1^1, ψ_1^1) and (φ_2^1, ψ_2^1) are satisfying (2.14). We write for $z \in B_{R_\varepsilon}(z_1)$ the following system

$$v_1(z) = \frac{1}{\gamma}\overline{u}(z - z_1) - \frac{1-\gamma}{\gamma\xi} G\left(\frac{\varepsilon z}{\tau}, z_2\right) - \frac{\ln\gamma}{\gamma} + h_1^1(z) + H_1^{\text{int},1}\left(\varphi_1^1, \psi_1^1; \frac{z - z_1}{R_\varepsilon}\right) + v_1^1(z),$$

$$v_2(z) = \frac{1}{\xi} G\left(\frac{\varepsilon z}{\tau}, z_2\right) + h_2^1(z) + H_2^{\text{int},1}\left(\varphi_2^1, \psi_2^1; \frac{z - z_1}{R_\varepsilon}\right) + v_2^1(z).$$

Using the fact that H^{int} is is bi-harmonic and the fact that $24 e^{\overline{u}} = \frac{384}{(1+r^2)^4}$, this amounts to solve the system

(2.16)

$$\begin{aligned} \mathbb{L} v_1^1 = {} & \frac{384}{\gamma(1+r^2)^4}\left[e^{\gamma(h_1^1 + H_1^{\text{int},1} + v_1^1) + (1-\gamma)(h_2^1 + H_2^{\text{int},1} + v_2^1)} - \gamma v_1^1 - 1\right] \\ & + \gamma_1 \left(\frac{\varepsilon}{\tau}\right)^2 \Delta \left(\frac{1}{\gamma}\overline{u} - \frac{1-\gamma}{\gamma\xi} G(z, z_2) + h_1^1 + H_1^{\text{int},1} + v_1^1\right) \\ & + \lambda_1 \left(\frac{\varepsilon}{\tau}\right)^{4-q} \left|\nabla \left(\frac{1}{\gamma}\overline{u} - \frac{1-\gamma}{\gamma\xi} G(z, z_2) + h_1^1 + H_1^{\text{int},1} + v_1^1\right)\right|^q - \Delta^2 h_1^1, \\ \Delta^2 v_2^1 = {} & \frac{384 C_\varepsilon \varepsilon^{8(\frac{\gamma+\xi-1}{\gamma})}}{(1+r^2)^{4\frac{(1-\xi)}{\gamma}}} e^{\xi(h_2^1 + H_2^{\text{int},1} + v_2^1) + \left[1 - \frac{(1-\xi)(1-\gamma)}{\gamma\xi}\right] G(z, z_2) + (1-\xi)\left(-\frac{\ln\gamma}{\gamma} + h_1^1 + H_1^{\text{int},1} + v_1^1\right)} \\ & + \gamma_2 \left(\frac{\varepsilon}{\tau}\right)^2 \Delta \left(\frac{1}{\xi} G(z, z_2) + h_2^1 + H_2^{\text{int},1} + v_2^1\right) \\ & + \lambda_2 \left(\frac{\varepsilon}{\tau}\right)^{4-q} \left|\nabla \left(\frac{1}{\xi} G(z, z_2) + h_2^1 + H_2^{\text{int},1} + v_2^1\right)\right|^q - \Delta^2 h_2^1, \end{aligned}$$

where $C_\varepsilon = [\tau(1+\varepsilon^2)]^{4(\frac{1-\gamma-\xi}{\gamma})}$.

Fix $\mu \in (1, 5-q)$ and $\delta \in \left(0, \min\left\{1, 4-q, \frac{\gamma+\xi-1}{\gamma}, \frac{\gamma+\xi-1}{\xi}\right\}\right)$. To find a solution of (2.16), it is enough to find a fixed point (v_1^1, v_2^1) in a small ball of $\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)$ solutions of

$$v_1^1 = \mathcal{G}_\mu \circ \xi_{\mu,R_\varepsilon} \circ \Re_1(v_1^1, v_2^1), \quad v_2^1 = \mathcal{K}_\delta \circ \xi_{\delta,R_\varepsilon} \circ \Re_2(v_1^1, v_2^1). \tag{2.17}$$

Here $\xi_{\sigma,R_\varepsilon}$ is defined in (2.3), $\mathcal{G}_\mu$ and $\mathcal{K}_\delta$ are defined after Propositions 2.4 and 2.5 and

$$\begin{aligned}
\Re_1(v_1^1, v_2^1) &= \frac{384}{\gamma(1+r^2)^4}\left[e^{\gamma(h_1^1+H_1^{\mathrm{int},1}+v_1^1)+(1-\gamma)(h_2^1+H_2^{\mathrm{int},1}+v_2^1)} - \gamma v_1^1 - 1\right] \\
&\quad + \gamma_1\left(\frac{\varepsilon}{\tau}\right)^2 \Delta\left(\frac{1}{\gamma}\overline{u} - \frac{1-\gamma}{\gamma\xi}G(z,z_2) + h_1^1 + H_1^{\mathrm{int},1} + v_1^1\right) \\
&\quad + \lambda_1\left(\frac{\varepsilon}{\tau}\right)^{4-q}\left|\nabla\left(\frac{1}{\gamma}\overline{u} - \frac{1-\gamma}{\gamma\xi}G(z,z_2) + h_1^1 + H_1^{\mathrm{int},1} + v_1^1\right)\right|^q - \Delta^2 h_1^1, \\
\Re_2(v_1^1, v_2^1) &= \frac{384 C_\varepsilon \varepsilon^{8(\frac{\gamma+\xi-1}{\gamma})}}{(1+r^2)^{4\frac{(1-\xi)}{\gamma}}} e^{\xi(h_2^1+H_2^{\mathrm{int},1}+v_2^1)+\left[1-\frac{(1-\xi)(1-\gamma)}{\gamma\xi}\right]G(z,z_2)+(1-\xi)\left(-\frac{\ln\gamma}{\gamma}+h_1^1+H_1^{\mathrm{int},1}+v_1^1\right)} \\
&\quad + \gamma_2\left(\frac{\varepsilon}{\tau}\right)^2 \Delta\left(\frac{1}{\xi}G(z,z_2) + h_2^1 + H_2^{\mathrm{int},1} + v_2^1\right) \\
&\quad + \lambda_2\left(\frac{\varepsilon}{\tau}\right)^{4-q}\left|\nabla\left(\frac{1}{\xi}G(z,z_2) + h_2^1 + H_2^{\mathrm{int},1} + v_2^1\right)\right|^q - \Delta^2 h_2^1.
\end{aligned}$$

We denote by $\aleph$ $(= \aleph_{\varepsilon,\tau,\varphi_j^1,\psi_j^1})$ and by Υ $(= \Upsilon_{\varepsilon,\tau,\varphi_j^1,\psi_j^1})$ the nonlinear operators appearing on the right-hand side of the two equations in (2.17).

Given $\kappa > 0$ (whose value will be fixed later on). We further assume that the functions $(\varphi_j^1, \psi_j^1) \in (\mathcal{C}^{4,\alpha} \times \mathcal{C}^{2,\alpha})$ for $j \in \{1,2\}$ satisfy

$$\|\varphi_j^1\|_{\mathcal{C}^{4,\alpha}(S^3)} \le \kappa r_\varepsilon^2 \quad \text{and} \quad \|\psi_j^1\|_{\mathcal{C}^{2,\alpha}(S^3)} \le \kappa r_\varepsilon^2. \tag{2.18}$$

Then we have the following result.

Lemma 2.15. *Let $\varphi^1 := (\varphi_1^1, \varphi_2^1) \in (\mathcal{C}^{4,\alpha}(S^3))^2$ and $\psi^1 := (\psi_1^1, \psi_2^1) \in (\mathcal{C}^{2,\alpha}(S^3))^2$ such that (φ_1^1, ψ_1^1) and (φ_2^1, ψ_2^1) satisfy* (2.14) *and* (2.18). *Given $\kappa > 0$, there exist $\varepsilon_\kappa > 0$ and $c_\kappa > 0$ and $\gamma_0 \in (0,1)$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\gamma \in (\gamma_0, 1)$, $\mu \in (1, 5-q)$ and $\delta \in \left(0, \min\left\{1, 4-q, \frac{\gamma+\xi-1}{\gamma}, \frac{\gamma+\xi-1}{\xi}\right\}\right)$,*

$$\|\aleph(0,0)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2, \quad \|\Upsilon(0,0)\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2,$$
$$\|\aleph(v_1^1, v_2^1) - \aleph(t_1^1, t_2^1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2 \|v_1^1 - t_1^1\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} + c_\kappa(1-\gamma)\|v_2^1 - t_2^1\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)}$$

and

$$\|\Upsilon(v_1^1, v_2^1) - \Upsilon(t_1^1, t_2^1)\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2 \|(v_1^1, v_2^1) - (t_1^1, t_2^1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)},$$

provided $(v_1^1, v_2^1), (t_1^1, t_2^1) \in \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)$ satisfying

$$\|(v_1^1, v_2^1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2, \quad \|(t_1^1, t_2^1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2. \tag{2.19}$$

Proof. The first estimate follows from the result of Lemma 2.13 together with the assumption on the norms of φ_j and ψ_j. Indeed, we have

$$\left\| H^{\text{int}}_{\varphi_j^1,\psi_j^1}\left(\frac{r}{R_\varepsilon}\,\cdot\right)\right\|_{\mathcal{C}^{4,\alpha}(\overline{B}_2(0)-B_1(0))} \le Cr^2R_\varepsilon^{-2}\big(\|\varphi_j^1\|_{\mathcal{C}^{4,\alpha}(S^3)} + \|\psi_j^1\|_{\mathcal{C}^{2,\alpha}(S^3)}\big)$$

for all $r \le R_\varepsilon/2$. Then by (2.18), we get $\|H^{\text{int}}_{\varphi_j^1,\psi_j^1}(\frac{r}{R_\varepsilon}\,\cdot)\|_{\mathcal{C}^{4,\alpha}(\overline{B}_2(0)-B_1(0))} \le c_\kappa\varepsilon^2r^2$.

On the other hand,

$$\begin{aligned}
&\sup_{r\le R_\varepsilon} r^{4-\mu}|\Re_1(0,0)| \\
&\le c_\kappa \sup_{r\le R_\varepsilon}\frac{384r^{4-\mu}}{\gamma(1+r^2)^4}\left|e^{\gamma(h_1^1+H_1^{\text{int},1})+(1-\gamma)(h_2^1+H_2^{\text{int},1})}-1\right| \\
&\quad + c_\kappa\sup_{r\le R_\varepsilon} r^{4-\mu}\gamma_1\left(\frac{\varepsilon}{\tau}\right)^2\left|\Delta\left(\frac{1}{\gamma}\overline{u}-\frac{1-\gamma}{\gamma\xi}G(z,z_2)+h_1^1+H_1^{\text{int},1}\right)\right| \\
&\quad + c_\kappa\sup_{r\le R_\varepsilon} r^{4-\mu}\left[\lambda_1\left(\frac{\varepsilon}{\tau}\right)^{4-q}\left|\nabla\left(\frac{1}{\gamma}\overline{u}-\frac{1-\gamma}{\gamma\xi}G(z,z_2)+h_1^1+H_1^{\text{int},1}\right)\right|^q-|\Delta^2h_1^1|\right] \\
&\le c_\kappa \sup_{r\le R_\varepsilon}\frac{384r^{4-\mu}}{\gamma(1+r^2)^4}\big(\gamma|h_1^1+H_1^{\text{int},1}|+(1+\gamma)|h_2^1+H_2^{\text{int},1}|\big) \\
&\quad + c_\kappa\gamma_1\left(\frac{\varepsilon}{\tau}\right)^2\sup_{r\le R_\varepsilon} r^{4-\mu}\left(\frac{2+r^2}{\gamma(1+r^2)^2}+Cr^{-2}+c_\kappa\varepsilon^2+|\Delta h_1^1|\right) \\
&\quad + c_\kappa\sup_{r\le R_\varepsilon} r^{4-\mu}\left[\lambda_1\left(\frac{\varepsilon}{\tau}\right)^{4-q}\left(\frac{r^q}{\gamma(1+r^2)^q}+Cr^{-q}+\varepsilon^{2q}r^q+|\nabla h_1^1|^q\right)-|\Delta^2h_1^1|\right] \\
&\le c_\kappa \sup_{r\le R_\varepsilon}\frac{384r^{4-\mu}}{\gamma(1+r^2)^4}\Big[\gamma\big(r^\mu\|h_1^1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}+r^2\|H_1^{\text{int},1}\|_{\mathcal{C}^{4,\alpha}_2(\overline{B}_1^*)}\big) \\
&\qquad\qquad + (1-\gamma)\big(r^\delta\|h_2^1\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}+r^2\|H_2^{\text{int},1}\|_{\mathcal{C}^{4,\alpha}_2(\overline{B}_1^*)}\big)\Big] \\
&\quad + 2c_\kappa\gamma_1\max\{\varepsilon^\mu r_\varepsilon^{2-\mu},\varepsilon^2\}+c_\kappa\gamma_1\varepsilon^\mu r_\varepsilon^{4-\mu}+c_\kappa\gamma_1 r_\varepsilon^2\|h_1^1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \\
&\quad + 2c_\kappa\lambda_1\max\{\varepsilon^\mu r_\varepsilon^{4-\mu-q},\varepsilon^{4-q}\}+c_\kappa\lambda_1\varepsilon^\mu r_\varepsilon^{4-\mu+q} \\
&\quad + c_\kappa\lambda_1\varepsilon^{-\mu(q-1)}r_\varepsilon^{4-\mu+(\mu-1)q}\|h_1^1\|^q_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}+c_\kappa\|h_1^1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \\
&\le c_\kappa \sup_{r\le R_\varepsilon}\frac{384r^{4-\mu}}{\gamma(1+r^2)^4}\Big[\gamma\big(r^\mu\|h_1^1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}+r^2\|H_1^{\text{int},1}\|_{\mathcal{C}^{4,\alpha}_2(\overline{B}_1^*)}\big) \\
&\qquad\qquad + (1-\gamma)\big(r^\delta\|h_2^1\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}+r^2\|H_2^{\text{int},1}\|_{\mathcal{C}^{4,\alpha}_2(\overline{B}_1^*)}\big)\Big] \\
&\quad + 2c_\kappa\gamma_1\max\{\varepsilon^\mu r_\varepsilon^{2-\mu},\varepsilon^2\}+c_\kappa\gamma_1\varepsilon^\mu r_\varepsilon^{4-\mu}+c_\kappa\gamma_1 r_\varepsilon^2\|h_1^1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \\
&\quad + 2c_\kappa\lambda_1\max\{\varepsilon^\mu r_\varepsilon^{4-\mu-q},\varepsilon^{4-q}\}+c_\kappa\lambda_1\varepsilon^\mu r_\varepsilon^{4-\mu+q} \\
&\quad + c_\kappa\lambda_1 r_\varepsilon^{6-q}(r_\varepsilon^{2+\mu}\varepsilon^{-\mu})^{q-1}+c_\kappa\|h_1^1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}.
\end{aligned}$$

Making use of Proposition 2.4 together with (2.4) and using the condition (A1) for $\mu \in (1, 5-q)$, we get that there exists $\overline{c}_\kappa > 0$ such that

$$\|\aleph(0,0)\|_{\mathcal{C}^{4,\alpha}_{\mu}(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2. \tag{2.20}$$

For the second estimate, we have

$$\begin{aligned}
&\sup_{r\le R_\varepsilon} r^{4-\delta}|\Re_2(0,0)| \\
&\le c_\kappa \sup_{r\le R_\varepsilon} \frac{384 C_\varepsilon \varepsilon^{8(\frac{\gamma+\xi-1}{\gamma})}}{(1+r^2)^{4\frac{(1-\xi)}{\gamma}}} r^{4-\delta} e^{\xi(h_2^1+H_2^{\mathrm{int},1})+\left[1-\frac{(1-\xi)(1-\gamma)}{\gamma\xi}\right]G(z,z_2)+(1-\xi)\left(h_1^1-\frac{\ln\gamma}{\gamma}+H_1^{\mathrm{int},1}\right)} \\
&\quad + c_\kappa \sup_{r\le R_\varepsilon} r^{4-\delta}\gamma_2\left(\frac{\varepsilon}{\tau}\right)^2\left|\Delta\left(\frac{1}{\xi}G(z,z_2)+h_2^1+H_2^{\mathrm{int},1}\right)\right| \\
&\quad + c_\kappa \sup_{r\le R_\varepsilon} r^{4-\delta}\left[\lambda_2\left(\frac{\varepsilon}{\tau}\right)^{4-q}\left|\nabla\left(\frac{1}{\xi}G(z,z_2)+h_2^1+H_2^{\mathrm{int},1}\right)\right|^q - |\Delta^2 h_2^1|\right] \\
&\le c_\kappa \sup_{r\le R_\varepsilon} \frac{384 C_\varepsilon \varepsilon^{8(\frac{\gamma+\xi-1}{\gamma})}}{(1+r^2)^{4\frac{(1-\xi)}{\gamma}}} r^{4-\delta}\Big(\xi\big(r^\delta\|h_2^1\|_{\mathcal{C}^{4,\alpha}_\delta} + r^2\|H_2^{\mathrm{int},1}\|_{\mathcal{C}^{4,\alpha}_2}\big) \\
&\qquad\qquad + (1-\xi)\big(r^\mu\|h_1^1\|_{\mathcal{C}^{4,\alpha}_\mu} + r^2\|H_1^{\mathrm{int},1}\|_{\mathcal{C}^{4,\alpha}_2}\big) + 1\Big) \\
&\quad + c_\kappa\gamma_2\varepsilon^2\big(R_\varepsilon^{2-\delta} + \varepsilon^2 R_\varepsilon^{4-\delta}\big) + c_\kappa\gamma_2 r_\varepsilon^2\|h_2^1\|_{\mathcal{C}^{4,\alpha}_\delta} \\
&\quad + c_\kappa\lambda_2\varepsilon^{4-q}\left(R_\varepsilon^{4-\delta-q} + R_\varepsilon^{4-\delta+q}\varepsilon^{2q} + \max\{R_\varepsilon^{4-\delta+(\delta-1)q}, c_1\}\|h_2^1\|^q_{\mathcal{C}^{4,\alpha}_\delta}\right) + c_\kappa\|h_2^1\|_{\mathcal{C}^{4,\alpha}_\delta} \\
&\le c_\kappa \sup_{r\le R_\varepsilon} \frac{384 C_\varepsilon \varepsilon^{8(\frac{\gamma+\xi-1}{\gamma})}}{(1+r^2)^{4\frac{(1-\xi)}{\gamma}}} r^{4-\delta}\Big(\xi\big(r^\delta\|h_2^1\|_{\mathcal{C}^{4,\alpha}_\delta} + r^2\|H_2^{\mathrm{int},1}\|_{\mathcal{C}^{4,\alpha}_2}\big) \\
&\qquad\qquad + (1-\xi)\big(r^\mu\|h_1^1\|_{\mathcal{C}^{4,\alpha}_\mu} + r^2\|H_1^{\mathrm{int},1}\|_{\mathcal{C}^{4,\alpha}_2}\big) + 1\Big) \\
&\quad + c_\kappa\gamma_2\big(\varepsilon^\delta r_\varepsilon^{2-\delta} + \varepsilon^\delta r_\varepsilon^{4-\delta} + r_\varepsilon^2\|h_2^1\|_{\mathcal{C}^{4,\alpha}_\delta}\big) \\
&\quad + c_\kappa\lambda_2\left(\varepsilon^\delta r_\varepsilon^{4-\delta-q} + \varepsilon^\delta r_\varepsilon^{4-\delta+q} + \max\{\varepsilon^{\delta(1-q)} r_\varepsilon^{4-\delta+(\delta-1)q}, \varepsilon^{4-q}\}\|h_2^1\|^q_{\mathcal{C}^{4,\alpha}_\delta}\right) + c_\kappa\|h_2^1\|_{\mathcal{C}^{4,\alpha}_\delta}.
\end{aligned}$$

Using the same argument as above, we get

$$\|\Upsilon(0,0)\|_{\mathcal{C}^{4,\alpha}_{\delta}(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2. \tag{2.21}$$

To derive the third estimate, for (v_1^1, v_2^1), (t_1^1, t_2^1) verifying (2.19), we have

$$\begin{aligned}
&\sup_{r\le R_\varepsilon} r^{4-\mu}|\Re_1(v_1^1, v_2^1) - \Re_1(t_1^1, t_2^1)| \\
&\le c_\kappa \sup_{r\le R_\varepsilon} \frac{384 r^{4-\mu}}{\gamma(1+r^2)^4}\Big| e^{\gamma(h_1^1+H_1^{\mathrm{int},1}+v_1^1)+(1-\gamma)(h_2^1+H_2^{\mathrm{int},1}+v_2^1)} - \gamma v_1^1 \\
&\qquad\qquad - e^{\gamma(h_1^1+H_1^{\mathrm{int},1}+t_1^1)+(1-\gamma)(h_2^1+H_2^{\mathrm{int},1}+t_2^1)} + \gamma t_1^1\Big|
\end{aligned}$$

$$
\begin{aligned}
&+ c_\kappa \gamma_1 \left(\frac{\varepsilon}{\tau}\right)^2 \sup_{r\le R_\varepsilon} r^{4-\mu}\Bigg|\Delta\left(\frac{1}{\gamma}\overline{u} - \frac{1-\gamma}{\gamma\xi}G(z,z_2) + h_1^1 + H_1^{\mathrm{int},1} + v_1^1\right) \\
&\qquad\qquad - \Delta\left(\frac{1}{\gamma}\overline{u} - \frac{1-\gamma}{\gamma\xi}G(z,z_2) + h_1^1 + H_1^{\mathrm{int},1} + t_1^1\right)\Bigg| \\
&+ c_\kappa \lambda_1 \left(\frac{\varepsilon}{\tau}\right)^{4-q} \sup_{r\le R_\varepsilon} r^{4-\mu}\Bigg(\left|\nabla\left(\frac{1}{\gamma}\overline{u} - \frac{1-\gamma}{\gamma\xi}G(z,z_2) + h_1^1 + H_1^{\mathrm{int},1} + v_1^1\right)\right|^q \\
&\qquad\qquad - \left|\nabla\left(\frac{1}{\gamma}\overline{u} - \frac{1-\gamma}{\gamma\xi}G(z,z_2) + h_1^1 + H_1^{\mathrm{int},1} + t_1^1\right)\right|^q\Bigg) \\
&\le c_\kappa \sup_{r\le R_\varepsilon} \frac{384 r^{4-\mu}}{\gamma(1+r^2)^4}\big(\gamma^2|(v_1^1)^2 - (t_1^1)^2| + (1-\gamma)|v_2^1 - t_2^1|\big) + c_\kappa\gamma_1\varepsilon^2 \sup_{r\le R_\varepsilon} r^{4-\mu}|\Delta(v_1^1 - t_1^1)| \\
&+ c_\kappa\lambda_1\varepsilon^{4-q} \sup_{r\le R_\varepsilon} r^{4-\mu}\Bigg(\left|\nabla\left(\frac{1}{\gamma}\overline{u} - \frac{1-\gamma}{\gamma\xi}G(z,z_2) + h_1^1 + H_1^{\mathrm{int},1} + t_1^1 + v_1^1 - t_1^1\right)\right|^q \\
&\qquad\qquad - \left|\nabla\left(\frac{1}{\gamma}\overline{u} - \frac{1-\gamma}{\gamma\xi}G(z,z_2) + h_1^1 + H_1^{\mathrm{int},1} + t_1^1\right)\right|^q\Bigg).
\end{aligned}
$$

Using Lemma 2.9 and the fact that a functions w in $\mathcal{C}^{k,\alpha}_\mu(\mathbb{R}^4)$ are bounded by a constant times $(1+r^2)^{\mu/2}$ and have their ℓ-th partial derivatives that are bounded by $(1+r^2)^{(\mu-\ell)/2}$, for $\ell = 1,\dots,k+\alpha$ (a.e. $|\nabla^\ell w| \le c_\kappa r^{\mu-\ell}\|w\|_{\mathcal{C}^{k,\alpha}_\mu(\mathbb{R}^4)}$, $(1+r^2)^{(\mu-\ell)/2} \backsim r^{\mu-\ell}$ for r very large) and provided $(h_1^1, h_2^1) \in \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)$ satisfy (2.9), we obtain

$$
\begin{aligned}
&\sup_{r\le R_\varepsilon} r^{4-\mu}|\Re_1(v_1^1, v_2^1) - \Re_1(t_1^1, t_2^1)| \\
&\le c_\kappa \sup_{r\le R_\varepsilon} \frac{384 r^{4-\mu}}{\gamma(1+r^2)^4}\big(\gamma^2 r^{2\mu}\big(\|v_1^1\|_{\mathcal{C}^{4,\alpha}_\mu} + \|t_1^1\|_{\mathcal{C}^{4,\alpha}_\mu}\big)\|v_1^1 - t_1^1\|_{\mathcal{C}^{4,\alpha}_\mu} + (1-\gamma) r^\delta \|v_2^1 - t_2^1\|_{\mathcal{C}^{4,\alpha}_\delta}\big) \\
&\quad + c_\kappa\gamma_1 r_\varepsilon^2 \|v_1^1 - t_1^1\|_{\mathcal{C}^{4,\alpha}_\mu} \\
&\quad + c_\kappa\lambda_1\varepsilon^{4-q} \sup_{r\le R_\varepsilon} r^{4-\mu}\Bigg(\left|\nabla\left(\frac{1}{\gamma}\overline{u} - \frac{1-\gamma}{\gamma\xi}G(z,z_2) + h_1^1 + H_1^{\mathrm{int},1} + t_1^1\right)\right|^{q-1} \\
&\qquad\qquad + |\nabla(v_1^1 - t_1^1)|^{q-1}\Bigg)|\nabla(v_1^1 - t_1^1)| \\
&\le c_\kappa \sup_{r\le R_\varepsilon} \frac{384 r^{4-\mu}}{\gamma(1+r^2)^4}\big(\gamma^2 r^{2\mu}\big(\|v_1^1\|_{\mathcal{C}^{4,\alpha}_\mu} + \|t_1^1\|_{\mathcal{C}^{4,\alpha}_\mu}\big)\|v_1^1 - t_1^1\|_{\mathcal{C}^{4,\alpha}_\mu} + (1-\gamma) r^\delta \|v_2^1 - t_2^1\|_{\mathcal{C}^{4,\alpha}_\delta}\big) \\
&\quad + c_\kappa\gamma_1 r_\varepsilon^2 \|v_1^1 - t_1^1\|_{\mathcal{C}^{4,\alpha}_\mu} \\
&\quad + c_\kappa\lambda_1\varepsilon^{4-q} \sup_{r\le R_\varepsilon} r^{4-\mu}\Bigg[\left(\frac{1}{\gamma}\right)^{q-1}|\nabla\overline{u}|^{q-1} - \left(\frac{1-\gamma}{\gamma\xi}\right)^{q-1}|\nabla G(z,z_2)|^{q-1} \\
&\qquad\qquad + |\nabla H_1^{\mathrm{int},1}|^{q-1} + |\nabla h_1^1|^{q-1} + |\nabla t_1^1|^{q-1} + |\nabla v_1^1|^{q-1}\Bigg]|\nabla(v_1^1 - t_1^1)| \\
&\le c_\kappa \sup_{r\le R_\varepsilon} \frac{384 r^{4-\mu}}{\gamma(1+r^2)^4}\big(\gamma^2 r^{2\mu}\big(\|v_1^1\|_{\mathcal{C}^{4,\alpha}_\mu} + \|t_1^1\|_{\mathcal{C}^{4,\alpha}_\mu}\big)\|v_1^1 - t_1^1\|_{\mathcal{C}^{4,\alpha}_\mu} + (1-\gamma) r^\delta \|v_2^1 - t_2^1\|_{\mathcal{C}^{4,\alpha}_\delta}\big) \\
&\quad + c_\kappa\gamma_1 r_\varepsilon^2 \|v_1^1 - t_1^1\|_{\mathcal{C}^{4,\alpha}_\mu}
\end{aligned}
$$

$$+ c_\kappa \lambda_1 \varepsilon^{4-q} \big[R_\varepsilon^{4-q} + R_\varepsilon^{2+q} \varepsilon^{2(q-1)} + R_\varepsilon^{4-q+\mu q-\mu} \big(\|h_1^1\|^{q-1}_{\mathcal{C}^{4,\alpha}_\mu} + \|t_1^1\|^{q-1}_{\mathcal{C}^{4,\alpha}_\mu} + \|v_1^1\|^{q-1}_{\mathcal{C}^{4,\alpha}_\mu} \big) \big] \|v_1^1 - t_1^1\|_{\mathcal{C}^{4,\alpha}_\mu}$$

$$\leq c_\kappa \sup_{r \leq R_\varepsilon} \frac{384 r^{4-\mu}}{\gamma(1+r^2)^4} \big(\gamma^2 r^{2\mu} \big(\|v_1^1\|_{\mathcal{C}^{4,\alpha}_\mu} + \|t_1^1\|_{\mathcal{C}^{4,\alpha}_\mu} \big) \|v_1^1 - t_1^1\|_{\mathcal{C}^{4,\alpha}_\mu} + (1-\gamma) r^\delta \|v_2^1 - t_2^1\|_{\mathcal{C}^{4,\alpha}_\delta} \big)$$

$$+ c_\kappa \gamma_1 r_\varepsilon^2 \|v_1^1 - t_1^1\|_{\mathcal{C}^{4,\alpha}_\mu} + c_\kappa \lambda_1 \big(r_\varepsilon^{4-q} + r_\varepsilon^{2+q} + r_\varepsilon^{4-q} \big(r_\varepsilon^{2+\mu} \varepsilon^{-\mu} \big)^{q-1} \big) \|v_1^1 - t_1^1\|_{\mathcal{C}^{4,\alpha}_\mu}.$$

Making use of Proposition 2.4 together with (2.4) and using the condition (A1) for $\mu \in (1, 5-q)$, we conclude that there exists $\bar{c}_\kappa > 0$ such that

$$\|\aleph(v_1^1, v_2^1) - \aleph(t_1^1, t_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leq c_\kappa r_\varepsilon^2 \|v_1^1 - t_1^1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} + c_\kappa (1-\gamma) \|v_2^1 - t_2^1\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}. \tag{2.22}$$

Similarly we get the estimate for

$$\|\Upsilon(v_1^1, v_2^1) - \Upsilon(t_1^1, t_2^1)\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \leq c_\kappa r_\varepsilon^2 \|(v_1^1, v_2^1) - (t_1^1, t_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4) \times \mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}. \tag{2.23}$$ □

Reducing ε_κ if necessary, we can assume that $c_\kappa r_\varepsilon^2 \leq 1/2$ for all $\varepsilon \in (0, \varepsilon_k)$. There exists also $\gamma_0 \in (0,1)$ such that $c_\kappa(1-\gamma) \leq 1/2$ for all $\gamma \in (\gamma_0, 1)$. Therefore (2.20), (2.21), (2.22) and (2.23) are enough to show that

$$(v_1^1, v_2^1) \mapsto (\aleph(v_1^1, v_2^1), \Upsilon(v_1^1, v_2^1))$$

is a contraction from the ball

$$\big\{ (v_1^1, v_2^1) \in \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4) \times \mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4) : \|(v_1^1, v_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4) \times \mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \leq 2 c_\kappa r_\varepsilon^2 \big\}$$

into itself and hence a unique fixed point (v_1^1, v_2^1) exists in this set. This fixed point is a solution of (2.17).

We may summarize the result obtained as follows.

Proposition 2.16. *Given $\kappa > 0$, there exist $\varepsilon_\kappa > 0$, $c_\kappa > 0$ and $\gamma_0 \in (0,1)$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\gamma \in (\gamma_0, 1)$, for all τ in some fixed compact subset of $[\tau^-, \tau^+] \subset (0, \infty)$ and for φ and ψ satisfying* (2.14) *and* (2.18)*, there exists a unique $(v_1^1, v_2^1) := (v_{1,\varepsilon,\tau,\varphi,\psi}, v_{2,\varepsilon,\tau,\varphi,\psi})$ solution of* (2.17) *such that*

$$\|(v_1^1, v_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4) \times \mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \leq 2 c_\kappa r_\varepsilon^2.$$

Hence

$$v_1(z) := \frac{1}{\gamma} \bar{u}(z - z_1) - \frac{1-\gamma}{\gamma \xi} G\left(\frac{\varepsilon z}{\tau}, z_2 \right) - \frac{\ln \gamma}{\gamma} + h_1^1(z) + H_1^{\mathrm{int},1}\left(\varphi_1^1, \psi_1^1; \frac{z - z_1}{R_\varepsilon} \right) + v_1^1(z),$$

$$v_2(z) := \frac{1}{\xi} G\left(\frac{\varepsilon z}{\tau}, z_2 \right) + h_2^1(z) + H_2^{\mathrm{int},1}\left(\varphi_2^1, \psi_2^1; \frac{z - z_1}{R_\varepsilon} \right) + v_2^1(z)$$

solve (2.6) *in $B_{R_\varepsilon}(z_1)$.*

Similarly, we prove

Proposition 2.17. *Given $\kappa > 0$, there exist $\varepsilon_\kappa > 0$, $c_\kappa > 0$ and $\xi_0 \in (0,1)$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\xi \in (\xi_0, 1)$, for all τ in some fixed compact subset of $[\tau^-, \tau^+] \subset (0, \infty)$ and for φ and ψ satisfying (2.14) and (2.18), there exists a unique $(v_1^2, v_2^2) := (v_{1,\varepsilon,\tau,\varphi,\psi}, v_{2,\varepsilon,\tau,\varphi,\psi})$ solution of (2.17) such that*

$$\|(v_1^2, v_2^2)\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2.$$

Hence

$$v_1(z) := \frac{1}{\gamma} G\left(\frac{\varepsilon z}{\tau}, z_1\right) + h_1^2(z) + H_1^{\text{int},2}\left(\varphi_1^2, \psi_1^2; \frac{z - z_2}{R_\varepsilon}\right) + v_1^2(z),$$
$$v_2(z) := \frac{1}{\xi}\overline{u}(z - z_2) - \frac{1-\xi}{\gamma\xi} G\left(\frac{\varepsilon z}{\tau}, z_1\right) - \frac{\ln\xi}{\xi} + h_2^2(z) + H_2^{\text{int},2}\left(\varphi_2^2, \psi_2^2; \frac{z - z_2}{R_\varepsilon}\right) + v_2^2(z)$$

solve (2.6) in $B_{R_\varepsilon}(z_2)$.

Remark also that the functions v_1^1, v_2^1, v_1^2 and v_2^2 obtained in the above propositions, depend continuously on the parameter τ.

2.3. The nonlinear exterior problem

Given $\widetilde{\mathbf{z}} := (\widetilde{z}_1, \widetilde{z}_2) \in \Omega^2$ close to $\mathbf{z} := (z_1, z_2)$, $\boldsymbol{\eta} := (\eta_1, \eta_2) \in \mathbb{R}^2$ close to 0, $\widetilde{\boldsymbol{\varphi}}_1 := (\widetilde{\varphi}_1^1, \widetilde{\varphi}_1^2) \in (\mathcal{C}^{4,\alpha}(S^3))^2$, $\widetilde{\boldsymbol{\varphi}}_2 := (\widetilde{\varphi}_2^1, \widetilde{\varphi}_2^2) \in (\mathcal{C}^{4,\alpha}(S^3))^2$, $\widetilde{\boldsymbol{\psi}}_1 := (\widetilde{\psi}_1^1, \widetilde{\psi}_1^2) \in (\mathcal{C}^{2,\alpha}(S^3))^2$ and $\widetilde{\boldsymbol{\psi}}_2 := (\widetilde{\psi}_2^1, \widetilde{\psi}_2^2) \in (\mathcal{C}^{2,\alpha}(S^3))^2$ satisfying (2.15), define

$$\widetilde{\mathbf{w}}_1(z) = \frac{1+\eta_1}{\gamma} G(z, \widetilde{z}_1) + \sum_{i=1}^{2} \chi_{r_0}(z - \widetilde{z}_i) H_1^{\text{ext}}\left(\widetilde{\varphi}_1^i, \widetilde{\psi}_1^i; \frac{z - \widetilde{z}_i}{r_\varepsilon}\right),$$
$$\widetilde{\mathbf{w}}_2(z) = \frac{1+\eta_2}{\xi} G(z, \widetilde{z}_2) + \sum_{i=1}^{2} \chi_{r_0}(z - \widetilde{z}_i) H_2^{\text{ext}}\left(\widetilde{\varphi}_2^i, \widetilde{\psi}_2^i; \frac{z - \widetilde{z}_i}{r_\varepsilon}\right).$$

Here χ_{r_0} is a cut-off function identically equal to 1 in $B_{r_0/2}(0)$ and identically equal to 0 outside $B_{r_0}(0)$. We would like to find a solution of the system

$$\begin{aligned} \Delta^2 u_1 - \gamma_1 \Delta u_1 - \lambda_1 |\nabla u_1|^q &= \rho^4 e^{\gamma u_1 + (1-\gamma) u_2}, \\ \Delta^2 u_2 - \gamma_2 \Delta u_2 - \lambda_2 |\nabla u_2|^q &= \rho^4 e^{\xi u_2 + (1-\xi) u_1} \end{aligned} \tag{2.24}$$

in the domain $\overline{\Omega}_{r_\varepsilon}(\widetilde{\mathbf{z}}) := \overline{\Omega}_{r_\varepsilon} \setminus \{(\widetilde{z}_1, \widetilde{z}_2)\}$ with $u_k = \widetilde{\mathbf{w}}_k + \widetilde{v}_k$ a perturbation of $\widetilde{\mathbf{w}}_k$, $k = 1, 2$. This amounts to solve in $\overline{\Omega}_{r_\varepsilon}(\widetilde{\mathbf{z}})$,

$$\begin{aligned} \Delta^2 \widetilde{v}_1 &= \rho^4 e^{\gamma(\widetilde{w}_1 + \widetilde{v}_1) + (1-\gamma)(\widetilde{w}_2 + \widetilde{v}_2)} + \gamma_1 \Delta(\widetilde{w}_1 + \widetilde{v}_1) + \lambda_1 |\nabla(\widetilde{w}_1 + \widetilde{v}_1)|^q - \Delta^2 \widetilde{w}_1, \\ \Delta^2 \widetilde{v}_2 &= \rho^4 e^{\xi(\widetilde{w}_2 + \widetilde{v}_2) + (1-\xi)(\widetilde{w}_1 + \widetilde{v}_1)} + \gamma_2 \Delta(\widetilde{w}_2 + \widetilde{v}_2) + \lambda_2 |\nabla(\widetilde{w}_2 + \widetilde{v}_2)|^q - \Delta^2 \widetilde{w}_2. \end{aligned} \tag{2.25}$$

For all $\sigma \in (0, r_0/2)$ and all $\widetilde{\mathbf{z}} = (\widetilde{z}_1, \widetilde{z}_2) \in \Omega^2$ such that $\|\mathbf{z} - \widetilde{\mathbf{z}}\| \le r_0/2$, where $\mathbf{z} = (z_1, z_2)$, we denote by $\widetilde{\xi}_{\sigma,\widetilde{\mathbf{z}}} \colon \mathcal{C}^{0,\alpha}_\nu(\overline{\Omega}_\sigma(\widetilde{\mathbf{z}})) \to \mathcal{C}^{0,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{z}}))$ the extension operator defined by

$$\begin{cases} \widetilde{\xi}_{\sigma,\widetilde{\mathbf{z}}}(f) \equiv f & \text{in } \overline{\Omega}(\widetilde{\mathbf{z}}), \\ \widetilde{\xi}_{\sigma,\widetilde{\mathbf{z}}}(f)(\widetilde{z}_j + z) = \widetilde{\chi}\big(\frac{|z|}{\sigma}\big) f\big(\widetilde{z}_j + \sigma \frac{z}{|z|}\big) & \text{in } B_\sigma(\widetilde{z}_j) \setminus B_{\sigma/2}(\widetilde{z}_j),\ \forall\, 1 \le j \le 2, \\ \widetilde{\xi}_{\sigma,\widetilde{\mathbf{z}}}(f) \equiv 0 & \text{in } B_{\sigma/2}(\widetilde{z}_1) \cup B_{\sigma/2}(\widetilde{z}_2). \end{cases}$$

Here $\widetilde{\chi}$ is a cut-off function over $\mathbb{R}_+$ which is equal to 1 for $t \ge 1$ and equal to 0 for $t \le 1/2$. Obviously, there exists a constant $\overline{c} = \overline{c}(\nu) > 0$ only depending on ν such that

$$\big\|\widetilde{\xi}_{\sigma,\widetilde{\mathbf{z}}}(w)\big\|_{\mathcal{C}^{0,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{z}}))} \le \overline{c}\|w\|_{\mathcal{C}^{0,\alpha}_\nu(\overline{\Omega}_\sigma(\widetilde{\mathbf{z}}))}. \tag{2.26}$$

We fix $\nu \in (-1, 0)$. In order to solve (2.25), it is enough to find $(\widetilde{v}_1, \widetilde{v}_2) \in (\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{z}})))^2$ solution of

$$\widetilde{v}_1 = \widetilde{\mathcal{K}}_\nu \circ \widetilde{\xi}_{r_\varepsilon,\widetilde{\mathbf{z}}} \circ \widetilde{S}_1(\widetilde{v}_1, \widetilde{v}_2) \quad \text{and} \quad \widetilde{v}_2 = \widetilde{\mathcal{K}}_\nu \circ \widetilde{\xi}_{r_\varepsilon,\widetilde{\mathbf{z}}} \circ \widetilde{S}_2(\widetilde{v}_1, \widetilde{v}_2), \tag{2.27}$$

where

$$\begin{aligned} \widetilde{S}_1(\widetilde{v}_1, \widetilde{v}_2) &= \rho^4 e^{\gamma(\widetilde{w}_1+\widetilde{v}_1)+(1-\gamma)(\widetilde{w}_2+\widetilde{v}_2)} + \gamma_1\Delta(\widetilde{w}_1 + \widetilde{v}_1) + \lambda_1|\nabla(\widetilde{w}_1 + \widetilde{v}_1)|^q - \Delta^2\widetilde{w}_1, \\ \widetilde{S}_2(\widetilde{v}_1, \widetilde{v}_2) &= \rho^4 e^{\xi(\widetilde{w}_2+\widetilde{v}_2)+(1-\xi)(\widetilde{w}_1+\widetilde{v}_1)} + \gamma_2\Delta(\widetilde{w}_2 + \widetilde{v}_2) + \lambda_2|\nabla(\widetilde{w}_2 + \widetilde{v}_2)|^q - \Delta^2\widetilde{w}_2. \end{aligned}$$

We denote by

$$\widetilde{\mathcal{N}}(\widetilde{v}_1, \widetilde{v}_2) = \widetilde{\mathcal{K}}_\nu \circ \widetilde{\xi}_{r_\varepsilon,\widetilde{\mathbf{z}}} \circ \widetilde{S}_1(\widetilde{v}_1, \widetilde{v}_2) \quad \text{and} \quad \widetilde{\mathcal{M}}(\widetilde{v}_1, \widetilde{v}_2) = \widetilde{\mathcal{K}}_\nu \circ \widetilde{\xi}_{r_\varepsilon,\widetilde{\mathbf{z}}} \circ \widetilde{S}_2(\widetilde{v}_1, \widetilde{v}_2).$$

Given $\kappa > 0$ (whose value will be fixed later on), we further assume that for $i, j \in \{1, 2\}$ the functions $\widetilde{\varphi}^i_j$, $\widetilde{\psi}^i_j$, the parameters η_i and the point $\widetilde{\mathbf{z}} = (\widetilde{z}_1, \widetilde{z}_2)$ satisfy

$$\|\widetilde{\varphi}^i_j\|_{\mathcal{C}^{4,\alpha}(S^3)} \le \kappa r_\varepsilon^2, \quad \|\widetilde{\psi}^i_j\|_{\mathcal{C}^{2,\alpha}(S^3)} \le \kappa r_\varepsilon^2, \tag{2.28}$$

$$|\eta_i| \le \kappa r_\varepsilon^2, \quad |\widetilde{z}_i - z_i| \le \kappa r_\varepsilon. \tag{2.29}$$

Then, the following result holds.

Lemma 2.18. *Under the above assumptions, there exists a constant $c_\kappa > 0$ such that*

$$\begin{gathered} \|\widetilde{\mathcal{N}}(0,0)\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{z}}))} \le c_\kappa r_\varepsilon^2, \quad \|\widetilde{\mathcal{M}}(0,0)\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{z}}))} \le c_\kappa r_\varepsilon^2, \\ \|\widetilde{\mathcal{N}}(\widetilde{v}_1, \widetilde{v}_2) - \widetilde{\mathcal{N}}(\widetilde{v}'_1, \widetilde{v}'_2)\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{z}}))} \le c_\kappa r_\varepsilon^2 \|(\widetilde{v}_1, \widetilde{v}_2) - (\widetilde{v}'_1, \widetilde{v}'_2)\|_{(\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{z}})))^2} \end{gathered}$$

and

$$\|\widetilde{\mathcal{M}}(\widetilde{v}_1, \widetilde{v}_2) - \widetilde{\mathcal{M}}(\widetilde{v}'_1, \widetilde{v}'_2)\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{z}}))} \le c_\kappa r_\varepsilon^2 \|(\widetilde{v}_1, \widetilde{v}_2) - (\widetilde{v}'_1, \widetilde{v}'_2)\|_{(\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{z}})))^2},$$

provided $\big(\widetilde{v}_1, \widetilde{v}_2, \widetilde{v}'_1, \widetilde{v}'_2\big) \in \big(\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{z}}))\big)^4$ *satisfy*

$$\|(\widetilde{v}_1, \widetilde{v}_2)\|_{(\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{z}})))^2} \le 2c_\kappa r_\varepsilon^2 \quad \textit{and} \quad \|(\widetilde{v}'_1, \widetilde{v}'_2)\|_{(\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{z}})))^2} \le 2c_\kappa r_\varepsilon^2. \tag{2.30}$$

Proof. As for the interior problem, the proof of the two first estimates follows from the asymptotic behavior of H^{ext} together with the assumption on the norm of boundary data $\widetilde{\varphi}_j^i$ and $\widetilde{\psi}_j^i$ given by (2.28). Indeed, let c_κ be a constant depending only on κ, by Lemma 2.13,

$$\left|H^{\text{ext}}\left(\widetilde{\varphi}_j^i, \widetilde{\psi}_j^i; \frac{z-\widetilde{z}_i}{r_\varepsilon}\right)\right| \le c_\kappa r_\varepsilon^3 r^{-1}. \tag{2.31}$$

On the other hand,

$$\widetilde{S}_1(0,0) = \rho^4 e^{\gamma \widetilde{\mathbf{w}}_1 + (1-\gamma_1)\widetilde{\mathbf{w}}_2} + \gamma_1 \Delta \widetilde{\mathbf{w}}_1 + \lambda_1 |\nabla \widetilde{\mathbf{w}}_1|^q - \Delta^2 \widetilde{\mathbf{w}}_1$$

and

$$\widetilde{S}_2(0,0) = \rho^4 e^{\xi \widetilde{\mathbf{w}}_2 + (1-\xi)\widetilde{\mathbf{w}}_1} + \gamma_2 \Delta \widetilde{\mathbf{w}}_2 + \lambda_2 |\nabla \widetilde{\mathbf{w}}_2|^q - \Delta^2 \widetilde{\mathbf{w}}_2.$$

We will estimate $\widetilde{S}_1(0,0)$ in different sub-regions of $\overline{\Omega}^*(\widetilde{\mathbf{z}})$.

• In $B_{r_0/2}(\widetilde{z}_1) \setminus B_{r_\varepsilon}(\widetilde{z}_1)$, we have $\chi_{r_0}(z-\widetilde{z}_1) = 1$, $\chi_{r_0}(z-\widetilde{z}_2) = 0$ and $\Delta^2 \widetilde{\mathbf{w}}_1 = 0$, so that

$$\widetilde{S}_1(0,0) = \rho^4 e^{\gamma \widetilde{\mathbf{w}}_1 + (1-\gamma_1)\widetilde{\mathbf{w}}_2} + \gamma_1 \Delta \widetilde{\mathbf{w}}_1 + \lambda_1 |\nabla \widetilde{\mathbf{w}}_1|^q.$$

We denote

$$\begin{aligned}
[\nabla, \chi_{r_0}]w = \nabla \chi_{r_0} \cdot w + \chi_{r_0} \cdot \nabla w, \quad [\Delta, \chi_{r_0}]w = w \Delta \chi_{r_0} + \chi_{r_0} \Delta w + 2 \nabla \chi_{r_0} \cdot \nabla w, \\
[\Delta^2, \chi_{r_0}]w = w \Delta^2 \chi_{r_0} + 2 \Delta w \Delta \chi_{r_0} + 4 \nabla (\Delta w) \cdot \nabla \chi_{r_0} \\
+ 4 \nabla w \cdot \nabla (\Delta \chi_{r_0}) + 4 \sum_{i,j=1}^4 \frac{\partial^2 \chi_{r_0}}{\partial z_i \partial z_j} \frac{\partial^2 w}{\partial z_i \partial z_j}.
\end{aligned}$$

Then

$$\begin{aligned}
|\widetilde{S}_1(0,0)| &\le c_\kappa \varepsilon^4 \left| e^{\gamma \left[\frac{1+\eta_1}{\gamma} G(z,\widetilde{z}_1) + H_1^{\text{ext}}\left(\widetilde{\varphi}_1^1, \widetilde{\psi}_1^1; \frac{z-\widetilde{z}_1}{r_\varepsilon}\right)\right]} e^{(1-\gamma)\left[\frac{1+\eta_2}{\xi} G(z,\widetilde{z}_2) + H_2^{\text{ext}}\left(\widetilde{\varphi}_2^1, \widetilde{\psi}_2^1; \frac{z-\widetilde{z}_1}{r_\varepsilon}\right)\right]} \right| \\
&\quad + c_\kappa \gamma_1 \left| \Delta \left(\frac{1+\eta_1}{\gamma} G(z,\widetilde{z}_1) + H_1^{\text{ext}}\left(\widetilde{\varphi}_1^1, \widetilde{\psi}_1^1; \frac{z-\widetilde{z}_1}{r_\varepsilon}\right)\right)\right| \\
&\quad + c_\kappa \lambda_1 \left| \nabla \left(\frac{1+\eta_1}{\gamma} G(z,\widetilde{z}_1) + H_1^{\text{ext}}\left(\widetilde{\varphi}_1^1, \widetilde{\psi}_1^1; \frac{z-\widetilde{z}_1}{r_\varepsilon}\right)\right)\right|^q \\
&\le c_\kappa \varepsilon^4 |z-\widetilde{z}_1|^{-8(1+\eta_1)} |z-\widetilde{z}_2|^{-8\frac{(1-\gamma)(1+\eta_2)}{\xi}} + c_\kappa \gamma_1 \frac{1+\eta_1}{\gamma} |z-\widetilde{z}_1|^{-2} \\
&\quad + c_\kappa \gamma_1 r_\varepsilon^3 |z-\widetilde{z}_1|^{-3} + c_\kappa \lambda_1 |z-\widetilde{z}_1|^{-q} + c_\kappa \lambda_1 r_\varepsilon^{3q} |z-\widetilde{z}_1|^{-2q} \\
&\le c_\kappa \varepsilon^4 |z-\widetilde{z}_1|^{-8(1+\eta_1)} + c_\kappa \gamma_1 \frac{1+\eta_1}{\gamma} |z-\widetilde{z}_1|^{-2} + c_\kappa \gamma_1 r_\varepsilon^3 |z-\widetilde{z}_1|^{-3} \\
&\quad + c_\kappa \lambda_1 |z-\widetilde{z}_1|^{-q} + c_\kappa \lambda_1 r_\varepsilon^{3q} |z-\widetilde{z}_1|^{-2q}.
\end{aligned}$$

Hence, for $\nu \in (-1,0)$, $q \in [1,4)$ and η_1 small enough, we get

$$\begin{aligned}
&\|\widetilde{S}_1(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(B_{r_0/2}(\widetilde{z}_1))} \\
&\le \sup_{r_\varepsilon \le r \le r_0/2} r^{4-\nu}|\widetilde{S}_1(0,0)| \\
&\le c_\kappa \varepsilon^4 \sup_{r_\varepsilon \le r \le r_0/2} r^{-4-\nu} + c_\kappa \gamma_1 \frac{1+\eta_1}{\gamma} \sup_{r_\varepsilon \le r \le r_0/2} r^{2-\nu} + c_\kappa \gamma_1 r_\varepsilon^3 \sup_{r_\varepsilon \le r \le r_0/2} r^{1-\nu} \\
&\quad + c_\kappa \lambda_1 \sup_{r_\varepsilon \le r \le r_0/2} r^{4-\nu-q} + c_\kappa \lambda_1 r_\varepsilon^{3q} \sup_{r_\varepsilon \le r \le r_0/2} r^{4-\nu-2q} \\
&\le c_\kappa \varepsilon^4 r_\varepsilon^{-4} + c_\kappa \gamma_1 + c_\kappa \gamma_1 r_\varepsilon^3 + c_\kappa \lambda_1 + c_\kappa \lambda_1 \max\left\{ r_\varepsilon^{3q}(r_0/2)^{4-\nu-2q}, r_\varepsilon^{4-\nu+q} \right\} \\
&\le c_\kappa r_\varepsilon^2.
\end{aligned}$$

• In $B_{r_0}(\widetilde{z}_1) \setminus B_{r_0/2}(\widetilde{z}_1)$, using the estimate (2.31) and taking into account that $\Delta^2 G(z,\widetilde{z}_1) = 0$, then we have

$$\begin{aligned}
|\widetilde{S}_1(0,0)| &\le \left| c_\kappa \varepsilon^4 e^{\gamma \widetilde{\mathbf{w}}_1 + (1-\gamma)\widetilde{\mathbf{w}}_2} + c_\kappa \gamma_1 \Delta \widetilde{\mathbf{w}}_1 + c_\kappa \lambda_1 |\nabla \widetilde{\mathbf{w}}_1|^q - \Delta^2 \widetilde{\mathbf{w}}_1 \right| \\
&\le c_\kappa \varepsilon^4 |z - \widetilde{z}_1|^{-8(1+\eta_1)} \\
&\quad + c_\kappa \gamma_1 \left(\frac{1+\eta_1}{\gamma} |\Delta G(z,\widetilde{z}_1)| + \sum_{i=1}^{2} \left| [\Delta, \chi_{r_0}(z-\widetilde{z}_i)] H_1^{\text{ext}} \left(\widetilde{\varphi}_1^i, \widetilde{\psi}_1^i; \frac{z-\widetilde{z}_i}{r_\varepsilon} \right) \right| \right) \\
&\quad + c_\kappa \lambda_1 \left| \frac{1+\eta_1}{\gamma} \nabla G(z,\widetilde{z}_1) + \sum_{i=1}^{2} [\nabla, \chi_{r_0}(z-\widetilde{z}_i)] H_1^{\text{ext}} \left(\widetilde{\varphi}_1^i, \widetilde{\psi}_1^i; \frac{z-\widetilde{z}_i}{r_\varepsilon} \right) \right|^q \\
&\quad + c_\kappa \sum_{i=1}^{2} \left| [\Delta^2, \chi_{r_0}(z-\widetilde{z}_i)] H_1^{\text{ext}} \left(\widetilde{\varphi}_1^i, \widetilde{\psi}_1^i; \frac{z-\widetilde{z}_i}{r_\varepsilon} \right) \right| \\
&\le c_\kappa \varepsilon^4 |z-\widetilde{z}_1|^{-8(1+\eta_1)} + c_\kappa \gamma_1 \frac{1+\eta_1}{\gamma} |z-\widetilde{z}_1|^{-2} + c_\kappa \gamma_1 r_\varepsilon^3 |z-\widetilde{z}_1|^{-3} \\
&\quad + c_\kappa \lambda_1 |z-\widetilde{z}_1|^{-q} + c_\kappa \lambda_1 r_\varepsilon^{3q} |z-\widetilde{z}_1|^{-2q}.
\end{aligned}$$

Hence, for $\nu \in (-1,0)$, $q \in [1,4)$ and η_1 small enough, we get

$$\begin{aligned}
\|\widetilde{S}_1(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(B_{r_0}(\widetilde{z}_1) - B_{r_0/2}(\widetilde{z}_1))} &\le \sup_{r_0/2 \le r \le r_0} r^{4-\nu} |\widetilde{S}_1(0,0)| \\
&\le c_\kappa \varepsilon^4 + c_\kappa \gamma_1 + c_\kappa \gamma_1 r_\varepsilon^3 + c_\kappa \lambda_1 \\
&\quad + c_\kappa \lambda_1 \max\left\{ r_\varepsilon^{3q} r_0^{4-\nu-2q}, r_\varepsilon^{3q} (r_0/2)^{4-\nu-2q} \right\} \\
&\le c_\kappa r_\varepsilon^2.
\end{aligned}$$

Similarly, for $\nu \in (-1,0)$ and η_2 small enough, we can prove the same result for $\widetilde{z}_2$.

• In $\Omega - \left(B_{r_0}(\widetilde{z}_1) \cup B_{r_0}(\widetilde{z}_2) \right)$, we have $\chi_{r_0}(z-\widetilde{z}_1) = 0$, $\chi_{r_0}(z-\widetilde{z}_2) = 0$ and $\Delta^2 \widetilde{\mathbf{w}}_1 = 0$.

Thus

$$\begin{aligned}|\widetilde{S}_1(0,0)| &\le c_\kappa \varepsilon^4 e^{\gamma\left(\frac{1+\eta_1}{\gamma}\right)G(z,\widetilde{z}_1)} e^{(1-\gamma)\left(\frac{1+\eta_2}{\xi}\right)G(z,\widetilde{z}_2)} \\ &\quad + c_\kappa\gamma_1 \frac{1+\eta_1}{\gamma}|\Delta G(z,\widetilde{z}_1)| + c_\kappa\lambda_1\left|\frac{1+\eta_1}{\gamma}\nabla G(z,\widetilde{z}_1)\right|^q \\ &\le c_\kappa\varepsilon^4|z-\widetilde{z}_1|^{-8(1+\eta_1)} + c_\kappa\gamma_1\frac{1+\eta_1}{\gamma}|z-\widetilde{z}_1|^{-2} + c_\kappa\lambda_1|z-\widetilde{z}_1|^{-q}.\end{aligned}$$

So for $\nu \in (-1,0)$, we have

$$\begin{aligned}\|\widetilde{S}_1(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-4}\left(\overline{\Omega}-\bigcup_{i=1}^2 B_{r_0}(\widetilde{z}_i)\right)} &\le \sup_{r\ge r_0} r^{4-\nu}|\widetilde{S}_1(0,0)| \\ &\le c_\kappa\varepsilon^4\sup_{r\ge r_0} r^{-4-\nu} + c_\kappa\gamma_1\frac{1+\eta_1}{\gamma}\sup_{r\ge r_0} r^{2-\nu} + c_\kappa\lambda_1\sup_{r\ge r_0} r^{4-\nu-q} \\ &\le c_\kappa\varepsilon^4 + c_\kappa\gamma_1 + c_\kappa\lambda_1 \le c_\kappa r_\varepsilon^2.\end{aligned}$$

Then with the previous three steps, we conclude that

$$\|\widetilde{S}_1(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(\overline{\Omega}_{r_0}(\widetilde{\mathbf{z}}))} \le c_\kappa r_\varepsilon^2.$$

Making use of Proposition 2.7 together with (2.26), we conclude that

$$\|\widetilde{\mathcal{N}}(0,0)\|_{\mathcal{C}^{4,\alpha}_{\nu}(\overline{\Omega}^*(\widetilde{\mathbf{z}}))} \le c_\kappa r_\varepsilon^2 \quad \text{and} \quad \|\widetilde{\mathcal{M}}(0,0)\|_{\mathcal{C}^{4,\alpha}_{\nu}(\overline{\Omega}^*(\widetilde{\mathbf{z}}))} \le c_\kappa r_\varepsilon^2.$$

For the proof of the third estimate, let $\widetilde{v}_1$, $\widetilde{v}_2$, $\widetilde{v}_1'$ and $\widetilde{v}_2' \in \mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*)$ satisfy (2.30), using Lemma 2.9 and the fact that for all $w \in \mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{z}))$, there exists $c > 0$ such that $|\nabla^i w| \le c r^{\nu-i}\|w\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{z}))}$, $i \ge 1$, we get

$$\begin{aligned}&\sup_{r\in\overline{\Omega}_{r_\varepsilon}} r^{4-\nu}\big|\widetilde{S}_1(\widetilde{v}_1,\widetilde{v}_2) - \widetilde{S}_1(\widetilde{v}_1',\widetilde{v}_2')\big| \\ &\le \sup_{r\in\overline{\Omega}_{r_\varepsilon}} r^{4-\nu}c_\kappa\varepsilon^4 e^{\gamma\widetilde{w}_1+(1-\gamma)\widetilde{w}_2}\big|e^{\gamma\widetilde{v}_1+(1-\gamma)\widetilde{v}_2} - e^{\gamma\widetilde{v}_1'+(1-\gamma)\widetilde{v}_2'}\big| + \sup_{r\in\overline{\Omega}_{r_\varepsilon}} r^{4-\nu}c_\kappa\gamma_1|\Delta(\widetilde{v}_1-\widetilde{v}_1')| \\ &\quad + \sup_{r\in\overline{\Omega}_{r_\varepsilon}} r^{4-\nu}c_\kappa\lambda_1\big(|\nabla\widetilde{w}_1+\nabla\widetilde{v}_1'|^{q-1} + |\nabla\widetilde{v}_1-\nabla\widetilde{v}_1'|^{q-1}\big)|\nabla(\widetilde{v}_1-\widetilde{v}_1')| \\ &\le c_\kappa\varepsilon^4\sup_{r\in\overline{\Omega}_{r_\varepsilon}} r^{4-\nu}|z-\widetilde{z}_1|^{-8(1+\eta_1)}|z-\widetilde{z}_2|^{-8\frac{(1-\gamma)(1+\eta_2)}{\xi}}\big(\gamma|\widetilde{v}_1-\widetilde{v}_1'| + (1-\gamma)|\widetilde{v}_2-\widetilde{v}_2'|\big) \\ &\quad + c_\kappa\gamma_1\sup_{r\in\overline{\Omega}_{r_\varepsilon}} r^{4-\nu}|\Delta(\widetilde{v}_1-\widetilde{v}_1')| \\ &\quad + c_\kappa\lambda_1\sup_{r\in\overline{\Omega}_{r_\varepsilon}} r^{4-\nu}\big(|\nabla\widetilde{w}_1|^{q-1} + |\nabla\widetilde{v}_1'|^{q-1} + |\nabla\widetilde{v}_1|^{q-1}\big)|\nabla(\widetilde{v}_1-\widetilde{v}_1')| \\ &\le c_\kappa\varepsilon^4\sup_{r\in\overline{\Omega}_{r_\varepsilon}} r^{4-\nu}r^{-8(1+\eta_1)}\big(\gamma r^\nu\|\widetilde{v}_1-\widetilde{v}_1'\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*)} + (1-\gamma)r^\nu\|\widetilde{v}_2-\widetilde{v}_2'\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*)}\big) \\ &\quad + c_\kappa\gamma_1\sup_{r\in\overline{\Omega}_{r_\varepsilon}} r^2\|\widetilde{v}_1-\widetilde{v}_1'\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*)}\end{aligned}$$

$$
\begin{aligned}
&+ c_\kappa \lambda_1 \sup_{r\in\overline{\Omega}_{r_\varepsilon}} r^{4-\nu}\Bigg[r^{\nu-1}\Bigg|\frac{1+\eta_1}{\gamma}\nabla G(z,\overline{z}_1)+\sum_{i=1}^{2}[\nabla,\chi_{r_0}(z-\widetilde{z}_i)]H_1^{\text{ext}}\left(\widetilde{\varphi}_1^i,\widetilde{\psi}_1^i;\frac{z-\widetilde{z}_i}{r_\varepsilon}\right)\Bigg|^{q-1}\\
&\qquad + r^{\nu-1}r^{(\nu-1)(q-1)}\big(\|\widetilde{v}_1\|^{q-1}_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*)}+\|\widetilde{v}'_1\|^{q-1}_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*)}\big)\Bigg]\|\widetilde{v}_1-\widetilde{v}'_1\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*)}\\
&\le c_\kappa\varepsilon^4 r_\varepsilon^{-4}\big(\|\widetilde{v}_1-\widetilde{v}'_1\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*)}+\|\widetilde{v}_2-\widetilde{v}'_2\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*)}\big)+c_\kappa\gamma_1\|\widetilde{v}_1-\widetilde{v}'_1\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*)}\\
&\quad + c_\kappa\lambda_1\left(\sup_{r\in\overline{\Omega}_{r_\varepsilon}} r^{4-q}+r_\varepsilon^{3(q-1)}\sup_{r\in\overline{\Omega}_{r_\varepsilon}} r^{5-2q}\right)\|\widetilde{v}_1-\widetilde{v}'_1\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*)}\\
&\quad + 2c_\kappa\lambda_1 r_\varepsilon^{2(q-1)}\sup_{r\in\overline{\Omega}_{r_\varepsilon}} r^{3+(\nu-1)(q-1)}\|\widetilde{v}_1-\widetilde{v}'_1\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*)}\\
&\le c_\kappa\varepsilon^4 r_\varepsilon^{-4}\big(\|\widetilde{v}_1-\widetilde{v}'_1\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*)}+\|\widetilde{v}_2-\widetilde{v}'_2\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*)}\big)+c_\kappa\gamma_1\|\widetilde{v}_1-\widetilde{v}'_1\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*)}\\
&\quad + c_\kappa\lambda_1\big(1+\max\{r_\varepsilon^{2+q},r_\varepsilon^{3(q-1)}\}+\max\{r_\varepsilon^{2+q+\nu(q-1)},r_\varepsilon^{2(q-1)}\}\big)\|\widetilde{v}_1-\widetilde{v}'_1\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*)}.
\end{aligned}
$$

So, for η_1 small enough and using the estimate (2.26), there exists $\overline{c}_\kappa$ (depending on κ) such that

$$
\|\widetilde{\mathcal{N}}(\widetilde{v}_1,\widetilde{v}_2)-\widetilde{\mathcal{N}}(\widetilde{v}'_1,\widetilde{v}'_2)\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{z}}))}\le \overline{c}_\kappa r_\varepsilon^2\big(\|\widetilde{v}_1-\widetilde{v}'_1\|_{C^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{z}}))}+\|\widetilde{v}_2-\widetilde{v}'_2\|_{C^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{z}}))}\big). \tag{2.32}
$$

Similarly we can use the same argument to prove

$$
\|\widetilde{\mathcal{M}}(\widetilde{v}_1,\widetilde{v}_2)-\widetilde{\mathcal{M}}(\widetilde{v}'_1,\widetilde{v}'_2)\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{z}}))}\le \overline{c}_\kappa r_\varepsilon^2\big(\|\widetilde{v}_1-\widetilde{v}'_1\|_{C^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{z}}))}+\|\widetilde{v}_2-\widetilde{v}'_2\|_{C^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{z}}))}\big). \tag{2.33}
$$

□

Reducing ε_κ if necessary, we can assume that $\overline{c}_\kappa r_\varepsilon^2 \le 1/2$ for all $\varepsilon\in(0,\varepsilon_\kappa)$. Then, (2.32) and (2.33) are enough to show that

$$
(\widetilde{v}_1,\widetilde{v}_2)\mapsto\big(\widetilde{\mathcal{N}}(\widetilde{v}_1,\widetilde{v}_2),\widetilde{\mathcal{M}}(\widetilde{v}_1,\widetilde{v}_2)\big)
$$

is a contraction from the ball

$$
\big\{(\widetilde{v}_1,\widetilde{v}_2)\in\big(\mathcal{C}^{4,\alpha}_\nu(\mathbb{R}^4)\big)^2 : \|(\widetilde{v}_1,\widetilde{v}_2)\|_{(\mathcal{C}^{4,\alpha}_\nu(\mathbb{R}^4))^2}\le 2\overline{c}_\kappa r_\varepsilon^2\big\}
$$

into itself. Hence there exists a unique fixed point $(\widetilde{v}_1,\widetilde{v}_2)$ in this set, which is a solution of (2.27). Applying a fixed point theorem for contraction mappings, we conclude that

Proposition 2.19. *Given $\kappa>0$, there exists $\varepsilon_\kappa>0$ (depending on κ) such that for any $\varepsilon\in(0,\varepsilon_\kappa)$, η_i and $\widetilde{z}_i$ satisfying* (2.29) *and functions $\widetilde{\varphi}^i_j$ and $\widetilde{\psi}^i_j$ satisfying* (2.15) *and* (2.28)*, there exists a unique $(\widetilde{v}_1,\widetilde{v}_2)$ ($:=(\widetilde{v}_{1,\varepsilon,\eta_1,\widetilde{\mathbf{z}},\widetilde{\varphi}^i_1,\widetilde{\psi}^i_1},\widetilde{v}_{2,\varepsilon,\eta_2,\widetilde{\mathbf{z}},\widetilde{\varphi}^i_2,\widetilde{\psi}^i_2})$) solution of* (2.27) *so that for (v_1,v_2) defined by*

$$
v_1(z):=\frac{1+\eta_1}{\gamma}G(z,\widetilde{z}_1)+\sum_{i=1}^{2}\chi_{r_0}(z-\widetilde{z}_i)H_1^{\text{ext}}\left(\widetilde{\varphi}^i_1,\widetilde{\psi}^i_1;\frac{z-\widetilde{z}_i}{r_\varepsilon}\right)+\widetilde{v}_1(z),
$$

$$
v_2(z):=\frac{1+\eta_2}{\xi}G(z,\widetilde{z}_2)+\sum_{i=1}^{2}\chi_{r_0}(z-\widetilde{z}_i)H_2^{\text{ext}}\left(\widetilde{\varphi}^i_2,\widetilde{\psi}^i_2;\frac{z-\widetilde{z}_i}{r_\varepsilon}\right)+\widetilde{v}_2(z)
$$

solve (2.24) *in* $\overline{\Omega}_{r_\varepsilon}(\widetilde{\mathbf{z}})$. *In addition, we have*

$$\|(\widetilde{v}_1, \widetilde{v}_2)\|_{(\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{z}})))^2} \le 2\overline{c}_\kappa r_\varepsilon^2.$$

2.4. The nonlinear Cauchy-data matching

We will gather the results of the previous sections. Using the previous notations, assume that $\widetilde{\mathbf{z}} := (\widetilde{z}_1, \widetilde{z}_2) \in \Omega^2$ are given close to $\mathbf{z} := (z_1, z_2)$. Assume also that

$$\boldsymbol{\tau} := (\tau_1, \tau_2) \in [\tau_1^-, \tau_1^+] \times [\tau_2^-, \tau_2^+] \subset (0, \infty)^2$$

are given (the values of τ_l^- and τ_l^+, $l = 1, 2$, will be fixed later). First, we consider some set of boundary data $\boldsymbol{\varphi}^i := (\varphi_1^i, \varphi_2^i) \in (\mathcal{C}^{4,\alpha}(S^3))^2$ and $\boldsymbol{\psi}^i := (\psi_1^i, \psi_2^i) \in (\mathcal{C}^{2,\alpha}(S^3))^2$. Given $\varepsilon \in (0, \varepsilon_\kappa)$ and according to the results of Propositions 2.16 and 2.17, we can find $u_{\text{int}} := (u_{\text{int},1}, u_{\text{int},2})$ a solution of (2.5) in $B_{r_\varepsilon}(\widetilde{z}_1) \cup B_{r_\varepsilon}(\widetilde{z}_2)$, which can be decomposed as

$$u_{\text{int},1}(z) := \begin{cases} \frac{1}{\gamma} u_{\varepsilon,\tau_1}(z - \widetilde{z}_1) - \frac{1-\gamma}{\gamma\xi} G(z, \widetilde{z}_2) - \frac{\ln\gamma}{\gamma} + h_1^1\big(\frac{R_\varepsilon^1(z-\widetilde{z}_1)}{r_\varepsilon}\big) & \\ \quad + v_1^1\big(\frac{R_\varepsilon^1(z-\widetilde{z}_1)}{r_\varepsilon}\big) + H_1^{\text{int},1}\big(\varphi_1^1, \psi_1^1; \frac{z-\widetilde{z}_1}{r_\varepsilon}\big) & \text{in } B_{r_\varepsilon}(\widetilde{z}_1), \\ \frac{1}{\gamma} G(z, \widetilde{z}_1) + h_1^2\big(\frac{R_\varepsilon^2(z-\widetilde{z}_2)}{r_\varepsilon}\big) + v_1^2\big(\frac{R_\varepsilon^2(z-\widetilde{z}_2)}{r_\varepsilon}\big) + H_1^{\text{int},2}\big(\varphi_1^2, \psi_1^2; \frac{z-\widetilde{z}_2}{r_\varepsilon}\big) & \text{in } B_{r_\varepsilon}(\widetilde{z}_2) \end{cases}$$

and

$$u_{\text{int},2}(z) := \begin{cases} \frac{1}{\xi} G(z, \widetilde{z}_2) + h_2^1\big(\frac{R_\varepsilon^1(z-\widetilde{z}_1)}{r_\varepsilon}\big) + v_2^1\big(\frac{R_\varepsilon^1(z-\widetilde{z}_1)}{r_\varepsilon}\big) + H_2^{\text{int},1}\big(\varphi_2^1, \psi_2^1; \frac{z-\widetilde{z}_1}{r_\varepsilon}\big) & \text{in } B_{r_\varepsilon}(\widetilde{z}_1), \\ \frac{1}{\xi} u_{\varepsilon,\tau_1}(z - \widetilde{z}_2) - \frac{1-\xi}{\gamma\xi} G(z, \widetilde{z}_1) - \frac{\ln\xi}{\xi} + h_2^2\big(\frac{R_\varepsilon^2(z-\widetilde{z}_2)}{r_\varepsilon}\big) & \\ \quad + v_2^2\big(\frac{R_\varepsilon^2(z-\widetilde{z}_2)}{r_\varepsilon}\big) + H_2^{\text{int},2}\big(\varphi_2^2, \psi_2^2; \frac{z-\widetilde{z}_2}{r_\varepsilon}\big) & \text{in } B_{r_\varepsilon}(\widetilde{z}_2), \end{cases}$$

where for $i, j \in \{1, 2\}$, $R_\varepsilon^i = \tau_i \frac{r_\varepsilon}{\varepsilon}$ and the functions h_j^i and v_j^i satisfy

$$\|(h_1^1, h_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4) \times \mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2, \quad \|(h_1^2, h_2^2)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4) \times \mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2$$

and

$$\|(v_1^1, v_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4) \times \mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2, \quad \|(v_1^2, v_2^2)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4) \times \mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2.$$

Similarly, given some boundary data $\widetilde{\varphi}_j^i \in C^{4,\alpha}(S^3)$, $\widetilde{\psi}_j^i \in C^{2,\alpha}(S^3)$ satisfying (2.15), $(\eta_1, \eta_2) \in \mathbb{R}^2$ satisfying (2.29), provided $\varepsilon \in (0, \varepsilon_\kappa)$, by Proposition 2.19, we find a solution $u_{\text{ext}} := (u_{\text{ext},1}, u_{\text{ext},2})$ of (2.5) in $\overline{\Omega} \setminus (B_{r_\varepsilon}(\widetilde{z}_1) \cup B_{r_\varepsilon}(\widetilde{z}_2))$ which can be decomposed as

$$u_{\text{ext},1}(z) := \frac{1+\eta_1}{\gamma} G(z, \widetilde{z}_1) + \sum_{i=1,2} \chi_{r_0}(z - \widetilde{z}_i) H_1^{\text{ext}}\left(\widetilde{\varphi}_1^i, \widetilde{\psi}_1^i; \frac{z - \widetilde{z}_i}{r_\varepsilon}\right) + \widetilde{v}_1(z),$$

$$u_{\text{ext},2}(z) := \frac{1+\eta_2}{\xi} G(z, \widetilde{z}_2) + \sum_{i=1,2} \chi_{r_0}(z - \widetilde{z}_i) H_2^{\text{ext}}\left(\widetilde{\varphi}_2^i, \widetilde{\psi}_2^i; \frac{z - \widetilde{z}_i}{r_\varepsilon}\right) + \widetilde{v}_2(z)$$

with $\widetilde{v}_1, \widetilde{v}_2 \in \mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{z}}))$ satisfying

$$\|(\widetilde{v}_1, \widetilde{v}_2)\|_{(\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\mathbf{z}})))^2} \leq 2\overline{c}_\kappa r_\varepsilon^2.$$

It remains to determine the parameters and the boundary data in such a way that the function which is equal to u_{int} in $B_{r_\varepsilon}(\widetilde{z}_1) \cup B_{r_\varepsilon}(\widetilde{z}_2)$ and to u_{ext} in $\overline{\Omega}_{r_\varepsilon}(\widetilde{\mathbf{z}})$ is a smooth function. This amounts to find the boundary data and the parameters so that, for each $i = 1, 2$,

$$\begin{aligned} u_{\text{int},i} &= u_{\text{ext},i}, & \partial_r u_{\text{int},i} &= \partial_r u_{\text{ext},i}, \\ \Delta u_{\text{int},i} &= \Delta u_{\text{ext},i}, & \partial_r \Delta u_{\text{int},i} &= \partial_r \Delta u_{\text{ext},i} \end{aligned} \tag{2.34}$$

on $\partial B_{r_\varepsilon}(\widetilde{z}_1)$ and $\partial B_{r_\varepsilon}(\widetilde{z}_2)$.

In other words, here we try to match the interior and exterior solutions obtained in the previous sections as well as their normal derivatives on the boundary of each small ball for the equations u_1 and u_2 respectively by choosing the suitable data at the boundaries.

Suppose that (2.34) is verified, this provides that for each ε small enough $u_\varepsilon \in \mathcal{C}^{4,\alpha}$ (which is obtained by matching together the functions u_{int} and the function u_{ext}), a weak solution of our system and elliptic regularity theory implies that this solution is in fact smooth. That will complete the proof since as ε tends to 0, the sequence of solutions we have obtained satisfies the required singular limit behavior, namely, u_ε strongly converges to $G(\,\cdot\,, z_i)$.

Before we proceed, the following remarks are due. First it will be convenient to observe that the function u_{ε,τ_i} can be expanded as

$$u_{\varepsilon,\tau_i}(z) = -4\ln\tau_i - 8\ln|z| + \mathcal{O}\left(\frac{\varepsilon^2\tau_i^{-2}}{|z|^2}\right) \quad \text{on } \partial B_{r_\varepsilon}(z_i).$$

- For z on $\partial B_{r_\varepsilon}(\widetilde{z}_1)$, we have

$$\begin{aligned} &(u_{\text{int},1} - u_{\text{ext},1})(z) \\ &= -\frac{4}{\gamma}\ln\tau_1 + \frac{8\eta_1}{\gamma}\ln|z-\widetilde{z}_1| - \frac{1-\gamma}{\gamma\xi}G(z,\widetilde{z}_2) - \frac{\ln\gamma}{\gamma} + h_1^1\left(R_\varepsilon^1\frac{z-\widetilde{z}_1}{r_\varepsilon}\right) \\ &\quad + v_1^1\left(R_\varepsilon^1\frac{z-\widetilde{z}_1}{r_\varepsilon}\right) + H_1^{\text{int},1}\left(\varphi_1^1, \psi_1^1, \frac{z-\widetilde{z}_1}{r_\varepsilon}\right) - H_1^{\text{ext}}\left(\widetilde{\varphi}_1^1, \widetilde{\psi}_1^1, \frac{z-\widetilde{z}_1}{r_\varepsilon}\right) \\ &\quad - \frac{1+\eta_1}{\gamma}H(z,\widetilde{z}_1) + \mathcal{O}\left(\frac{\varepsilon^2\tau_1^{-2}}{|z-\widetilde{z}_1|^2}\right) + \mathcal{O}(r_\varepsilon^2). \end{aligned} \tag{2.35}$$

Next, even though all functions are defined on $\partial B_{r_\varepsilon}(\widetilde{z}_1)$ in (2.34), it will be more convenient to solve on S^3 the following set of equations

$$\begin{aligned} (u_{\text{int},1} - u_{\text{ext},1})(\widetilde{z}_1 + r_\varepsilon\,\cdot\,) &= 0, & \partial_r(u_{\text{int},1} - u_{\text{ext},1})(\widetilde{z}_1 + r_\varepsilon\,\cdot\,) &= 0, \\ \Delta(u_{\text{int},1} - u_{\text{ext},1})(\widetilde{z}_1 + r_\varepsilon\,\cdot\,) &= 0 \quad \text{and} & \partial_r\Delta(u_{\text{int},1} - u_{\text{ext},1})(\widetilde{z}_1 + r_\varepsilon\,\cdot\,) &= 0. \end{aligned} \tag{2.36}$$

Since the boundary data are chosen to satisfy (2.14) or (2.15), we decompose

$$\varphi_1^1 = \varphi_{1,0}^1 + \varphi_{1,1}^1 + \varphi_1^{1,\perp}, \quad \psi_1^1 = 8\varphi_{1,0}^1 + 12\varphi_{1,1}^1 + \psi_1^{1,\perp},$$
$$\widetilde{\varphi}_1^1 = \widetilde{\varphi}_{1,0}^1 + \widetilde{\varphi}_{1,1}^1 + \widetilde{\varphi}_1^{1,\perp}, \quad \widetilde{\psi}_1^1 = \widetilde{\psi}_{1,1}^1 + \widetilde{\psi}_1^{1,\perp},$$

where $\varphi_{1,0}^1, \widetilde{\varphi}_{1,0}^1 \in \mathbb{E}_0 = \mathbb{R}$ are constant on S^3, $\varphi_{1,1}^1$, $\widetilde{\varphi}_{1,1}^1$, $\widetilde{\psi}_{1,1}^1$ belong to $\mathbb{E}_1 = \mathrm{Span}\{e_1, e_2, e_3, e_4\}$ and $\varphi_1^{1,\perp}$, $\widetilde{\varphi}_1^{1,\perp}$, $\psi_1^{1,\perp}$, $\widetilde{\psi}_1^{1,\perp}$ are $L^2(S^3)$ orthogonal to $\mathbb{E}_0$ and $\mathbb{E}_1$.

Using (2.35), we have for $z \in S^3$,

$$\begin{aligned}
&(u_{\mathrm{int},1} - u_{\mathrm{ext},1})(\widetilde{z}_1 + r_\varepsilon z) \\
&= -\frac{4}{\gamma}\ln\tau_1 + \frac{8\eta_1}{\gamma}\ln(r_\varepsilon|z|) - \frac{1}{\gamma}\left(H(\widetilde{z}_1,\widetilde{z}_1) + \frac{1-\gamma}{\xi}G(\widetilde{z}_1,\widetilde{z}_2)\right) \\
&\quad - \frac{\ln\gamma}{\gamma} + H_1^{\mathrm{int},1}(\varphi_1^1,\psi_1^1,z) - H_1^{\mathrm{ext}}(\widetilde{\varphi}_1^1,\widetilde{\psi}_1^1,z) - \frac{\eta_1}{\gamma}H(\widetilde{z}_1,\widetilde{z}_1) + \mathcal{O}(r_\varepsilon^2).
\end{aligned}$$

Then, the projection of the set equations (2.36) over $\mathbb{E}_0$ will yield

$$\begin{aligned}
-4\ln\tau_1 + 8\eta_1\ln r_\varepsilon - \ln\gamma + \gamma\varphi_{1,0}^1 - \gamma\widetilde{\varphi}_{1,0}^1 - \mathcal{E}_1(\widetilde{z}_1,\widetilde{\mathbf{z}}) + \mathcal{O}(r_\varepsilon^2) &= 0, \\
8\eta_1 + 2\gamma\varphi_{1,0}^1 + 2\gamma\widetilde{\varphi}_{1,0}^1 + \mathcal{O}(r_\varepsilon^2) &= 0, \\
16\eta_1 + 8\gamma\varphi_{1,0}^1 + \mathcal{O}(r_\varepsilon^2) &= 0, \\
-32\eta_1 + \mathcal{O}(r_\varepsilon^2) &= 0,
\end{aligned} \tag{2.37}$$

where

$$\mathcal{E}_1(\,\cdot\,,\widetilde{\mathbf{z}}) := H(\,\cdot\,,\widetilde{z}_1) + \frac{1-\gamma}{\xi}G(\,\cdot\,,\widetilde{z}_2).$$

The system (2.37) can be simply written as

$$\eta_1 = \mathcal{O}(r_\varepsilon^2), \quad \varphi_{1,0}^1 = \mathcal{O}(r_\varepsilon^2), \quad \widetilde{\varphi}_{1,0}^1 = \mathcal{O}(r_\varepsilon^2) \quad \text{and} \quad \frac{1}{\ln r_\varepsilon}\big[4\ln\tau_1 + \ln\gamma + \mathcal{E}_1(\widetilde{z}_1,\widetilde{\mathbf{z}})\big] = \mathcal{O}(r_\varepsilon^2).$$

We are now in a position to define τ_1^- and τ_1^+. In fact, according to the above analysis, as ε tends to 0, we expect that $\widetilde{z}_i$ will converge to z_i for $i \in \{1,2\}$ and τ_1 will converge to τ_1^* satisfying

$$4\ln\tau_1^* = -\ln\gamma - \mathcal{E}_1(z_1,\mathbf{z}).$$

Hence it is enough to choose τ_1^- and τ_1^+ in such a way that

$$4\ln(\tau_1^-) < -\ln\gamma - \mathcal{E}_1(z_1,\mathbf{z}) < 4\ln(\tau_1^+).$$

Consider now the projection of (2.36) over $\mathbb{E}_1$. Give a smooth function f defined in Ω, we identify its gradient $\nabla f = (\partial_{x_1} f, \ldots, \partial_{x_4} f)$ with the element of $\mathbb{E}_1$ as

$$\overline{\nabla} f = \sum_{i=1}^4 \partial_{x_i} f e_i.$$

With these notations in mind, we obtain the system of equations

$$\begin{aligned}
\varphi^1_{1,1} - \widetilde{\varphi}^1_{1,1} - \frac{1}{\gamma}\overline{\nabla}\mathcal{E}_1(\widetilde{z}^1, \widetilde{\mathbf{z}}) + \mathcal{O}(r_\varepsilon^2) &= 0,\\
3\varphi^1_{1,1} + 3\widetilde{\varphi}^1_{1,1} + \frac{1}{2}\widetilde{\psi}^1_{1,1} - \frac{1}{\gamma}\overline{\nabla}\mathcal{E}_1(\widetilde{z}_1, \widetilde{\mathbf{z}}) + \mathcal{O}(r_\varepsilon^2) &= 0,\\
15\varphi^1_{1,1} - 3\widetilde{\varphi}^1_{1,1} - \widetilde{\psi}^1_{1,1} + \mathcal{O}(r_\varepsilon^2) &= 0,\\
15\varphi^1_{1,1} + 15\widetilde{\varphi}^1_{1,1} + \frac{18}{4}\widetilde{\psi}^1_{1,1} + \mathcal{O}(r_\varepsilon^2) &= 0,
\end{aligned}$$

which can be simplified as follows

$$\varphi^1_{1,1} = \mathcal{O}(r_\varepsilon^2), \quad \psi^1_{1,1} = \mathcal{O}(r_\varepsilon^2), \quad \widetilde{\varphi}^1_{1,1} = \mathcal{O}(r_\varepsilon^2), \quad \widetilde{\psi}^1_{1,1} = \mathcal{O}(r_\varepsilon^2) \quad \text{and} \quad \overline{\nabla}\mathcal{E}_1(\widetilde{z}_1, \widetilde{\mathbf{z}}) = \mathcal{O}(r_\varepsilon^2).$$

Finally, we consider the projection onto $L^2(S^3)^\perp$. This is the step in the proof which makes use of the nondegeneracy condition assumed on the critical point of the functional $\mathcal{F}$, see also Remark 2.20 at the end of the section. This assumption is needed in order to obtain the order r_ε^2 in the following estimates. For more details on this condition, we refer the reader to [5]. This yields the system

$$\begin{aligned}
\varphi_1^{1,\perp} - \widetilde{\varphi}_1^{1,\perp} + \mathcal{O}(r_\varepsilon^2) &= 0,\\
\partial_r\big(H^{\text{int}}_{\varphi_1^{1,\perp},\psi_1^{1,\perp}} - H^{\text{ext}}_{\widetilde{\varphi}_1^{1,\perp},\widetilde{\psi}_1^{1,\perp}}\big) + \mathcal{O}(r_\varepsilon^2) &= 0,\\
\psi_1^{1,\perp} - \widetilde{\psi}_1^{1,\perp} + \mathcal{O}(r_\varepsilon^2) &= 0,\\
\partial_r\Delta\big(H^{\text{int}}_{\varphi_1^{1,\perp},\psi_1^{1,\perp}} - H^{\text{ext}}_{\widetilde{\varphi}_1^{1,\perp},\widetilde{\psi}_1^{1,\perp}}\big) + \mathcal{O}(r_\varepsilon^2) &= 0.
\end{aligned}$$

Applying Lemma 2.14, this last system can be rewritten as

$$\varphi_1^{1,\perp} = \mathcal{O}(r_\varepsilon^2), \quad \widetilde{\varphi}_1^{1,\perp} = \mathcal{O}(r_\varepsilon^2), \quad \psi_1^{1,\perp} = \mathcal{O}(r_\varepsilon^2) \quad \text{and} \quad \widetilde{\psi}_1^{1,\perp} = \mathcal{O}(r_\varepsilon^2).$$

If we define the parameter $t_1 \in \mathbb{R}$ by

$$t_1 := \frac{1}{\ln r_\varepsilon}\big[4\ln\tau_1 + \ln\gamma + \mathcal{E}_1(\widetilde{z}_1, \widetilde{\mathbf{z}})\big],$$

then the systems found by projecting (2.36) gather in this equality

$$T^1_\varepsilon = \big(t_1, \eta_1, \varphi^1_{1,0}, \widetilde{\varphi}^1_{1,0}, \varphi^1_{1,1}, \widetilde{\varphi}^1_{1,1}, \widetilde{\psi}^1_{1,1}, \overline{\nabla}\mathcal{E}_1(\widetilde{z}_1, \widetilde{\mathbf{z}}), \varphi_1^{1,\perp}, \widetilde{\varphi}_1^{1,\perp}, \psi_1^{1,\perp}, \widetilde{\psi}_1^{1,\perp}\big) = \mathcal{O}_1(r_\varepsilon^2). \tag{2.38}$$

As usual, the terms $\mathcal{O}_1(r_\varepsilon^2)$ depend nonlinearly on all the variables on the left side, but are bounded (in the appropriate norm) by a constant (independent of ε and κ) times r_ε^2, provided $\varepsilon \in (0, \varepsilon_\kappa)$.

- On the other hand, on $\partial B_{r_\varepsilon}(\widetilde{z}_1)$, we have

$$\begin{aligned}
(u_{\text{int},2} - u_{\text{ext},2})(z) = &-\frac{\eta_2}{\xi}G(z, \widetilde{z}_2) + h^1_2\left(R^1_\varepsilon\frac{z - \widetilde{z}_1}{r_\varepsilon}\right) + H_2^{\text{int},1}\left(\varphi^1_2, \psi^1_2, \frac{z - \widetilde{z}_1}{r_\varepsilon}\right)\\
&- H_2^{\text{ext}}\left(\widetilde{\varphi}^1_2, \widetilde{\psi}^1_2, \frac{z - \widetilde{z}_1}{r_\varepsilon}\right) + v^1_2\left(R^1_\varepsilon\frac{z - \widetilde{z}_1}{r_\varepsilon}\right) + \mathcal{O}(r_\varepsilon^2).
\end{aligned}$$

In the same manner as above, we will solve on S^3 the following system

$$(2.39)\qquad \begin{aligned} &(u_{\mathrm{int},2}-u_{\mathrm{ext},2})(\widetilde{z}_1+r_\varepsilon\,\cdot\,)=0, && \partial_r(u_{\mathrm{int},2}-u_{\mathrm{ext},2})(\widetilde{z}_1+r_\varepsilon\,\cdot\,)=0,\\ &\Delta(u_{\mathrm{int},2}-u_{\mathrm{ext},2})(\widetilde{z}_1+r_\varepsilon\,\cdot\,)=0 \quad \text{and} && \partial_r\Delta(u_{\mathrm{int},2}-u_{\mathrm{ext},2})(\widetilde{z}_1+r_\varepsilon\,\cdot\,)=0.\end{aligned}$$

We decompose

$$\begin{aligned} \varphi_2^1&=\varphi_{2,0}^1+\varphi_{2,1}^1+\varphi_2^{1,\perp}, \quad \psi_2^1=8\varphi_{2,0}^1+12\varphi_{2,1}^1+\psi_2^{1,\perp},\\ \widetilde{\varphi}_2^1&=\widetilde{\varphi}_{2,0}^1+\widetilde{\varphi}_{2,1}^1+\widetilde{\varphi}_2^{1,\perp}, \quad \widetilde{\psi}_2^1=\widetilde{\psi}_{2,1}^1+\widetilde{\psi}_2^{1,\perp},\end{aligned}$$

where $\varphi_{2,0}^1,\widetilde{\varphi}_{2,0}^1\in\mathbb{E}_0$, $\varphi_{2,1}^1,\widetilde{\varphi}_{2,1}^1,\widetilde{\psi}_{2,1}^1\in\mathbb{E}_1$ and $\varphi_2^{1,\perp}$, $\widetilde{\varphi}_2^{1,\perp}$, $\psi_2^{1,\perp}$, $\widetilde{\psi}_2^{1,\perp}$ belong to $(L^2(S^3))^\perp$.

Projecting the set of equations (2.39) over $\mathbb{E}_0$, we get

$$\begin{aligned} \varphi_{2,0}^1-\widetilde{\varphi}_{2,0}^1+\mathcal{O}(r_\varepsilon^2)&=0,\\ 2\varphi_{2,0}^1+2\widetilde{\varphi}_{2,0}^1+\mathcal{O}(r_\varepsilon^2)&=0,\\ 8\varphi_{2,0}^1+\mathcal{O}(r_\varepsilon^2)&=0.\end{aligned}$$

From the L^2-projection of (2.39) over $\mathbb{E}_1$, we obtain the system of equations

$$\begin{aligned} \varphi_{2,1}^1-\widetilde{\varphi}_{2,1}^1+\mathcal{O}(r_\varepsilon^2)&=0,\\ 3\varphi_{2,1}^1+3\widetilde{\varphi}_{2,1}^1+\frac{1}{2}\widetilde{\psi}_{2,1}^1+\mathcal{O}(r_\varepsilon^2)&=0,\\ 15\varphi_{2,1}^1-3\widetilde{\varphi}_{2,1}^1-\widetilde{\psi}_{2,1}^1+\mathcal{O}(r_\varepsilon^2)&=0,\\ 15\varphi_{2,1}^1+15\widetilde{\varphi}_{2,1}^1+\frac{18}{4}\widetilde{\psi}_{2,1}^1+\mathcal{O}(r_\varepsilon^2)&=0.\end{aligned}$$

Finally, we consider the L^2-projection onto $(L^2(S^3))^\perp$. This yields the system

$$\begin{aligned} \varphi_2^{1,\perp}-\widetilde{\varphi}_2^{1,\perp}+\mathcal{O}(r_\varepsilon^2)&=0,\\ \partial_r\big(H^{\mathrm{int}}_{\varphi_2^{1,\perp},\psi_2^{1,\perp}}-H^{\mathrm{ext}}_{\widetilde{\varphi}_2^{1,\perp},\widetilde{\psi}_2^{1,\perp}}\big)+\mathcal{O}(r_\varepsilon^2)&=0,\\ \psi_2^{1,\perp}-\widetilde{\psi}_2^{1,\perp}+\mathcal{O}(r_\varepsilon^2)&=0,\\ \partial_r\Delta\big(H^{\mathrm{int}}_{\varphi_2^{1,\perp},\psi_2^{1,\perp}}-H^{\mathrm{ext}}_{\widetilde{\varphi}_2^{1,\perp},\widetilde{\psi}_2^{1,\perp}}\big)+\mathcal{O}(r_\varepsilon^2)&=0.\end{aligned}$$

Using Lemma 2.14 again, the above system can be rewritten as

$$\varphi_2^{1,\perp}=\mathcal{O}(r_\varepsilon^2), \quad \widetilde{\varphi}_2^{1,\perp}=\mathcal{O}(r_\varepsilon^2), \quad \psi_2^{1,\perp}=\mathcal{O}(r_\varepsilon^2) \quad \text{and} \quad \widetilde{\psi}_2^{1,\perp}=\mathcal{O}(r_\varepsilon^2).$$

Then, the systems found by projecting (2.39) gather in this equality

$$\big(\varphi_{2,0}^1,\widetilde{\varphi}_{2,0}^1,\varphi_{2,1}^1,\widetilde{\varphi}_{2,1}^1,\widetilde{\psi}_{2,1}^1,\varphi_2^{1,\perp},\widetilde{\varphi}_2^{1,\perp},\psi_2^{1,\perp},\widetilde{\psi}_2^{1,\perp}\big)=\mathcal{O}(r_\varepsilon^2).$$

- On $\partial B_{r_\varepsilon}(\widetilde{z}_2)$, we have

$$(u_{\mathrm{int},1} - u_{\mathrm{ext},1})(z) = -\frac{\eta_1}{\gamma} G(z, \widetilde{z}_1) + h_1^2 \left(R_\varepsilon^2 \frac{z - \widetilde{z}_2}{r_\varepsilon} \right) + H_1^{\mathrm{int},2} \left(\varphi_1^2, \psi_1^2, \frac{z - \widetilde{z}_2}{r_\varepsilon} \right) - H_1^{\mathrm{ext}} \left(\widetilde{\varphi}_1^2, \widetilde{\psi}_1^2, \frac{z - \widetilde{z}_2}{r_\varepsilon} \right) + v_1^2 \left(R_\varepsilon^2 \frac{z - \widetilde{z}_2}{r_\varepsilon} \right) + \mathcal{O}(r_\varepsilon^2).$$

Next, even though all functions are defined on $\partial B_{r_\varepsilon}(\widetilde{z}_2)$ in (2.34), it will be more convenient to solve on S^3 the following set of equations

$$\begin{aligned} (u_{\mathrm{int},1} - u_{\mathrm{ext},1})(\widetilde{z}_2 + r_\varepsilon \cdot\,) &= 0, & \partial_r (u_{\mathrm{int},1} - u_{\mathrm{ext},1})(\widetilde{z}_2 + r_\varepsilon \cdot\,) &= 0, \\ \Delta (u_{\mathrm{int},1} - u_{\mathrm{ext},1})(\widetilde{z}_2 + r_\varepsilon \cdot\,) &= 0 \quad \text{and} & \partial_r \Delta (u_{\mathrm{int},1} - u_{\mathrm{ext},1})(\widetilde{z}_2 + r_\varepsilon \cdot\,) &= 0. \end{aligned} \tag{2.40}$$

We decompose

$$\begin{aligned} \varphi_1^2 &= \varphi_{1,0}^2 + \varphi_{1,1}^2 + \varphi_1^{2,\perp}, \quad \psi_1^2 = 8\varphi_{1,0}^2 + 12\varphi_{1,1}^2 + \psi_1^{2,\perp}, \\ \widetilde{\varphi}_1^2 &= \widetilde{\varphi}_{1,0}^2 + \widetilde{\varphi}_{1,1}^2 + \widetilde{\varphi}_1^{2,\perp}, \quad \widetilde{\psi}_1^2 = \widetilde{\psi}_{1,1}^2 + \widetilde{\psi}_1^{2,\perp}, \end{aligned}$$

where $\varphi_{1,0}^2, \widetilde{\varphi}_{1,0}^2 \in \mathbb{E}_0$, $\varphi_{1,1}^2, \widetilde{\varphi}_{1,1}^2, \widetilde{\psi}_{1,1}^2 \in \mathbb{E}_1 = \ker(\Delta_{S^3} + 1) = \mathrm{Span}\{e_1, e_2, e_3, e_4\}$ and $\varphi_1^{2,\perp}$, $\widetilde{\varphi}_1^{2,\perp}$, $\psi_1^{2,\perp}$, $\widetilde{\psi}_1^{2,\perp}$ belong to $(L^2(S^3))^\perp$.

Projecting the set of equations (2.40) over $\mathbb{E}_0$, we get

$$\begin{aligned} \varphi_{1,0}^2 - \widetilde{\varphi}_{1,0}^2 + \mathcal{O}(r_\varepsilon^2) &= 0, \\ 2\varphi_{1,0}^2 + 2\widetilde{\varphi}_{1,0}^2 + \mathcal{O}(r_\varepsilon^2) &= 0, \\ 8\varphi_{1,0}^2 + \mathcal{O}(r_\varepsilon^2) &= 0. \end{aligned}$$

From the L^2-projection of (2.40) over $\mathbb{E}_1$, we obtain the system of equations

$$\begin{aligned} \varphi_{1,1}^2 - \widetilde{\varphi}_{1,1}^2 + \mathcal{O}(r_\varepsilon^2) &= 0, \\ 3\varphi_{1,1}^2 + 3\widetilde{\varphi}_{1,1}^2 + \frac{1}{2}\widetilde{\psi}_{1,1}^2 + \mathcal{O}(r_\varepsilon^2) &= 0, \\ 15\varphi_{1,1}^2 - 3\widetilde{\varphi}_{1,1}^2 - \widetilde{\psi}_{1,1}^2 + \mathcal{O}(r_\varepsilon^2) &= 0, \\ 15\varphi_{1,1}^2 + 15\widetilde{\varphi}_{1,1}^2 + \frac{18}{4}\widetilde{\psi}_{1,1}^2 + \mathcal{O}(r_\varepsilon^2) &= 0. \end{aligned}$$

Finally, we consider the L^2-projection onto $(L^2(S^3))^\perp$. This yields the system

$$\begin{aligned} \varphi_1^{2,\perp} - \widetilde{\varphi}_1^{2,\perp} + \mathcal{O}(r_\varepsilon^2) &= 0, \\ \partial_r \big(H^{\mathrm{int}}_{\varphi_1^{2,\perp}, \psi_1^{2,\perp}} - H^{\mathrm{ext}}_{\widetilde{\varphi}_1^{2,\perp}, \widetilde{\psi}_1^{2,\perp}} \big) + \mathcal{O}(r_\varepsilon^2) &= 0, \\ \psi_1^{2,\perp} - \widetilde{\psi}_1^{2,\perp} + \mathcal{O}(r_\varepsilon^2) &= 0, \\ \partial_r \Delta \big(H^{\mathrm{int}}_{\varphi_1^{2,\perp}, \psi_1^{2,\perp}} - H^{\mathrm{ext}}_{\widetilde{\varphi}_1^{2,\perp}, \widetilde{\psi}_1^{2,\perp}} \big) + \mathcal{O}(r_\varepsilon^2) &= 0. \end{aligned}$$

Applying Lemma 2.14, this last system can be written as

$$\varphi_1^{2,\perp} = \mathcal{O}(r_\varepsilon^2), \quad \widetilde{\varphi}_1^{2,\perp} = \mathcal{O}(r_\varepsilon^2), \quad \psi_1^{2,\perp} = \mathcal{O}(r_\varepsilon^2) \quad \text{and} \quad \widetilde{\psi}_1^{2,\perp} = \mathcal{O}(r_\varepsilon^2).$$

Then, the systems found by projecting (2.40) gather in this equality

$$\big(\varphi_{1,0}^2, \widetilde{\varphi}_{1,0}^2, \varphi_{1,1}^2, \widetilde{\varphi}_{1,1}^2, \widetilde{\psi}_{1,1}^2, \varphi_1^{2,\perp}, \widetilde{\varphi}_1^{2,\perp}, \psi_1^{2,\perp}, \widetilde{\psi}_1^{2,\perp}\big) = \mathcal{O}(r_\varepsilon^2).$$

• Similarly, in $\partial B_{r_\varepsilon}(\widetilde{z}_2)$ when ε tends to 0, we expect that $\widetilde{z}_2$ converges to z_2 and τ_2 converges to τ_2^* satisfying

$$4 \ln \tau_2^* = -\ln \xi - \mathcal{E}_2(\widetilde{z}_2, \widetilde{\mathbf{z}}).$$

So we choose τ_2^- and τ_2^+ to satisfy

$$4 \ln(\tau_2^-) < -\ln \xi - \mathcal{E}_2(\widetilde{z}_2, \widetilde{\mathbf{z}}) < 4 \ln(\tau_2^+),$$

where

$$\mathcal{E}_2(\,\cdot\,, \widetilde{\mathbf{z}}) := H(\,\cdot\,, \widetilde{z}_2) + \frac{1-\xi}{\gamma} G(\,\cdot\,, \widetilde{z}_1).$$

Using the decomposition $\mathbb{E}_0 \oplus \mathbb{E}_1 \oplus (L^2(S^3))^\perp$,

$$\begin{aligned} \varphi_2^2 &= \varphi_{2,0}^2 + \varphi_{2,1}^2 + \varphi_2^{2,\perp}, \quad \psi_2^2 = 8\varphi_{2,0}^2 + 12\varphi_{2,1}^2 + \psi_2^{2,\perp}, \\ \widetilde{\varphi}_2^2 &= \widetilde{\varphi}_{2,0}^2 + \widetilde{\varphi}_{2,1}^2 + \widetilde{\varphi}_2^{2,\perp}, \quad \widetilde{\psi}_2^2 = \widetilde{\psi}_{2,1}^2 + \widetilde{\psi}_2^{2,\perp}. \end{aligned}$$

We can prove that

$$\begin{aligned} (u_{\text{int},2} - u_{\text{ext},2})(\widetilde{z}_2 + r_\varepsilon\,\cdot\,) = 0, \qquad & \partial_r(u_{\text{int},2} - u_{\text{ext},2})(\widetilde{z}_2 + r_\varepsilon\,\cdot\,) = 0, \\ \Delta(u_{\text{int},2} - u_{\text{ext},2})(\widetilde{z}_2 + r_\varepsilon\,\cdot\,) = 0 \quad \text{and} \quad & \partial_r \Delta(u_{\text{int},2} - u_{\text{ext},2})(\widetilde{z}_2 + r_\varepsilon\,\cdot\,) = 0 \end{aligned}$$

near S^3 yield to

$$T_\varepsilon^2 = \big(t_2, \eta_2, \varphi_{2,0}^2, \widetilde{\varphi}_{2,0}^2, \varphi_{2,1}^2, \widetilde{\varphi}_{2,1}^2, \widetilde{\psi}_{2,1}^2, \overline{\nabla}\mathcal{E}_2(\widetilde{z}_2, \widetilde{\mathbf{z}}), \varphi_2^{2,\perp}, \widetilde{\varphi}_2^{2,\perp}, \psi_2^{2,\perp}, \widetilde{\psi}_2^{2,\perp}\big) = \mathcal{O}_2(r_\varepsilon^2), \tag{2.41}$$

where

$$t_2 := \frac{1}{\ln r_\varepsilon}\big[4 \ln \tau_2 + \ln \xi + \mathcal{E}_2(\widetilde{z}_2, \widetilde{\mathbf{z}})\big].$$

Finally, recall that $\mathbf{d} = r_\varepsilon(\widetilde{\mathbf{z}} - \mathbf{z})$, in addition the previous systems can be written as

$$\big(\mathbf{d}, t_i, \eta_i, \varphi^i, \widetilde{\varphi}^i, \psi^i, \widetilde{\psi}^i, \overline{\nabla}\mathcal{E}_i\big) = \mathcal{O}(r_\varepsilon^2).$$

Combining (2.38) and (2.41), we have

$$T_\varepsilon = (T_\varepsilon^1, T_\varepsilon^2) = \big(\mathcal{O}_1(r_\varepsilon^2), \mathcal{O}_2(r_\varepsilon^2)\big) = \mathcal{O}(r_\varepsilon^2). \tag{2.42}$$

Then the nonlinear mapping which appears on the right-hand side of (2.42) is continuous and compact. In addition, reducing ε_κ if necessary, this nonlinear mapping sends the ball of radius κr_ε^2 (for the natural product norm) into itself, provided κ is fixed large enough. Applying Schauder's fixed point theorem in the ball of radius κr_ε^2 in the product space where the entries live, (now τ is fixed and $\tau = \tau_\kappa$), we obtain the existence of a solution of equation (2.42).

This completes the proof of Theorem 1.6.

Remark 2.20. In order to inverse problem (2.38) and (2.41), we remark that the fact that z_i is a nondegenerate critical point of $\mathcal{E}_i(\,\cdot\,,\mathbf{z})$, $i \in \{1,2\}$, is equivalent to say that (z_1,z_2) is a nondegenerate critical point of the function $\mathcal{F}$ defined by

$$\mathcal{F}(z_1,z_2) = \frac{1-\xi}{2\gamma}H(z_1,z_1) + \frac{1-\gamma}{2\xi}H(z_2,z_2) + \frac{1-\gamma}{\xi}\frac{1-\xi}{\gamma}G(z_1,z_2).$$

Indeed, we have

$$\nabla\mathcal{F}(z_1,z_2) = \left(\frac{\partial\mathcal{F}}{\partial z_1}(z_1,z_2), \frac{\partial\mathcal{F}}{\partial z_2}(z_1,z_2)\right).$$

On the other hand, $\mathcal{E}_1(z,\mathbf{z}) = H(z,\widetilde{z}_1) + \frac{1-\gamma}{\xi}G(z,\widetilde{z}_2)$ and $\mathcal{E}_2(z,\mathbf{z}) = H(z,\widetilde{z}_2) + \frac{1-\xi}{\gamma}G(z,\widetilde{z}_1)$, then

$$\frac{\partial\mathcal{E}_1}{\partial z}(z_1,\mathbf{z}) = \frac{\partial H}{\partial z}(z_1,z_1) + \frac{1-\gamma}{\xi}\frac{\partial G}{\partial z}(z_1,z_2) = \frac{\partial\mathcal{F}}{\partial z_1}(z_1,z_2)$$

and

$$\frac{\partial\mathcal{E}_2}{\partial z}(z_2,\mathbf{z}) = \frac{\partial H}{\partial z}(z_2,z_2) + \frac{1-\xi}{\gamma}\frac{\partial G}{\partial z}(z_1,z_2) = \frac{\partial\mathcal{F}}{\partial z_2}(z_1,z_2).$$

References

[1] G. Arioli, F. Gazzola, H.-C. Grunau and E. Mitidieri, *A semilinear fourth order elliptic problem with exponential nonlinearity*, SIAM J. Math. Anal. **36** (2005), no. 4, 1226–1258.

[2] S. Baraket, I. Bazarbacha, R. Chetouane and A. Messaoudi, *Singular limit solutions for a* 4*-dimensional semilinear elliptic system of Liouville type*, Taiwanese J. Math. **24** (2020), no. 4, 855–909.

[3] S. Baraket, I. Bazarbacha and N. Trabelsi, *Construction of singular limits for four-dimensional elliptic problems with exponentially dominated nonlinearity*, Bull. Sci. Math. **131** (2007), no. 7, 670–685.

[4] S. Baraket, I. Ben Omrane, T. Ouni and N. Trabelsi, *Singular limits for* 2*-dimensional elliptic problem with exponentially dominated nonlinearity and singular data*, Commun. Contemp. Math. **13** (2011), no. 4, 697–725.

[5] S. Baraket, M. Dammak, T. Ouni and F. Pacard, *Singular limits for a* 4*-dimensional semilinear elliptic problem with exponential nonlinearity*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **24** (2007), no. 6, 875–895.

[6] S. Baraket and F. Pacard, *Construction of singular limits for a semilinear elliptic equation in dimension* 2, Calc. Var. Partial Differential Equations **6** (1998), no. 1, 1–38.

[7] S. Baraket and D. Ye, *Singular limit solutions for two-dimensional elliptic problems with exponentially dominated nonlinearity*, Chinese Ann. Math. Ser. B **22** (2001), no. 3, 287–296.

[8] S.-Y. A. Chang, *On a fourth-order partial differential equation in conformal geometry*, in: *Harmonic Analysis and Partial Differential Equations (Chicago, IL, 1996)*, 127–150, Chicago Lectures in Math., Univ. Chicago Press, Chicago, IL, 1999.

[9] Z. Chen and W. Zou, *On linearly coupled Schrödinger systems*, Proc. Amer. Math. Soc. **142** (2014), no. 1, 323–333.

[10] M. Clapp, C. Muñoz and M. Musso, *Singular limits for the bi-Laplacian operator with exponential nonlinearity in* $\mathbb{R}^4$, Ann. Inst. H. Poincaré C Anal. Non Linéaire **25** (2008), no. 5, 1015–1041.

[11] M. Dammak and T. Ouni, *Singular limits for* 4*-dimensional semilinear elliptic problems with exponential nonlinearity adding a singular source term given by Dirac masses*, Differential Integral Equations **21** (2008), no. 11-12, 1019–1036.

[12] Z. Djadli and A. Malchiodi, *Existence of conformal metrics with constant* Q*-curvature*, Ann. of Math. (2) **168** (2008), no. 3, 813–858.

[13] P. Esposito, *Blow up solutions for a Liouville equation with singular data*, in: *Recent Advances in Elliptic and Parabolic Problems*, 61–79, World Sci. Publ., Hackensack, NJ, 2005.

[14] Y. Kuramoto and T. Tsuzuki, *Persistent propagation of concentration waves in dissipative media far from thermal equilibrium*, Prog. Theor. Phys. **55** (1976), no. 2, 356–369.

[15] A. C. Lazer and P. J. Mckenna, *Large-amplitude periodic oscillations in suspension bridges: Some new connections with nonlinear analysis*, SIAM Rev. **32** (1990), no. 4, 537–578.

[16] C.-S. Lin, *A classification of solutions of a conformally invariant fourth order equation in* $\mathbb{R}^4$, Comment. Math. Helv. **73** (1998), no. 2, 206–231.

[17] F. Lin and Y. Yang, *Nonlinear non-local elliptic equation modelling electrostatic actuation*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **463** (2007), no. 2081, 1323–1337.

[18] J. Liouville, *Sur l'équation aux différences partielles* $\frac{\partial^2 \log \lambda}{\partial u \partial v} \pm \frac{\lambda}{2a^2} = 0$, J. Math. Pures Appl. **18** (1853), 71–72.

[19] F. Mahmoudi and A. Malchiodi, *Concentration on minimal submanifolds for a singularly perturbed Neumann problem*, Adv. Math. **209** (2007), no. 2, 460–525.

[20] D. Mugnai, *A note on an exponential semilinear equation of the fourth order*, Differential Integral Equations **17** (2004), no. 1-2, 45–52.

[21] T. Ouni, S. Baraket and M. Khtaifi, *Singular limits for* 4*-dimensional general stationary q-Kuramoto–Sivashinsky equation (q-KSE) with exponential nonlinearity*, An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. **24** (2016), no. 3, 295–337.

[22] C. V. Pao, *On fourth-order elliptic boundary value problems*, Proc. Amer. Math. Soc. **128** (2000), no. 4, 1023–1030.

[23] Y. Rébaï, *Weak solutions of nonlinear elliptic equations with prescribed singular set*, J. Differential Equations **127** (1996), no. 2, 439–453.

[24] G. I. Sivashinsky, *Nonlinear analysis of hydrodynamic instability in laminar flames I: Derivation of basic equations*, Acta Astronaut. **4** (1977), no. 11-12, 1177–1206.

[25] ______, *Weak turbulence in periodic flows*, Phys. D **17** (1985), no. 2, 243–255.

[26] T. Suzuki, *Two-dimensional Emden–Fowler equation with exponential nonlinearity*, in: *Nonlinear Diffusion Equations and Their Equilibrium States, 3 (Gregynog, 1989)*, 493–512, Progr. Nonlinear Differential Equations Appl. **7**, Birkhäuser Boston, Boston, MA, 1992.

[27] G. Tarantello, *A note on a semilinear elliptic problem*, Differential Integral Equations **5** (1992), no. 3, 561–565.

[28] J. Wei, *Asymptotic behavior of a nonlinear fourth order eigenvalue problem*, Comm. Partial Differential Equations **21** (1996), no. 9-10, 1451–1467.

[29] J. Wei and D. Ye, *Nonradial solutions for a conformally invariant fourth order equation in* $\mathbb{R}^4$, Calc. Var. Partial Differential Equations **32** (2008), no. 3, 373–386.

Lilia Larbi
University of Tunis El Manar, Departement of Mathematics, Faculty of Sciences of Tunis, Campus of University 2092 Tunis El Manar, Tunisia
E-mail address: `lilialarbi1994@gmail.com`

Nihed Trabelsi
University of Sfax, Departement of Mathematics, Faculty of Sciences of Sfax, Soukra Road Km4 BP 1171-3000, Tunisia
E-mail address: `nihed.trabelsi78@gmail.com`