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On Sobolev-type Inequalities on Morrey Spaces of an Integral Form

Takao Ohno* and Tetsu Shimomura

Abstract. We prove Sobolev-type inequalities for modified Riesz potentials of functions in Morrey spaces of an integral form over non-doubling metric measure spaces. Our results are new even for the doubling metric measure setting. In particular, our results extend the previous results in Morrey spaces of an integral form in the Euclidean case.

1. Introduction

For $0 < \alpha < N$ and a locally integrable function f on $\mathbf{R}^N$ the Riesz potential $U_\alpha f$ of order α is defined by

$$U_\alpha f(x) = \int_{\mathbf{R}^N} |x-y|^{\alpha-N} f(y)\, dy.$$

The classical Sobolev inequality says that the Riesz potential $U_\alpha f$ of order α with $f \in L^p(\mathbf{R}^N)$ belongs to $L^{p^*}(\mathbf{R}^N)$ when $1 < p < \infty$ and $1/p^* = 1/p - \alpha/N > 0$ (see, e.g. [2, Theorem 3.1.4(b)]). Morrey spaces were introduced by C. B. Morrey [17] in 1938 to study the existence and regularity of partial differential equations. Sobolev's inequality for Morrey spaces was studied by D. R. Adams [1]. We also refer to [4, 12–14, 18, 22], etc.

In [15], the second author and Mizuta studied a Sobolev-type inequality for $U_\alpha f$ for locally integrable functions f on $\mathbf{R}^N$ satisfying

$$\sup_{x\in G} \left(\int_0^{d_G} r^{\nu-N} \varphi(r) \left(\int_{B(x,r)} |f(y)|^p\, dy \right) \frac{dr}{r} \right)^{1/p} < \infty, \tag{1.1}$$

where $0 < \nu \le N$, G is a bounded open set in $\mathbf{R}^N$, $d_G = \sup\{d(x,y) : x, y \in G\}$ and φ is positive monotone functions on the interval $(0,\infty)$ satisfying the conditions (φ) and (i) in [15].

We denote by (X, d, μ) a metric measure space, where X is a bounded set, d is a metric on X and μ is a nonnegative complete Borel regular outer measure on X which is finite in every bounded set. We often write X instead of (X, d, μ). For $x \in X$ and $r > 0$, we denote by $B(x,r)$ the open ball in X centered at x with radius r and $d_X = \sup\{d(x,y) : x, y \in X\}$.

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*Corresponding author.

We assume that $d_X < \infty$, $\mu(\{x\}) = 0$ for $x \in X$ and $0 < \mu(B(x,r)) < \infty$ for $x \in X$ and $r > 0$ for simplicity. We do not assume that μ has a so-called doubling condition. Recall that a Radon measure μ is said to be doubling if there exists a constant $c_0 > 0$ such that $\mu(B(x,2r)) \le c_0\mu(B(x,r))$ for all $x \in \mathrm{supp}(\mu)$ $(= X)$ and $r > 0$ (see [3]). Otherwise μ is said to be non-doubling. For examples of non-doubling metric measure spaces we refer to [21, 26]. In connection with the $5r$-covering lemma, the doubling condition had been a key condition in harmonic analysis. However, Nazarov, Treil and Volberg showed that the doubling condition is not necessary by using the modified maximal operator [19, 20]. In this paper, we show that this is the case for the modified Riesz potential operator.

For $\alpha > 0$ and $\tau \ge 1$, we define the (modified) Riesz potential of order α for a locally integrable function f on X by

$$I_{\alpha,\tau} f(x) = \int_X \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y)$$

(e.g., see [6,16,23,26]). Note here that we can not reduce the number τ any more (see [25]), which is based on the idea of Stempak [27]. This is equal to $U_\alpha f$ when $X = \mathbf{R}^N$ and $\mu = dx$. In the doubling metric measure setting, we use $I_{\alpha,1} f$. For another type of Riesz potentials like

$$I_\eta f(x) = \int_X \frac{f(y)}{\mu(B(x,d(x,y)))^{1-\eta}}\, d\mu(y),$$

see [5, 10].

To obtain general results, we consider a weight function $\omega(r)\colon (0,\infty) \to (0,\infty)$ satisfying the following conditions:

(ω0) $\omega(\cdot)$ is continuous on $(0,\infty)$;

(ω1) $\omega(\cdot)$ is almost increasing on $(0,\infty)$, namely there exists a constant $\widetilde{c}_1 \ge 1$ such that

$$\omega(r_1) \le \widetilde{c}_1 \omega(r_2) \quad \text{whenever } 0 < r_1 < r_2 < \infty;$$

(ω2) there exists a constant $\widetilde{c}_2 > 1$ such that

$$\widetilde{c}_2^{\,-1} \omega(r) \le \omega(2r) \le \widetilde{c}_2 \omega(r) \quad \text{whenever } r > 0;$$

(ω3) there exist constants $\omega_0 > 0$ and $\widetilde{c}_3 \ge 1$ such that

$$\widetilde{c}_3^{\,-1} r^{\omega_0} \le \omega(r) \le \widetilde{c}_3 \quad \text{for all } 0 < r \le 2d_X.$$

Example 1.1. Let $0 < \sigma < \omega_0$ and $\beta \in \mathbf{R}$. Then

$$\omega(r) = r^\sigma (\log(e + 1/r))^\beta$$

satisfies (ω0), (ω1), (ω2) and (ω3).

Recall that f is a locally integrable function on X if f is an integrable function on all balls B in X. Let $p \geq 1$ and $\theta \geq 1$. In connection with (1.1), given $\omega(r)$ as above, we define the $\mathcal{L}^{p,\omega,\theta}$ norm by

$$\|f\|_{\mathcal{L}^{p,\omega,\theta}(X)} = \inf \left\{ \lambda > 0; \sup_{x\in X} \left(\int_0^{2d_X} \frac{\omega(r)}{\mu(B(x,\theta r))} \left(\int_{B(x,r)} \left(\frac{|f(y)|}{\lambda} \right)^p d\mu(y) \right) \frac{dr}{r} \right) \leq 1 \right\}.$$

The space of all measurable functions f on X with $\|f\|_{\mathcal{L}^{p,\omega,\theta}(X)} < \infty$ is denoted by $\mathcal{L}^{p,\omega,\theta}(X)$. The space $\mathcal{L}^{p,\omega,\theta}(X)$ is called a Morrey space of an integral form. Here note that $2d_X$ can be replaced by κd_X with $\kappa > 1$.

Our aim in this paper is to give a general version of Sobolev-type inequality for Riesz potentials $I_{\alpha,\tau}f$ of functions in Morrey spaces $\mathcal{L}^{p,\omega,\theta}(X)$ of an integral form over non-doubling metric measure spaces X (see Theorem 3.3), as an extension of [15, Theorem 5.4] in the Euclidean case. Our results are new even for the doubling metric measure setting. To this end, we apply Hedberg's trick [8] by the use of the Hardy–Littlewood maximal operator M_λ adapted to our setting (see Theorem 2.4). See Section 2 for the definition of M_λ and Remarks 2.1 and 2.2 on the number λ.

2. Boundedness of the maximal operator

Throughout the paper, we let C denote various constants independent of the variables in question and $C(a, b, \ldots)$ be a constant that depends on $a, b, \ldots$ only.

For a locally integrable function f on X and $\lambda \geq 1$, the Hardy–Littlewood maximal function $M_\lambda f$ is defined by

$$M_\lambda f(x) = \sup_{r>0} \frac{1}{\mu(B(x,\lambda r))} \int_{B(x,r)} |f(y)|\, d\mu(y).$$

For $\lambda \geq 1$, we say that X satisfies $(M\lambda)$ if there exists a constant $C > 0$ such that

$$\mu(\{x \in X : M_\lambda f(x) > k\}) \leq \frac{C}{k} \int_X |f(y)|\, d\mu(y) \tag{2.1}$$

for all measurable functions $f \in L^1(X)$ and $k > 0$.

Remark 2.1. In (2.1), we can not reduce the number λ any more (Stempak [27]).

As for the precise value of λ, we know the following.

Remark 2.2. By a covering argument, Nazarov, Treil and Volberg [19, 20] proved that X satisfies $(M3)$ if X is a separable metric space. Meanwhile X satisfies $(M\lambda)$ for any $\lambda > 0$ if μ satisfies the doubling condition (see [9]). Terasawa [29] showed that X satisfies $(M\lambda)$

for $\lambda \geq 2$ if $\mu(B(x,r))$ is continuous in the variable $r > 0$ when $x \in X$ is fixed. In [24], Sawano showed that X satisfies $(M\lambda)$ for $\lambda \geq 2$ if X is a separable metric space. Another remarkable example of the Poincaré disc can be found in [28] where (2.1) with $\lambda = 1$ is established despite the fact that the corresponding Riemannian volume μ is non-doubling. As in [7, Appendix], the Euclidean space $\mathbf{R}^N$, endowed with $\mu \equiv e^{|x|^2}\,dx$, fails to satisfy $(M1)$.

We know the following result.

Lemma 2.3. *Let $1 < p_0 < \infty$ and let $\lambda \geq 1$. Suppose X satisfies $(M\lambda)$. Then there exists a constant $C > 0$ such that*

$$\int_X \{M_\lambda f(x)\}^{p_0}\,d\mu(x) \leq C$$

for all measurable functions f on X with $\|f\|_{L^{p_0}(X)} \leq 1$.

Now we are ready to show the boundedness of the maximal operator M_λ.

Theorem 2.4. *Let $1 \leq \theta_1 < \theta_2$ and $\lambda > \theta_1(\theta_2+1)/(\theta_2-\theta_1)$. Assume that X satisfies $(M\lambda)$. Further suppose*

$(\omega 1')$ *$r \mapsto r^{-\varepsilon_1}\omega(r)$ is almost increasing in $(0, d_X]$ for some $\varepsilon_1 > 0$.*

If $p > 1$, then there is a constant $C > 0$ such that

$$\|M_\lambda f\|_{\mathcal{L}^{p,\omega,\theta_2}(X)} \leq C\|f\|_{\mathcal{L}^{p,\omega,\theta_1}(X)}$$

for all $f \in \mathcal{L}^{p,\omega,\theta_1}(X)$.

Proof. Let f be a nonnegative measurable function on X with $\|f\|_{\mathcal{L}^{p,\omega,\theta_1}(X)} \leq 1$. Let $z \in X$ and $0 < r \leq 2d_X$. For $\kappa_1 = \theta_2/\theta_1 > 1$, write

$$f(y) = f(y)\chi_{B(z,\kappa_1 r)}(y) + f(y)\chi_{X\setminus B(z,\kappa_1 r)}(y) := f_1(y) + f_2(y),$$

where χ_E is the characteristic function of E.

By Lemma 2.3 and $(\omega 2)$, we have

$$\begin{aligned}
&\int_0^{2d_X} \frac{\omega(r)}{\mu(B(z,\theta_2 r))}\left(\int_{B(z,r)} \{M_\lambda f_1(x)\}^p\,d\mu(y)\right)\frac{dr}{r} \\
&\leq \int_0^{2d_X} \frac{\omega(r)}{\mu(B(z,\theta_2 r))}\left(\int_X \{M_\lambda f_1(x)\}^p\,d\mu(y)\right)\frac{dr}{r} \\
&\leq C\int_0^{2d_X} \frac{\omega(r)}{\mu(B(z,\theta_2 r))}\left(\int_X f_1(y)^p\,d\mu(y)\right)\frac{dr}{r}
\end{aligned}$$

$$
\begin{aligned}
(2.2) \qquad &\le C\int_0^{2d_X} \frac{\omega(r)}{\mu(B(z,\theta_2 r))}\left(\int_{B(z,\kappa_1 r)} f(y)^p\,d\mu(y)\right)\frac{dr}{r}\\
&\le C\int_0^{2\kappa_1 d_X} \frac{\omega(\kappa_1^{-1} r)}{\mu(B(z,\theta_2 \kappa_1^{-1} r))}\left(\int_{B(z,r)} f(y)^p\,d\mu(y)\right)\frac{dr}{r}\\
&\le C\int_0^{2d_X} \frac{\omega(r)}{\mu(B(z,\theta_1 r))}\left(\int_{B(z,r)} f(y)^p\,d\mu(y)\right)\frac{dr}{r}\\
&\le C.
\end{aligned}
$$

Next we treat f_2. Let

$$
\kappa_2 = \lambda\left(1-\frac{1}{\kappa_1}\right)-\frac{1}{\kappa_1} = \lambda\left(1-\frac{\theta_1}{\theta_2}\right)-\frac{\theta_1}{\theta_2}.
$$

Then note that $\kappa_2 > \theta_1$ and

$$
B(z,\kappa_2 d(z,y)) \subset B(x,\lambda d(x,y)) \tag{2.3}
$$

for $x \in B(z,r)$ and $y \in X \setminus B(z,\kappa_1 r)$. Indeed, when $w \in B(z,\kappa_2 d(z,y))$, we have

$$
\begin{aligned}
d(w,x) &\le d(w,z)+d(z,x) < \kappa_2 d(z,y) + \frac{1}{\kappa_1}d(z,y) = \lambda\left(1-\frac{1}{\kappa_1}\right)d(z,y)\\
&\le \lambda\left(1-\frac{1}{\kappa_1}\right)\left(1-\frac{1}{\kappa_1}\right)^{-1} d(x,y) = \lambda d(x,y).
\end{aligned}
$$

For $\gamma = \kappa_2\theta_1^{-1} > 1$, let j_0 be the smallest integer $\kappa_1\gamma^{j_0/2}r \ge d_X$. For $x \in B(z,r)$ and $0<\varepsilon<\varepsilon_1$, we see from Hölder's inequality and (2.3) that

$$
\begin{aligned}
M_\lambda f_2(x) &= \sup_{\rho>0}\frac{1}{\mu(B(x,\lambda\rho))}\int_{B(x,\rho)} f_2(y)\,d\mu(y)\\
&\le \sup_{\rho>0}\left(\frac{1}{\mu(B(x,\lambda\rho))}\int_{B(x,\rho)} f_2(y)^p\,d\mu(y)\right)^{1/p}\\
&\le C\left(\int_{X\setminus B(z,\kappa_1 r)} \frac{1}{\mu(B(x,\lambda d(x,y)))} f(y)^p\,d\mu(y)\right)^{1/p}\\
&\le C\left(\int_{X\setminus B(z,\kappa_1 r)} \frac{1}{\mu(B(z,\kappa_2 d(z,y)))} f(y)^p\,d\mu(y)\right)^{1/p}.
\end{aligned}
$$

We decompose

$$
M_\lambda f_2(x) \le C\left(\sum_{j=1}^{j_0}\int_{B(z,\kappa_1\gamma^{j/2}r)\setminus B(z,\kappa_1\gamma^{(j-1)/2}r)} \frac{1}{\mu(B(z,\kappa_2 d(z,y)))} f(y)^p\,d\mu(y)\right)^{1/p}
$$

$$
\begin{aligned}
&\le C\sum_{j=1}^{j_0}\left(\int_{B(z,\kappa_1\gamma^{j/2}r)\setminus B(z,\kappa_1\gamma^{(j-1)/2}r)}\frac{1}{\mu(B(z,\kappa_2 d(z,y)))}f(y)^p\,d\mu(y)\right)^{1/p}\\
&\le C\sum_{j=1}^{j_0}\left(\frac{1}{\mu(B(z,\kappa_1\kappa_2\gamma^{(j-1)/2}r))}\int_{B(z,\kappa_1\gamma^{j/2}r)}f(y)^p\,d\mu(y)\right)^{1/p}\\
&= C\sum_{j=1}^{j_0}\left(\frac{1}{\mu(B(z,\kappa_1\gamma^{(j+1)/2}\theta_1 r))}\int_{B(z,\kappa_1\gamma^{j/2}r)}f(y)^p\,d\mu(y)\right)^{1/p}\\
&\le C\left(\sum_{j=1}^{j_0}\left((\kappa_1\gamma^{j/2}r)^{\varepsilon/p}\omega(\kappa_1\gamma^{j/2}r)^{-1/p}\right)^{p'}\right)^{1/p'}\\
&\quad\times\left(\sum_{j=1}^{j_0}\frac{(\kappa_1\gamma^{j/2}r)^{-\varepsilon}\omega(\kappa_1\gamma^{j/2}r)}{\mu(B(z,\kappa_1\gamma^{(j+1)/2}\theta_1 r))}\int_{B(z,\kappa_1\gamma^{j/2}r)}f(y)^p\,d\mu(y)\right)^{1/p}
\end{aligned}
\tag{2.4}
$$

by Hölder's inequality, where $1/p+1/p'=1$. Here note from $(\omega 1')$ that

$$
\begin{aligned}
&\left(\sum_{j=1}^{j_0}\left((\kappa_1\gamma^{j/2}r)^{\varepsilon/p}\omega(\kappa_1\gamma^{j/2}r)^{-1/p}\right)^{p'}\right)^{1/p'}\\
&\le Cr^{\varepsilon_1/p}\omega(r)^{-1/p}\left(\sum_{j=1}^{j_0}(\kappa_1\gamma^{j/2}r)^{(\varepsilon/p-\varepsilon_1/p)p'}\right)^{1/p'}\le Cr^{\varepsilon/p}\omega(r)^{-1/p}
\end{aligned}
\tag{2.5}
$$

for $0<\varepsilon<\varepsilon_1$. Further note from $(\omega 2)$ that

$$
\begin{aligned}
&\sum_{j=1}^{j_0}\frac{(\kappa_1\gamma^{j/2}r)^{-\varepsilon}\omega(\kappa_1\gamma^{j/2}r)}{\mu(B(z,\kappa_1\gamma^{(j+1)/2}\theta_1 r))}\int_{B(z,\kappa_1\gamma^{j/2}r)}f(y)^p\,d\mu(y)\\
&\le C\sum_{j=1}^{j_0}\int_{\kappa_1\gamma^{j/2}r}^{\kappa_1\gamma^{(j+1)/2}r}\frac{t^{-\varepsilon}\omega(t)}{\mu(B(z,\theta_1 t))}\left(\int_{B(z,t)}f(y)^p\,d\mu(y)\right)\frac{dt}{t}\\
&\le C\int_{\kappa_1\gamma^{1/2}r}^{\gamma d_X}\frac{t^{-\varepsilon}\omega(t)}{\mu(B(z,\theta_1 t))}\left(\int_{B(z,t)}f(y)^p\,d\mu(y)\right)\frac{dt}{t}\\
&\le C\int_{r}^{2d_X}\frac{t^{-\varepsilon}\omega(t)}{\mu(B(z,\theta_1 t))}\left(\int_{B(z,t)}f(y)^p\,d\mu(y)\right)\frac{dt}{t}.
\end{aligned}
\tag{2.6}
$$

By (2.4), (2.5) and (2.6), we have

$$
M_\lambda f_2(x)\le Cr^{\varepsilon/p}\omega(r)^{-1/p}\left(\int_r^{2d_X}\frac{t^{-\varepsilon}\omega(t)}{\mu(B(z,\theta_1 t))}\left(\int_{B(z,t)}f(y)^p\,d\mu(y)\right)\frac{dt}{t}\right)^{1/p}.
$$

Hence, we obtain

$$
\begin{aligned}
&\int_0^{2d_X} \frac{\omega(r)}{\mu(B(z,\theta_2 r))} \left(\int_{B(z,r)} \{M_\lambda f_2(x)\}^p \, d\mu(y) \right) \frac{dr}{r} \\
&\le C \int_0^{2d_X} r^{\varepsilon} \left(\int_r^{2d_X} \frac{t^{-\varepsilon}\omega(t)}{\mu(B(z,\theta_1 t))} \left(\int_{B(z,t)} f(y)^p \, d\mu(y) \right) \frac{dt}{t} \right) \frac{dr}{r} \\
&\le C \int_0^{2d_X} \frac{t^{-\varepsilon}\omega(t)}{\mu(B(z,\theta_1 t))} \left(\int_{B(z,t)} f(y)^p \, d\mu(y) \right) \left(\int_0^t r^{\varepsilon} \frac{dr}{r} \right) \frac{dt}{t} \\
&\le C \int_0^{2d_X} \frac{\omega(t)}{\mu(B(z,\theta_1 t))} \left(\int_{B(z,t)} f(y)^p \, d\mu(y) \right) \frac{dt}{t} \\
&\le C.
\end{aligned} \tag{2.7}
$$

Thus, in view of (2.2) and (2.7), we complete the proof. □

Remark 2.5. Note that $(\omega 1')$ implies $(\omega 1)$. Let $\omega(r) = r^{\sigma}(\log(e + 1/r))^{\beta}$ be as in Example 1.1. Then note that $(\omega 1')$ holds for $0 < \varepsilon_1 < \sigma$.

3. Sobolev-type inequality

We recall the following lemma.

Lemma 3.1. [11, Lemma 5.1(3)] *Set*

$$\omega^{-1}(r) = \sup\{s > 0 \mid \omega(s) < r\}$$

for $r > 0$. Then

$$\omega(\omega^{-1}(r)) = r$$

for all $r > 0$ with $\omega^{-1}(r) < \infty$.

We consider the following condition:

$(\omega\alpha)$ for $\alpha > 0$, there exist constants $\varepsilon_2 > 0$ and $A_1 \ge 1$ such that

$$r_2^{\varepsilon_2+\alpha}\omega(r_2)^{-1/p} \le A_1 r_1^{\varepsilon_2+\alpha}\omega(r_1)^{-1/p}$$

whenever $0 < r_1 < r_2 < d_X$.

Lemma 3.2. *Let $1 \le \theta < \tau$. Assume that $(\omega\alpha)$ holds. Then there exists a constant $C > 0$ such that*

$$\int_{X\setminus B(x,\delta)} \frac{d(x,y)^{\alpha} f(y)}{\mu(B(x,\tau d(x,y)))} \, d\mu(y) \le C\delta^{\alpha}\omega(\delta)^{-1/p}$$

for all $x \in X$, $0 < \delta < d_X/2$ and nonnegative $f \in \mathcal{L}^{p,\omega,\theta}(X)$ with $\|f\|_{\mathcal{L}^{p,\omega,\theta}(X)} \le 1$.

Proof. Let f be a nonnegative measurable function with $\|f\|_{\mathcal{L}^{p,\omega,\theta}(X)} \le 1$. Let $x \in X$ and $0 < \delta < d_X/2$. We find by $(\omega\alpha)$

$$
\begin{aligned}
&\int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y) \\
&\le \int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha}{\mu(B(x,\tau d(x,y)))} \omega(d(x,y))^{-1/p}\, d\mu(y) \\
&\quad + \int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))} \cdot \frac{f(y)^{p-1}}{\omega(d(x,y))^{-(p-1)/p}}\, d\mu(y) \\
&\le \int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha}{\mu(B(x,\tau d(x,y)))} \omega(d(x,y))^{-1/p}\, d\mu(y) \\
&\quad + C\delta^\alpha \omega(\delta)^{-1/p} \int_{X\setminus B(x,\delta)} \frac{\omega(d(x,y))}{\mu(B(x,\tau d(x,y)))} f(y)^p\, d\mu(y) \\
&= I_1 + C I_2.
\end{aligned}
$$

Let j_0 be the smallest integer such that $\tau^{j_0}\delta \ge d_X$. By $(\omega 1)$, $(\omega 2)$ and $(\omega\alpha)$, we have

$$
\begin{aligned}
I_1 &= \sum_{j=1}^{j_0} \int_{B(x,\tau^j\delta)\setminus B(x,\tau^{j-1}\delta)} \frac{d(x,y)^\alpha}{\mu(B(x,\tau d(x,y)))} \omega(d(x,y))^{-1/p}\, d\mu(y) \\
&\le C \sum_{j=1}^{j_0} (\tau^j\delta)^\alpha \omega(\tau^j\delta)^{-1/p} \le C \int_\delta^{\tau d_X} \rho^\alpha \omega(\rho)^{-1/p}\, \frac{d\rho}{\rho} \\
&\le C \int_\delta^{d_X} \rho^\alpha \omega(\rho)^{-1/p}\, \frac{d\rho}{\rho} \le C\delta^\alpha \omega(\delta)^{-1/p}.
\end{aligned}
$$

Next, for $\gamma = \tau\theta^{-1} > 1$, let j_1 be the smallest positive integer such that $\gamma^{j_1/2}\delta \ge d_X$. Then we have by $(\omega 1)$ and $(\omega 2)$

$$
\begin{aligned}
I_2 &= C\delta^\alpha \omega(\delta)^{-1/p} \sum_{j=1}^{j_1} \int_{B(x,\gamma^{j/2}\delta)\setminus B(x,\gamma^{(j-1)/2}\delta)} \frac{\omega(d(x,y))}{\mu(B(x,\tau d(x,y)))} f(y)^p\, d\mu(y) \\
&\le C\delta^\alpha \omega(\delta)^{-1/p} \sum_{j=1}^{j_1} \frac{\omega(\gamma^{j/2}\delta)}{\mu(B(x,\gamma^{(j+1)/2}\theta\delta))} \int_{B(x,\gamma^{j/2}\delta)} f(y)^p\, d\mu(y) \\
&\le C\delta^\alpha \omega(\delta)^{-1/p} \sum_{j=1}^{j_1} \int_{\gamma^{j/2}\delta}^{\gamma^{(j+1)/2}\delta} \frac{\omega(t)}{\mu(B(x,\theta t))} \left(\int_{B(x,t)} f(y)^p\, d\mu(y) \right) \frac{dt}{t}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
I_2 &\le C\delta^\alpha \omega(\delta)^{-1/p} \int_{\gamma^{1/2}\delta}^{\gamma d_X} \frac{\omega(t)}{\mu(B(x,\theta t))} \left(\int_{B(x,t)} f(y)^p\, d\mu(y) \right) \frac{dt}{t} \\
&\le C\delta^\alpha \omega(\delta)^{-1/p} \int_0^{2d_X} \frac{\omega(t)}{\mu(B(x,\theta t))} \left(\int_{B(x,t)} f(y)^p\, d\mu(y) \right) \frac{dt}{t} \\
&\le C\delta^\alpha \omega(\delta)^{-1/p}
\end{aligned}
$$

since we have by ($\omega1$) and ($\omega2$)

$$\begin{aligned}
&\int_{2d_X}^{\gamma d_X} \frac{\omega(t)}{\mu(B(x,\theta t))} \left(\int_{B(x,t)} f(y)^p \, d\mu(y) \right) \frac{dt}{t} \\
&\le C \frac{\omega(d_X)}{\mu(X)} \int_X f(y)^p \, d\mu(y) \\
&\le C \int_{d_X}^{2d_X} \frac{\omega(t)}{\mu(B(x,\theta t))} \left(\int_{B(x,t)} f(y)^p \, d\mu(y) \right) \frac{dt}{t} \\
&\le C \int_0^{2d_X} \frac{\omega(t)}{\mu(B(x,\theta t))} \left(\int_{B(x,t)} f(y)^p \, d\mu(y) \right) \frac{dt}{t}
\end{aligned}$$

when $\gamma > 2$. Thus we obtain the required result. $\square$

Before stating the main theorem we give the assumptions for the function in Sobolev-type inequalities. We consider a function

$$\Psi(t)\colon [0,\infty) \to [0,\infty)$$

satisfies

($\Psi1$) $\Psi(\cdot)$ is continuous on $[0,\infty)$;

($\Psi2$) $t \mapsto \Psi(t)/t$ is almost increasing on $(0,\infty)$, namely there exists a constant $A_2 \ge 1$ such that

$$\Psi(t_1)/t_1 \le A_2 \Psi(t_2)/t_2 \quad \text{whenever } 0 < t_1 < t_2;$$

($\Psi3$) there exists a constant $A_3 \ge 1$ such that

$$\Psi\big(t\omega^{-1}(t^{-p})^{\alpha}\big) \le A_3 t^p \quad \text{for all } t \ge 1.$$

We write $\overline{\psi}(t) = \sup_{0<s\le t}(\Psi(s)/s)$ and

$$\overline{\Psi}(t) = \int_0^t \overline{\psi}(r) \, dr$$

for $t \ge 0$. Then $\overline{\Psi}(\cdot)$ is convex and

$$\Psi(t/2) \le \overline{\Psi}(t) \le A_2 \Psi(t)$$

for all $t \ge 0$.

Let $\theta \ge 1$. Given $\Psi(t)$ and $\omega(r)$ as above, we define the $\mathcal{L}^{\Psi,\omega,\theta}$ norm by

$$\|f\|_{\mathcal{L}^{\Psi,\omega,\theta}(X)} = \inf \Bigg\{ \lambda > 0; \\ \sup_{x\in X} \left(\int_0^{2d_X} \frac{\omega(r)}{\mu(B(x,\theta r))} \left(\int_{B(x,r)} \overline{\Psi}\big(|f(y)|/\lambda\big) \, d\mu(y) \right) \frac{dr}{r} \right) \le 1 \Bigg\}.$$

The space of all measurable functions f on X with $\|f\|_{\mathcal{L}^{\Psi,\omega,\theta}(X)} < \infty$ is denoted by $\mathcal{L}^{\Psi,\omega,\theta}(X)$.

As an application of M_λ, we establish a Sobolev-type inequality for $I_{\alpha,\tau} f$ of functions in $\mathcal{L}^{p,\omega,\theta_1}(X)$ in the non-doubling setting.

Theorem 3.3. *Let X be a non-doubling metric measure space. Let $1 \le \theta_1 < \theta_2$ and $\theta_1(\theta_2+1)/(\theta_2-\theta_1) < \lambda < \tau$. Assume that X satisfies $(M\lambda)$ and $(\omega 1')$ and $(\omega\alpha)$ hold. If $p > 1$, then there exists a constant $C > 0$ such that*

$$\|I_{\alpha,\tau} f\|_{\mathcal{L}^{\Psi,\omega,\theta_2}(X)} \le C \|f\|_{\mathcal{L}^{p,\omega,\theta_1}(X)}$$

for all $f \in \mathcal{L}^{p,\omega,\theta_1}(X)$.

Proof. Let f be a nonnegative measurable function on X such that $\|f\|_{\mathcal{L}^{p,\omega,\theta_1}(X)} \le 1$. Let $x \in X$ and $0 < \delta < d_X/2$. If $M_\lambda f(x) < 1$, then

$$I_{\alpha,\tau} f(x) = \int_{B(x,d_X)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y) \le C d_X^\alpha M_\lambda f(x) \le C$$

by $\lambda < \tau$ (see [21, p. 134]). If $\omega^{-1}\big(\{M_\lambda f(x)\}^{-p}\big) \ge d_X/2$ and $M_\lambda f(x) \ge 1$, then

$$\begin{aligned} I_{\alpha,\tau} f(x) &= \int_{B(x,d_X)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y) \chi_{\{y \in X : M_\lambda f(y) \ge 1\}}(x) \\ &\le C d_X^\alpha M_\lambda f(x) \chi_{\{y \in X : M_\lambda f(y) \ge 1\}}(x) \\ &\le C M_\lambda f(x) \omega^{-1}\big(\{M_\lambda f(x)\}^{-p}\big)^\alpha \chi_{\{y \in X : M_\lambda f(y) \ge 1\}}(x). \end{aligned}$$

By Lemma 3.2, we find

$$\begin{aligned} I_{\alpha,\tau} f(x) &= \int_{B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y) + \int_{X \setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y) \\ &\le C\big\{\delta^\alpha M_\lambda f(x) + \delta^\alpha \omega(\delta)^{-1/p}\big\}. \end{aligned}$$

If $\omega^{-1}\big(\{M_\lambda f(x)\}^{-p}\big) < d_X/2$ and $M_\lambda f(x) \ge 1$, then take $\delta = \omega^{-1}\big(\{M_\lambda f(x)\}^{-p}\big)$. Then we have

$$I_{\alpha,\tau} f(x) \le C M_\lambda f(x) \omega^{-1}\big(\{M_\lambda f(x)\}^{-p}\big)^\alpha \chi_{\{y \in X : M_\lambda f(y) \ge 1\}}(x)$$

by Lemma 3.1. Therefore we obtain

$$I_{\alpha,\tau} f(x) \le C_1' \max\big\{M_\lambda f(x) \omega^{-1}\big(\{M_\lambda f(x)\}^{-p}\big)^\alpha \chi_{\{y \in X : M_\lambda f(y) \ge 1\}}(x), 1\big\},$$

so that by $(\Psi 2)$ and $(\Psi 3)$, we have

$$\begin{aligned} \Psi\big(I_{\alpha,\tau} f(x)/C_1'\big) &\le C\big\{\Psi\big(M_\lambda f(x) \omega^{-1}\big(\{M_\lambda f(x)\}^{-p}\big)^\alpha\big) \chi_{\{y \in X : M_\lambda f(y) \ge 1\}}(x) + 1\big\} \\ &\le C\big[\{M_\lambda f(x)\}^p + 1\big]. \end{aligned}$$

Therefore we obtain by Theorem 2.4,

$$\begin{aligned}
&\int_0^{2d_X} \frac{\omega(r)}{\mu(B(z,\theta_2 r))} \left(\int_{B(z,r)} \Psi\big(I_{\alpha,\tau} f(x)/C_1'\big)\, d\mu(x) \right) \frac{dr}{r} \\
&\le C \left\{ \int_0^{2d_X} \frac{\omega(r)}{\mu(B(z,\theta_2 r))} \left(\int_{B(z,r)} \{M_\lambda f(x)\}^p\, d\mu(x) \right) \frac{dr}{r} + \int_0^{2d_X} \omega(r)\, \frac{dr}{r} \right\} \\
&\le C
\end{aligned}$$

for all $z \in X$ since

$$\int_0^{2d_X} \omega(r)\, \frac{dr}{r} = \int_0^{2d_X} r^{-\varepsilon_1} \omega(r) \cdot r^{\varepsilon_1}\, \frac{dr}{r} \le C \int_0^{2d_X} r^{\varepsilon_1}\, \frac{dr}{r} \le C$$

by $(\omega 1')$ and $(\omega 3)$. This completes the proof of the theorem. □

As in the proof of Theorem 3.3, we can prove the following theorem for the doubling metric measure case.

Theorem 3.4. *Let X be a doubling metric measure space. Assume that $(\omega 1')$ and $(\omega\alpha)$ hold. If $p > 1$, then there exists a constant $C > 0$ such that*

$$\|I_{\alpha,1} f\|_{\mathcal{L}^{\Psi,\omega,1}(X)} \le C \|f\|_{\mathcal{L}^{p,\omega,1}(X)}$$

for all $f \in \mathcal{L}^{p,\omega,1}(X)$.

4. Corollaries

In this section, we give consequences of Theorems 3.3 and 3.4.

Let

$$\omega(r) = r^\sigma (\log(e + 1/r))^\beta$$

be as in Example 1.1 and set

$$\Psi(t) = \left\{ t(\log(e+t))^{\alpha\beta/\sigma} \right\}^{p^*},$$

where $1/p^* = 1/p - \alpha/\sigma$. If $1 < p < \sigma/\alpha$, then $\omega(r)$ satisfies condition $(\omega\alpha)$ and $\Psi(t)$ satisfies condition $(\Psi 3)$.

Example 4.1. Let $X_1 = \{(x,0) \in \mathbf{R}^2 : 0 \le x < 1\}$ and $X_2 = \{(x,y) \in \mathbf{R}^2 : |x| < 1, x_1 < 0\}$ and define $(X,d,\mu) = (X_1,d_2,m_1) \cup (X_2,d_2,m_2)$, where d_2 denotes the 2-dimensional Euclidean distance and m_i denotes the i-dimensional Lebesgue measure. It is easy to show that μ is non-doubling. Since X is a separable metric space, X satisfies $(M\lambda)$ for $\lambda \ge 2$ (see Remark 2.2).

Let $\theta \geq 1$ and $p \geq 1$. Consider the function

$$f(y) = d_2(0,y)^{-a}\chi_{X_2}(y)$$

for $a < \min\{2/p, \sigma/p\}$. Then note that

$$\begin{aligned}
&\int_0^4 \frac{\omega(r)}{\mu(B(x,\theta r))}\left(\int_{B(x,r)} |f(y)|^p \, d\mu(y)\right) \frac{dr}{r} \\
&\leq C \int_0^4 \frac{\omega(r)}{\mu(B(x,\theta r))}\left(\int_{B(0,r)\cap X_2} |f(y)|^p \, d\mu(y)\right) \frac{dr}{r} \\
&\leq C \int_0^4 \frac{\omega(r)}{\mu(B(x,\theta r))} r^{2-ap} \, \frac{dr}{r} \\
&\leq C \int_0^4 r^{\sigma-ap} (\log(e+1/r))^{\beta} \, \frac{dr}{r} < \infty
\end{aligned}$$

for all $x \in X$ since $\mu(B(x,\theta r)) \geq Cr^2$ for all $x \in X$ and $0 < r < 4$. Therefore $f \in \mathcal{L}^{p,\omega,\theta}(X)$, so that $\mathcal{L}^{p,\omega,\theta}(X) \neq \{0\}$.

Corollary 4.2. *Let X be a non-doubling metric measure space. Let $1 \leq \theta_1 < \theta_2$ and $\theta_1(\theta_2+1)/(\theta_2-\theta_1) < \lambda < \tau$. Assume that X satisfies $(M\lambda)$. If $1 < p < \sigma/\alpha$, then there exists a constant $C > 0$ such that*

$$\|I_{\alpha,\tau} f\|_{\mathcal{L}^{\Psi,\omega,\theta_2}(X)} \leq C\|f\|_{\mathcal{L}^{p,\omega,\theta_1}(X)}$$

for all $f \in \mathcal{L}^{p,\omega,\theta_1}(X)$.

Corollary 4.3. *Let X be a doubling metric measure space. If $1 < p < \sigma/\alpha$, then there exists a constant $C > 0$ such that*

$$\|I_{\alpha,1} f\|_{\mathcal{L}^{\Psi,\omega,1}(X)} \leq C\|f\|_{\mathcal{L}^{p,\omega,1}(X)}$$

for all $f \in \mathcal{L}^{p,\omega,1}(X)$.

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Takao Ohno
Faculty of Education, Oita University, Dannoharu Oita-city 870-1192, Japan
E-mail address: t-ohno@oita-u.ac.jp

Tetsu Shimomura
Department of Mathematics, Graduate School of Humanities and Social Sciences, Hiroshima University, Higashi-Hiroshima 739-8524, Japan
E-mail address: tshimo@hiroshima-u.ac.jp