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On Nodal Properties for Some Nonlinear Conformable Fractional Differential Equations

Wei-Chuan Wang and Yan-Hsiou Cheng*

Abstract. A class of conformable fractional differential equations

$$D_x^\alpha D_x^\alpha u(x) + \omega(x) f(u(x)) = 0 \quad \text{on } (0,1)$$

is considered. We first give a sufficient condition for the existence of sign-changing solutions with the prescribed number of zeros to this problem. On the basis of this result, we turn to a specific case of the above problem and give a uniqueness theorem related to the function ω. Essentially, the main methods using in this work are properties of conformable fractional calculus, the scaling argument and Prüfer-type substitutions.

1. Introduction

Fractional calculus has arisen in 1600s and developed as a pure theoretical field of mathematics. In the last few decades, fractional derivatives have been applied in various fields: physics (classic and quantum mechanics, thermodynamics, etc.), chemistry, biology, economics, engineering, signal and image processing, and control theory. For the above, the readers can refer to a survey paper [25] and their bibliographies. Several definitions of fractional derivatives, such as Riemann–Liouville, Caputo, Riesz, Riesz–Caputo, Weyl, Grunwald–Letnikov, Hadamard, and Chen derivatives, are used in the existent literature. Later, a simple solution to the discrepancies between known definitions was presented with the introduction of a new fractional notion, called the conformable fractional derivative (CFD) [19]. For simple and similar to the standard derivative, one can say that the conformable derivative combines the good characteristics of known fractional derivatives [1]. Now this subject is under strong development: see [3, 6, 7] and references therein.

The aim of the present paper is to study some solution properties related to a class of conformable fractional differential equations. First, we intend to establish some sufficient conditions on the existence of solutions having a prescribed number of zeros in a

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*Corresponding author.

finite interval. By considering sign-changing solutions to conformable fractional differential equations, we are led to study the following nonlinear problem:

$$D_x^\alpha D_x^\alpha u(x) + \omega(x) f(u(x)) = 0, \tag{1.1}$$

$$u(0) = u(1) = 0, \tag{1.2}$$

where D_x^α is the *CFD* of order α. Note that the details of *CFD* will be discussed in Section 2. Throughout the paper we assume the following conditions hold:

(C_1) $\alpha \in (0,1]$, $\omega \in C^1(\mathbb{R}^+)$, and $\omega > \epsilon_1$ on $[0,\infty)$, where ϵ_1 is a constant,

(C_2) $f \in C^1(\mathbb{R})$ with $sf(s) > 0$ for $s \neq 0$.

Note that the condition (C_2) implies $f(0) = 0$. For $\alpha = 1$, many cases of BVP (1.1)–(1.2) have been investigated in numerous papers using various methods and techniques (see e.g., [4, 12, 14, 15, 17, 18, 38, 42, 43]). Most results in the above mentioned papers are about the existence of one or more positive solutions. Erbe [13] initiated the idea of connecting BVP (1.1)–(1.2) with the eigenvalues of its corresponding linear Sturm–Liouville problem (1.3) (see below). Later, Nato and Tanaka [30] established the precise condition concerning the behavior of the ratio $f(s)/s$ at infinity and zero for the existence of solutions with prescribed numbers of zeros. Besides, much effort has focussed on a fractional generalization ($0 < \alpha \leq 1$) of the well known Sturm–Liouville problems recently (see [2, 20, 22, 33]). These types of problem arise in various areas of science and in many fields in engineering, we refer to [5, 26, 28, 36]. Here, we employ the following eigenvalue problem

$$D_x^\alpha D_x^\alpha y(x) + \lambda\omega(x) y(x) = 0 \quad \text{and} \quad y(0) = y(1) = 0. \tag{1.3}$$

It is known that (cf. [11, 29]) the problem (1.3) has a countable number of eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$ satisfying

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_k < \cdots \to \infty,$$

and the corresponding eigenfunction $y_k(x)$ has exactly $k-1$ zeros in $(0,1)$. Now (1.3) is known as a conformable fractional Sturm–Liouville problem.

Motivated by the idea in [30] and some previous results [23, 39, 40], we intend to give a sufficient condition to the existence of sign-changing solutions with prescribed numbers of zeros. To the best of the authors' knowledge, there are few results reported on this topic related to conformable fractional differential equations. The following is our first result.

Theorem 1.1. *Assume that there exists an integer $k \in \mathbb{N}$ such that*

$$\limsup_{s\to 0^+} \frac{f(s)}{s} < \lambda_k < \liminf_{s\to\infty} \frac{f(s)}{s}. \tag{1.4}$$

Then, the problem (1.1)–(1.2) *has a solution with exactly $k-1$ zeros in* $(0,1)$.

Under the derivation of Theorem 1.1, we next turn to a typical case of (1.1)–(1.2):

$$\begin{cases} D_x^\alpha D_x^\alpha u(x) + \omega(x)|u(x)|^{q-1}u(x) = 0, \\ u(0) = 0, \quad D_x^\alpha u(0) = \gamma > 0. \end{cases} \tag{1.5}$$

By Theorem 1.1 and its proof, we can obtain the following result immediately. It is related to the existence of nodal solutions depending on the increasing initial parameters.

Corollary 1.2 (Sturmian theory)**.** *Assume that* (C_1) *and* $q > 1$ *hold. Then there exists a strictly increasing sequence of positive numbers* $\{\gamma_n\}_{n=1}^\infty$*, such that the solution* $u(x;\gamma_n)$ *of the initial value problem* (1.5) *satisfies the right boundary condition* $u(1) = 0$*, and* $u(x;\gamma_n)$ *has exactly* $n-1$ *zeros in* $(0,1)$ *for* $n \in \mathbb{N}$*.*

Note that the sequence $\{\gamma_n\}_{n=1}^\infty$ obtained in Corollary 1.2 may not be unique, that is, the problem (1.5) may has two solutions having exactly $n-1$ zeros in $(0,1)$. In [37], the author showed that such a sequence is unique, provided $\omega \in C^2(\mathbb{R})$ and $([\omega(x)]^{-1/2})'' \leq 0$ on $\mathbb{R}$ with $\alpha = 1$. For the above result, one can treat the initial data as a version of energy for the nonlinear problem (1.5), and the initial data play a role similar to the spectral parameters. Now for each n denote $\{x_k^{(n)}\}_{k=1}^{n-1}$ to be the zero set (or nodal set) of $u(x;\gamma_n)$ and let $x_0^{(n)} = 0$, $x_n^{(n)} = 1$ and $u(x_k^{(n)}) = 0$ for $0 \leq k \leq n$. The denseness of the nodal set $\{x_k^{(n)}\}_{k=0}^n$ will be proven in Section 4. The rest of this work is to investigate that knowledge of the nodal points and some other information can determine the function ω. This issue is well studied now and called *the inverse nodal problem* (cf. [9, 10, 16, 24, 27, 35, 41] etc.). For the problem (1.5), we have the following

Theorem 1.3 (Uniqueness)**.** *Assume that* (C_1) *and* $q > 1$ *hold. Let* $\omega(x)$ *and* $\overline{\omega}(x)$ *be two functions in* (1.5) *associated with the initial values and nodal data* $(\gamma_n, x_k^{(n)})$ *and* $(\overline{\gamma}_n, \overline{x}_k^{(n)})$ *respectively. Assume that* $\omega(0) = \overline{\omega}(0)$*,* $\gamma_n = \overline{\gamma}_n$ *and* $x_k^{(n)} = \overline{x}_k^{(n)}$*,* $1 \leq k \leq n-1$*, for sufficiently large* n*. Then,* $\omega = \overline{\omega}$ *in* $[0,1]$*.*

Our plan for this paper is as follows. In Section 2, we first discuss the basic definitions and elementary properties for conformable fractional calculus. Later, we prove the uniqueness and global existence of solutions to the initial value problem (1.1) in this section. In Section 3, we prepare some technical lemmas and give the proof of Theorem 1.1. Finally in Section 4 we derive the denseness of nodal solutions to (1.5) and give the proof of Theorem 1.3.

2. Basic definitions and some preliminaries

The fractional calculus [1, 19, 21, 31, 34] attracted much attention and well studied in the last and present centuries. In this section, we first recall the elementary definitions and properties of conformable fractional calculus for the readers' convenience.

Definition 2.1. (cf. [1, 19]) Let $0 < \alpha \leq 1$ and $f\colon [0,\infty) \to \mathbb{R}$.

(i) The conformable fractional derivative of f of order α at $x > 0$ is defined by

$$D_x^\alpha f(x) = \lim_{\epsilon \to 0} \frac{f(x + \epsilon x^{1-\alpha}) - f(x)}{\epsilon},$$

and the conformable fractional derivative at 0 is defined as $D_x^\alpha f(0) = \lim_{x \to 0^+} D_x^\alpha f(x)$. Note that if f is differentiable, then

$$D_x^\alpha f(x) = x^{1-\alpha} f'(x),$$

where $' = \frac{d}{dx}$ is the ordinary derivative with respect to x. If $D_x^\alpha f(x_0)$ exists, one can say that f is α-differentiable at x_0.

(ii) The conformable fractional integral of f of order α is defined by

$$I_\alpha f(x) = \int_0^x t^{\alpha-1} f(t)\, dt \quad \text{for } x > 0.$$

Proposition 2.2. (cf. [1, 19, 29])

(i) *Let $f\colon [0,\infty) \to \mathbb{R}$ be any continuous function. Then, for all $x > 0$ we have*

$$D_x^\alpha I_\alpha f(x) = f(x).$$

(ii) *Let $f\colon (0,b) \to \mathbb{R}$ be differentiable. Then, for $x > 0$ we have*

$$I_\alpha D_x^\alpha f(x) = f(x) - f(0). \tag{2.1}$$

(iii) *For all $p \in \mathbb{R}$, $D_x^\alpha (x^p) = p x^{p-\alpha}$.*

(iv) *Let $f, g\colon (0,\infty) \to \mathbb{R}$ be α-differentiable. Then,*

$$D_x^\alpha (fg) = (D_x^\alpha f) g + f (D_x^\alpha g) \quad \textit{and} \quad D_x^\alpha \left(\frac{f}{g} \right) = \frac{(D_x^\alpha f) g - f (D_x^\alpha g)}{g^2} \quad \textit{with } g \neq 0.$$

(v) *(α-chain rule) Let $f, g\colon (0,\infty) \to \mathbb{R}$ be α-differentiable and $h(x) = f(g(x))$. Then, $h(x)$ is α-differentiable and for all x with $x \neq 0$ and $g(x) \neq 0$ we have*

$$D_x^\alpha h(x) = D_x^\alpha f(g(x)) \cdot D_x^\alpha g(x) \cdot g(x)^{\alpha-1}. \tag{2.2}$$

(vi) *(α-integration by parts) Let $f, g\colon [a,b] \to \mathbb{R}$ be two functions such that fg is differentiable. Then,*

$$\int_a^b x^{\alpha-1} f(x) D_x^\alpha g(x)\, dx = f(x) g(x) \Big|_a^b - \int_a^b x^{\alpha-1} g(x) D_x^\alpha f(x)\, dx. \tag{2.3}$$

Now we consider the initial value problem consisting of (1.1) coupled with

$$u(0) = 0, \quad D_x^\alpha u(0) = \gamma, \tag{2.4}$$

where γ is a positive parameter. We first employ a previous result to show the existence and uniqueness of a solution to (1.1) and (2.4).

Theorem 2.3. *Assume that* (C_1) *and* (C_2) *hold. Then there exists a local solution of* (1.1) *and* (2.4)*. Moreover, this solution is unique in a neighborhood* $J = [0, a]$ *for some* $a > 0$.

Proof. By (C_1), (C_2) and (2.1), the initial value problem (1.1) and (2.4) is equivalent to

$$D_x^\alpha u(x) = \gamma - \int_0^x t^{\alpha-1}\omega(t) f(u(t))\, dt := F_\gamma(x, u(x)) \quad \text{with } u(0) = 0. \tag{2.5}$$

Define $B(u, r) = \left\{v \in C([0, \overline{a}], \mathbb{R}) : \max_{x\in[0,\overline{a}]} |v(x)-u(x)| \leq r\right\}$ a closed ball in $C([0, \overline{a}], \mathbb{R})$. Then, $F_\gamma\colon [0, \overline{a}] \times B(0, r) \to \mathbb{R}$ is a continuous function satisfying $|F_\gamma(x, u)| \leq M$ for all $x \in [0, \overline{a}]$, $u \in B(0, r)$ and some $M > 0$. Moreover, $F_\gamma(x, \cdot\,)$ is Lipschitz continuous for all $x \in [0, \overline{a}]$. Now, by a previous result [32, Theorem 1] there exists a unique solution of (2.5) defined on some interval $J = [0, a]$. This completes the proof. □

Later, we introduce an energy identity related to the local solution $u(x) = u(x; \gamma)$ of (1.1) and (2.4) in $J = [0, a]$, and apply it to derive the global existence of the solutions. A priori estimates for the energy will be given. It is crucial to the proof of the main theorem. Assume that an energy function $E[u](x, \gamma)$ is defined by

$$E[u](x, \gamma) \equiv \frac{1}{2}\big(D_x^\alpha u(x)\big)^2 + \omega(x) F(u(x)) \quad \text{for } x \in [0, a], \tag{2.6}$$

where $F(s) = \int_0^s f(\sigma)\, d\sigma$. In particular,

$$E[u](0, \gamma) = \frac{1}{2}\gamma^2. \tag{2.7}$$

Then, the following a priori bound for the energy function $E[u](x, \gamma)$.

Proposition 2.4. *Assume that* $b > 0$ *and* $\gamma > 0$ *are arbitrary. For* $x \in [0, b]$*, the energy function* $E[u](x, \gamma)$ *defined as in* (2.6)–(2.7) *satisfies the following estimates*

$$\frac{1}{2}\gamma^2 e^{-b\kappa_b} \leq E[u](x, \gamma) \leq \frac{1}{2}\gamma^2 e^{b\kappa_b}, \tag{2.8}$$

where $\kappa_b = \max\left\{\frac{|\omega'(x)|}{\omega(x)} : x \in [0, b]\right\}$.

Proof. In view of (1.1), (2.2) and (2.6), we find that

$$
\begin{aligned}
D_x^\alpha E[u](x,\gamma) &= (D_x^\alpha u(x))^{2-\alpha}\cdot D_x^\alpha D_x^\alpha u(x)\cdot (D_x^\alpha u(x))^{\alpha-1}\\
&\quad + D_x^\alpha \omega(x)\cdot F(u(x)) + \omega(x)\cdot x^{1-\alpha}\frac{d}{dx}\left(\int_0^{u(x)} f(\sigma)\,d\sigma\right)\\
&= D_x^\alpha u(x)[D_x^\alpha D_x^\alpha u(x) + \omega(x) f(u(x))] + D_x^\alpha \omega(x)\cdot F(u(x))\\
&= D_x^\alpha \omega(x)\cdot F(u(x)).
\end{aligned} \tag{2.9}
$$

That is,

$$\frac{d}{dx}E[u](x,\gamma) \le \frac{|\omega'(x)|}{\omega(x)}\cdot \omega(x)F(u(x)).$$

Then,

$$\frac{d}{dx}E[u](x,\gamma) \le \kappa_a E[u](x,\gamma),$$

where $\kappa_a = \max\left\{\frac{|\omega'(x)|}{\omega(x)} : x\in J=[0,a]\right\}$. Hence, for any $x\in[0,a]$

$$E[u](x,\gamma) \le E[u](0,\gamma)e^{\kappa_a x} \le \frac{1}{2}\gamma^2 e^{a\kappa_a}$$

by (2.7) and the continuity of $E[u](x,\gamma)$ in x. This means that both $u(x;\gamma)$ and $D_x^\alpha u(x;\gamma)$ are bounded as long as the solution exists. Thus, by a standard argument, we conclude that $u(x;\gamma)$ can be extended to the whole interval $[0,b]$ for any $b>0$. Hence, for $x\in[0,b]$

$$E[u](x,\gamma) \le \frac{1}{2}\gamma^2 e^{b\kappa_b},$$

where $\kappa_b = \max\left\{\frac{|\omega'(x)|}{\omega(x)} : x\in[0,b]\right\}$. Similarly, in view of (2.9), for $x\in[0,b]$ one has

$$\frac{d}{dx}E[u](x,\gamma) \ge -\kappa_b\cdot\omega(x)F(u(x)) \ge -\kappa_b E[u](x,\gamma).$$

The above linear differential inequality implies that

$$E[u](x,\gamma) \ge E[u](0,\gamma)e^{-b\kappa_b} = \frac{1}{2}\gamma^2 e^{-b\kappa_b}$$

for $x\in[0,b]$. Therefore, the proof is complete. □

Corollary 2.5. *For the solution $u(x;\gamma)$ of the initial value problem* (1.1) *and* (2.4)*, it can be extended to all $x\ge 0$.*

Proof. Suppose that $u(x;\gamma)$ does not exist on the whole real axis. Without loss of generality, we assume that $u(x;\gamma)$ exists on a maximal right interval $[0,c)$ for some finite $c>0$. Then, $u(x;\gamma)$ is unbounded on $[0,c)$, $\lim_{x\to c^-}|u(x;\gamma)|=\infty$. Otherwise, if $u(x;\gamma)$ is bounded on $[0,c)$, then $\lim_{x\to c^-} D_x^\alpha u(x;\gamma)$ exists by taking the conformable fractional integral on (1.1) over the interval $[0, c)$. This implies that $u(x;\gamma)$ can be extended through c. On the other hand, since $\lim_{x\to c^-} |u(x;\gamma)| = \infty$, there exists a sequence $x_n \to c^-$ such that $|u(x_n;\gamma)| \to \infty$. Hence, by (2.6) and the conditions on ω and f one can obtain

$$\lim_{n\to\infty} E[u](x_n,\gamma) = \infty.$$

But this contradicts the right inequality in (2.8). Therefore, the solution $u(x;\gamma)$ exists for all $x \geq 0$. □

3. Sturmian comparison theorem and some technical lemmas

In this section some elementary lemmas are derived to prepare for the proof of Theorem 1.1. First consider a pair of conformable fractional differential equations

$$D_x^\alpha D_x^\alpha u(x) + Q_1(x)u(x) = 0, \quad x > 0, \tag{3.1}$$

$$D_x^\alpha D_x^\alpha v(x) + Q_2(x)v(x) = 0, \quad x > 0, \tag{3.2}$$

where $Q_2(x) \geq Q_1(x)$ for $x > 0$ are given continuous functions. Let us recall the Sturm comparison theorem for conformable fractional differential equations and derive some technical lemmas.

Lemma 3.1. [32, Theorem 3] *Let u be a nontrivial solution of (3.1) satisfying $u(s_1) = u(s_2) = 0$ when $0 < s_1 < s_2$. Then every nontrivial solution v of (3.2) has a zero in (s_1, s_2).*

Lemma 3.2. *Let $\{t_i\}_{i=1}^{k-1}$ be zeros of an eigenfunction y_k for (1.3) corresponding to λ_k satisfying $0 = t_0 < t_1 < t_2 < \cdots < t_{k-1} < t_k = 1$.*

(i) *Assume $\lambda > \lambda_k$. For each $i \in \{1, 2, 3, \ldots, k\}$, there is a solution z_i of*

$$D_x^\alpha D_x^\alpha z(x) + \lambda\omega(x)z(x) = 0, \tag{3.3}$$

which has at least two zeros in (t_{i-1}, t_i).

(ii) *Assume $\lambda < \lambda_k$. For each $i \in \{1, 2, 3, \ldots, k\}$, there exists a solution $\overline{z}_i$ of (3.3) satisfying $\overline{z}_i(x) > 0$ on $[t_{i-1}, t_i]$.*

Proof. We give a proof of (i) here. The proof of (ii) is similar, so we omit it. Consider the initial conditions

$$z(t_{i-1} + \epsilon) = 0 \quad \text{and} \quad D_x^\alpha z(t_{i-1} + \epsilon) = 1 \tag{3.4}$$

with $\epsilon \geq 0$ for fixed $i \in \{2, 3, 4, \ldots, k\}$. By Lemma 3.1, the solution of (3.3) and (3.4) with $\epsilon = 0$ has a zero $\overline{t}_0$ in (t_{i-1}, t_i). For a small $\epsilon > 0$ we claim that the solution of (3.3)–(3.4) has a zero t_ϵ near $\bar{t}_0$. Here we quote the Prüfer phase equation of (3.3) from (3.10) below. By the continuous dependence of solutions on initial conditions, the above claim holds. Since $z(t_{i-1}+\epsilon)=0$, we have that for sufficiently small $\epsilon>0$ the solution of (3.3)–(3.4) has two zeros $t_{i-1}+\epsilon$ and t_ϵ in (t_{i-1},t_i). For the first interval (t_0,t_1), consider the initial condition on t_1 as follows: $z(t_1-\epsilon)=0$ and $D_x^\alpha z(t_1-\epsilon)=-1$ with $\epsilon\geq 0$. Then applying the similar arguments, the proof for this subinterval is also valid. □

Lemma 3.3. *Let $M>0$, $\omega^*\equiv\max\{\omega(x):x\in[0,1]\}$ and γ satisfy $\frac{1}{2}\gamma^2e^{-\kappa_1}>\omega^*F(M)$. Define δ by*

$$\delta\equiv M(\gamma^2e^{-\kappa_1}-2\omega^*F(M))^{-1/2}. \tag{3.5}$$

Then the solution $u(x;\gamma)$ of (1.1) *and* (2.4) *has the following properties:*

(i) *If $u(x;\gamma)$ has no zero in (x_1,x_2) and satisfies $|u(x;\gamma)|\leq M$ on $[x_1,x_2]$ for some $x_1,x_2\in[0,1]$, then we have*
$$x_2-x_1\leq\delta.$$

(ii) *If $u(x;\gamma)$ has no zero in (x_1,x_2) for some $x_1,x_2\in[0,1]$ satisfying $x_2-x_1>2\delta$, then $|u(x;\gamma)|>M$ for $x\in(x_1+\delta,x_2-\delta)$.*

Proof. (i) For simplicity, we denote $u(x;\gamma)$ by $u(x)$. Without loss of generality, assume $u(x)>0$ on (x_1,x_2). Then we have

$$0\leq u(x)\leq M\quad\text{for } x_1\leq x\leq x_2, \tag{3.6}$$

and
$$E[u](x,\gamma)\leq\frac{1}{2}\big(D_x^\alpha u(x)\big)^2+\omega^*F(M)\quad\text{on } [x_1,x_2].$$
By Proposition 2.4,
$$\frac{1}{2}\gamma^2e^{-\kappa_1}\leq E[u](x,\gamma)\leq\frac{1}{2}\big(D_x^\alpha u(x)\big)^2+\omega^*F(M)\quad\text{on } [x_1,x_2].$$
This implies that
$$|D_x^\alpha u(x)|\geq(\gamma^2e^{-\kappa_1}-2\omega^*F(M))^{1/2}=\frac{M}{\delta}\quad\text{on } [x_1,x_2],$$
where δ is defined in (3.5). Therefore we have either $D_x^\alpha u(x)\geq M/\delta$ or $D_x^\alpha u(x)\leq -M/\delta$ on $[x_1,x_2]$. If $D_x^\alpha u(x)\geq M/\delta$ on $[x_1,x_2]$, then (3.6) implies that
$$M\geq u(x_2)=u(x_1)+\int_{x_1}^{x_2}t^{\alpha-1}D_t^\alpha u(t)\,dt\geq\frac{M}{\delta}\int_{x_1}^{x_2}t^{\alpha-1}\,dt=\frac{M}{\alpha\delta}(x_2^\alpha-x_1^\alpha),$$
and hence $x_2^\alpha-x_1^\alpha\leq\alpha\delta$. Also,
$$x_2^\alpha-x_1^\alpha=\alpha c^{\alpha-1}(x_2-x_1)\leq\alpha\delta$$

for some $c \in (x_1, x_2)$. Therefore,

$$x_2 - x_1 \le c^{1-\alpha}\delta \le \delta.$$

The proof for the case $D_x^\alpha u(x) \le -M/\delta$ on $[x_1, x_2]$ is similar.

(ii) Without loss of generality, let $u(x) > 0$ on (x_1, x_2). In view of (1.1) we see that

$$D_x^\alpha D_x^\alpha u(x) \le 0 \tag{3.7}$$

on $[x_1, x_2]$. Assume, on the contrary, that there exists $z \in (x_1 + \delta, x_2 - \delta)$ such that $u(z) \le M$. First suppose that $D_x^\alpha u(z) \ge 0$. By (3.7), one can obtain

$$D_x^\alpha u(x) \ge D_x^\alpha u(z) \ge 0,$$

where $x \in [x_1, z]$. Hence, u is nondecreasing on $[x_1, z]$. Then we have $0 \le u(x) \le u(z) \le M$ on $[x_1, z]$. By (i), we have $z - x_1 \le \delta$. This contradicts $z \in (x_1 + \delta, x_2 - \delta)$. Next, if $D_x^\alpha u(z) < 0$ and employ (3.7), one can obtain

$$D_x^\alpha u(x) \le D_x^\alpha u(z) < 0,$$

where $x \in [z, x_2]$. This implies that u is nonincreasing on $[z, x_2]$. Then $0 < u(x) \le u(z) \le M$ for $x \in [z, x_2]$. By (i) again, one also obtains $x_2 - z \le \delta$. This also contradicts $z \in (x_1 + \delta, x_2 - \delta)$. Therefore the proof is complete. □

Now, we employ a Prüfer-type substitution for the solution $u(x) = u(x;\gamma)$ of (1.1) and (2.4). Define

$$u(x) = r(x)\sin(\theta(x)) \quad \text{and} \quad D_x^\alpha u(x) = r(x)\cos(\theta(x)).$$

Then,

$$\frac{D_x^\alpha u(x)}{u(x)} = \frac{\cos(\theta(x))}{\sin(\theta(x))}. \tag{3.8}$$

Taking the conformable fractional derivative D_x^α on both sides of (3.8), and applying (1.1) and (2.4), one can obtain

$$\begin{aligned} \theta'(x) &= x^{\alpha-1}\left(\cos^2(\theta(x)) + \omega(x)\frac{f(u(x))}{u(x)}\sin^2(\theta(x))\right), \\ \frac{r'(x)}{r(x)} &= x^{\alpha-1}\left(1 - \omega(x)\frac{f(u(x))}{u(x)}\right)\cos(\theta(x))\sin(\theta(x)) \end{aligned} \tag{3.9}$$

with $\theta(0) = 0$ and $r(0) = \gamma$. Similarly, the Prüfer phase function for (1.3) with $\lambda = \lambda_k$ satisfies

$$\begin{cases} \phi_k'(x) = x^{\alpha-1}[\cos^2(\phi_k(x)) + \lambda_k\omega(x)\sin^2(\phi_k(x))] := G_k(x;\phi_k), \\ \phi_k(0) = 0, \quad \phi_k(1) = k\pi. \end{cases} \tag{3.10}$$

Now we are ready to derive the following main lemma which is crucial to the proof of Theorem 1.1.

Lemma 3.4. *Let $\{\lambda_k\}_{k=1}^{\infty}$ be the eigenvalues of* (1.3).

(i) *Assume that* $\limsup_{s\to 0^+} \frac{f(s)}{s} < \lambda_k$ *for some* $k \in \mathbb{N}$. *Then there exists* $\gamma_* > 0$ *such that* $\theta(1;\gamma) < k\pi$ *for all* $\gamma \in (0,\gamma_*]$. *That is, the solution* $u(x;\gamma)$ *has at most* $k-1$ *zeros in* $(0,1)$ *for* $\gamma \in (0,\gamma_*]$.

(ii) *Assume that* $\liminf_{s\to\infty} \frac{f(s)}{s} > \lambda_k$ *for some* $k \in \mathbb{N}$. *Then there exists* $\gamma^* > 0$ *such that the solution* $u(x;\gamma)$ *has at least* k *zeros in* $(0,1)$ *for* $\gamma \in [\gamma^*,\infty)$.

Proof. (i) By assumption, there exist $\delta > 0$ and $\lambda > 0$ such that

$$\frac{f(s)}{s} < \lambda < \lambda_k \quad \text{for } 0 < s < \delta.$$

Since $u \equiv 0$ satisfies (1.1), by the continuous dependence of solutions on initial conditions, there exists $\gamma_* > 0$ such that $|u(x;\gamma)| < \delta$ for $\gamma < \gamma_*$ and $x \in [0,1]$. By (3.9) and (3.10), for $\gamma < \gamma_*$ and $x \in [0,1]$, one gets

$$\theta'(x;\gamma) < x^{\alpha-1}[\cos^2(\theta(x)) + \lambda_k\omega(x)\sin^2(\theta(x))] = G_k(x;\theta).$$

Recall that ϕ_k is the Prfüer angle of the k-th eigenfunction of (1.3) satisfying (3.10). Thus $\phi_k(1) = k\pi$. By the comparison theorem [8, p. 30], one can obtain that $\theta(1;\gamma) < \phi_k(1)$ for $\gamma < \gamma_*$.

(ii) By assumption, there exist $\lambda > \lambda_k$ and $M > 0$ such that

$$\frac{f(s)}{s} > \lambda > \lambda_k \quad \text{for } s \geq M. \tag{3.11}$$

Let y_k be the k-th eigenfunction of (1.3) corresponding to λ_k and $\{x_i\}_{i=1}^{k-1}$ be zeros of y_k with $x_0 = 0$ and $x_k = 1$. By Lemma 3.2(i), for each $i \in \{1,2,\ldots,k\}$, there exists a solution z_i of (3.3) having at least two zeros in (x_{i-1},x_i). Now fix $i \in \{1,2,\ldots,k\}$. Let t_1 and t_2 be zeros of z_i satisfying $x_{i-1} < t_1 < t_2 < x_i$. Recall that δ is defined as in (3.5) and remark that δ tends to zero as γ tends to infinity. For this i, one can choose $\gamma_i > 0$ so large that, $x_i - x_{i-1} > 2\delta_i$ and $[t_1,t_2] \subset (x_{i-1}+\delta_i, x_i-\delta_i)$, where γ_i and δ_i are consistent with (3.5). Now let $\gamma \geq \gamma_i$. We claim that $u(x;\gamma)$ has at least one zero in (x_{i-1},x_i). Assume, on the contrary, that $u(x;\gamma)$ has no zero in (x_{i-1},x_i). By Lemma 3.3(ii), one obtains $|u(x;\alpha)| > M$ for $x \in (r_{i-1}+\delta, r_i-\delta)$. From (3.11), we have

$$\lambda\omega(x) < \frac{\omega(x)f(u(x;\alpha))}{u(x;\alpha)^{(p-1)}} \quad \text{for } x \in [t_1,t_2] \subset (r_{i-1}+\delta, r_i-\delta).$$

Then Lemma 3.1 implies that $u(x;\alpha)$ has at least one zero in (t_1,t_2). This leads a contradiction. Thus $u(x;\alpha)$ with $\alpha \geq \alpha_i$ has at least one zero in (r_{i-1},r_i). Set $\alpha^* = \max\{\alpha_i : i = 1,2,\ldots,k\}$. If $\alpha \geq \alpha^*$, then $u(x;\alpha)$ has at least one zero in (r_{i-1},r_i) for each $i = 1,2,3,\ldots,k$, which means that $u(x;\alpha)$ has at least k zeros in $(0,1)$ for $\alpha \in [\alpha^*,\infty)$. $\square$

Proof of Theorem 1.1. Assume that the condition (1.4) holds. Then, by Lemma 3.4(i), there exists $\gamma_* > 0$ such that $\theta(1;\gamma) < k\pi$ for $\gamma \le \gamma_*$. Also, Lemma 3.4(ii) implies that there exists $\gamma^* > 0$ such that $\theta(1;\gamma) > k\pi$ for $\gamma \ge \gamma^*$. Since $\theta(1;\gamma)$ is continuous in $\gamma \in (0,\infty)$, there exists γ_k such that $\theta(1;\gamma_k) = k\pi$. This completes the proof. □

4. An application: Nodal property and uniqueness

In this section, we give the proof of Theorem 1.3 and employ the classical methods, the scaling argument and Prüfer-type substitutions, to achieve the goal. Note that $x^{\alpha(1+q)}\omega(x) = O(x^{\alpha(1+q)})$ and $q > 1$. Suppose that $\{\gamma_i\}$ is a positively and strictly increasing sequence which tends to infinity. Now, define the sequence $\{\mu_i\}$ to satisfy the following relation for the scaling argument:

$$\mu_i = \max\left\{s > 0 : s^{\alpha(1+q)}\omega(s) = \gamma_i^{1-q}\right\} \tag{4.1}$$

for $i \in \mathbb{N}$. Hence, if $\{\gamma_i\}$ is a positively increasing sequence which tends to infinity, then the corresponding sequence $\{\mu_i\}$ satisfying (4.1) decreases to zero. Then, the *scaled function* $v_i(t)$ is defined by

$$v_i(x) = \frac{u(\mu_i x;\gamma_i)}{\mu_i^\alpha \gamma_i}. \tag{4.2}$$

By (1.5), a direct calculation yields that v_i satisfies

$$\begin{cases} D_x^\alpha D_x^\alpha v_i(x) + \frac{\omega(\mu_i x)}{\omega(\mu_i)}|v_i(x)|^{q-1}v_i(x) = 0, \\ v_i(0) = 0, \quad D_x^\alpha v_i(0) = 1 \quad \text{and} \quad v_i(\mu_i^{-1}) = 0. \end{cases} \tag{4.3}$$

Remark 4.1. For (4.3), one can define a functional $E[v_i](x)$ for v_i by

$$E[v_i](x) = \frac{1}{2}\big(D_x^\alpha v_i(x)\big)^2 + \frac{\omega(\mu_i x)}{(q+1)\omega(\mu_i)}|v_i(x)|^{q+1} \quad \text{with } E[v_i](0) = \frac{1}{2}.$$

Under the similar derivation as in Proposition 2.4, one can obtain that

$$E[v_i](x) \le E[v_i](0)e^{\mu_i kx} \le \frac{1}{2}e^{kR}$$

for some constant k and $x \in [0,R]$.

Then, we have the following

Lemma 4.2. *Suppose that $R > 0$ is arbitrary and μ_i defined as in* (4.1) *tends to zero as $i \to \infty$. Then $v_i(x)$ converges to the function $V(x)$ uniformly on any compact subinterval of $(0,R]$ as $i \to \infty$, where $V(x)$ solves*

$$\begin{cases} D_x^\alpha D_x^\alpha V(x) + |V(x)|^{q-1}V(x) = 0, \\ V(0) = 0, \quad D_x^\alpha V(0) = 1. \end{cases} \tag{4.4}$$

Remark 4.3. (i) The uniqueness and global existence of solutions for (4.3) and (4.4) are valid by using the similar arguments as in Theorem 2.3 and Corollary 2.5.

(ii) Define a functional $E[V](x) = \frac{1}{2}\big(D_x^\alpha V(x)\big)^2 + \frac{1}{q+1}|V(x)|^{q+1}$ for the solution of (4.4). It is easy to obtain that $E[V](x) = E[V](0) = 1/2$ for all $x \in \mathbb{R}^+$. This shows the uniform boundedness of $D_x^\alpha V(x)$ and $V(x)$.

Proof. Applying a standard argument after, one can express (4.3) and (4.4) as the first order systems,

$$\begin{aligned} v_i'(x) &= x^{\alpha-1} z_i(x), \\ z_i'(x) &= -x^{\alpha-1}|v_i(x)|^{q-1}v_i(x) + x^{\alpha-1}\left(1 - \frac{\omega(\mu_i x)}{\omega(\mu_i)}\right)|v_i(x)|^{q-1}v_i(x), \end{aligned} \tag{4.5}$$

and

$$V'(x) = x^{\alpha-1}Z(x), \quad Z'(x) = -x^{\alpha-1}|V(x)|^{q-1}V(x)$$

with the same initial conditions. Note that for $x \in [0,R]$ the term $|v_i(x)|^{q-1}v_i(x)$ is uniformly bounded by Remark 4.1. Hence, the term $x^{\alpha-1}\big(1 - \frac{\omega(\mu_i x)}{\omega(\mu_i)}\big)|v_i(x)|^{q-1}v_i(x)$ in (4.5) tends to zero uniformly in any compact subinterval of $(0,R]$ as i tends to infinity. This implies that v_i converges to V in the C^1-sense. Therefore, the proof is complete. $\square$

For Corollary 1.2, we will introduce a Prüfer-type substitution to derive the asymptotic estimates for the initial parameter γ_n and the nodal data of solutions. Define

$$u(x) = R(x)\sin(m\psi(x)) \quad \text{and} \quad D_x^\alpha u(x) = gR(x)\cos(m\psi(x)), \tag{4.6}$$

where m, g are some positive constants which will be specified later. Take

$$D_x^\alpha\left(\frac{D_x^\alpha u(x)}{u(x)}\right) = D_x^\alpha\left(\frac{g\cos(m\psi(x))}{\sin(m\psi(x))}\right).$$

Then, one can obtain the phase equation

$$\psi'(x) = x^{\alpha-1}\left(\frac{g}{m}\cos^2(m\psi(x)) + \frac{\omega(x)}{mg}|u(x)|^{q-1}\sin^2(m\psi(x))\right).$$

Applying the scaling argument (4.1)–(4.2) and taking $m = g = \mu^{-\alpha}$ (cf. (4.1)), one can get the modified phase equation

$$\psi'(x) = x^{\alpha-1}\left(\cos^2(\mu^{-\alpha}\psi(x)) + \frac{\omega(x)}{\omega(\mu)}\left|v\left(\frac{x}{\mu}\right)\right|^{q-1}\sin^2(\mu^{-\alpha}\psi(x))\right). \tag{4.7}$$

By Corollary 1.2, integrating (4.7) over $[0,1]$ with respect to $\gamma = \gamma_n$, one can obtain

$$\frac{n\pi}{\mu_n^{-\alpha}} = \int_0^1 t^{\alpha-1}\left(\cos^2(\mu_n^{-\alpha}\psi_n(t)) + \frac{\omega(t)}{\omega(\mu_n)}\left|v_n\left(\frac{t}{\mu_n}\right)\right|^{q-1}\sin^2(\mu_n^{-\alpha}\psi_n(t))\right) dt. \tag{4.8}$$

Note that $v_n\big(\frac{t}{\mu_n}\big)$ and $\sin(\mu_n^{-\alpha}\phi_n(t))$ vanish at the same point by (4.2) and (4.6). And if $\sin(\mu_n^{-\alpha}\psi_n(t))$ tends to zero, $|\cos(\mu_n^{-\alpha}\psi_n(t))|$ will approach to one obviously. Now one can conclude that the right-hand side of (4.8) never vanishes and is bounded. Hence, for sufficiently large n,

$$\mu_n^{-\alpha} = O(n). \tag{4.9}$$

By (4.1) and (C_1),

$$\gamma_n = O\big(n^{\frac{q+1}{q-1}}\big)$$

for sufficiently large n. Now, for $\gamma = \gamma_n$ rewrite the phase equation (4.7) as

$$\psi_n'(t) = t^{\alpha-1} - t^{\alpha-1}\left(\frac{\omega(t)}{\omega(\mu_n)}\left|v_n\left(\frac{t}{\mu_n}\right)\right|^{q-1} - 1\right)\sin^2(\mu_n^{-\alpha}\psi_n(t)). \tag{4.10}$$

Also, (4.7) implies that $\psi_n'(t)$ never vanishes for $t \in [0,1]$ and any fixed n by the same explanation for (4.8). Next, we turn to derive the nodal property for $u(x;\gamma_n)$. Integrating (4.10) over $[0, x_k^{(n)}]$, one can obtain

$$\mu_n^\alpha k\pi = \frac{1}{\alpha}(x_k^{(n)})^\alpha - \int_0^{x_k^{(n)}} t^{\alpha-1}\left(\frac{\omega(t)}{\omega(\mu_n)}\left|v_n\left(\frac{t}{\mu_n}\right)\right|^{q-1} - 1\right)\sin^2(\mu_n^{-\alpha}\psi_n(t))\,dt.$$

Then,

$$\begin{aligned}\mu_n^\alpha\pi &= \frac{1}{\alpha}\left((x_{k+1}^{(n)})^\alpha - (x_k^{(n)})^\alpha\right)\\ &\quad - \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} t^{\alpha-1}\left(\frac{\omega(t)}{\omega(\mu_n)}\left|v_n\left(\frac{t}{\mu_n}\right)\right|^{q-1} - 1\right)\sin^2(\mu_n^{-\alpha}\psi_n(t))\,dt,\end{aligned}$$

i.e.,

$$\mu_n^\alpha\pi = c_k^{\alpha-1}\left(x_{k+1}^{(n)} - x_k^{(n)}\right) - \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} t^{\alpha-1}\left(\frac{\omega(t)}{\omega(\mu_n)}\left|v_n\left(\frac{t}{\mu_n}\right)\right|^{q-1} - 1\right)\sin^2(\mu_n^{-\alpha}\psi_n(t))\,dt$$

for some $c_k \in (x_k^{(n)}, x_{k+1}^{(n)})$. Take a change of variables $\mu_n^{-\alpha}\psi_n(t) = \sigma$ with $t = \eta_n(\sigma)$, where η_n is its inverse function. Then, one has $\mu_n^{-\alpha}\psi_n'(t)\,dt = d\sigma$. And define $\ell_k^{(n)} = x_{k+1}^{(n)} - x_k^{(n)}$ for $k = 1,2,3,\ldots,n-1$, the nodal length of $u(x;\gamma_n)$. Hence,

$$\mu_n^\alpha\pi = c_k^{\alpha-1}\ell_k^{(n)} - \int_{k\pi}^{(k+1)\pi} \big(\eta_n(\sigma)\big)^{\alpha-1}\left(\frac{\omega(\eta_n(\sigma))}{\omega(\mu_n)}\left|v_n\left(\frac{\eta_n(\sigma)}{\mu_n}\right)\right|^{q-1} - 1\right)\frac{\sin^2(\sigma)}{\mu_n^{-\alpha}\psi_n'(\eta_n(\sigma))}\,d\sigma. \tag{4.11}$$

By (4.9), (4.10) and (4.11), for sufficiently large n one can obtain

$$\ell_k^{(n)} = O\left(\frac{1}{n}\right).$$

Now, we conclude the above results as follows.

Proposition 4.4. *For sufficiently large n and $1 \le k \le n-1$, the following estimates are valid.*

(i) *The initial parameter γ_n satisfies*

$$\gamma_n = O\big(n^{\frac{q+1}{q-1}}\big).$$

Moreover, the scaling parameter μ_n satisfies

$$\mu_n^{-\alpha} = O(n).$$

(ii) *The nodal length $\ell_k^{(n)}$ of $u(x;\gamma_n)$ satisfies*

$$\ell_k^{(n)} = O\left(\frac{1}{n}\right).$$

This shows that the nodal set $\mathbb{X}_n = \{x_k^{(n)} : k = 0,1,2,3,\dots,n; n \in \mathbb{N}\}$ of $u(x;\gamma_n)$ is a dense subset of $(0,1)$ for sufficiently large n.

Remark 4.5. For sufficiently large n, $\mu^{-1}(x_{k+1}^{(n)} - x_k^{(n)})$ is the kth nodal length of $V(x)$ by (4.2) and Lemma 4.2. One can also obtain the phase equation

$$\varphi'(x) = x^{\alpha-1}\big(\cos^2(\varphi(x)) + |V(x)|^{q-1}\sin^2(\varphi(x))\big)$$

for $V(x)$ by applying a Prüfer substitution: $V(x) = \rho(x)\sin(\varphi(x))$ and $D_x^\alpha V(x) = \rho(x)\cos(\varphi(x))$. By the nice property for $V(x)$ (cf. Remark 4.3) and the similar argument as in (4.8), one can conclude that the interval $[\mu_n^{-1}x_k^{(n)}, \mu_n^{-1}x_{k+1}^{(n)}]$ is uniformly bounded and never vanishes for sufficiently large n.

Now, it suffices to prove Theorem 1.3.

Proof of Theorem 1.3. Assume that u_n and $\overline{u}_n$ are two solutions associated with the initial data γ_n and $\overline{\gamma}_n$ corresponding to the functions ω and $\overline{\omega}$ in (1.5) respectively, i.e.,

$$\begin{aligned} D_x^\alpha D_x^\alpha u_n + \omega(x)|u_n|^{q-1}u_n = 0,\\ D_x^\alpha D_x^\alpha \overline{u}_n + \overline{\omega}(x)|\overline{u}_n|^{q-1}\overline{u}_n = 0. \end{aligned}$$

And μ_n and $\overline{\mu}_n$ are indicated the scaling parameters satisfying (4.1) associated with γ_n and $\overline{\gamma}_n$ respectively. Applying a version of conformable fractional Lagrange's identity (cf. (2.3)) on any subinterval $[x_k^{(n)}, x_{k+1}^{(n)}]$, $1 \le k \le n$, one can obtain

$$\begin{aligned} &(D_x^\alpha u_n \cdot \overline{u}_n - D_x^\alpha \overline{u}_n \cdot u_n)\Big|_{x_k^{(n)}}^{x_{k+1}^{(n)}} - \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} t^{\alpha-1}\left(D_x^\alpha u_n \cdot D_x^\alpha \overline{u}_n - D_x^\alpha \overline{u}_n \cdot D_x^\alpha u_n\right) dt \\ &+ \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} t^{\alpha-1}\overline{u}_n u_n\left(\omega|u_n|^{q-1} - \overline{\omega}|\overline{u}_n|^{q-1}\right) dt = 0. \end{aligned}$$

Then,

$$I_k := \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} t^{\alpha-1}\overline{u}_n u_n \big(\omega|u_n|^{q-1} - \overline{\omega}|\overline{u}_n|^{q-1}\big)\, dt = 0.$$

By the scaling argument (4.2), the above integral can be written as

$$\begin{aligned} I_k &= \overline{\mu}_n^\alpha \overline{\gamma}_n \mu_n^\alpha \gamma_n \\ &\quad \times \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} t^{\alpha-1}\overline{v}_n\left(\frac{t}{\overline{\mu}_n}\right) v_n\left(\frac{t}{\mu_n}\right)\left\{[\omega(t)-\overline{\omega}(t)](\overline{\mu}_n^\alpha\overline{\gamma}_n)^{q-1}\left|\overline{v}_n\left(\frac{t}{\overline{\mu}_n}\right)\right|^{q-1}\right. \\ &\quad \left. + \omega(t)(\mu_n^\alpha\gamma_n)^{q-1}\left[\left|v_n\left(\frac{t}{\mu_n}\right)\right|^{q-1} - \left(\frac{\overline{\mu}_n^\alpha\overline{\gamma}_n}{\mu_n^\alpha\gamma_n}\right)^{q-1}\left|\overline{v}_n\left(\frac{t}{\overline{\mu}_n}\right)\right|^{q-1}\right]\right\} dt = 0. \end{aligned}$$

Applying the change of variables $s = \frac{t}{\mu_n}$, one can obtain

$$\begin{aligned} I_k &= \overline{\mu}_n^\alpha \overline{\gamma}_n \mu_n^\alpha \gamma_n \\ &\quad \times \int_{\mu_n^{-1}x_k^{(n)}}^{\mu_n^{-1}x_{k+1}^{(n)}} (\mu_n s)^{\alpha-1}\overline{v}_n\left(\frac{\mu_n s}{\overline{\mu}_n}\right) v_n(s)\left\{[\omega(\mu_n s)-\overline{\omega}(\mu_n s)](\overline{\mu}_n^\alpha\overline{\gamma}_n)^{q-1}\left|\overline{v}_n\left(\frac{\mu_n s}{\overline{\mu}_n}\right)\right|^{q-1}\right. \\ &\quad \left. + \omega(\mu_n s)(\mu_n^\alpha\gamma_n)^{q-1}\left[|v_n(s)|^{q-1} - \left(\frac{\overline{\mu}_n^\alpha\overline{\gamma}_n}{\mu_n^\alpha\gamma_n}\right)^{q-1}\left|\overline{v}_n\left(\frac{\mu_n s}{\overline{\mu}_n}\right)\right|^{q-1}\right]\right\}\mu_n\, ds = 0. \end{aligned}$$

Now, by the assumption $\lim_{n\to\infty} \frac{\gamma_n}{\overline{\gamma}_n} = 1$ and $\omega(0) = \overline{\omega}(0)$, this implies $\lim_{n\to\infty} \frac{\mu_n}{\overline{\mu}_n} = 1$. Form Lemma 4.2 and Remark 4.3, one can obtain

$$\begin{aligned} I_k &= (\mu_n^\alpha\gamma_n)^{q+1}\mu_n^\alpha \\ &\quad \times \int_{\mu_n^{-1}x_k^{(n)}}^{\mu_n^{-1}x_{k+1}^{(n)}} s^{\alpha-1}\big\{[\omega(\mu_n s)-\overline{\omega}(\mu_n s)]|V(s)|^{q+1} + \omega(\mu_n s)[|V(s)|^{q+1}\cdot o(1)]\big\}\, ds \end{aligned} \tag{4.12}$$

for sufficiently large n. Here, $\lim_{n\to\infty}(\mu_n^\alpha\gamma_n)^{q+1}\mu_n^\alpha = \infty$ by Proposition 4.4. Note that $\mu_n^{-1}x_k^{(n)}$ and $\mu_n^{-1}x_{k+1}^{(n)}$ are two consecutive zeros of v_n. So they are almost zeros of V for sufficiently large n. Also, we may admit $|V(s)|^{q+1} \geq 0$ in $[\mu_n^{-1}x_k^{(n)}, \mu_n^{-1}x_{k+1}^{(n)}]$. Applying the mean value theorem for integrals to the first term of the integrand in (4.12), one can obtain

$$\int_{\mu_n^{-1}x_k^{(n)}}^{\mu_n^{-1}x_{k+1}^{(n)}} s^{\alpha-1}[\omega(\mu_n s)-\overline{\omega}(\mu_n s)]|V(s)|^{q+1}\, ds = [\omega(\widehat{x})-\overline{\omega}(\widehat{x})]\int_{\mu_n^{-1}x_k^{(n)}}^{\mu_n^{-1}x_{k+1}^{(n)}} s^{\alpha-1}|V(s)|^{q+1}\, ds$$

for some $\widehat{x} \in [x_k^{(n)}, x_{k+1}^{(n)}]$. This implies that $\omega(\widehat{x}) = \overline{\omega}(\widehat{x})$. By the denseness property of the nodal set $\mathbb{X}_n = \{x_k^{(n)}\}_{k=1}^{n-1}$ (see Proposition 4.4(ii)) and the conditions of ω and $\overline{\omega}$, this completes the proof. □

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Wei-Chuan Wang
Center for General Education, National Quemoy University, Kinmen 892, Taiwan
E-mail addresses: `wangwc@nqu.edu.tw`, `wangwc72@gmail.com`

Yan-Hsiou Cheng
Department of Mathematics and Information Education, National Taipei University of Education, Taipei 106, Taiwan
E-mail address: `yhcheng@tea.ntue.edu.tw`