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A Note on Embedding Inequalities for Weighted Sobolev and Besov Spaces

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Abstract. In this paper, we establish two embedding inequalities for the weighted Sobolev space and the weighted homogeneous endpoint Besov space by using the weighted Hausdorff capacity. To do this, we shall determine the dual spaces of weighted Choquet and weighted homogeneous Besov spaces.

1. Introduction

The purpose of this paper is to establish embedding theorems on weighted Sobolev and weighted Besov spaces. We first give a background to the problem. Let n be the spatial dimension. Adams proved in [1] the following inequality: for any $k \in \mathbb{N}$, $1 \leq k < n$,

$$\int_{\mathbb{R}^n} |f| \, \mathrm{d}H^{n-k} \leq C \|\nabla^k f\|_{L^1}, \quad f \in C_0^\infty(\mathbb{R}^n),$$

where H^d, $0 < d < n$, is the Hausdorff capacity of dimension d and the integral is taken in the Choquet sense; the quantity $\nabla^k f$ denotes the vector of all kth order derivatives of f, and $|\nabla^k f|$, the Euclidean length of that vector, $\|\nabla^k f\|_{L^1}$, the L^1-norm of $|\nabla^k f|$; and $C_0^\infty(\mathbb{R}^n)$ is the class of all infinitely differentiable functions having compact support in the Euclidean space $\mathbb{R}^n$. In [11], Xiao extended Adams' inequality to fractional derivatives by using the homogeneous endpoint Besov spaces $\dot{B}^s_{11}$: for any $s \in \mathbb{R}$, $0 < s < n$,

$$\int_{\mathbb{R}^n} |f| \, \mathrm{d}H^{n-s} \leq C \|f\|_{\dot{B}^s_{11}}, \quad f \in C_0^\infty(\mathbb{R}^n).$$

In this paper, we investigate some weighted analogues of these embedding inequalities. By weights we will always mean non-negative, locally integrable functions on $\mathbb{R}^n$ which are positive on a set of positive measure. Given a measurable set E and a weight w, we set $w(E) = \int_E w(x) \, \mathrm{d}x$, and $|E|$ denotes the Lebesgue measure of E.

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For a set $E \subset \mathbb{R}^n$, the d-dimensional weighted Hausdorff capacity(content) H^d_w of E is defined by

$$H^d_w(E) = \inf \left\{ \sum_{j=1}^{\infty} r_j^d \fint_{B(x_j,r_j)} w \, \mathrm{d}y : E \subset \bigcup_{j=1}^{\infty} B(x_j, r_j), \right\},$$

where the infimum is taken over all coverings of E by countable families of balls $B(x_j, r_j)$, centered at x_j and radius r_j, see [10]. When $w \equiv 1$, we simply denote by H^d, which is the d-dimensional Hausdorff capacity. For a non-negative function f, the integral of f with respect to H^d_w is taken in the Choquet sense,

$$\int_{\mathbb{R}^n} f \, \mathrm{d}H^d_w = \int_0^{\infty} H^d_w(\{x \in \mathbb{R}^n : f(x) > t\}) \, \mathrm{d}t.$$

A useful variant of weighted Hausdorff capacity is the dyadic version of H^d_w, denoted by $\widetilde{H}^d_w$. Let $\mathcal{D}$ be the set of all dyadic cubes in $\mathbb{R}^n$, that is,

$$\mathcal{D} := \left\{ 2^{-k}(m + [0,1)^n) : k \in \mathbb{Z}, m \in \mathbb{Z}^n \right\}.$$

The weighted dyadic Hausdorff capacity is defined by

$$\widetilde{H}^d_w(E) = \inf \left\{ \sum_{j=1}^{\infty} \ell(Q_j)^d \fint_{Q_j} w \, \mathrm{d}y : E \subset \bigcup_{j=1}^{\infty} Q_j, Q_j \in \mathcal{D} \right\},$$

where the infimum is taken over all coverings of E by countable families of dyadic cubes Q_j. It can be shown that H^d_w and $\widetilde{H}^d_w$ are equivalent when w is doubling, see [10, Proposition 3.4.2]. We say that w satisfies the doubling condition if $w(B(x, 2r)) \le Cw(B(x, r))$, for any $x \in \mathbb{R}^n$, $r > 0$. We summarize some elementary properties of $\widetilde{H}^d_w$ in the next section.

Let $1 \le p < \infty$ and let w be an arbitrary weight. We define the weighted Lebesgue space $L^p(\mathbb{R}^n, w) = L^p_w$ to be a Banach space equipped with the norm

$$\|f\|_{L^p_w} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) \, \mathrm{d}x \right)^{1/p}.$$

The weight w is in Muckenhoupt's class of A_1 if there exists a constant C such that

$$\fint_B w(z) \, \mathrm{d}z \le C \inf_{y \in B} w(y)$$

for any balls B, where the barred integral $\fint_B w$ stands for the usual integral average of w over B. The infimum of all such C is denoted by $[w]_{A_1}$. Also, w is in the reverse Hölder class RH_∞ if there exists a constant C such that

$$w(x) \le C \inf_{x \in B} \fint_B w(z) \, \mathrm{d}z$$

for almost every x, see [3]. The infimum of all such C is denoted by $[w]_{\mathrm{RH}_\infty}$.

We can extend Adams' inequality to the following.

Theorem 1.1. *Let k be an integer such that $1 \le k < n$. Suppose that w is in A_1 and w satisfies that*

$$\lim_{r\to\infty} \frac{w(B(x,r))}{r^k} = \infty \tag{1.1}$$

for every $x \in \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} |f| \,\mathrm{d}H_w^{n-k} \le C\|\nabla^k f\|_{L^1_w}, \quad f \in C_0^\infty(\mathbb{R}^n).$$

Remark 1.2. To prove this theorem, we need the left continuity of the dyadic Hausdorff capacity. The condition (1.1) implies the left continuity of $\widetilde{H}_w^{n-k}$, see [10, Proposition 3.4.22]. Turesson gave an example of a weight which does not satisfy (1.1) and $\widetilde{H}_w^d$ is not left continuous.

To extend Xiao's result to the weighted Besov spaces, we shall introduce some notions. Let $\mathcal{S}$ be the Schwartz class of rapidly decreasing functions and $\mathcal{S}'$, its dual, is the space of tempered distributions. The Fourier transform and the inverse Fourier transform are defined by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\,\mathrm{d}x \quad \text{and} \quad \mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} f(\xi)e^{ix\cdot\xi}\,\mathrm{d}\xi, \quad f \in \mathcal{S}.$$

We denote by Φ the set of all sequences $(\phi_j)_{j\in\mathbb{Z}} \subset \mathcal{S}$ satisfying the following two properties:

(i) $\operatorname{supp}\widehat{\phi_j} \subset \{x \in \mathbb{R}^n : 2^{j-1} \le |x| \le 2^{j+1}\}$, $j \in \mathbb{Z}$;

(ii) $\sum_{j\in\mathbb{Z}} \widehat{\phi_j} = 1$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

We also denote the space of all polynomials by $\mathcal{P}$.

Definition 1.3. (cf. [6]) Let $s \in \mathbb{R}$, $0 < p, q \le \infty$ and $(\phi_j) \in \Phi$. The weighted homogeneous Besov spaces corresponding to these indices are defined by

$$\|f\|_{\dot{B}^{s,w}_{pq}} := \left(\sum_{j\in\mathbb{Z}} 2^{jsq}\|\phi_j * f\|^q_{L^p_w}\right)^{1/q}$$

and $\dot{B}^{s,w}_{pq}$ is the set of all $f \in \mathcal{S}'/\mathcal{P}$ for which the norm $\|f\|_{\dot{B}^{s,w}_{pq}}$ is finite. Here $\left(\sum_{j\in\mathbb{Z}}(\cdot)^q\right)^{1/q}$ is interpreted as $\sup_{j\in\mathbb{Z}}(\cdot)$ if $q = \infty$.

Xiao's inequality can be extended to the following.

Theorem 1.4. *Let $s \in (0,n)$. Suppose that $w \in A_1 \cap \mathrm{RH}_\infty$ and w satisfies that*

$$\lim_{r\to\infty} \frac{w(B(x,r))}{r^s} = \infty$$

for every $x \in \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} |f| \,\mathrm{d}H_w^{n-s} \lesssim \|f\|_{\dot{B}^{s,w}_{11}}, \quad f \in C_0^\infty.$$

Example 1.5. It is easily seen that $w(x) = \min(|x|^a, 1)$ for $s - n < a \leq 0$ satisfies the condition $w \in A_1 \cap \mathrm{RH}_\infty$ and (1.1) for $k = s$.

In what follows, the letter C will be used for unimportant constants that may change from one occurrence to another. We write $A \lesssim B$, $B \gtrsim A$ if there is a independent constant C such that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, then we say that A and B are equivalent each other and denote by $A \approx B$.

This paper is organized as follows. In Section 2, we introduce the weighted Hausdorff capacity and summarize its essential properties. The weighted Morrey space consisting of signed Radon measures is also defined. In Section 3, it is shown that the dual space of $L^1(\widetilde{H}^d_w)$ can be identified with the weighted Morrey space of signed Radon measures. Using the duality, we extend Adams' embedding theorem. In Section 4, we introduce the weighted homogeneous Besov space and characterize the dual space of $\dot{B}^{s,w}_{11}$. To prove Theorem 1.4, we establish the lifting property of the Riesz potential on the weighted Morrey spaces. Finally, we show the equivalence between the weighted Morrey space and a variant of weighted Besov space $\dot{\mathcal{B}}^{-s,w}_{\infty\infty}$. To do this, we require the assumption that the weight w is in the Muckenhoupt A_1 and the Reverse Hölder class RH_∞. Combining these results, we prove the weighted version of Xiao's theorem.

2. Weighted Hausdorff capacity

In this section, following [7, 8], we first summarize some important properties of the weighted Hausdorff capacity and the dyadic Hausdorff capacity. These two types of weighted Hausdorff capacity are equivalent each other.

Proposition 2.1. [10, Proposition 3.4.2] *Let $0 < d \leq n$. If w is doubling, then for every set $E \subset \mathbb{R}^n$,*

$$H^d_w(E) \approx \widetilde{H}^d_w(E),$$

where the equivalence constants depend only on d, n and doubling constant of w.

We next emphasis that the set function $\widetilde{H}^d_w$ is strong subadditive (cf. [7,8]), that is,

$$\widetilde{H}^d_w(E \cup F) + \widetilde{H}^d_w(E \cap F) \leq \widetilde{H}^d_w(E) + \widetilde{H}^d_w(F), \quad E, F \subset \mathbb{R}^n.$$

Thanks to the strong subadditivity of the set function $\widetilde{H}^d_w$, its Choquet integral is sublinear, that is, for nonnegative functions f and g we have

$$\int_{\mathbb{R}^n} (f+g)\, \mathrm{d}\widetilde{H}^d_w \leq \int_{\mathbb{R}^n} f\, \mathrm{d}\widetilde{H}^d_w + \int_{\mathbb{R}^n} g\, \mathrm{d}\widetilde{H}^d_w.$$

This implies that the quantity

$$\|f\|_{L^p(\widetilde{H}^d_w)} := \left(\int_{\mathbb{R}^n} |f|^p \, \mathrm{d}\widetilde{H}^d_w \right)^{1/p}$$

is the norm when $1 \le p < \infty$.

The next proposition gives a sufficient condition for $\widetilde{H}^d_w$ to be left continuous.

Proposition 2.2. [10, Proposition 3.4.22] *Let $0 \le \alpha < n$. Let w be a doubling weight. Assume that*

$$\lim_{r\to\infty} \frac{w(B(x,r))}{r^\alpha} = \infty \tag{2.1}$$

for every $x \in \mathbb{R}^n$. If $E_1 \subset E_2 \subset \cdots$ is an increasing sequence of subsets of $\mathbb{R}^n$, then

$$\widetilde{H}^{n-\alpha}_w \left(\bigcup_{j=1}^{\infty} E_j \right) = \lim_{j\to\infty} \widetilde{H}^{n-\alpha}_w (E_j).$$

We define the Choquet space $L^p(\widetilde{H}^d_w)$, $1 \le p < \infty$, by the completion of the set of all continuous functions having compact support $C_0(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_{L^p(\widetilde{H}^d_w)} = \left(\int_{\mathbb{R}^n} |f|^p \, \mathrm{d}\widetilde{H}^d_w \right)^{1/p}.$$

We next introduce the weighted maximal operator of order d of a (signed) Radon measure μ is defined by

$$M^d_w \mu(x) = \sup_{r>0} \frac{|\mu|(B(x,r))}{r^d \fint_{B(x,r)} w(y) \, \mathrm{d}y}.$$

If we let $w \equiv 1$, $\mathrm{d}|\mu| = |f| \, \mathrm{d}x$ for a locally integrable function f and the Lebesgue measure $\mathrm{d}x$, $0 < d < n$, then this operator is just the fractional integral operator of f with the order $n - d$. The precise definition of the fractional integral operator is given in Section 5. We also define a variant of Morrey spaces denoted by $\mathbb{L}^{\infty,d}_w$ the set of all Radon measures μ satisfying

$$|||\mu|||_{d,w} := \sup_{x\in\mathbb{R}^n} M^d_w \mu(x) < \infty. \tag{2.2}$$

Remark 2.3. When $w \equiv 1$, in [1], the set of all Radon measures μ satisfying (2.2) is denoted by $\mathbf{L}^{1,d}$ and is referred to as the Morrey space.

3. Proof of Theorem 1.1

The key ingredient of the proof of Theorem 1.1 is the dual of the Choquet space $L^1(\widetilde{H}^d_w)$.

Theorem 3.1. *Let $w \in A_1$. Then the dual space of $L^1(\widetilde{H}^d_w)$ is $\mathbb{L}^{\infty,d}_w$. More precisely, for $\mu \in \mathbb{L}^{\infty,d}_w$, there exists a unique linear functional F_μ on $L^1(\widetilde{H}^d_w)$ such that*

$$F_\mu(\phi) = \int_{\mathbb{R}^n} \phi \, \mathrm{d}\mu, \quad \phi \in C_0(\mathbb{R}^n),$$

and conversely any continuous linear functional F on $L^1(\widetilde{H}^d_w)$ is realized as $F = F_\mu$ for some $\mu \in \mathbb{L}^{\infty,d}_w$. Moreover, $\|F\| \approx |||\mu|||_{d,w}$ holds.

Proof. For any $\mu \in \mathbb{L}_w^{\infty,d}$, we set

$$F_\mu(\phi) = \int_{\mathbb{R}^n} \phi \,\mathrm{d}\mu, \quad \phi \in C_0(\mathbb{R}^n).$$

The functional F_μ is obviously linear. We first observe that

$$|F_\mu(\phi)| \le \int_{\mathbb{R}^n} |\phi| \,\mathrm{d}|\mu| = \int_0^\infty |\mu|(\{|\phi| > t\}) \,\mathrm{d}t.$$

Taking an arbitrary covering of $\{|\phi| > t\}$ by open balls $\{B_j\}$, we observe that

$$|\mu|(|\phi| > t) \le \sum_{j=1}^\infty |\mu|(B_j) = \sum_{j=1}^\infty \frac{|\mu|(B_j)}{r_j^d \fint_{B_j} w} \cdot r_j^d \fint_{B_j} w \le \sup_{x \in \mathbb{R}^n} M_w^d \mu(x) \sum_{j=1}^\infty r_j^d \fint_{B_j} w.$$

Therefore,

$$|\mu|(|\phi| > t) \le \|\|\mu\|\|_{d,w} H_w^d(|\phi| > t) \le C\|\|\mu\|\|_{d,w} \widetilde{H}_w^d(|\phi| > t), \tag{3.1}$$

and hence

$$|F_\mu(\phi)| \le C\|\|\mu\|\|_{d,w} \int_0^\infty \widetilde{H}_w^d(\{|\phi| > t\}) \,\mathrm{d}t = C\|\|\mu\|\|_{d,w} \int_{\mathbb{R}^n} |\phi| \,\mathrm{d}\widetilde{H}_w^d.$$

By definition, for each $f \in L^1(\widetilde{H}_w^d)$, there is a sequence $\phi_j \in C_0(\mathbb{R}^n)$ such that

$$\|f - \phi_j\|_{L^1(\widetilde{H}_w^d)} \to 0, \quad j \to \infty.$$

Since we can easily see that $\{F_\mu(\phi_j)\}_j$ forms Cauchy sequence of $\mathbb{R}$,

$$F_\mu(f) := \lim_{j \to \infty} F_\mu(\phi_j)$$

is well-defined. Moreover, since $\|\cdot\|_{L^1(\widetilde{H}_w^d)}$ is a norm, we have

$$|F_\mu(f)| \le C\|\|\mu\|\|_{d,w} \|f\|_{L^1(\widetilde{H}_w^d)}.$$

This implies that F_μ can be extended to a functional on $L^1(\widetilde{H}_w^d)$ and $\|F_\mu\| \le C\|\|\mu\|\|_{d,w}$.

Conversely, let $F \in L^1(\widetilde{H}_w^d)^*$. Since $C_0(\mathbb{R}^n) \subset L^1(\widetilde{H}_w^d)$, by Riesz's representation theorem, there exists a Radon measure μ such that

$$F(\phi) = \int_{\mathbb{R}^n} \phi \,\mathrm{d}\mu, \quad \phi \in C_0(\mathbb{R}^n).$$

For any $\psi \in C_0(\mathbb{R}^n)$, we notice that

$$\begin{aligned}
\int_{\mathbb{R}^n} |\psi| \,\mathrm{d}|\mu| &= \sup \left\{ \int_{\mathbb{R}^n} \phi \,\mathrm{d}\mu : \phi \in C_0(\mathbb{R}^n), |\phi| \le |\psi| \right\} \\
&\le \|F\| \sup \left\{ \|\phi\|_{L^1(\widetilde{H}_w^d)} : \phi \in C_0(\mathbb{R}^n), |\phi| \le |\psi| \right\} \\
&\le \|F\| \|\psi\|_{L^1(\widetilde{H}_w^d)} \le C\|F\| \|\psi\|_{L^1(H_w^d)}.
\end{aligned}$$

Thus, if $\psi = 1$ on $B(x,r)$ and $\psi = 0$ on $B(x,r+\varepsilon)^c$, then

$$|\mu|(B(x,r)) \le C\|F\| H_w^d(B(x,r+\varepsilon)) \le C\|F\|(r+\varepsilon)^d \fint_{B(x,r+\varepsilon)} w.$$

Hence $\mu \in \mathbb{L}_w^{\infty,d}$ and $|||\mu|||_{d,w} \le C\|F\|$. $\square$

Corollary 3.2. *Let $0 < d < n$ and let $w \in A_1$. Suppose that w satisfies* (2.1) *for $\alpha = n-d$. If f is a nonnegative lower semi-continuous function on $\mathbb{R}^n$, then*

$$\int_{\mathbb{R}^n} f \,\mathrm{d}H_w^d \approx \sup\left\{\int_{\mathbb{R}^n} f\,\mathrm{d}\mu : |||\mu|||_{d,w} \le 1, \mu \in \mathbb{L}_{w,+}^{\infty,d}\right\},$$

where $\mathbb{L}_{w,+}^{\infty,d}$ is the non-negative elements in $\mathbb{L}_w^{\infty,d}$.

Proof. As mentioned above, $\|\cdot\|_{L^1(\widetilde{H}_w^d)}$ becomes a norm and $L^1(\widetilde{H}_w^d)$ is a normed space. By the previous theorem, the canonical map of $L^1(\widetilde{H}_w^d)$ into its second dual has norm

$$\int_{\mathbb{R}^n} |f|\,\mathrm{d}\widetilde{H}_w^d \approx \sup\left\{\left|\int_{\mathbb{R}^n} f\,\mathrm{d}\mu\right| : |||\mu|||_{d,w} \le 1\right\}$$

for $f \in L^1(\widetilde{H}_w^d)$. For a non-negative lower semi-continuous f, we approximate from below by a non-negative sequence $\{\phi_j\} \subset C_0(\mathbb{R}^n)$. Then

$$\begin{aligned}\int_{\mathbb{R}^n} \phi_j\,\mathrm{d}\widetilde{H}_w^d &\approx \sup\left\{\int_{\mathbb{R}^n} \phi_j\,\mathrm{d}\mu : |||\mu|||_{d,w} \le 1, \mu \in \mathbb{L}_{w,+}^{\infty,d}\right\}\\ &\le \sup\left\{\int_{\mathbb{R}^n} f\,\mathrm{d}\mu : |||\mu|||_{d,w} \le 1, \mu \in \mathbb{L}_{w,+}^{\infty,d}\right\}.\end{aligned}$$

By the left continuity of $\widetilde{H}_w^d$ provided by (2.1), we get

$$\widetilde{H}_w^d(\{x \in \mathbb{R}^n : \phi_j(x) \ge \lambda\}) \to \widetilde{H}_w^d(\{x \in \mathbb{R}^n : f(x) \ge \lambda\}),$$

and hence

$$\int_{\mathbb{R}^n} f\,\mathrm{d}\widetilde{H}_w^d \lesssim \sup\left\{\int_{\mathbb{R}^n} f\,\mathrm{d}\mu : |||\mu|||_{d,w} \le 1, \mu \in \mathbb{L}_{w,+}^{\infty,d}\right\}. \tag{3.2}$$

On the other hand, by (3.1) and again the left continuity of $\widetilde{H}_w^d$, we obtain

$$|\mu|(f > t) \le C|||\mu|||_{d,w}\widetilde{H}_w^d(f > t).$$

Integrating over $t \in [0,\infty]$, we have

$$\int_{\mathbb{R}^n} f\,\mathrm{d}\mu \le C|||\mu|||_{d,w}\int_{\mathbb{R}^n} f\,\mathrm{d}\widetilde{H}_w^d$$

and taking supremum over $|||\mu|||_{d,w} \le 1$, we get

$$\sup\left\{\int_{\mathbb{R}^n} f\,\mathrm{d}\mu : |||\mu|||_{d,w} \le 1, \mu \in \mathbb{L}_{w,+}^{\infty,d}\right\} \le C\int_{\mathbb{R}^n} f\,\mathrm{d}\widetilde{H}_w^d. \tag{3.3}$$

Thus, combining (3.2) and (3.3), the result follows from the equivalence $\widetilde{H}_w^d \approx H_w^d$. $\square$

The following estimate is a weighted version of [5, Theorem 1, p. 24].

Theorem 3.3. [10, Theorem 2.6.3] *Let $w \in A_1$, and let k be an integer such that $1 \leq k < n$. Suppose that μ is a positive Radon measure, satisfying*

$$M := |||\mu|||_{n-k,w} < \infty.$$

Then the inequality

$$\int_{\mathbb{R}^n} |u| \, \mathrm{d}\mu \leq C \int_{\mathbb{R}^n} |\nabla^k u| w \, \mathrm{d}x \tag{3.4}$$

holds for every $u \in C_0^\infty(\mathbb{R}^n)$ with $C = C'M$, where C' only depends on k, n, and $[w]_{A_1}$. Conversely, if there exists a constant C such that (3.4) *holds for every $u \in C_0^\infty(\mathbb{R}^n)$, then $C \geq C'M$, with C' as before. In particular, M is finite.*

Proof of Theorem 1.1. By Corollary 3.2, we have

$$\int_{\mathbb{R}^n} |f| \, \mathrm{d}H_w^{n-k} \approx \sup \left\{ \int_{\mathbb{R}^n} |f| \, \mathrm{d}\mu : |||\mu|||_{n-k,w} \leq 1, \mu \in \mathbb{L}_{w,+}^{\infty,d} \right\}$$

for $f \in C_0^\infty(\mathbb{R}^n)$. On the other hand, by Theorem 3.3, we get

$$\sup \left\{ \int_{\mathbb{R}^n} |f| \, \mathrm{d}\mu : |||\mu|||_{n-k,w} \leq 1, \mu \in \mathbb{L}_{w,+}^{\infty,d} \right\} \leq C \|\nabla^k f\|_{L_w^1},$$

which completes the proof. □

4. Weighted homogeneous Besov spaces

We introduce a variant of weighted homogeneous Besov spaces.

Definition 4.1. Let $1 \leq p < \infty$, $1 \leq q \leq \infty$, and let $s \in \mathbb{R}$. We denote by $l_s^q(L_w^p)$ the set of sequence of measurable functions (f_j) satisfying

$$\sum_{j \in \mathbb{Z}} 2^{jsq} \|f_j\|_{L_w^p}^q < \infty.$$

If $p = \infty$, we also define $l_s^q(L_w^\infty)$ the set of (f_j) such that

$$\sum_{j \in \mathbb{Z}} 2^{jsq} \|f_j / w\|_{L^\infty}^q < \infty.$$

As usual, $\sum_j (\cdot)^q$ is interpreted as $\sup_j (\cdot)$ if $q = \infty$.

Definition 4.2. [6] Let $s \in \mathbb{R}$. We denote by $\dot{\mathcal{B}}^{s,w}_{\infty\infty}$ the space of those tempered distributions $f \in \mathcal{S}'/\mathcal{P}$ for which there exist $(\phi_j)_{j\in\mathbb{Z}} \in \Phi$ and $(f_j)_{j\in\mathbb{Z}} \in l^\infty_s(L^\infty_w)$ such that

$$f = \sum_{j\in\mathbb{Z}} \phi_j * f_j \quad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

We define

$$\|f\|_{\dot{\mathcal{B}}^{s,w}_{\infty\infty}} := \inf \left\{ \sup_{j\in\mathbb{Z}} 2^{js} \|f_j/w\|_{L^\infty} \right\}$$

where the infimum is taken over all possible representations of f.

Again the key tool of our argument is also the dual space of $\dot{B}^{s,w}_{11}$. The following theorem is the homogeneous analogue of Theorem 2.10 in [6]. For the reader's convenience we shall give a complete proof.

Theorem 4.3. *The following identification holds*

$$(\dot{B}^{s,w}_{11})^* = \dot{\mathcal{B}}^{-s,w}_{\infty\infty}.$$

In order to prove this, we establish the following lemma.

Lemma 4.4. *For any continuous linear functional $\Psi \in (l^1_s(L^1_w))^*$, there exists $(g_j) \in l^\infty_{-s}(L^\infty_w)$ such that*

$$\Psi(f) = \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} f_j(x) g_j(x)\,\mathrm{d}x$$

for every $f = (f_j) \in l^1_s(L^1_w)$, and $\|\Psi\| = \|(g_j)\|_{l^\infty_{-s}(L^\infty_w)}$.

Proof. Let $\Psi \in (l^1_s(L^1_w))^*$. For any $f = (f_j) \in l^1_s(L^1_w)$, we define

$$F_j(x) := 2^{js} f_j(x) w(x),$$

then we easily see that

$$F = (F_j) \in l^1(L^1), \quad \|(F_j)\|_{l^1(L^1)} = \|(f_j)\|_{l^1_s(L^1_w)}.$$

Now, another functional

$$L(F) := \Psi(f)$$

defines a continuous linear functional on $l^1(L^1)$. Indeed, we observe

$$|L(F)| = |\Psi(f)| \le \|\Psi\| \|(f_j)_j\|_{l^1_s(L^1_w)} = \|\Psi\| \|(F_j)_j\|_{l^1(L^1)}$$

and $\|L\| \le \|\Psi\|$. By a well-known duality between $l^1(L^1)$ and $l^\infty(L^\infty)$ (e.g., [9, Proposition 2.11.1]), there exists $G = (G_j)_j \in l^\infty(L^\infty)$ such that

$$L(F) = \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} F_j(x) G_j(x)\,\mathrm{d}x, \quad \text{and} \quad \|L\| = \|(G_j)_j\|_{l^\infty(L^\infty)}.$$

Again, we set $g_j(x) := 2^{js}G_j(x)w(x)$, then it follows that $g = (g_j)_j \in l^{\infty}_{-s}(L^{\infty}_w)$ and

$$\|L\| = \|\{G_j\}\|_{l^\infty(L^\infty)} = \sup_{j\in\mathbb{Z}} 2^{-js}\|g_j/w\|_{L^\infty} = \|(g_j)_j\|_{l^{\infty}_{-s}(L^{\infty}_w)}.$$

Therefore,

$$\Psi(f) = L(F) = \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} F_j(x)G_j(x)\,\mathrm{d}x = \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} f_j(x)g_j(x)\,\mathrm{d}x.$$

Finally, we obtain

$$|\Psi(f)| \le \sup_{j\in\mathbb{Z}} 2^{-js}\|g_j/w\|_{L^\infty} \sum_{j\in\mathbb{Z}} 2^{js} \int_{\mathbb{R}^n} f_j(x)w(x)\,\mathrm{d}x = \|L\|\|(f_j)_j\|_{l^1_s(L^1_w)}$$

and $\|\Psi\| \le \|L\|$. Combining the previous estimate, we complete the proof. □

We are now in a position to prove Theorem 4.3. The proof is due to H. Triebel in [9, Theorem 2.11.2].

Proof of Theorem 4.3. We first prove that

$$\dot{\mathcal{B}}^{-s,w}_{\infty\infty} \subset (\dot{B}^{s,w}_{11})^*.$$

Let $f \in \dot{\mathcal{B}}^{-s,w}_{\infty\infty}$. By definition, we find sequences $(\varphi_j) \in \Phi$ and $(f_j) \in l^{\infty}_{-s}(L^{\infty}_w)$ such that

$$f = \sum_{j\in\mathbb{Z}} \varphi_j * f_j \ \text{ in } \ \mathcal{S}'(\mathbb{R}^n), \quad \text{and} \quad \sup_{j\in\mathbb{Z}} 2^{-js}\|f_j/w\|_{L^\infty} < \infty.$$

If $\phi \in \mathcal{S}$ then we have

$$\langle f, \phi\rangle = \sum_{j\in\mathbb{Z}} \langle \varphi_j * f_j, \phi\rangle = \sum_{j\in\mathbb{Z}} \langle f_j, \mathcal{F}[\widehat{\varphi_j}\mathcal{F}^{-1}[\phi]]\rangle$$

and hence

$$\begin{aligned}
|\langle f, \phi\rangle| &\le \sum_{j\in\mathbb{Z}} 2^{-sj}\|f_j/w\|_{L^\infty} 2^{sj}\|\mathcal{F}[\widehat{\varphi_j}\mathcal{F}^{-1}[\phi]]\|_{L^1_w} \\
&\le \sup_{j\in\mathbb{Z}} 2^{-sj}\|f_j/w\|_{L^\infty} \sum_{j\in\mathbb{Z}} 2^{sj}\|\mathcal{F}[\widehat{\varphi_j}\mathcal{F}^{-1}[\phi]]\|_{L^1_w} \\
&\le C \sup_{j\in\mathbb{Z}} 2^{-sj}\|f_j/w\|_{L^\infty}\|\phi\|_{\dot{B}^{s,w}_{11}}.
\end{aligned}$$

Taking infimum over all possible f on the right-hand side, we obtain

$$|\langle f, \phi\rangle| \le C\|f\|_{\dot{\mathcal{B}}^{-s,w}_{\infty\infty}}\|\phi\|_{\dot{B}^{s,w}_{11}}$$

and this implies $f \in (\dot{B}^{s,w}_{11})^*$.

We prove the converse. Let $(\varphi_j)_{j\in\mathbb{Z}} \in \Phi$. Because

$$\dot{B}^{s,w}_{11} \ni f \mapsto (\varphi_j * f)_{j\in\mathbb{Z}}$$

is a one-to-one mapping from $\dot{B}^{s,w}_{11}$ onto a subspace of $l^1_s(L^1_w)$, every functional $\Psi \in (B^{s,w}_{11})^*$ can be interpreted as a functional on that subspace. By the Hahn–Banach theorem, Ψ can be extended to a continuous linear functional on $l^1_s(L^1_w)$, where the norm of Ψ is preserved. By Lemma 4.4, there exists $(g_j)_{j\in\mathbb{Z}} \in l^\infty_{-s}(L^\infty_w)$ such that

$$\Psi(f) = \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} g_j(x)\varphi_j * f(x)\,\mathrm{d}x \tag{4.1}$$

and

$$\sup_{j\in\mathbb{Z}} 2^{-js}\|g_j/w\|_{L^\infty} = \|\Psi\|.$$

So (4.1) can be written as

$$\Psi(f) = \sum_{j\in\mathbb{Z}} (\varphi_j(-\cdot) * g_j)(f).$$

These are the desired results. □

5. Proof of Theorem 1.4

In order to prove Theorem 1.4, we write

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}y, \quad x \in \mathbb{R}^n,$$

which is referred to as the fractional integral operator or Riesz potential with order $\alpha \in (0,n)$ of a function f defined on $\mathbb{R}^n$. The following theorem is due to [6, Theorem 2.8 and Remark 2.9] which is known as the lifting property.

Theorem 5.1. [6] *Suppose that $\alpha, s \in (0,n)$. Then we have*

$$\dot{B}^{s,w}_{11} = I_\alpha(\dot{B}^{s-\alpha,w}_{11}).$$

In other words, $f \in \dot{B}^{s,w}_{11}$ if and only if there is a $g \in \dot{B}^{s-\alpha,w}_{11}$ such that $f = I_\alpha g$ and $\|f\|_{\dot{B}^{s,w}_{11}} \approx \|g\|_{\dot{B}^{s-\alpha,w}_{11}}$.

To prove Theorem 1.4, we need more two lemmas.

Lemma 5.2. *Let $w \in A_1$. Suppose that $d, \alpha \in (0,n)$ with $0 < d+\alpha < n$. If $\mu \in \mathbb{L}^{\infty,d}_{w,+}$, then $I_\alpha\mu \in \mathbb{L}^{\infty,d+\alpha}_{w,+}$.*

Proof. We suppose that $\mu \in \mathbb{L}^{\infty,d}_{w,+}$, that is,

$$\|\!|\mu|\!\|_{d,w} = \sup_{x\in\mathbb{R}^n, r>0} \frac{\mu(B(x,r))}{r^d \fint_{B(x,r)} w} < \infty.$$

Now we show that $I_\alpha \mu \in \mathbb{L}^{\infty,d+\alpha}_{w,+}$, and

$$\|\!|I_\alpha \mu|\!\|_{d+\alpha,w} \lesssim \|\!|\mu|\!\|_{d,w}.$$

For any open ball $B(x,r) \subset \mathbb{R}^n$,

$$\begin{aligned}
I_\alpha\mu(B(x,r)) &= \int_{B(x,r)} I_\alpha\mu(z)\,\mathrm{d}z = \int_{B(x,r)} \int_{\mathbb{R}^n} \frac{\mathrm{d}\mu(y)}{|z-y|^{n-\alpha}}\,\mathrm{d}z \\
&= \int_{B(x,r)} \left(\int_{B(z,2r)} + \int_{B(z,2r)^c} \right) \frac{1}{|y-z|^{n-\alpha}}\,\mathrm{d}\mu(y)\mathrm{d}z \\
&\lesssim \int_{B(x,r)} \left(\int_{B(x,3r)} \frac{\mathrm{d}\mu(y)}{|y-z|^{n-\alpha}} \right) \mathrm{d}z \\
&\quad + \int_{B(x,r)} \left(\sum_{j=1}^\infty \int_{2^j r \le |y-z| < 2^{j+1} r} \frac{\mathrm{d}\mu(y)}{|y-z|^{n-\alpha}} \right) \mathrm{d}z \\
&= (\mathrm{I}) + (\mathrm{II}).
\end{aligned}$$

Since $z \in (x,r)$ and $y \in B(x,3r)$ implies $z \in B(y,4r)$, we have

$$(\mathrm{I}) \le \int_{B(x,3r)} \left(\int_{B(y,4r)} \frac{\mathrm{d}z}{|y-z|^{n-\alpha}} \right) \mathrm{d}\mu(y) = \frac{4^\alpha \nu_n}{\alpha} r^\alpha \mu(B(x,3r)),$$

where ν_n is the volume of the unit ball. By the definition of $\|\!|\mu|\!\|_{d,w}$ and the A_1-condition, it follows that

$$\begin{aligned}
r^\alpha \mu(B(x,3r)) &\le r^\alpha \|\!|\mu|\!\|_{d,w} (3r)^d \fint_{B(x,3r)} w(z)\,\mathrm{d}z \\
&\le 3^d [w]_{A_1} \|\!|\mu|\!\|_{d,w} r^{d+\alpha} \fint_{B(x,r)} w(z)\,\mathrm{d}z.
\end{aligned}$$

To estimate (II), we observe that

$$\begin{aligned}
(\mathrm{II}) &= \int_{B(x,r)} \left(\sum_{j=1}^\infty \int_{2^j r \le |y-z| < 2^{j+1} r} \frac{\mathrm{d}\mu(y)}{|y-z|^{n-\alpha}} \right) \mathrm{d}z \\
&\le \int_{B(x,r)} \left(\sum_{j=1}^\infty \mu(B(z, 2^{j+1} r)) (2^j r)^{\alpha-n} \right) \mathrm{d}z \\
&\le \|\!|\mu|\!\|_{d,w} \int_{B(x,r)} \left(\sum_{j=1}^\infty (2^{j+1} r)^d (2^j r)^{\alpha-n} \fint_{B(z,2^{j+1} r)} w(y)\,\mathrm{d}y \right) \mathrm{d}z.
\end{aligned}$$

By the A_1-condition, we get

$$\begin{aligned} &\leq [w]_{A_1} |||\mu|||_{d,w} \sum_{j=1}^{\infty} (2^{j+1}r)^d (2^j r)^{\alpha-n} \int_{B(x,r)} w(z)\,\mathrm{d}z \\ &= \nu_n [w]_{A_1} |||\mu|||_{d,w} r^{d+\alpha} \fint_{B(x,r)} w(z)\,\mathrm{d}z \sum_{j=1}^{\infty} (2^{j+1})^d (2^j)^{\alpha-n}, \end{aligned}$$

and the last series converges as $d+\alpha<n$.

Combining these estimates, we obtain

$$I_\alpha \mu(x) \leq C[w]_{A_1} |||\mu|||_{d,w} r^{d+\alpha} \fint_{B(x,r)} w(z)\,\mathrm{d}z$$

and this implies

$$|||I_\alpha \mu|||_{d+\alpha,w} \lesssim |||\mu|||_{d,w}$$

and hence $I_\alpha \mu \in \mathbb{L}^{\infty,d+\alpha}_{w,+}$. □

The following theorem is the weighted homogeneous variation of [2, Remark 2, p. 90], and also see [11, p. 834].

Lemma 5.3. *Suppose that $w \in A_1 \cap \mathbb{R}H_\infty$. Let $s \in (0,n)$ be a real number. Then, $u \in \mathbb{L}^{\infty,n-s}_w$ if and only if $u \in \mathcal{B}^{-s,w}_{\infty\infty}$, and*

$$|||u|||_{n-s,w} \approx \|u\|_{\dot{\mathcal{B}}^{-s,w}_{\infty\infty}}$$

for any positive $u \in \mathcal{S}'/\mathcal{P}$.

Proof. We first assume $u \in \mathbb{L}^{\infty,n-s}_w$. By Theorem 4.3, it suffices to show that $u \in (\dot{B}^{s,w}_{11})^*$, in other words, we shall show $|\langle u,g\rangle| \leq C\|g\|_{\dot{B}^{s,w}_{11}}$ for $g \in \mathcal{S}$. To prove this, we take an arbitrary $g \in \mathcal{S}$. For $(\phi_j)_{j\in\mathbb{Z}} \in \Phi$ and $\chi_j = \phi_{j-1}+\phi_j+\phi_{j+1}$, we observe that

$$\begin{aligned} \langle u,g\rangle &= \sum_{j\in\mathbb{Z}} \langle \phi_j * u, g\rangle \approx \sum_{j\in\mathbb{Z}} \langle \phi_j * u, \chi_j * g\rangle \\ &= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} 2^{-sj} \frac{\phi_j * u(x)}{w(x)} \cdot 2^{sj} \chi_j * g(x) w(x)\,\mathrm{d}x, \end{aligned}$$

and hence

$$|\langle u,g\rangle| \lesssim \sup_{j\in\mathbb{Z},x\in\mathbb{R}^n} 2^{-sj} \left|\frac{\phi_j * u(x)}{w(x)}\right| \|g\|_{\dot{B}^{s,w}_{11}}.$$

If η is the characteristic function for the unit ball $B(0,1)$, then $\widehat{\eta}(\xi) \neq 0$ on a neighborhood of 0 (see the proof of Corollary 4.1.6 in [2]). By Wiener's theorem (see e.g. Lemma VIII.6.3 in [4] and also Theorem 4.1.5 in [2]), $\eta_j(x) = 2^{jn}\eta(2^j x)$ divides ϕ_j in the sense that there is $\psi \in L^1$ and $\psi_j(x) = 2^{jn}\psi(2^j x)$ such that

$$\widehat{\phi}_j(\xi) = \widehat{\psi}_j(\xi)\widehat{\eta}_j(\xi), \quad \xi \in \operatorname{supp}\widehat{\phi}_j.$$

Thus we have $\phi_j = \psi_j * \eta_j$ and

$$\eta_j * u(x) = 2^{jn} u(B(x, 2^{-j})) \approx \frac{u(B(x,r))}{r^n}, \quad 2^{-j} \approx r > 0.$$

Therefore, for fixed $j \in \mathbb{Z}$ we observe that

$$\begin{aligned} 2^{-sj} \left| \frac{\phi_j * u(x)}{w(x)} \right| &\lesssim \int_{\mathbb{R}^n} |\psi_j(x-y)| \frac{|u|(B(y,r))}{r^{n-s} w(x)} \, \mathrm{d}y \\ &= \int_{B(x,r)} + \sum_{k=1}^{\infty} \int_{B(x,2^k r) \setminus B(x, 2^{k-1} r)} =: (\mathrm{I}) + \sum_{k=1}^{\infty} (\mathrm{II})_k. \end{aligned}$$

Since $y \in B(x,r)$,

$$\begin{aligned} (\mathrm{I}) &= \int_{B(x,r)} \psi_j(x-y) \frac{u(B(y,r))}{r^{n-s} w(x)} \, \mathrm{d}y \\ &\le [w]_{A_1} \int_{B(x,r)} \psi_j(x-y) \frac{u(B(y,r))}{r^{n-s} \fint_{B(y,r)} w(z) \, \mathrm{d}z} \, \mathrm{d}y \\ &\le [w]_{A_1} |||u|||_{n-s,w} \int_{B(x,r)} \psi_j(x-y) \, \mathrm{d}y, \end{aligned}$$

where we have used the fact that

$$\fint_{B(y,r)} w(z) \, \mathrm{d}z \le [w]_{A_1} w(x), \quad x \in B(y,r).$$

To estimate $(\mathrm{II})_k$, combining the reverse Hölder condition, we have

$$\fint_{B(y,r)} w(z) \, \mathrm{d}z \le [w]_{A_1} w(y) \le [w]_{A_1} [w]_{\mathrm{RH}_\infty} \fint_{B(y,2^k r)} w(z) \, \mathrm{d}z \le [w]_{A_1}^2 [w]_{\mathrm{RH}_\infty} w(x),$$

and this implies

$$(\mathrm{II})_k \le [w]_{A_1}^2 [w]_{\mathrm{RH}_\infty} |||u|||_{n-s,w} \int_{B(x,2^k r) \setminus B(x,2^{k-1} r)} \psi_j(x-y) \, \mathrm{d}y.$$

Noticing $\|\psi_j\|_{L^1} = \|\psi\|_{L^1}$, we obtain

$$|\langle u, g \rangle| \le C |||u|||_{n-s,w} \|g\|_{\dot{B}^{s,w}_{11}}.$$

This means $u \in (\dot{B}^{s,w}_{11})^* = \dot{\mathcal{B}}^{-s,w}_{\infty\infty}$ and $\|u\|_{\dot{\mathcal{B}}^{-s,w}_{\infty\infty}} \lesssim |||u|||_{n-s,w}$.

We prove the converse. Assume that $u \in \dot{\mathcal{B}}^{-s,w}_{\infty\infty}$ and $\zeta \in \dot{B}^{s,w}_{11}$. To estimate

$$\sup_{x \in \mathbb{R}^n, r>0} \frac{u(B(x,r))}{r^{n-s} \fint_{B(x,r)} w(z) \, \mathrm{d}z},$$

we set $\eta = \mathbf{1}_{B(0,1)}$ and $\eta_r(x) := r^{-n} \eta(x/r)$, then we have

$$\sup_{x \in \mathbb{R}^n, r>0} \frac{u(B(x,r))}{r^{n-s} \fint_{B(x,r)} w(z) \, \mathrm{d}z} = \sup_{x \in \mathbb{R}^n, r>0} r^s \frac{\eta_r * u(x)}{\fint_{B(x,r)} w(z) \, \mathrm{d}z}.$$

By the definition of $u \in \dot{\mathcal{B}}^{-s,w}_{\infty\infty}$, there exist $(\varphi_j)_{j\in\mathbb{Z}} \in \Phi$ and $(u_j)_{j\in\mathbb{Z}} \in l^\infty_{-s}(L^\infty_w)$ such that

$$u = \sum_{j\in\mathbb{Z}} \varphi_j * u_j.$$

Since we assume u is positive, we can approximate η by a smooth ζ with $\eta \le \zeta$. Now it suffices to estimate

$$r^{-s} \sum_{j\in\mathbb{Z}} \frac{\zeta_r * \varphi_j * u_j(x)}{\fint_{B(x,r)} w(z)\,\mathrm{d}z}$$

for fixed $x \in \mathbb{R}^n$ and $r > 0$. To this end, we observe that

$$\begin{aligned}\frac{\zeta_r * \varphi_j * u_j(x)}{\fint_{B(x,r)} w(z)\,\mathrm{d}z} &= \int_{\mathbb{R}^n} \zeta_r * \varphi_j(x-y) \frac{u_j(y)}{\fint_{B(x,r)} w(z)\,\mathrm{d}z}\,\mathrm{d}y \\ &= \left(\int_{B(x,r)} + \sum_{k=1}^{\infty} \int_{B(x,2^k r)\setminus B(x,2^{k-1}r)} \right).\end{aligned}$$

For the fist integral, by using the RH_∞ condition we can estimate

$$\frac{u_j(y)}{\fint_{B(x,r)} w(z)\,\mathrm{d}z} \le [w]_{\mathrm{RH}_\infty} \frac{u_j(y)}{w(y)}, \quad y \in B(x,r).$$

For the latter integral, we notice that

$$w(y) \le [w]_{\mathrm{RH}_\infty} \fint_{B(x,2^k r)} w(z)\,\mathrm{d}z \le [w]_{A_1} [w]_{\mathrm{RH}_\infty} \fint_{B(x,r)} w(z)\,\mathrm{d}z$$

holds for $k = 1, 2, \dots$ and $y \in B(x, 2^k r) \setminus B(x, 2^{k-1} r)$. Combining these estimates, we obtain

$$\frac{\zeta_r * \varphi_j * u_j(x)}{\fint_{B(x,r)} w(z)\,\mathrm{d}z} \le C \|\varphi_j * \zeta_r\|_{L^1} \|u_j/w\|_{L^\infty}$$

for any $j \in \mathbb{Z}$ and $r > 0$. Taking $i \in \mathbb{Z}$ with $2^{-i} \approx r$, we can easily see that

$$\|\varphi_j * \zeta_r\|_{L^1} \approx \|\varphi_j * \zeta_i\|_{L^1} = \|\varphi_{j-i} * \zeta\|_{L^1},$$

and hence

$$\begin{aligned} r^{-s} \sum_{j\in\mathbb{Z}} \frac{\zeta_r * \varphi_j * u_j(x)}{\fint_{B(x,r)} w(z)\,\mathrm{d}z} &\lesssim \sum_{j\in\mathbb{Z}} 2^{-is} \|\varphi_{j-i} * \zeta\|_{L^1} \|u_j/w\|_{L^\infty} \\ &= \sum_{j\in\mathbb{Z}} 2^{(j-i)s} \|\varphi_{j-i} * \zeta\|_{L^1} 2^{-js} \|u_j/w\|_{L^\infty} \\ &\le \|\zeta\|_{\dot{B}^s_{11}} \|u\|_{\dot{\mathcal{B}}^{-s,w}_{\infty\infty}}, \end{aligned}$$

where $\dot{B}^s_{11}$ is the usual homogeneous Besov space. Taking supremum over $r > 0$ and $x \in \mathbb{R}^n$ on the left-hand side, we get

$$\|u\|_{n-s,w} \lesssim \|u\|_{\dot{\mathcal{B}}^{-s,w}_{\infty\infty}}.$$

This completes the proof. $\square$

Proof of Theorem 1.4. By Corollary 3.2, we have

$$\int_{\mathbb{R}^n} |f| \, \mathrm{d}H_w^{n-s} \approx \sup \left\{ \int_{\mathbb{R}^n} |f| \, \mathrm{d}\mu : \|\|\mu\|\|_{n-s,w} \leq 1, \mu \in \mathbb{L}_{w,+}^{\infty,n-s} \right\}$$

for $f \in C_0^\infty(\mathbb{R}^n)$. Together with this and Theorem 5.1, it suffices to show that

$$\sup \left\{ \int_{\mathbb{R}^n} |I_\alpha g| \, \mathrm{d}\mu : \|\|\mu\|\|_{n-s,w} \leq 1, \mu \in \mathbb{L}_{w,+}^{\infty,n-s} \right\} \lesssim \|g\|_{\dot{B}_{11}^{s-\alpha,w}}, \quad g \in C_0^\infty, \ s > \alpha. \tag{5.1}$$

By Lemma 5.3, $I_\alpha\mu$ and $(I_\alpha\mu)\operatorname{sgn} g$ belong to the dual space $(\dot{B}_{11}^{s-\alpha,w})^* = \dot{\mathcal{B}}_{\infty\infty}^{\alpha-s,w}$, and it follows that

$$\|(I_\alpha\mu)\operatorname{sgn} g\|_{\dot{\mathcal{B}}_{\infty\infty}^{\alpha-s,w}} \lesssim \|I_\alpha\mu\|_{\dot{\mathcal{B}}_{\infty\infty}^{\alpha-s,w}} \approx \|\|I_\alpha\mu\|\|_{n-s+\alpha,w} \lesssim \|\|\mu\|\|_{n-s,w},$$

where we have used Lemma 5.2 in the last inequality. Therefore, a further use of Fubini's theorem yields

$$\begin{aligned} \int_{\mathbb{R}^n} |I_\alpha g(y)| \, \mathrm{d}\mu(y) &\leq \int_{\mathbb{R}^n} I_\alpha\mu(x)|g(x)| \, \mathrm{d}x \\ &\leq \|g\|_{\dot{B}_{11}^{s-\alpha,w}} \|(I_\alpha\mu)\operatorname{sgn} g\|_{\dot{\mathcal{B}}_{\infty\infty}^{\alpha-s,w}} \lesssim \|g\|_{\dot{B}_{11}^{s-\alpha,w}} \|\|\mu\|\|_{n-s,w} \end{aligned}$$

and so (5.1). This proves the theorem. □

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