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On the Existence of Auslander-Reiten $(d+2)$-angles in $(d+2)$-angulated Categories

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Abstract. Let $\mathcal{C}$ be a $(d+2)$-angulated category. In this note, we show that if $\mathcal{C}$ is locally finite, then $\mathcal{C}$ has Auslander-Reiten $(d+2)$-angles. This extends a result of Xiao and Zhu for triangulated categories.

1. Introduction

Auslander-Reiten theory was introduced by Auslander and Reiten in [1, 2]. Since then, Auslander-Reiten theory has become a fundamental tool for studying the representation theory of Artin algebras. Later it has been generalized to the situations such as exact categories [9], triangulated categories [6, 15] and its subcategories [3, 10], as well as some additive categories [10, 12, 16] by researchers. Extriangulated categories were introduced by Nakaoka and Palu [13] as a simultaneous generalization of exact categories and triangulated categories. Hence, many results hold on exact categories and triangulated categories can be unified in the same framework. Iyama, Nakaoka and Palu [7] introduced the notions of almost split extensions and Auslander-Reiten-Serre duality for extriangulated categories. Meanwhile, they gave explicit connections between these notions and the classical notion of dualizing k-varieties. Xiao and Zhu [17, 18] showed that if a triangulated category $\mathcal{C}$ is locally finite, then $\mathcal{C}$ has Auslander-Reiten triangles. Recently, Zhu and Zhuang [20] proved that if an extriangulated category $\mathcal{C}$ is locally finite, then $\mathcal{C}$ has Auslander-Reiten $\mathbb{E}$-triangles.

In [5], Geiss, Keller and Oppermann introduced a new type of categories, called $(d+2)$-angulated categories, which generalize triangulated categories: the classical triangulated categories are the special case $d=1$. These categories appear for instance when considering certain d-cluster tilting subcategories of triangulated categories. Iyama and Yoshino [8] defined Auslander-Reiten $(d+2)$-angles in special $(d+2)$-angulated categories. Later,

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Fedele [4] defined Auslander-Reiten $(d+2)$-angles in additive subcategories of $(d+2)$-angulated categories which are closed under d-extensions, an example of which is a wide subcategory. He also proved that there are Auslander-Reiten $(d+2)$-angles in certain additive subcategories of $(d+2)$-angulated categories. Recently, the author [19] showed that a $(d+2)$-angulated category $\mathcal{C}$ has Auslander-Reiten $(d+2)$-angles if and only if $\mathcal{C}$ has a Serre functor.

In this note, we continue to study Auslander-Reiten $(d+2)$-angles in $(d+2)$-angulated categories. We will generalize Xiao and Zhu's result to $(d+2)$-angulated categories. Moreover, our proof is different from the usual triangulated case.

Our main result is the following.

Theorem 1.1. (see Theorem 3.8 for details) *Let $\mathcal{C}$ be a locally finite $(d+2)$-angulated category. If $X \in \mathcal{C}$ is an indecomposable object, then there are an Auslander-Reiten $(d+2)$-angle ending at X, and an Auslander-Reiten $(d+2)$-angle starting at X. In this case, we say that $\mathcal{C}$ has Auslander-Reiten $(d+2)$-angles.*

This article is organized as follows: In Section 2, we review some elementary concepts to be used later, including $(d+2)$-angulated categories and Auslander-Reiten $(d+2)$ angles. In Section 3, we prove our main result.

2. Preliminaries

In this section, we first recall some definitions and basic properties of $(d+2)$-angulated categories from [5]. Let $\mathcal{C}$ be an additive category with an automorphism $\Sigma^d\colon \mathcal{C} \to \mathcal{C}$, where d is an integer no less than one.

A $(d+2)$-Σ^d-*sequence* in $\mathcal{C}$ is a sequence of objects and morphisms

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{d-1}} A_n \xrightarrow{f_d} A_{d+1} \xrightarrow{f_{d+1}} \Sigma^d A_0.$$

Its *left rotation* is the $(d+2)$-Σ^d-sequence

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \cdots \xrightarrow{f_d} A_{d+1} \xrightarrow{f_{d+1}} \Sigma^d A_0 \xrightarrow{(-1)^d \Sigma^d f_0} \Sigma^d A_1.$$

A *morphism* of $(d+2)$-Σ^d-sequences is a sequence of morphisms $\varphi = (\varphi_0, \varphi_1, \dots, \varphi_{d+1})$ such that the following diagram

$$\begin{array}{ccccccccccc}
A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_d} & A_{d+1} & \xrightarrow{f_{d+1}} & \Sigma^d A_0 \\
\downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & & & \downarrow{\scriptstyle \varphi_{d+1}} & & \downarrow{\scriptstyle \Sigma^d \varphi_0} \\
B_0 & \xrightarrow{g_0} & B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_d} & B_{d+1} & \xrightarrow{g_{d+1}} & \Sigma^d B_0
\end{array}$$

commutes, where each row is a $(d+2)$-Σ^d-sequence. It is an *isomorphism* if $\varphi_0, \varphi_1, \varphi_2, \dots,$ φ_{d+1} are all isomorphisms in $\mathcal{C}$.

Definition 2.1. [5, Definition 2.1] A $(d+2)$-*angulated category* is a triple $(\mathcal{C}, \Sigma^d, \Theta)$, where $\mathcal{C}$ is an additive category, Σ^d is an automorphism of $\mathcal{C}$ (Σ^d is called the d-suspension functor), and Θ is a class of $(d+2)$-Σ^d-sequences (whose elements are called $(d+2)$-angles), which satisfies the following axioms:

(N1) (a) The class Θ is closed under isomorphisms, direct sums and direct summands.

(b) For each object $A \in \mathcal{C}$, the trivial sequence

$$A \xrightarrow{1_A} A \to 0 \to 0 \to \cdots \to 0 \to \Sigma^d A$$

belongs to Θ.

(c) Each morphism $f_0 \colon A_0 \to A_1$ in $\mathcal{C}$ can be extended to $(d+2)$-Σ^d-sequence:

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{d-1}} A_d \xrightarrow{f_d} A_{d+1} \xrightarrow{f_{d+1}} \Sigma^d A_0.$$

(N2) A $(d+2)$-Σ^d-sequence belongs to Θ if and only if its left rotation belongs to Θ.

(N3) Each solid commutative diagram

$$\begin{array}{ccccccccccc}
A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_d} & A_{d+1} & \xrightarrow{f_{d+1}} & \Sigma^d A_0 \\
\downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & & & \downarrow{\scriptstyle \varphi_{d+1}} & & \downarrow{\scriptstyle \Sigma^d \varphi_0} \\
B_0 & \xrightarrow{g_0} & B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_d} & B_{d+1} & \xrightarrow{g_{d+1}} & \Sigma^d B_0
\end{array}$$

with rows in Θ, the dotted morphisms exist and give a morphism of $(d+2)$-Σ^d-sequences.

(N4) In the situation of (N3), the morphisms $\varphi_2, \varphi_3, \ldots, \varphi_{d+1}$ can be chosen such that the mapping cone

$$A_1 \oplus B_0 \xrightarrow{\begin{pmatrix} -f_1 & 0 \\ \varphi_1 & g_0 \end{pmatrix}} A_2 \oplus B_1 \xrightarrow{\begin{pmatrix} -f_2 & 0 \\ \varphi_2 & g_1 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} -f_{d+1} & 0 \\ \varphi_{d+1} & g_d \end{pmatrix}} \Sigma^n A_0 \oplus B_{d+1}$$
$$\xrightarrow{\begin{pmatrix} -\Sigma^d f_0 & 0 \\ \Sigma^d \varphi_1 & g_{d+1} \end{pmatrix}} \Sigma^d A_1 \oplus \Sigma^d B_0$$

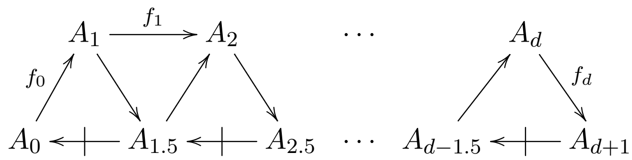
belongs to Θ.

Now we give an example of $(d+2)$-angulated categories.

Example 2.2. We recall the standard construction of $(d+2)$-angulated categories given by Geiss-Keller-Oppermann [5, Theorem 1]. Let $\mathcal{C}$ be a triangulated category and $\mathcal{T}$ a d-cluster tilting subcategory which is closed under Σ^d, where Σ is the shift functor of $\mathcal{C}$. Then $(\mathcal{T}, \Sigma^d, \Theta)$ is a $(d+2)$-angulated category, where Θ is the class of all sequences

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{d-1}} A_d \xrightarrow{f_d} A_{d+1} \xrightarrow{f_{d+1}} \Sigma^d A_0$$

such that there exists a diagram



with $A_i \in \mathcal{T}$ for all $i \in \mathbb{Z}$, such that all oriented triangles are triangles in $\mathcal{C}$, all non-oriented triangles commute, and f_{d+1} is the composition along the lower edge of the diagram.

The following two lemmas are very useful which are needed in the sequel.

Lemma 2.3. [4, Lemma 3.13] *Let $\mathcal{C}$ be a $(d+2)$-angulated category, and*

$$A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{\alpha_{d+1}} \Sigma^d A_0 \tag{2.1}$$

a $(d+2)$-angle in $\mathcal{C}$. Then the following statements are equivalent:

(1) *α_0 is a section;*

(2) *α_d is a retraction;*

(3) $\alpha_{d+1} = 0$.

If a $(d+2)$-angle (2.1) *satisfies one of the above equivalent conditions, it is called* split.

Lemma 2.4. [11, Corollary 3.4] *Let $\mathcal{C}$ be a $(d+2)$-angulated category, and*

$$A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{\alpha_{d+1}} \Sigma^d A_0$$

a $(d+2)$-angle in $\mathcal{C}$. Then for any morphism $\varphi_0\colon A_0 \to B_0$, there exists the following commutative diagram of $(d+2)$-angles

$$\begin{array}{ccccccccccccc}
A_0 & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{d-1}} & A_d & \xrightarrow{\alpha_d} & A_{d+1} & \xrightarrow{\alpha_{d+1}} & \Sigma^d A_0 \\
\downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & & & \downarrow{\scriptstyle \varphi_d} & & \| & & \downarrow{\scriptstyle \Sigma^d \varphi_0} \\
B_0 & \xrightarrow{\beta_0} & B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_{d-1}} & B_d & \xrightarrow{\beta_d} & A_{d+1} & \xrightarrow{\beta_{d+1}} & \Sigma^d B_0
\end{array}$$

such that

$$A_0 \xrightarrow{\begin{pmatrix} -\alpha_0 \\ \varphi_0 \end{pmatrix}} A_1 \oplus B_0 \xrightarrow{\begin{pmatrix} -\alpha_1 & 0 \\ \varphi_1 & \beta_0 \end{pmatrix}} A_2 \oplus B_1 \xrightarrow{\begin{pmatrix} -\alpha_2 & 0 \\ \varphi_2 & \beta_1 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} -\alpha_{d-1} & 0 \\ \varphi_{d-1} & \beta_{d-2} \end{pmatrix}} A_d \oplus B_{d-1}$$
$$\xrightarrow{(\varphi_d, \beta_{d-1})} B_d \xrightarrow{(-1)^d \alpha_{d+1} \beta_d} \Sigma^d A_0$$

is a $(d+2)$-angle in $\mathcal{C}$.

Now we recall the Auslander-Reiten $(d+2)$ theory in $(d+2)$-angulated categories.

We denote by $\mathrm{rad}_{\mathcal{C}}$ the Jacobson radical of $\mathcal{C}$. Namely, $\mathrm{rad}_{\mathcal{C}}$ is an ideal of $\mathcal{C}$ such that $\mathrm{rad}_{\mathcal{C}}(A, A)$ coincides with the Jacobson radical of the endomorphism ring $\mathrm{End}(A)$ for any $A \in \mathcal{C}$.

Definition 2.5. (see [8, Definition 3.8] and [4, Definition 5.1]) Let $\mathcal{C}$ be a $(d+2)$-angulated category. A $(d+2)$-angle

$$A_\bullet: \ A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{\alpha_{d+1}} \Sigma^d A_0$$

in $\mathcal{C}$ is called an *Auslander-Reiten $(d+2)$-angle* if α_0 is left almost split, α_d is right almost split and $\alpha_1, \alpha_2, \ldots, \alpha_{d-1}$ are in $\mathrm{rad}_{\mathcal{C}}$ when $d > 1$.

Remark 2.6. [4, Remark 5.2] Assume $A_\bullet$ as in Definition 2.5 is an Auslander-Reiten $(d+2)$-angle. Since α_0 is left almost split, we obtain that $\mathrm{End}(A_0)$ is local and hence A_0 is indecomposable. Similarly, since α_d is right almost split, it follows that $\mathrm{End}(A_{d+1})$ is local and hence A_{d+1} is indecomposable. Moreover, when $d = 1$, we have α_0 and α_d in $\mathrm{rad}_{\mathcal{C}}$, so that α_d is right minimal and α_0 is left minimal. When $d > 1$, since $\alpha_{d-1} \in \mathrm{rad}_{\mathcal{C}}$, we have that α_d is right minimal and similarly α_0 is left minimal.

Remark 2.7. [4, Lemma 5.3] Let $\mathcal{C}$ be a $(d+2)$-angulated category and

$$A_\bullet: \ A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{\alpha_{d+1}} \Sigma^d A_0$$

a $(d+2)$-angle in $\mathcal{C}$. Then the following statements are equivalent:

(1) $A_\bullet$ is an Auslander-Reiten $(d+2)$-angle;

(2) $\alpha_0, \alpha_1, \ldots, \alpha_{d-1}$ are in $\mathrm{rad}_{\mathcal{C}}$ and α_d is right almost split;

(3) $\alpha_1, \alpha_2, \ldots, \alpha_d$ are in $\mathrm{rad}_{\mathcal{C}}$ and α_0 is left almost split.

Lemma 2.8. [4, Lemma 5.4] *Let $\mathcal{C}$ be a $(d+2)$-angulated category and*

$$A_\bullet: \ A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{\alpha_{d+1}} \Sigma^d A_0$$

a $(d+2)$-angle in $\mathcal{C}$. Assume that α_d is right almost split and, if $d > 1$, also that $\alpha_1, \alpha_2, \ldots, \alpha_{d-1}$ are in $\mathrm{rad}_{\mathcal{C}}$. Then the following statements are equivalent:

(1) $A_\bullet$ *is an Auslander-Reiten $(d+2)$-angle;*

(2) $\mathrm{End}(A_0)$ *is local;*

(3) α_{d+1} *is left minimal;*

(4) α_0 *is in* $\mathrm{rad}_{\mathcal{C}}$.

In the case $d=1$, so in the case of a triangulated category, a morphism can be extended to a triangle in a unique way up to isomorphism. On the other hand, for $d>1$, a morphism can be extendend to a $(d+2)$-angle in different non-isomorphic ways. However, we still have a unique "minimal" $(d+2)$-angle extending any given morphism.

Lemma 2.9. (see [14, Lemma 5.18] and [4, Lemma 3.14]) *Let $d>1$ and $h\colon A_{d+1}\to\Sigma^d A_0$ be any morphism in a $(d+2)$-angulated category $\mathcal{C}$. Then, up to isomorphism, there exists a unique $(d+2)$-angle of the form*

$$A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-2}} A_{d-1} \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{h} \Sigma^d A_0$$

with $\alpha_1,\alpha_2,\dots,\alpha_{d-1}$ in $\mathrm{rad}_{\mathcal{C}}$.

3. Proof of the main result

In this section, let k be a field. We always assume that $\mathcal{C}$ is a k-linear Hom-finite Krull-Schmidt $(d+2)$-angulated category. We denote by $\mathrm{ind}(\mathcal{C})$ the set of isomorphism classes of indecomposable objects in $\mathcal{C}$. For any $X\in\mathrm{ind}(\mathcal{C})$, we denote by $\mathrm{Supp}\,\mathrm{Hom}_{\mathcal{C}}(X,-)$ the subcategory of $\mathcal{C}$ generated by objects Y in $\mathrm{ind}(\mathcal{C})$ with $\mathrm{Hom}_{\mathcal{C}}(X,Y)\neq 0$. Similarly, $\mathrm{Supp}\,\mathrm{Hom}_{\mathcal{C}}(-,X)$ denotes the subcategory generated by objects Y in $\mathrm{ind}(\mathcal{C})$ with $\mathrm{Hom}_{\mathcal{C}}(Y,X)\neq 0$. If $\mathrm{Supp}\,\mathrm{Hom}_{\mathcal{C}}(X,-)$ ($\mathrm{Supp}\,\mathrm{Hom}_{\mathcal{C}}(-,X)$, respectively) contains only finitely many indecomposable objects, we say that $|\mathrm{Supp}\,\mathrm{Hom}_{\mathcal{C}}(X,-)|<\infty$ ($|\mathrm{Supp}\,\mathrm{Hom}_{\mathcal{C}}(-,X)|<\infty$ respectively).

Based on the definition of locally finite triangulated categories [17, 18], we define the notion of locally finite $(d+2)$-angulated categories.

Definition 3.1. A $(d+2)$-angulated category $\mathcal{C}$ is called *locally finite* if $|\mathrm{Supp}\,\mathrm{Hom}_{\mathcal{C}}(X,-)|<\infty$ and $|\mathrm{Supp}\,\mathrm{Hom}_{\mathcal{C}}(-,X)|<\infty$, for any object $X\in\mathrm{ind}(\mathcal{C})$.

We know that the derived categories of finite dimensional hereditary algebras of finite type and the stable module categories of finite dimensional self-injective algebras of finite type are examples of locally finite triangulated categories, see [17, 18]. In those locally finite triangulated categories, we take a d-cluster titling subcategory which is closed under the d-th power of the shift functor. By Example 2.2, we obtain some locally finite $(d+2)$-angulated categories.

Definition 3.2. Let $\mathcal{C}$ be a $(d+2)$-angulated category and $X\in\mathrm{ind}(\mathcal{C})$. We define a set of $(d+2)$-angles as follows:

$$\begin{aligned} S(X):=\big\{A_\bullet: A \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} X \xrightarrow{\alpha_{d+1}} \Sigma^d A_0 \;\big|\; & \\ A_\bullet \text{ is a non-split } (d+2)\text{-angle with } A\in\mathrm{ind}(\mathcal{C}), \text{ and when } d>1, & \\ \alpha_1,\alpha_2,\dots,\alpha_{d-1} \text{ in } \mathrm{rad}_{\mathcal{C}}\big\}. & \end{aligned}$$

Dually, we can define a set of $(d+2)$-angles as follows:

$$\begin{aligned} T(X) := \big\{ A_\bullet : X \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A \xrightarrow{\alpha_{d+1}} \Sigma^d A_0 \mid \\ A_\bullet \text{ is a non-split } (d+2)\text{-angle with } A \in \operatorname{ind}(\mathcal{C}), \text{ and when } d > 1, \\ \alpha_1, \alpha_2, \ldots, \alpha_{d-1} \text{ in } \operatorname{rad}_{\mathcal{C}} \big\}. \end{aligned}$$

Lemma 3.3. *Let $\mathcal{C}$ be a $(d+2)$-angulated category and $X \in \operatorname{ind}(\mathcal{C})$. Then $S(X)$ and $T(X)$ are non-empty sets.*

Proof. It is enough to prove that $S(X)$ is non-empty set because we can prove the statement on $T(X)$ by duality.

Since $X \in \operatorname{ind}(\mathcal{C})$, there is an object $A \in \mathcal{C}$ such that $\operatorname{Hom}_{\mathcal{C}}(X, \Sigma^d A) \neq 0$. Thus there exists a non-split $(d+2)$-angle:

$$B_\bullet : \quad A \xrightarrow{\alpha_0} B_1 \xrightarrow{\alpha_1} B_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-2}} B_{d-1} \xrightarrow{\alpha_{d-1}} B_d \xrightarrow{\alpha_d} X \xrightarrow{h} \Sigma^d A.$$

We decompose A into a direct sum of indecomposable objects, i.e., $A = \bigoplus_{i=1}^n A_i$. Without loss of generality, we can assume that $A = U \oplus V$ where U and V are indecomposable objects. By Lemma 2.4, for the morphism $(1,0)\colon U \oplus V \to U$, there exists the following commutative diagram of $(d+2)$-angles

$$\begin{array}{ccccccccccccccc}
U \oplus V & \xrightarrow{(u,v)} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{d-1}} & A_d & \xrightarrow{\alpha_d} & X & \xrightarrow{h} & \Sigma^d U \oplus \Sigma^d V \\
\downarrow{\scriptstyle (1,0)} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow & & & & \downarrow & & \| & & \downarrow{\scriptstyle (1,0)} \\
U & \xrightarrow{\beta_0} & C_1 & \xrightarrow{\beta_1} & C_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_{d-1}} & C_d & \xrightarrow{\beta_d} & X & \xrightarrow{\beta_{d+1}} & \Sigma^d U.
\end{array}$$

Similarly, for the morphism $(0,1)\colon U \oplus V \to V$, there exists the following commutative diagram

$$\begin{array}{ccccccccccccccc}
U \oplus V & \xrightarrow{(u,v)} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{d-1}} & A_d & \xrightarrow{\alpha_d} & X & \xrightarrow{h} & \Sigma^d U \oplus \Sigma^d V \\
\downarrow{\scriptstyle (0,1)} & & \downarrow{\scriptstyle \psi_1} & & \downarrow & & & & \downarrow & & \| & & \downarrow{\scriptstyle (0,1)} \\
V & \xrightarrow{\gamma_0} & D_1 & \xrightarrow{\gamma_1} & D_2 & \xrightarrow{\gamma_2} & \cdots & \xrightarrow{\gamma_{d-1}} & D_d & \xrightarrow{\gamma_d} & X & \xrightarrow{\gamma_{d+1}} & \Sigma^d V
\end{array}$$

of $(d+2)$-angles. We assert that at least one of the following two $(d+2)$-angles

$$U \xrightarrow{\beta_0} C_1 \xrightarrow{\beta_1} C_2 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_{d-1}} C_d \xrightarrow{\beta_d} X \xrightarrow{\beta_{d+1}} \Sigma^d U,$$

$$V \xrightarrow{\gamma_0} D_1 \xrightarrow{\gamma_1} D_2 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_{d-1}} D_d \xrightarrow{\gamma_d} X \xrightarrow{\gamma_{d+1}} \Sigma^d V$$

is non-split. Otherwise, we obtain $\beta_{d+1} = 0 = \gamma_{d+1}$ by Lemma 2.3. By (N3), we have the following commutative diagram

$$\begin{array}{ccccccccccccccc}
U \oplus V & \xrightarrow{(u,v)} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{d-1}} & A_d & \xrightarrow{\alpha_d} & X & \xrightarrow{h} & \Sigma^d U \oplus \Sigma^d V \\
\| & & \downarrow{\scriptstyle \binom{\varphi_1}{\psi_1}} & & \downarrow & & & & \downarrow & & \downarrow & & \| \\
U \oplus V & \xrightarrow{\delta_0} & C_1 \oplus D_1 & \xrightarrow{\delta_1} & C_2 \oplus D_2 & \xrightarrow{\delta_2} & \cdots & \xrightarrow{\delta_{d-1}} & C_d \oplus D_d & \xrightarrow{\delta_d} & X \oplus X & \xrightarrow{\delta_{d+1}} & \Sigma^d U \oplus \Sigma^d V
\end{array}$$

of $(d+2)$-angles, where $\delta_i = \begin{pmatrix} \beta_i & 0 \\ 0 & \gamma_i \end{pmatrix}$. It follows that $h = 0$. This is a contradiction since $B_\bullet$ is non-split.

For the morphism $\beta_{d+1} \neq 0$ or $\gamma_{d+1} \neq 0$, by Lemma 2.9, we can find a $(d+2)$-angle as desired. This shows that $S(X)$ is a non-empty set. $\square$

Definition 3.4. Let $\mathcal{C}$ be a $(d+2)$-angulated category, and

$$A_\bullet : \ A \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} X \xrightarrow{\alpha_{d+1}} \Sigma^d A_0,$$
$$B_\bullet : \ B \xrightarrow{\beta_0} B_1 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_{d-1}} B_d \xrightarrow{\beta_d} X \xrightarrow{\beta_{d+1}} \Sigma^d B_0$$

two $(d+2)$-angles in $S(X)$. We say that $A_\bullet > B_\bullet$ if there are morphisms $\varphi_i \in \mathrm{Hom}_{\mathcal{C}}(A_i, B_i)$, $(i = 0, 1, \ldots, d)$ such that the following diagram

$$\begin{array}{ccccccccccccc}
A & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{d-1}} & A_d & \xrightarrow{\alpha_d} & X & \xrightarrow{\alpha_{d+1}} & \Sigma^d A_0 \\
\downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & & & \downarrow{\scriptstyle \varphi_d} & & \| & & \downarrow{\scriptstyle \Sigma^d \varphi_0} \\
B & \xrightarrow{\beta_0} & B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_{d-1}} & B_d & \xrightarrow{\beta_d} & X & \xrightarrow{\beta_{d+1}} & \Sigma^d B_0
\end{array}$$

commutes. We say that $A_\bullet \sim B_\bullet$ if φ_0 is an isomorphism.

Dually, let

$$A_\bullet : \ X \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A \xrightarrow{\alpha_{d+1}} \Sigma^d A_0,$$
$$B_\bullet : \ X \xrightarrow{\beta_0} B_1 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_{d-1}} B_d \xrightarrow{\beta_d} B \xrightarrow{\beta_{d+1}} \Sigma^d B_0$$

be two $(d+2)$-angles in $T(X)$. We say that $A_\bullet > B_\bullet$ if there are morphisms $\varphi_i \in \mathrm{Hom}_{\mathcal{C}}(A_i, B_i)$, $(i = 1, 2, \ldots, d+1)$ such that the following diagram

$$\begin{array}{ccccccccccccc}
X & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{d-1}} & A_d & \xrightarrow{\alpha_d} & A & \xrightarrow{\alpha_{d+1}} & \Sigma^d X \\
\| & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & & & \downarrow{\scriptstyle \varphi_d} & & \downarrow{\scriptstyle \varphi_{d+1}} & & \| \\
X & \xrightarrow{\beta_0} & B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_{d-1}} & B_d & \xrightarrow{\beta_d} & B & \xrightarrow{\beta_{d+1}} & \Sigma^d X
\end{array}$$

commutes. We say that $A_\bullet \sim B_\bullet$ if φ_{d+1} is an isomorphism.

Lemma 3.5. *Both $S(X)$ and $T(X)$ are direct ordered sets with the relations defined in Definition* 3.4.

Proof. We only prove that the first statement is true for $S(X)$, and the second statement for $T(X)$ can be proved similarly.

Assume that

$$A_\bullet : \ A \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} X \xrightarrow{\alpha_{d+1}} \Sigma^d A_0,$$
$$B_\bullet : \ B \xrightarrow{\beta_0} B_1 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_{d-1}} B_d \xrightarrow{\beta_d} X \xrightarrow{\beta_{d+1}} \Sigma^d B_0$$

belong to $S(X)$.

We first show that if $A_\bullet > B_\bullet$ and $B_\bullet > A_\bullet$, then $A_\bullet \sim B_\bullet$.

Since $A_\bullet > B_\bullet$ and $B_\bullet > A_\bullet$, we have the following two commutative diagrams

$$\begin{array}{ccccccccccccc}
A & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{d-1}} & A_d & \xrightarrow{\alpha_d} & X & \xrightarrow{\alpha_{d+1}} & \Sigma^d A_0 \\
\downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & & & \downarrow{\scriptstyle \varphi_d} & & \| & & \downarrow{\scriptstyle \Sigma^d\varphi_0} \\
B & \xrightarrow{\beta_0} & B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_{d-1}} & B_d & \xrightarrow{\beta_d} & X & \xrightarrow{\beta_{d+1}} & \Sigma^d B_0,
\end{array}$$

$$\begin{array}{ccccccccccccc}
B & \xrightarrow{\beta_0} & B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_{d-1}} & B_d & \xrightarrow{\beta_d} & X & \xrightarrow{\beta_{d+1}} & \Sigma^d B_0 \\
\downarrow{\scriptstyle \psi_0} & & \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2} & & & & \downarrow{\scriptstyle \psi_d} & & \| & & \downarrow{\scriptstyle \Sigma^d\psi_0} \\
A & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{d-1}} & A_d & \xrightarrow{\alpha_d} & X & \xrightarrow{\alpha_{d+1}} & \Sigma^d A_0.
\end{array}$$

Since A is indecomposable, we have that $\operatorname{End}(A)$ is local. This implies that $\psi_0\varphi_0$ is nilpotent or an isomorphism. If $\psi_0\varphi_0$ is nilpotent, there exists a positive integer m such that $(\psi_0\varphi_0)^m = 0$. We write $\omega_i = \psi_i\varphi_i$. Thus we have the following commutative diagram

$$\begin{array}{ccccccccccccccc}
A & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{d-2}} & A_{d-1} & \xrightarrow{\alpha_{d-1}} & A_d & \xrightarrow{\alpha_d} & X & \xrightarrow{\alpha_{d+1}} & \Sigma^d A_0 \\
\downarrow{\scriptstyle (\psi_0\varphi_0)^m} & & \downarrow{\scriptstyle (\omega_1)^m} & & \downarrow{\scriptstyle (\omega_2)^m} & & & & \downarrow{\scriptstyle (\omega_{d-1})^m} & & \downarrow{\scriptstyle (\varphi_d)^m} & & \| & & \downarrow{\scriptstyle \Sigma^d(\psi_0\varphi_0)^m} \\
A & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{d-2}} & A_{d-1} & \xrightarrow{\alpha_{d-1}} & A_d & \xrightarrow{\alpha_d} & X & \xrightarrow{\alpha_{d+1}} & \Sigma^d A_0.
\end{array}$$

Then $\alpha_{d+1} = \Sigma^d(\psi_0\varphi_0)^m\alpha_{d+1} = 0$. This is a contradiction since $A_\bullet$ is non-split. Hence $\psi_0\varphi_0$ is an isomorphism. Similarly, we can also obtain that $\varphi_0\psi_0$ is an isomorphism. This shows that φ_0 is an isomorphism. So $A_\bullet \sim B_\bullet$.

It is clear that if $A_\bullet > B_\bullet$ and $B_\bullet > C_\bullet$, then $A_\bullet \sim C_\bullet$.

Now we show that if $A_\bullet, B_\bullet \in S(X)$, then there exists $C_\bullet \in S(X)$ such that $A_\bullet > C_\bullet$ and $B_\bullet \sim C_\bullet$.

For the morphism $\beta_d\colon B_d \to X$, by the duality of Lemma 2.4, there exists the following commutative diagram of $(d+2)$-angles

$$\begin{array}{ccccccccccccc}
A & \xrightarrow{\gamma_0} & D_1 & \xrightarrow{\gamma_1} & D_2 & \xrightarrow{\gamma_2} & \cdots & \xrightarrow{\gamma_{d-1}} & D_d & \xrightarrow{\gamma_d} & B_d & \xrightarrow{\gamma_{d+1}} & \Sigma^d A \\
\| & & \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2} & & & & \downarrow{\scriptstyle \psi_d} & & \downarrow{\scriptstyle \gamma_d} & & \| \\
A & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{d-1}} & A_d & \xrightarrow{\alpha_d} & X & \xrightarrow{\alpha_{d+1}} & \Sigma^d A
\end{array}$$

such that

$$M_\bullet:\quad D_1 \to M_1 \to M_2 \to \cdots \to M_{d-1} \to B_d \oplus A_d \xrightarrow{(\beta_d,\alpha_d)} X \xrightarrow{h} \Sigma^d D_1$$

is a $(d+2)$-angle in $\mathcal{C}$, where $M_i = D_{i+1} \oplus A_i$ $(i = 1, 2, \ldots, d-1)$. Since neither β_d nor α_d is a retraction, we have that (β_d, α_d) is also not a retraction. Otherwise, there exists a morphism $\binom{s}{t}\colon X\to B_d\oplus A_d$ such that $(\beta_d,\alpha_d)\binom{s}{t}=1_X$ and then $\beta_d s+\alpha_d t=1_X$. Since X is indecomposable, we have that $\operatorname{End}(X)$ is local. This implies that either $\beta_d s$ or $\alpha_d t$ is an isomorphism. Thus either β_d or α_d is a retraction, a contradiction. That is, $M_\bullet$ is non-split.

Assume that $D_1=U\oplus V$ where U and V are indecomposable objects. By Lemma 2.4, for the morphism $(1,0)\colon U\oplus V\to U$, there exists the following commutative diagram of $(d+2)$-angles

$$\begin{array}{ccccccccccccc}
U\oplus V & \xrightarrow{(u,v)} & M_1 & \longrightarrow & M_2 & \longrightarrow & \cdots & \longrightarrow & B_d\oplus A_d & \longrightarrow & X & \longrightarrow & \Sigma^d U\oplus\Sigma^d V\\
\downarrow{\scriptstyle(1,0)} & & \downarrow{\scriptstyle\varphi_1} & & \downarrow & & & & \downarrow & & \| & & \downarrow{\scriptstyle(1,0)}\\
U & \xrightarrow{\delta_0} & L_1 & \longrightarrow & L_2 & \longrightarrow & \cdots & \longrightarrow & L_d & \longrightarrow & X & \xrightarrow{h} & \Sigma^d U.
\end{array}$$

Similarly, for the morphism $(0,1)\colon U\oplus V\to V$, there exists the following commutative diagram of $(d+2)$-angles

$$\begin{array}{ccccccccccccc}
U\oplus V & \xrightarrow{(u,v)} & M_1 & \longrightarrow & M_2 & \longrightarrow & \cdots & \longrightarrow & B_d\oplus A_d & \longrightarrow & X & \longrightarrow & \Sigma^d U\oplus\Sigma^d V\\
\downarrow{\scriptstyle(0,1)} & & \downarrow{\scriptstyle\psi_1} & & \downarrow & & & & \downarrow & & \| & & \downarrow{\scriptstyle(0,1)}\\
V & \xrightarrow{\eta_0} & N_1 & \longrightarrow & N_2 & \longrightarrow & \cdots & \longrightarrow & N_d & \longrightarrow & X & \longrightarrow & \Sigma^d V.
\end{array}$$

By the similar arguments as in the proof of Lemma 3.3, we conclude that the at least one of the following two $(d+2)$-angles is non-split

$$U\xrightarrow{\delta_0}L_1\longrightarrow L_2\longrightarrow\cdots\longrightarrow L_d\longrightarrow X\xrightarrow{h}\Sigma^d U,$$

$$V\xrightarrow{\eta_0}N_1\longrightarrow N_2\longrightarrow\cdots\longrightarrow N_d\longrightarrow X\longrightarrow\Sigma^d V.$$

Without loss of generality, we assume that

$$U\xrightarrow{\delta_0}L_1\longrightarrow L_2\longrightarrow\cdots\longrightarrow L_d\longrightarrow X\xrightarrow{h}\Sigma^d U$$

is non-split. By Lemma 2.9, we can find a non-split $(d+2)$-angle

$$C_\bullet:\quad U\xrightarrow{\lambda_0}C_1\xrightarrow{\lambda_1}C_2\xrightarrow{\lambda_2}\cdots\xrightarrow{\lambda_{d-1}}C_d\xrightarrow{\lambda_d}X\xrightarrow{h}\Sigma^d U$$

with $\lambda_1,\lambda_2,\dots,\lambda_{d-1}$ in $\operatorname{rad}_{\mathcal{C}}$. By (N3), we have the following commutative diagram

$$\begin{array}{ccccccccccccc}
A & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{d-1}} & A_d & \xrightarrow{\alpha_d} & X & \xrightarrow{\alpha_{d+1}} & \Sigma^d A_0\\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow{\scriptstyle\binom{0}{1}} & & \| & & \downarrow\\
U\oplus V & \xrightarrow{(u,v)} & M_1 & \longrightarrow & M_2 & \longrightarrow & \cdots & \longrightarrow & B_d\oplus A_d & \xrightarrow{(\beta_d,\alpha_d)} & X & \longrightarrow & \Sigma^d U\oplus\Sigma^d V\\
\downarrow{\scriptstyle(1,0)} & & \downarrow{\scriptstyle\varphi_1} & & \downarrow & & & & \downarrow & & \| & & \downarrow{\scriptstyle(1,0)}\\
U & \xrightarrow{\delta_0} & L_1 & \longrightarrow & L_2 & \longrightarrow & \cdots & \longrightarrow & L_d & \longrightarrow & X & \xrightarrow{h} & \Sigma^d U\\
\| & & \downarrow & & \downarrow & & & & \downarrow & & \| & & \|\\
U & \xrightarrow{\lambda_0} & C_1 & \xrightarrow{\lambda_1} & C_2 & \xrightarrow{\lambda_2} & \cdots & \xrightarrow{\lambda_{d-1}} & C_d & \xrightarrow{\lambda_d} & X & \xrightarrow{h} & \Sigma^d U
\end{array}$$

of $(d+2)$-angles. This shows that $A_\bullet > C_\bullet$.

By (N3), we have the following commutative diagram

$$
\begin{array}{ccccccccccccc}
B & \xrightarrow{\beta_0} & B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_{d-1}} & B_d & \xrightarrow{\beta_d} & X & \xrightarrow{\beta_{d+1}} & \Sigma^d B_0 \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow {\scriptstyle \binom{1}{0}} & & \Vert & & \downarrow \\
U\oplus V & \xrightarrow{(u,v)} & M_1 & \longrightarrow & M_2 & \longrightarrow & \cdots & \longrightarrow & B_d\oplus A_d & \xrightarrow{(\beta_d,\alpha_d)} & X & \longrightarrow & \Sigma^d U\oplus\Sigma^d V \\
\downarrow {\scriptstyle (1,0)} & & \downarrow {\scriptstyle \varphi_1} & & \downarrow & & & & \downarrow & & \Vert & & \downarrow {\scriptstyle (1,0)} \\
U & \xrightarrow{\delta_0} & L_1 & \longrightarrow & L_2 & \longrightarrow & \cdots & \longrightarrow & L_d & \longrightarrow & X & \xrightarrow{h} & \Sigma^d U \\
\Vert & & \downarrow & & \downarrow & & & & \downarrow & & \Vert & & \Vert \\
U & \xrightarrow{\lambda_0} & C_1 & \xrightarrow{\lambda_1} & C_2 & \xrightarrow{\lambda_2} & \cdots & \xrightarrow{\lambda_{d-1}} & C_d & \xrightarrow{\lambda_d} & X & \xrightarrow{h} & \Sigma^d U
\end{array}
$$

of $(d+2)$-angles. This shows that $B_\bullet > C_\bullet$. □

Lemma 3.6. *Let $\mathcal{C}$ be a locally finite $(d+2)$-angulated category and $X \in \mathrm{ind}(\mathcal{C})$. Then $S(X)$ has a minimal element, and $T(X)$ has a minimal element.*

Proof. We only prove the first statement, the second statement can be proved similarly.

Since $X \in \mathrm{ind}(\mathcal{C})$, there is an object $A \in \mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(X, \Sigma^d A) \neq 0$. Then there exists a non-split $(d+2)$-angle

$$A_\bullet:\ \ A \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-2}} A_{d-1} \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} X \xrightarrow{h} \Sigma^d A.$$

We decompose B_d into a direct sum of indecomposable objects, i.e., $A_d = \bigoplus_{k=1}^n B_k$. Thus $A_\bullet$ can be written as

$$A_\bullet:\ \ A \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-2}} A_{d-1} \xrightarrow{\alpha_{d-1}} \bigoplus_{k=1}^{n} B_k \xrightarrow{(b_1,b_2,\ldots,b_n)} X \xrightarrow{h} \Sigma^d A$$

where $b_k \in \mathrm{rad}_{\mathcal{C}}(B_k, X)$, $k = 1, 2, \ldots, n$.

Since $\mathcal{C}$ is locally finite, there are only finitely many objects $X_i \in \mathrm{ind}(\mathcal{C})$, $i = 1, 2, \ldots, m$ such that $\mathrm{Hom}_{\mathcal{C}}(X_i, X) \neq 0$. We assume that λ_{ij} $(1 \leq j \leq q_i)$ form a basis of the k-vector space $\mathrm{rad}_{\mathcal{C}}(B_k, X)$. Put $M := \big(\bigoplus_{k=1}^n B_k\big) \oplus \big(\bigoplus_{i=1}^n (X_i)^{\oplus q_i}\big)$, considering the morphism

$$\delta := (b_1, b_2, \ldots, b_n, \lambda_{11}, \ldots, \lambda_{ij}, \ldots, \lambda_{mq_m}) \in \mathrm{rad}_{\mathcal{C}}(M, X)$$

which is not a retraction, it can be embedded in a $(d+2)$-angle:

$$M_\bullet:\ \ B \to M_1 \to M_2 \to \cdots \to M_{d-1} \to M \xrightarrow{\delta} X \to \Sigma^d B.$$

Thus $M_\bullet$ is non-split since δ is not a retraction. Assume that $B = U \oplus V$ where U and V are indecomposable objects. By Lemma 2.4, for the morphism $(1,0)\colon U \oplus V \to U$, there exists the following commutative diagram of $(d+2)$-angles

$$\begin{array}{ccccccccccccc}
U\oplus V & \xrightarrow{(u,v)} & M_1 & \longrightarrow & M_2 & \longrightarrow & \cdots & \longrightarrow & M & \longrightarrow & X & \longrightarrow & \Sigma^d U\oplus\Sigma^d V \\
\downarrow{\scriptstyle (1,0)} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow & & & & \downarrow & & \| & & \downarrow{\scriptstyle (1,0)} \\
U & \xrightarrow{\theta_0} & L_1 & \longrightarrow & L_2 & \longrightarrow & \cdots & \longrightarrow & L_d & \longrightarrow & X & \xrightarrow{f} & \Sigma^d U.
\end{array}$$

Similarly, for the morphism $(0,1)\colon U\oplus V\to V$, there exists the following commutative diagram of $(d+2)$-angles

$$\begin{array}{ccccccccccccc}
U\oplus V & \xrightarrow{(u,v)} & M_1 & \longrightarrow & M_2 & \longrightarrow & \cdots & \longrightarrow & M & \longrightarrow & X & \longrightarrow & \Sigma^d U\oplus\Sigma^d V \\
\downarrow{\scriptstyle (0,1)} & & \downarrow{\scriptstyle \psi_1} & & \downarrow & & & & \downarrow & & \| & & \downarrow{\scriptstyle (0,1)} \\
V & \xrightarrow{\eta_0} & L_1 & \longrightarrow & L_2 & \longrightarrow & \cdots & \longrightarrow & L_d & \longrightarrow & X & \longrightarrow & \Sigma^d V.
\end{array}$$

By the similar arguments as in the proof of Lemma 3.3, we conclude that at least one of the following two $(d+2)$-angles

$$U\xrightarrow{\theta_0}L_1\longrightarrow L_2\longrightarrow\cdots\longrightarrow L_d\longrightarrow X\xrightarrow{f}\Sigma^d U,$$

$$V\xrightarrow{\eta_0}N_1\longrightarrow N_2\longrightarrow\cdots\longrightarrow N_d\longrightarrow X\longrightarrow\Sigma^d V$$

is non-split. Without loss of generality, we assume that

$$U\xrightarrow{\theta_0}L_1\longrightarrow L_2\longrightarrow\cdots\longrightarrow L_d\longrightarrow X\xrightarrow{f}\Sigma^d U$$

is non-split. By Lemma 2.9, we can find a non-split $(d+2)$-angle

$$C_\bullet:\ \ U\xrightarrow{\omega_0}C_1\xrightarrow{\omega_1}C_2\xrightarrow{\omega_2}\cdots\xrightarrow{\omega_{d-1}}C_d\xrightarrow{\omega_d}X\xrightarrow{f}\Sigma^d U$$

with $\omega_1,\omega_2,\dots,\omega_{d-1}$ in $\mathrm{rad}_{\mathcal{C}}$. Then $C_\bullet\in S(X)$. By (N3), we have the following commutative diagram

$$\begin{array}{ccccccccccccc}
U\oplus V & \xrightarrow{(u,v)} & M_1 & \longrightarrow & M_2 & \longrightarrow & \cdots & \longrightarrow & M & \xrightarrow{\delta} & X & \longrightarrow & \Sigma^d U\oplus\Sigma^d V \\
\downarrow{\scriptstyle (1,0)} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow & & & & \downarrow & & \| & & \downarrow{\scriptstyle (1,0)} \\
U & \xrightarrow{\delta_0} & L_1 & \longrightarrow & L_2 & \longrightarrow & \cdots & \longrightarrow & L_d & \longrightarrow & X & \xrightarrow{f} & \Sigma^d U \\
\| & & \downarrow & & \downarrow & & & & \downarrow & & \| & & \| \\
U & \xrightarrow{\omega_0} & C_1 & \xrightarrow{\omega_1} & C_2 & \xrightarrow{\omega_2} & \cdots & \xrightarrow{\omega_{d-1}} & C_d & \xrightarrow{\omega_d} & X & \xrightarrow{f} & \Sigma^d U
\end{array}$$

of $(d+2)$-angles.

For any $D_\bullet\in S(X)$, it can be written as

$$D_\bullet:\ \ D\to D_1\to D_2\to\cdots\to D_{d-1}\to\bigoplus_{i=1}^{p}L_i\xrightarrow{d=(d_1,d_2,\dots,d_p)}X\to\Sigma^d D$$

with $d_i \in \mathrm{rad}_{\mathcal{C}}(L_i, X)$, $i = 1, 2, \dots, p$. Since $D_\bullet \in S(X)$ is non-split, we get that d is not a retraction which implies $d \in \mathrm{rad}_{\mathcal{C}}\left(\bigoplus_{i=1}^p L_i, X\right)$. By the definitions of λ_{ij} and δ, there exists a morphism $\rho\colon \bigoplus_{i=1}^p L_i \to M$ such that $d = \delta\rho$. By (N3), we have the following commutative diagram

$$\begin{array}{ccccccccccccccc}
D & \longrightarrow & D_1 & \longrightarrow & D_2 & \longrightarrow & \cdots & \longrightarrow & D_{d-1} & \longrightarrow & \bigoplus_{i=1}^p L_i & \xrightarrow{d} & X & \longrightarrow & \Sigma^d D \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow{\rho} & & \| & & \downarrow \\
B & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & \cdots & \longrightarrow & M_{d-1} & \longrightarrow & M & \xrightarrow{\delta} & X & \longrightarrow & \Sigma^d B
\end{array}$$

of $(d+2)$-angles, where $B = U \oplus V$. Thus we get the following commutative diagram

$$\begin{array}{ccccccccccccccc}
D & \longrightarrow & D_1 & \longrightarrow & D_2 & \longrightarrow & \cdots & \longrightarrow & D_{d-1} & \longrightarrow & \bigoplus_{i=1}^p L_i & \xrightarrow{d} & X & \longrightarrow & \Sigma^d D \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow{\rho} & & \| & & \downarrow \\
U & \xrightarrow{\omega_0} & C_1 & \xrightarrow{\omega_1} & C_2 & \xrightarrow{\omega_2} & \cdots & \xrightarrow{\omega_{d-2}} & C_{d-1} & \xrightarrow{\omega_{d-1}} & C_d & \xrightarrow{\omega_d} & X & \xrightarrow{f} & \Sigma^d U
\end{array}$$

of $(d+2)$-angles. This shows that $C_\bullet$ is a minimal element in $S(X)$. □

Remark 3.7. If there exists a minimal element in $S(X)$ or $T(X)$, then it is unique up to isomorphism by Lemma 2.9.

We are now in a position to prove our main result.

Theorem 3.8. *Let $\mathcal{C}$ be a locally finite $(d+2)$-angulated category. If $X \in \mathrm{ind}(\mathcal{C})$, then there are an Auslander-Reiten $(d+2)$-angle ending at X, and an Auslander-Reiten $(d+2)$-angle starting at X. In this case, we say that $\mathcal{C}$ has Auslander-Reiten $(d+2)$-angles.*

Proof. Since $X \in \mathrm{ind}(\mathcal{C})$, we know that $S(X)$ is non-empty by Lemma 3.3. By Lemma 3.6, there exists a $(d+2)$-angle

$$A_\bullet : \quad A \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} X \xrightarrow{\alpha_{d+1}} \Sigma^d A_0$$

which is a minimal element in $S(X)$. Since $A_\bullet \in S(X)$, we have that $\alpha_1, \alpha_2, \dots, \alpha_{d_{d-1}} \in \mathrm{rad}_{\mathcal{C}}$ and A is indecomposable. Then $\mathrm{End}(A)$ is local.

We need to prove that $A_\bullet$ is an Auslander-Reiten $(d+2)$-angle. By Lemma 2.8, it suffices to show that α_d is right almost split.

Assume that $g\colon M \to X$ is not a retraction. By the duality of Lemma 2.4, there exists the following commutative diagram of $(d+2)$-angles

$$\begin{array}{ccccccccccccc}
A & \xrightarrow{\gamma_0} & B_1 & \xrightarrow{\gamma_1} & B_2 & \xrightarrow{\gamma_2} & \cdots & \xrightarrow{\gamma_{d-1}} & B_d & \xrightarrow{\gamma_d} & M & \xrightarrow{\gamma_{d+1}} & \Sigma^d A \\
\| & & \downarrow{\psi_1} & & \downarrow{\psi_2} & & & & \downarrow{\psi_d} & & \downarrow{\gamma_d} & & \| \\
A & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{d-1}} & A_d & \xrightarrow{\alpha_d} & X & \xrightarrow{\alpha_{d+1}} & \Sigma^d A
\end{array}$$

such that

$$N_\bullet: \quad B_1 \to N_1 \to N_2 \to \cdots \to N_{d-1} \to M \oplus A_d \xrightarrow{(g,\alpha_d)} X \xrightarrow{h} \Sigma^d B_1$$

is a $(d+2)$-angle in $\mathcal{C}$, where $N_i = B_{i+1} \oplus A_i$, $i = 1, 2, \ldots, d-1$. Since g and α_d are not retractions, we have that (g, α_d) is also not a retraction by the similar arguments as in the proof of Lemma 3.5. That is, $N_\bullet$ is non-split.

Without loss of generality, we can assume that $B_1 = U \oplus V$ where U and V are indecomposable objects. By Lemma 2.4, for the morphism $(1,0)\colon U \oplus V \to U$, there exists the following commutative diagram of $(d+2)$-angles

$$\begin{array}{ccccccccccccc}
U \oplus V & \xrightarrow{(u,v)} & N_1 & \longrightarrow & N_2 & \longrightarrow & \cdots & \longrightarrow & M \oplus A_d & \longrightarrow & X & \longrightarrow & \Sigma^d U \oplus \Sigma^d V \\
\downarrow{\scriptstyle (1,0)} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow & & & & \downarrow & & \| & & \downarrow{\scriptstyle (1,0)} \\
U & \xrightarrow{\delta_0} & L_1 & \longrightarrow & L_2 & \longrightarrow & \cdots & \longrightarrow & L_d & \longrightarrow & X & \xrightarrow{f} & \Sigma^d U.
\end{array}$$

Similarly, for the morphism $(0,1)\colon U \oplus V \to V$, there exists the following commutative diagram of $(d+2)$-angles

$$\begin{array}{ccccccccccccc}
U \oplus V & \xrightarrow{(u,v)} & N_1 & \longrightarrow & N_2 & \longrightarrow & \cdots & \longrightarrow & M \oplus A_d & \longrightarrow & X & \longrightarrow & \Sigma^d U \oplus \Sigma^d V \\
\downarrow{\scriptstyle (0,1)} & & \downarrow{\scriptstyle \psi_1} & & \downarrow & & & & \downarrow & & \| & & \downarrow{\scriptstyle (0,1)} \\
V & \xrightarrow{\eta_0} & Q_1 & \longrightarrow & Q_2 & \longrightarrow & \cdots & \longrightarrow & Q_d & \longrightarrow & X & \longrightarrow & \Sigma^d V.
\end{array}$$

By the similar arguments as in the proof of Lemma 3.3, we conclude that at least one of the following two $(d+2)$-angles

$$U \xrightarrow{\delta_0} L_1 \longrightarrow L_2 \longrightarrow \cdots \longrightarrow L_d \longrightarrow X \xrightarrow{f} \Sigma^d U,$$
$$V \xrightarrow{\eta_0} Q_1 \longrightarrow Q_2 \longrightarrow \cdots \longrightarrow Q_d \longrightarrow X \longrightarrow \Sigma^d V$$

is non-split. Without loss of generality, we assume that

$$U \xrightarrow{\delta_0} L_1 \longrightarrow L_2 \longrightarrow \cdots \longrightarrow L_d \longrightarrow X \xrightarrow{f} \Sigma^d U$$

is non-split. By Lemma 2.9, we can find a non-split $(d+2)$-angle

$$C_\bullet: \quad U \xrightarrow{\lambda_0} C_1 \xrightarrow{\lambda_1} C_2 \xrightarrow{\lambda_2} \cdots \xrightarrow{\lambda_{d-1}} C_d \xrightarrow{\lambda_d} X \xrightarrow{f} \Sigma^d U$$

with $\lambda_1, \lambda_2, \ldots, \lambda_{d-1}$ in $\mathrm{rad}_{\mathcal{C}}$. By (N3), we have the following commutative diagram

$$\begin{array}{ccccccccccccc}
A & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{d-1}} & A_d & \xrightarrow{\alpha_d} & X & \xrightarrow{\alpha_{d+1}} & \Sigma^d A_0 \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow{\scriptstyle \binom{0}{1}} & & \| & & \downarrow \\
U\oplus V & \xrightarrow{(u,v)} & N_1 & \longrightarrow & N_2 & \longrightarrow & \cdots & \longrightarrow & M\oplus A_d & \xrightarrow{(g,\alpha_d)} & X & \longrightarrow & \Sigma^d U\oplus\Sigma^d V \\
\downarrow{\scriptstyle (1,0)} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow & & & & \downarrow & & \| & & \downarrow{\scriptstyle (1,0)} \\
U & \xrightarrow{\delta_0} & L_1 & \longrightarrow & L_2 & \longrightarrow & \cdots & \longrightarrow & L_d & \longrightarrow & X & \xrightarrow{f} & \Sigma^d U \\
\| & & \downarrow & & \downarrow & & & & \downarrow & & \| & & \| \\
U & \xrightarrow{\lambda_0} & C_1 & \xrightarrow{\lambda_1} & C_2 & \xrightarrow{\lambda_2} & \cdots & \xrightarrow{\lambda_{d-1}} & C_d & \xrightarrow{\lambda_d} & X & \xrightarrow{f} & \Sigma^d U
\end{array}$$

of $(d+2)$-angles. We obtain that $A_\bullet > C_\bullet$, which implies $A_\bullet \sim C_\bullet$, since $A_\bullet$ is the minimal element in $S(X)$. Thus there exists the following commutative diagram

$$\begin{array}{ccccccccccccc}
U & \xrightarrow{\lambda_0} & C_1 & \xrightarrow{\lambda_1} & C_2 & \xrightarrow{\lambda_2} & \cdots & \xrightarrow{\lambda_{d-1}} & C_d & \xrightarrow{\lambda_d} & X & \xrightarrow{f} & \Sigma^d U \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \| & & \downarrow \\
A & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{d-1}} & A_d & \xrightarrow{\alpha_d} & X & \xrightarrow{\alpha_{d+1}} & \Sigma^d A_0
\end{array}$$

of $(d+2)$-angles. Hence we get the following commutative diagram

$$\begin{array}{ccccccccccccc}
U\oplus V & \xrightarrow{(u,v)} & N_1 & \longrightarrow & N_2 & \longrightarrow & \cdots & \longrightarrow & M\oplus A_d & \xrightarrow{(g,\alpha_d)} & X & \longrightarrow & \Sigma^d U\oplus\Sigma^d V \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow{\scriptstyle (a,b)} & & \| & & \downarrow \\
A & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{d-1}} & A_d & \xrightarrow{\alpha_d} & X & \xrightarrow{\alpha_{d+1}} & \Sigma^d A_0
\end{array}$$

of $(d+2)$-angles. It follows that $g = a\alpha_d$. This shows that α_d is right almost split.

Similarly, we can show that if $X \in \mathrm{ind}(\mathcal{C})$, then there exists an Auslander-Reiten $(d+2)$-angle starting at X. This completes the proof. □

Remark 3.9. As a special case of Theorem 3.8 when $d = 1$, that is, if $\mathcal{C}$ is a locally finite triangulated category, then $\mathcal{C}$ has Auslander-Reiten triangles, see [17, 18].

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