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Radial Limits of Nonparametric PMC Surfaces with Intermediate Boundary Curvature

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Abstract. The influence of the geometry of the domain on the behavior of generalized solutions of Dirichlet problems for elliptic partial differential equations has been an important subject for over a century. We investigate the boundary behavior of variational solutions f of Dirichlet problems for prescribed mean curvature equations in a domain $\Omega \subset \mathbb{R}^2$ near a point $\mathcal{O} \in \partial\Omega$ under different assumptions about the curvature of $\partial\Omega$ on each side of $\mathcal{O}$. We prove that the radial limits at $\mathcal{O}$ of f exist under different assumptions about the Dirichlet boundary data ϕ, depending on the curvature properties of $\partial\Omega$ near $\mathcal{O}$.

1. Introduction

Let Ω be a bounded, locally Lipschitz domain in $\mathbb{R}^2$ and define $Nf = \nabla \cdot Tf = \operatorname{div}(Tf)$, where $f \in C^2(\Omega)$ and $Tf = \frac{\nabla f}{\sqrt{1+|\nabla f|^2}}$. Let $H \in C^{1,\lambda}(\overline{\Omega})$ for some $\lambda \in (0,1)$ and satisfy the condition
$$\left|\int_\Omega H\eta\,dx\right| \leq \frac{1}{2}\int_\Omega |D\eta|\,dx \quad \text{for all } \eta \in C_0^1(\Omega)$$
(e.g., see [14, (16.60)] and [15]). We wish to study the following

Dirichlet problem. *Let $\phi \in L^\infty(\partial\Omega)$. Find a function $f \in C^2(\Omega)\cap C^0(\overline{\Omega})$ which satisfies*

$$Nf = 2H \quad \text{in } \Omega, \tag{1.1}$$
$$f = \phi \quad \text{on } \partial\Omega. \tag{1.2}$$

If $\phi \in C^0(\partial\Omega)$ and a function $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$ exists which satisfies (1.1)–(1.2), this is a *classical solution* and, in this case, every appropriate *approximate solution* (e.g., Perron solutions, variational solution, viscosity solutions) will equal this classical solution. The geometry of Ω plays a critical role with regard to the existence of classical solutions when $\phi \in C^0(\partial\Omega)$. For some choices of domain Ω and boundary data ϕ, no classical

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solution of (1.1)–(1.2) exists; when $H \equiv 0$, much of the history (up to 1985) of this topic can be found in Nitsche's book [24] (e.g., §285, 403–418) and, for general H, one might consult [26]. (Appropriate "smallness of ϕ" conditions can imply the existence of classical solutions when Ω is not convex in the $H \equiv 0$ case (e.g., [24, §285 & §412] and [17, 25, 27, 28]) or when $\partial\Omega$ does not satisfy appropriate curvature conditions in the general case (e.g., [1, 16, 22]); however see [24, §411]. In [2], Bourni assumes $\partial\Omega$ and ϕ are ($C^{1,\alpha}$) smooth, ignores the geometry of Ω and characterizes the "graph" of a variational solution which may include portions of the boundary cylinder $\partial\Omega \times \mathbb{R}$; in comparison, we do not assume any regularity for our boundary data ϕ and focus on the closure in $\overline{\Omega} \times \mathbb{R}$ of the graph of f over Ω.)

We wish to investigate the effects of the geometry of Ω on the behavior of a variational solution f of (1.1)–(1.2) near a point $\mathcal{O} \in \partial\Omega$; $\partial\Omega$ might be smooth or have a corner at $\mathcal{O}$ or ϕ might be discontinuous at $\mathcal{O}$. For convenience, we assume $\mathcal{O} = (0, 0)$. In many cases, the approximate solution is unique since if $f, g \in C^2(\Omega)$ both satisfy (1.1) and $f = g$ almost everywhere on $\partial\Omega$, then $f = g$ in Ω (e.g., [12, Theorem 5.1]); see, for example, [7, 645-6]) for a discussion of when Perron and variational solutions exist.

Let α and β, $\alpha < \beta < \alpha + 2\pi$, be the angles which the tangent rays to $\partial\Omega$ at $\mathcal{O}$ make with the positive x-axis such that

$$\{(r\cos\theta, r\sin\theta) : 0 < r < \epsilon(\theta), \alpha < \theta < \beta\} \subset \Omega \cap B_\delta(\mathcal{O})$$

for some $\delta > 0$ and some function $\epsilon(\cdot)\colon (\alpha, \beta) \to (0, \delta)$, and $\beta - \alpha \in (0, 2\pi)$ is the size of the "corner" at $\mathcal{O}$ of $\partial\Omega$. Here and throughout this note, we adopt the sign convention that the curvature of $\partial\Omega$ is nonnegative when Ω is convex and we denote by $\Lambda(\mathbf{x})$ the curvature of $\partial\Omega$ at points $\mathbf{x} \in \partial\Omega$ at which $\partial\Omega$ is smooth. Our primary interest is in the existence and behavior of the radial limits at $\mathcal{O}$,

$$Rf(\theta) = \lim_{r \downarrow 0} f(r\cos\theta, r\sin\theta),$$

of a solution $f \in C^2(\Omega)$ of (1.1); the existence of radial limits when $H \equiv 0$ was established in [18] (see also [6, 8, 19]) and this was extended to general H in [7] (see also [9, 10, 21]). When Γ is a $C^{2,\lambda}$ open subset of $\partial\Omega$ for some $\lambda \in (0, 1)$, $\mathcal{O} \in \Gamma$, $H \equiv 0$ and f is a variational solution of (1.1)–(1.2), the following is known:

(i) If $\Lambda(\mathcal{O}) < 0$, then $Rf(\theta)$ exists for every $\theta \in [\alpha, \beta]$ no matter how badly discontinuous ϕ is at $\mathcal{O}$ (see [11, Theorem 1.1]);

(ii) If $\Lambda(\mathcal{O}) > 0$, then there exist $\phi \in L^\infty(\partial\Omega) \cap C^\infty(\partial\Omega \setminus \{\mathcal{O}\})$ such that $Rf(\theta)$ does not exist for any $\theta \in (\alpha, \beta)$ (i.e., [20]);

(iii) If $Rf(\theta)$ exists for some $\theta_0 \in [\alpha,\beta]$, then $Rf(\theta)$ exists for every $\theta \in (\alpha,\beta)$ (i.e., [9, Theorem 2]);

(iv) If $\Lambda(\mathcal{O}) = 0$, then the existence of $Rf(\theta)$ is unknown (but see [11, Theorem 1.1]);

(v) If ϕ is continuous at $\mathcal{O}$ or has a jump discontinuity at $\mathcal{O}$ and $\Lambda(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \Gamma$ near $\mathcal{O}$, then $Rf(\theta)$ exists for every $\theta \in [\alpha,\beta]$ (see [18, 19]).

The equivalent statements when $H \not\equiv 0$ are

(vi) If $\Lambda(\mathcal{O}) < -2|H(\mathcal{O})|$, then $Rf(\theta)$ exists for every $\theta \in [\alpha,\beta]$ no matter how badly discontinuous ϕ is at $\mathcal{O}$ (see [11, Theorem 1.1]);

(vii) If $\Lambda(\mathcal{O}) > 2|H(\mathcal{O})|$, then there exist $\phi \in L^\infty(\partial\Omega) \cap C^\infty(\partial\Omega \setminus \{\mathcal{O}\})$ such that $Rf(\theta)$ does not exist for any $\theta \in (\alpha,\beta)$ (i.e., [21, Theorem 3]);

(viii) If $Rf(\theta)$ exists for some $\theta_0 \in [\alpha,\beta]$, then $Rf(\theta)$ exists for every $\theta \in (\alpha,\beta)$ (i.e., [9, Theorem 2]);

(ix) If $-2|H(\mathcal{O})| \leq \Lambda(\mathcal{O}) < 2|H(\mathcal{O})|$, then the existence of $Rf(\theta)$ is unknown;

(x) If ϕ is continuous at $\mathcal{O}$ or has a jump discontinuity at $\mathcal{O}$ and $\Lambda(\mathbf{x}) \geq |H(\mathbf{x})|$ for all $\mathbf{x} \in \Gamma$ near $\mathcal{O}$, then $Rf(\theta)$ exists for every $\theta \in [\alpha,\beta]$ (see [7, 19]).

Our goals here are to determine what happens in case (ix) when $\partial\Omega$ is smooth near $\mathcal{O}$ and to determine the effects of the value of the curvature $\Lambda(\mathbf{x})$ on each side of $\mathcal{O}$ when $\partial\Omega$ is not smooth at $\mathcal{O}$ (i.e., $\beta - \alpha \neq \pi$).

2. Preliminaries and theorems

Let us assume that $B_\delta(\mathcal{O}) \cap \partial\Omega \setminus \{\mathcal{O}\}$ consists of two components, $\partial^-\Omega$ and $\partial^+\Omega$, which are smooth (i.e., $C^{2,\lambda}$ for some $\lambda \in (0,1)$) curves, $\partial^-\Omega$ is tangent to the ray $\theta = \alpha$ at $\mathcal{O}$ and $\partial^+\Omega$ is tangent to the ray $\theta = \beta$ at $\mathcal{O}$; here (r,θ) represents polar coordinates about $\mathcal{O}$ and $B_\delta(\mathcal{O}) = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x} - \mathcal{O}| < \delta\}$. We assume $\partial^-\Omega$ is an (open) subset of a $C^{2,\lambda}$-curve Σ^- which contains $\mathcal{O}$ as an interior point and $\partial^+\Omega$ is an (open) subset of a $C^{2,\lambda}$-curve Σ^+ which contains $\mathcal{O}$ as an interior point; if $\beta - \alpha = \pi$, we assume $\Sigma^- = \Sigma^+$ (see Figure 2.1).

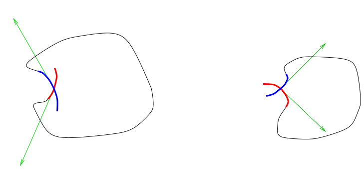

Figure 2.1: $\Sigma^\pm$ when $\beta - \alpha > \pi$ (left); $\Sigma^\pm$ when $\beta - \alpha < \pi$ (right).

Let $f \in BV(\Omega) \cap C^2(\Omega)$ minimize the functional

$$J(h) = \int_\Omega \sqrt{1+|Dh|^2} + \int_\Omega 2Hh\,d\mathbf{x} + \int_{\partial\Omega} |u-\phi|\,dH_1 \tag{2.1}$$

for $h \in BV(\Omega)$, so that f is the variational solution of (1.1)–(1.2). (Our focus is local (near $\mathcal{O}$); if Ω was not bounded, we would consider $f \in C^2(\Omega)$ to be a generalized variational solution or just argue as in [21].) Let $Rf(\theta)$ denote the radial limit of f at $\mathcal{O}$ in the direction $\theta \in (\alpha, \beta)$,

$$Rf(\theta) = \lim_{r\downarrow 0} f(r\cos\theta, r\sin\theta),$$

and set $Rf(\alpha) = \lim_{\partial^-\Omega\ni\mathbf{x}\to\mathcal{O}} f^*(\mathbf{x})$ and $Rf(\beta) = \lim_{\partial^+\Omega\ni\mathbf{x}\to\mathcal{O}} f^*(\mathbf{x})$ when these limits exist, where f^* denotes the trace of f on $\partial\Omega$. In [9] (together with [5]), the following two results were proven.

Proposition 2.1. (see [9, Theorem 1] and [5]) *Let $f \in C^2(\Omega) \cap L^\infty(\Omega)$ satisfy* (1.1) *and suppose $\beta - \alpha > \pi$. Then for each $\theta \in (\alpha, \beta)$, $Rf(\theta)$ exists and $Rf(\cdot)$ is a continuous function on (α, β) which behaves in one of the following ways:*

(i) *Rf is a constant function and all nontangential limits of f at $\mathcal{O}$ exist.*

(ii) *There exist $\alpha_1, \alpha_2 \in [\alpha, \beta]$ with $\alpha_1 < \alpha_2$ such that*

$$Rf(\theta) \text{ is } \begin{cases} \text{constant} & \text{for } \alpha < \theta \le \alpha_1, \\ \text{strictly monotonic} & \text{for } \alpha_1 \le \theta \le \alpha_2, \\ \text{constant} & \text{for } \alpha_2 \le \theta < \beta. \end{cases}$$

(iii) *There exist α_1, α_2 and θ_0 with $\alpha \le \alpha_1 < \theta_0 < \theta_0 + \pi < \alpha_2 \le \beta$ such that*

$$Rf(\theta) \text{ is } \begin{cases} \text{constant} & \text{for } \alpha < \theta \le \alpha_1, \\ \text{strictly increasing (decreasing)} & \text{for } \alpha_1 \le \theta \le \theta_0, \\ \text{constant} & \text{for } \theta_0 \le \theta \le \theta_0 + \pi, \\ \text{strictly decreasing (resp. increasing)} & \text{for } \theta_0 + \pi \le \theta \le \alpha_2, \\ \text{constant} & \text{for } \alpha_2 \le \theta < \beta. \end{cases}$$

Proposition 2.2. (see [9, Theorem 2] and [5]) *Let $f \in C^2(\Omega) \cap L^\infty(\Omega)$ satisfy* (1.1) *and suppose $m = \lim_{\partial^-\Omega\ni\mathbf{x}\to\mathcal{O}} f(\mathbf{x})$ exists. Then for each $\theta \in (\alpha, \beta)$, $Rf(\theta)$ exists and $Rf(\cdot)$ is a continuous function on $[\alpha, \beta)$, where $Rf(\alpha) := m$. If $\beta - \alpha \le \pi$, Rf can behave as in* (i) *or* (ii) *in Proposition* 2.1. *If $\beta - \alpha > \pi$, Rf can behave as in* (i), (ii) *or* (iii) *in Proposition* 2.1.

We shall prove

Theorem 2.3. *Let f be the variational solution of* (1.1)–(1.2)*. Suppose $\Gamma \subset \partial\Omega$ is a $C^{2,\lambda}$ (open) curve for some $\lambda \in (0,1)$, $\mathcal{O} \in \Gamma$, H is non-negative or non-positive in a neighborhood of $\mathcal{O}$ and $\Lambda(\mathcal{O}) < 2|H(\mathcal{O})|$. Then $Rf(\theta)$ exists for each $\theta \in (\alpha,\beta)$, $Rf \in C^0((\alpha,\beta))$ and Rf behaves as in* (i) *or* (ii) *in Proposition* 2.1*. Further, if $\Lambda(\mathcal{O}) < -2|H(\mathcal{O})|$, then $Rf(\alpha)$ and $Rf(\beta)$ both exist, $Rf \in C^0([\alpha,\beta])$, and, in case* (i) *in Proposition* 2.1*, $f \in C^0(\Omega \cup \{\mathcal{O}\})$.*

Example 2.4. Let $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1, x > 0\}$ and set $H = 1/2$ and $\phi(x,y) = \sin\left(\frac{\pi}{x^2+y^2}\right)$ for $(x,y) \neq \mathcal{O} = (0,0)$. Let $f \in C^2(\Omega)$ minimize (2.1) over $BV(\Omega)$. Then Theorem 2.3 implies that the radial limit $Rf(\theta)$ exists for each $\theta \in (-\pi/2,\pi/2)$ even though ϕ has no limit at $\mathcal{O}$. The symmetry of the problem then implies that Proposition 2.1(ii) cannot hold and so the radial limits are all the same and f has a nontangential limit at $\mathcal{O}$.

Theorem 2.5. *Let $f \in L^\infty(\Omega) \cap C^2(\Omega)$ minimize* (2.1) *over $BV(\Omega)$ (i.e., f is the variational solution of* (1.1)–(1.2)*). Suppose H is non-negative or non-positive in a neighborhood of $\mathcal{O}$,*

$$\limsup_{\partial^{\pm}\Omega \ni \mathbf{x} \to \mathcal{O}} \left(\Lambda(\mathbf{x}) - 2|H(\mathbf{x})|\right) < 0 \tag{2.2}$$

and, if $\beta - \alpha < \pi$, $m = \lim_{\partial^{-}\Omega \ni \mathbf{x} \to \mathcal{O}} f(\mathbf{x})$ exists. Then $Rf(\theta)$ exists for each $\theta \in (\alpha,\beta)$ and $Rf \in C^0(\alpha,\beta)$.

(a) *If $\beta - \alpha \leq \pi$, Rf behaves as in* (i) *or* (ii) *in Proposition* 2.1*.*

(b) *If $\beta - \alpha > \pi$, Rf behaves as in* (i), (ii) *or* (iii) *in Proposition* 2.1*.*

If, in addition, $\limsup_{\partial^{\pm}\Omega \ni \mathbf{x} \to \mathcal{O}}(\Lambda(\mathbf{x}) + 2|H(\mathbf{x})|) < 0$, then $Rf(\alpha)$ and $Rf(\beta)$ both exist, $Rf \in C^0([\alpha,\beta])$, and, in case (i) *in Proposition* 2.1*, $f \in C^0(\Omega \cup \{\mathcal{O}\})$.*

As noted previously (e.g., [9]), the "gliding hump" construction (which depends on the existence of classical solutions of (1.1)–(1.2)) cannot be successfully used when $\beta - \alpha > \pi$. When $\beta - \alpha < \pi$ and (2.2) holds, local barriers for (1.1)–(1.2) do not exist on $\partial^{\pm}\Omega$ and the "gliding hump" construction in [20] and [21, Theorem 3] cannot be directly used in Ω. One easily sees that this construction can be used to obtain a solution $g \in C^2(\Omega_0)$ of (1.1) such that none of the radial limits $Rg(\theta)$ of g at $\mathcal{O}$ exist whenever $\mathcal{O} \in \partial\Omega_0$ and Ω_0 is a domain for which the Dirichlet problem has local barriers at each point of $\partial\Omega_0$; let us assume as in [21, Theorem 3] that $H = 1/2$ and Ω_0 is the disk of radius 1 centered at $(1,0)$. If we rotate Ω about $\mathcal{O}$ so $\Omega \cap B_\delta(\mathcal{O}) \subset \Omega_0$ and $\beta = -\alpha < \pi/2$, define $\phi = g$ on $\partial(B_\delta(\mathcal{O}) \cap \Omega)$ and set $f = g$ in $\Omega \cap B_\delta(\mathcal{O})$, then $f \in C^2(\Omega \cap B_\delta(\mathcal{O}))$ satisfies (1.1),

$\phi \in L^\infty(\partial(B_\delta(\mathcal{O}) \cap \Omega)) \cap C^\infty(\partial(B_\delta(\mathcal{O}) \cap \Omega) \setminus \{\mathcal{O}\})$ and none of the radial limits $Rf(\theta)$, $\alpha \le \theta \le \beta$, of f at $\mathcal{O}$ exist (see Figure 2.2(a)). This shows the necessity of the assumption that $\lim_{\partial^-\Omega \ni \mathbf{x} \to \mathcal{O}} f(\mathbf{x})$ exists when $\beta - \alpha < \pi$, although if $Rf(\sigma)$ exists for any $\sigma \in (\alpha, \beta)$, we can split Ω into two pieces (see Figure 2.2(b)), apply [9, Theorem 2] twice and see that $Rf(\theta)$ exists for all $\theta \in (\alpha, \beta)$ (which justifies (iii) & (viii) in §1).

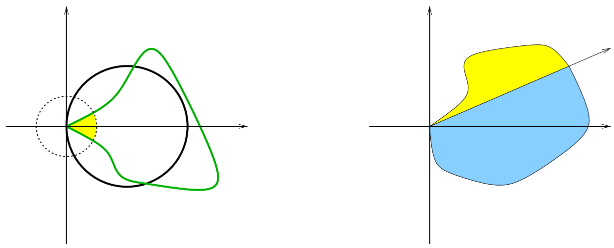

Figure 2.2: (a) $\Omega \cap B_\delta(\mathcal{O}) \subset \Omega_0$, (b) $Rf(\sigma)$ exists, $\sigma > 0$.

3. Proofs

Let Q be the operator on $C^2(\Omega)$ given by

$$Qf(\mathbf{x}) := Nf(\mathbf{x}) - 2H(\mathbf{x}), \quad \mathbf{x} \in \Omega. \tag{3.1}$$

Let ν be the exterior unit normal to $\partial\Omega$, defined almost everywhere on $\partial\Omega$. At every point $\mathbf{y} \in \partial\Omega$ for which $\partial\Omega$ is a C^1 curve in a neighborhood of $\mathbf{y}$, $\widehat{\nu}$ denotes a continuous extension of ν to a neighborhood of $\mathbf{y}$. Finally we adopt the convention used in [3, p. 178] with regard to the meaning of phrases like "$T\psi(\mathbf{y}) \cdot \nu(\mathbf{y}) = 1$ at a point $\mathbf{y} \in \partial\Omega$" and the notation, definitions and conventions used in [11], including upper and lower Bernstein pairs $(U^\pm, \psi^\pm)$, which we quote below.

Definition 3.1. Given a locally Lipschitz domain Ω, an *upper Bernstein pair* (U^+, ψ^+) for a curve $\Gamma \subset \partial\Omega$ and a function H in (3.1) is a domain U^+ and a function $\psi^+ \in C^2(U^+) \cap C^0(\overline{U^+})$ such that $\Gamma \subset \partial U^+$, ν is the exterior unit normal to ∂U^+ at each point of Γ (i.e., U^+ and Ω lie on the same side of Γ), $Q\psi^+ \le 0$ in U^+, and $T\psi^+ \cdot \nu = 1$ almost everywhere on Γ in the same sense as in [3]; that is, for almost every $\mathbf{y} \in \Gamma$,

$$\lim_{U^+ \ni \mathbf{x} \to \mathbf{y}} \frac{\nabla\psi^+(\mathbf{x}) \cdot \widehat{\nu}(\mathbf{x})}{\sqrt{1 + |\nabla\psi^+(\mathbf{x})|^2}} = 1.$$

Definition 3.2. Given a domain Ω as above, a *lower Bernstein pair* (U^-, ψ^-) for a curve $\Gamma \subset \partial\Omega$ and a function H in (3.1) is a domain U^- and a function $\psi^- \in C^2(U^-) \cap C^0(\overline{U^-})$ such that $\Gamma \subset \partial U^-$, ν is the exterior unit normal to ∂U^- at each point of Γ (i.e., U^- and Ω lie on the same side of Γ), $Q\psi^- \ge 0$ in U^-, and $T\psi^- \cdot \nu = -1$ almost everywhere on Γ (in the same sense as above).

The argument which establishes [14, Corollary 14.13], together with boundary regularity results (e.g., [2, 23]), are noted in [11, Remark 1] and imply the following

Lemma 3.3. *Suppose Δ is a $C^{2,\lambda}$ domain in $\mathbb{R}^2$ for some $\lambda \in (0,1)$, $\mathbf{y} \in \partial\Omega$ and $\Lambda(\mathbf{y}) < 2|H(\mathbf{y})|$, where $\Lambda(\mathbf{y})$ denotes the curvature of $\partial\Delta$ at $\mathbf{y}$. If H is non-negative in $U \cap \Omega$ for some neighborhood U of $\mathbf{y}$, then there exist $\tau > 0$ and an upper Bernstein pair (U^+, ψ^+) for (Γ, H), where $\Gamma = B_\tau(\mathbf{y}) \cap \partial\Omega$ and $U^+ = B_\tau(\mathbf{y}) \cap \Omega$. If H is non-positive in $U \cap \Omega$ for some neighborhood U of $\mathbf{y}$, then there exist $\tau > 0$ and a lower Bernstein pair (U^-, ψ^-) for (Γ, H), where $\Gamma = B_\tau(\mathbf{y}) \cap \partial\Omega$ and $U^- = B_\tau(\mathbf{y}) \cap \Omega$.*

Proof of Theorem 2.3. The claims in the last sentence of the theorem follow from [11, Theorem 1.1]. Since the remainder of the conclusion of the theorem concerns interior radial limits, we may assume that $f \in C^0(\overline{\Omega} \setminus \{\mathcal{O}\})$ (i.e., $f \in C^2(\Omega)$ and, if necessary, we could replace Ω by a set $U \subset \Omega$ such that $\partial U \cap \partial\Omega = \{\mathcal{O}\}$, ∂U has the same tangent rays at $\mathcal{O}$ as does $\partial\Omega$ and the curvature Λ^* of ∂U satisfies $\Lambda^*(\mathcal{O}) < 2|H(\mathcal{O})|$).

Let $z_1 = \liminf_{\Omega \ni \mathbf{x} \to \mathcal{O}} f(\mathbf{x})$ and $z_2 = \limsup_{\Omega \ni \mathbf{x} \to \mathcal{O}} f(\mathbf{x})$; if $z_1 = z_2$, then Proposition 2.1(i) holds and thus we assume $z_1 < z_2$. Set $S_0 = \{(\mathbf{x}, f(\mathbf{x})) : \mathbf{x} \in \Omega\}$. Since f minimizes J in (2.1), we see that the area of S_0 is finite; let M_0 denote this area. For $\delta \in (0,1)$, set

$$p(\delta) = \sqrt{\frac{8\pi M_0}{\ln\left(\frac{1}{\delta}\right)}}.$$

Let $E = \{(u,v) : u^2 + v^2 < 1\}$. As in [6, 21], there is a parametric description of the surface S_0,

$$Y(u,v) = (a(u,v), b(u,v), c(u,v)) \in C^2(E : \mathbb{R}^3),$$

which has the following properties:

(a_1) Y is a diffeomorphism of E onto S_0.

(a_2) Set $G(u,v) = (a(u,v), b(u,v))$, $(u,v) \in E$. Then $G \in C^0(\overline{E} : \mathbb{R}^2)$.

(a_3) Set $\sigma(\mathcal{O}) = G^{-1}(\partial\Omega \setminus \{\mathcal{O}\})$; then $\sigma(\mathcal{O})$ is a connected (open) arc of ∂E and G maps $\sigma(\mathcal{O})$ onto $\partial\Omega \setminus \{\mathcal{O}\}$. We may assume the endpoints of $\sigma(\mathcal{O})$ are $\mathbf{o}_1$ and $\mathbf{o}_2$. (Note that $\mathbf{o}_1$ and $\mathbf{o}_2$ are not assumed to be distinct.)

(a_4) Y is conformal on E: $Y_u \cdot Y_v = 0$, $Y_u \cdot Y_u = Y_v \cdot Y_v$ on E.

(a_5) $\triangle Y := Y_{uu} + Y_{vv} = 2H(Y) Y_u \times Y_v$ on E.

Let $\zeta(\mathcal{O}) = \partial E \setminus \sigma(\mathcal{O})$; then $G(\zeta(\mathcal{O})) = \{\mathcal{O}\}$ and $\mathbf{o}_1$ and $\mathbf{o}_2$ are the endpoints of $\zeta(\mathcal{O})$.

Suppose first that $\mathbf{o}_1 \neq \mathbf{o}_2$. From the Courant-Lebesgue Lemma (e.g., Lemma 3.1 in [4]), we see that there exists $\rho = \rho(\delta, \mathbf{w}) \in \left(\delta, \sqrt{\delta}\right)$ such that the arc length $l_\rho = l_{\rho(\delta,\mathbf{w})}$

of $Y(C_{\rho(\delta,\mathbf{w})}(\mathbf{w}))$ is less than $p(\delta)$, for each $\delta \in (0,1)$ and $\mathbf{w} \in \partial E$; here $C_r(\mathbf{w}) = \{(u,v) \in E : |(u,v) - \mathbf{w}| = r\}$. Set $E_r(\mathbf{w}) = \{(u,v) \in E : |(u,v) - \mathbf{w}| < r\}$, $E'_r(\mathbf{w}) = G(E_r(\mathbf{w}))$ and $C'_r(\mathbf{w}) = G(C_r(\mathbf{w}))$. Choose $\delta_1 > 0$ such that $2\sqrt{\delta_1} < |\mathbf{o}_1 - \mathbf{o}_2|$. Let $\mathbf{w}_0 \in \zeta(\mathcal{O})$ be the "midpoint" of $\mathbf{o}_1$ and $\mathbf{o}_2$, so that $\sqrt{\delta_1} < |\mathbf{w}_0 - \mathbf{o}_1| = |\mathbf{w}_0 - \mathbf{o}_2|$. Set $\mathcal{C} = C'_{\rho(\delta_1,\mathbf{w}_0)}(\mathbf{w}_0)$; then $\{(\mathbf{x}, f(\mathbf{x})) : \mathbf{x} \in \mathcal{C}\}$ $(= Y(C_{\rho(\delta_1,\mathbf{w}_0)}(\mathbf{w}_0)))$ is a curve of finite length $l_{\rho(\delta_1,\mathbf{w}_0)}$ with endpoints $(\mathcal{O}, z_a)$ and $(\mathcal{O}, z_b)$ for some $z_a, z_b \in \mathbb{R}$. Notice, in particular, that the graph of f over $\mathcal{C}$ is either continuous at $\mathcal{O}$ (if $z_a = z_b$) or has a jump discontinuity at $\mathcal{O}$ (if $z_a \neq z_b$).

We may now argue as in [19]. Let $\Omega_0 = G(E_{\rho(\delta_1,\mathbf{w}_0)}(\mathbf{w}_0)) = E'_{\rho(\delta_1,\mathbf{w}_0)}(\mathbf{w}_0)$, so that $\partial\Omega_0 = \mathcal{C} \cup \{\mathcal{O}\}$. From the Courant-Lebesgue Lemma and the general comparison principle (see [12, Theorem 5.1]), we see that Y is uniformly continuous on $E_{\rho(\delta_1,\mathbf{w}_0)}(\mathbf{w}_0)$ and so extends to a continuous function on the closure of $E_{\rho(\delta_1,\mathbf{w}_0)}(\mathbf{w}_0)$. From Steps 2, 4 and 5 of [21] and with [5] replacing Step 3 of [21], we see that there exist $\alpha_0, \beta_0 \in [\alpha, \beta]$ with $\alpha_0 < \beta_0$ such that

$$\{r(\cos\theta, \sin\theta) : 0 < r < \epsilon_0(\theta), \alpha_0 < \theta < \beta_0\} \subset \Omega_0 \cap B_{\delta_0}(\mathcal{O})$$

for some function $\epsilon_0(\cdot)\colon (\alpha,\beta) \to (0,\delta_0)$ and the radial limits $Rf(\theta)$ of f at $\mathcal{O}$ exist for $\alpha_0 \leq \theta \leq \beta_0$. Since $\partial\Omega$ is $(C^{2,\lambda})$ smooth near $\mathcal{O}$, we have $\beta - \alpha = \pi$ and so $\beta_0 - \alpha_0 \leq \pi$. (We note that $z_a = z_b$ when $\mathbf{o}_1 \neq \mathbf{o}_2$ and $\beta_0 - \alpha_0 \leq \pi$ implies $f \in C^0(\overline{\Omega})$, a contradiction, and so $z_a \neq z_b$.) The existence of $Rf(\cdot)$ on (α,β) now follows from two applications of [9, Theorem 2], one in the domain $\{(r\cos\theta, r\sin\theta) \in \Omega : r > 0, (\alpha_0 + \beta_0)/2 < \theta < \beta\}$ and one in the domain $\{(r\cos\theta, r\sin\theta) \in \Omega : r > 0, \alpha < \theta < (\alpha_0 + \beta_0)/2\}$.

Suppose second that $\mathbf{o} = \mathbf{o}_1 = \mathbf{o}_2$ and $\zeta(\mathcal{O}) = \{\mathbf{o}\}$. Let us assume that H is non-negative in a neighborhood of $\mathcal{O}$; here $H(Y(u,v))$ means $H(a(u,v), b(u,v))$. From Lemma 3.3, we see that an upper Bernstein pair (U^+, ψ^+) for (Γ_1, H) exists, where $U^+ = \Omega \cap B_\tau(\mathcal{O})$ and $\Gamma_1 = \Gamma \cap B_\tau(\mathcal{O})$ for some $\tau > 0$; let q denote a modulus of continuity for ψ^+. Then $T\psi^+ \cdot \nu = +1$ (in the sense of [3]) on Γ_1 and, for each $C \in \mathbb{R}$, $Q(\psi^+ + C) = Q(\psi^+) \leq 0$ on $\Omega \cap U^+$ or equivalently

$$N(\psi^+ + C)(\mathbf{x}) \leq 2H(\mathbf{x}) = Nf(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega \cap U^+. \tag{3.2}$$

From the Courant-Lebesgue Lemma, we see that there exists $\rho = \rho(\delta, \mathbf{w}) \in \left(\delta, \sqrt{\delta}\right)$ such that the arc length $l_\rho = l_{\rho(\delta,\mathbf{w})}$ of $Y(C_{\rho(\delta,\mathbf{w})}(\mathbf{w}))$ is less than $p(\delta)$, for each $\delta \in (0,1)$ and $\mathbf{w} \in \partial E$.

Let us assume that $\delta \in (0,1)$ is small enough that $p(\delta) < \tau$, so that $G(\mathbf{w}) \in U^+$ for each $\mathbf{w} \in E$ with $|\mathbf{w} - \mathbf{o}| \leq \sqrt{\delta}$ and $G(\mathbf{w}) \in \Gamma_1$ for each $\mathbf{w} \in \partial E$ with $|\mathbf{w} - \mathbf{o}| \leq \sqrt{\delta}$. Now $\psi^+ - \psi^+(\mathbf{x}) \leq q(p(\delta))$ in $E'_{\rho(\delta,\mathbf{o})}(\mathbf{o})$ for any $\mathbf{x} \in E'_{\rho(\delta,\mathbf{o})}(\mathbf{o})$ and using (3.2) in conjunction with Finn's general comparison principle (see [12, Theorem 5.1]) implies that if $U \subset E'_{\rho(\delta,\mathbf{o})}(\mathbf{o})$ is an open set, then

$$f \leq \sup_{\Omega\cap\partial U} f + \psi^+ - \inf_{\Omega\cap\partial U} \psi^+ \leq \sup_{\Omega\cap\partial U} f + q(p(\delta)) \quad \text{in } U. \tag{3.3}$$

Set

$$k(\delta) = \inf_{\mathbf{u}\in C_{\rho(\delta,\mathbf{o})}(\mathbf{o})} c(\mathbf{u}) = \inf_{\mathbf{x}\in C'_{\rho(\delta,\mathbf{o})}(\mathbf{o})} f(\mathbf{x}).$$

Now $f \le k(\delta) + p(\delta)$ on $C'_{\rho(\delta,\mathbf{o})}(\mathbf{o})$ and $\psi^+ - \inf_{C'_{\rho(\delta,\mathbf{o})}(\mathbf{o})} \psi^+ \le q(p(\delta))$ in $E'_{\rho(\delta,\mathbf{o})}(\mathbf{o})$ and so (3.3) implies

$$f \le k(\delta) + p(\delta) + \psi^+ - \inf_{C'_{\rho(\delta,\mathbf{o})}(\mathbf{o})} \psi^+ \le k(\delta) + p(\delta) + q(p(\delta))$$

or

$$\sup_{E'_{\rho(\delta,\mathbf{o})}(\mathbf{o})} f \le \inf_{C'_{\rho(\delta,\mathbf{o})}(\mathbf{o})} f + p(\delta) + q(p(\delta)).$$

Since $\sup_{E'_{\rho(\delta,\mathbf{o})}(\mathbf{o}))} f \ge z_2$,

$$\inf_{C'_{\rho(\delta,\mathbf{o})}(\mathbf{o})} f \ge z_2 - p(\delta) - q(p(\delta)) = z_2 - o(\delta) \quad \text{for each } \delta > 0. \tag{3.4}$$

Let $z(\delta) = z_2 - 2p(\delta) - q(p(\delta))$ and

$$M(\delta) = \{\mathbf{x} \in E'_{\rho(\delta,\mathbf{o})}(\mathbf{o}) : f(\mathbf{x}) > z(\delta)\}.$$

(Recall $f \in C^0(\overline{\Omega}\setminus\{\mathcal{O}\})$ and $c \in C^0(\overline{E}\setminus\{\mathbf{o}\})$.) Then for each $\delta \in (0, p^{-1}(\tau))$, (3.4) implies $f \ge z_2 - p(\delta) - q(p(\delta)) > z(\delta)$ on $C'_{\rho(\delta,\mathbf{o})}(\mathbf{o})$ and so

$$C'_{\rho(\delta,\mathbf{o})}(\mathbf{o}) \subset M(\delta) \quad \text{and} \quad \mathcal{O} \in \overline{M(\delta)}.$$

Let $V(\delta)$ denote the component of $M(\delta)$ which contains $C'_{\rho(\delta,\mathbf{o})}(\mathbf{o})$. We claim that $\mathcal{O} \in \overline{V(\delta)}$. Suppose otherwise; then there is a curve $\mathcal{I}$ in $E'_{\rho(\delta,\mathbf{o})}(\mathbf{o})$ (with endpoints $\mathbf{x}^- \in \partial^-\Omega$ and $\mathbf{x}^+ \in \partial^+\Omega$) such that $f \le z(\delta)$ on $\mathcal{I}$. Let $\Omega(\mathcal{I})$ be the component of $\Omega \setminus \mathcal{I}$ whose closure contains $\mathcal{O}$. Then (3.3) implies that

$$f \le \sup_{\mathcal{I}} f + q(p(\delta)) \le z(\delta) + q(p(\delta)) = z_2 - 2p(\delta) \quad \text{in } \Omega(\mathcal{I})$$

and so $\limsup_{E\ni\mathbf{w}\to\mathbf{o}} c(\mathbf{w}) \le z_2 - 2p(\delta) < z_2$, which is a contradiction; hence no such curve $\mathcal{I}$ exists and $\mathcal{O} \in \overline{V(\delta)}$.

Now $f \ge z(\delta)$ in $V(\delta)$ for each $\delta \in (0, p^{-1}(\tau))$. Let $\mathcal{C}$ be any curve in Ω which starts at a point $\mathbf{x}_0 \in C'_{\rho(p^{-1}(\tau),\mathbf{o})}(\mathbf{o})$ and ends at $\mathcal{O}$ such that

$$\mathcal{C} \subset V(\delta) \quad \text{for each } \delta \in (0, p^{-1}(\tau)).$$

Since $\liminf_{\mathcal{C}\ni\mathbf{x}\to\mathcal{O}} f(\mathbf{x}) \ge \lim_{\delta\downarrow 0} z(\delta) = z_2$ and $z_2 = \limsup_{\Omega\ni\mathbf{x}\to\mathcal{O}} f(\mathbf{x})$, we see that

$$\lim_{\mathcal{C}\ni\mathbf{x}\to\mathcal{O}} f(\mathbf{x}) = z_2.$$

We may, if we wish, extend $\mathcal{C}$ by adding to $\mathcal{C}$ a curve from $\mathbf{x}_0$ to a point on $\partial\Omega \setminus \overline{E'_{\rho(p^{-1}(\tau),\mathbf{o})}(\mathbf{o})}$.

Now we modify the argument in the proof of [9, Theorem 2] to show that $Rf(\theta) = z_2$ for all $\theta \in (\alpha, \beta)$; that is, we shall show that the nontangential limit of f at $\mathcal{O}$ exists and equals z_2. Let $\alpha', \beta' \in (\alpha, \beta)$ with $\alpha' < \beta'$.

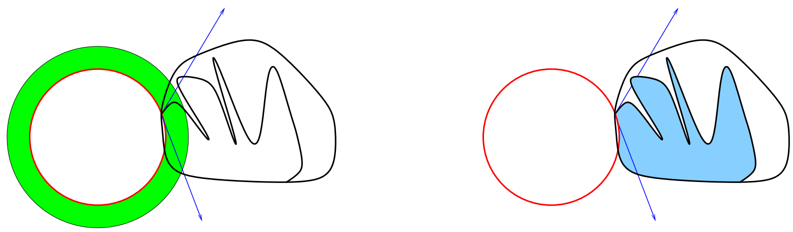

Figure 3.1: Ω, $\mathcal{A}_-$ and $\mathcal{C}$ (left); Ω_2 (right).

Let $H_0 = \sup_{B_{\delta_0}(\mathcal{O})\cap\Omega} H$ and fix $c_0 \in \left(-\frac{1}{4c_0H_0}, 0\right)$. Set $r_1 = \frac{1-\sqrt{1+4c_0H_0}}{2H_0}$ and $r_2 = \frac{1+\sqrt{1+4c_0H_0}}{2H_0}$ (see [21, p. 171], [13]). Let $\mathcal{A}_\pm$ be annuli with inner boundaries $\partial_1\mathcal{A}_\pm$ with equal radii r_1 and outer boundaries $\partial_2\mathcal{A}_\pm$ with equal radii r_2 such that $\mathcal{O} \in \partial_1\mathcal{A}_+ \cap \partial_1\mathcal{A}_-$, $\partial_1\mathcal{A}_+$ is tangent to the ray $\theta = \beta'$ at $\mathcal{O}$, $\partial_1\mathcal{A}_-$ is tangent to the ray $\theta = \alpha'$ at $\mathcal{O}$ and $\partial_1\mathcal{A}_\pm \cap \{(r\cos\theta, r\sin\theta) : 0 < r < \delta_0, \alpha' < \theta < \beta'\} = \emptyset$ (see Figure 3.1). Let $h_\pm = h(\widehat{r}_\pm)$ denote unduloid surfaces defined respectively on $\mathcal{A}_\pm$ with constant mean curvature $-H_0$ which become vertical at $\widehat{r}_\pm = r_1, r_2$ and make contact angles of π and 0 with the vertical cylinders $\widehat{r}_\pm = r_2$ and $\widehat{r}_\pm = r_1$ respectively, where $\widehat{r}_+(\mathbf{x}) = |\mathbf{x} - \mathbf{c}_+|$, $\widehat{r}_-(\mathbf{x}) = |\mathbf{x} - \mathbf{c}_-|$, $\mathbf{c}_+$ denotes the center of the annulus $\mathcal{A}_+$ and $\mathbf{c}_-$ denotes the center of the annulus $\mathcal{A}_-$. With respect to the upward direction, the graphs of $h_\pm$ over $\mathcal{A}_\pm$ have constant mean curvature $-H_0$ and the graphs of $-h_\pm$ over $\mathcal{A}_\pm$ have constant mean curvature H_0.

Set $\tau_1 = \min\{\tau, r_2 - r_1\}$. Let $\delta \in (0, p^{-1}(\tau_1))$. Since $\mathcal{C}$ is a curve in Ω with $\mathcal{O}$ as an endpoint, there exists $\mathbf{x}(\delta) \in \mathcal{C} \cap C'_{\rho(\delta,\mathbf{o})}(\mathbf{o})$ such that the portion $\mathcal{C}(\delta)$ of $\mathcal{C}$ between $\mathcal{O}$ and $\mathbf{x}(\delta)$ lies in $E'_{\rho(\delta,\mathbf{o})}(\mathbf{o})$ and divides $E'_{\rho(\delta,\mathbf{o})}(\mathbf{o})$ into two components. Let U_+ be the component of $E'_{\rho(\delta,\mathbf{o})}(\mathbf{o}) \setminus \mathcal{C}(\delta)$ whose closure contains a portion of $\partial^+\Omega$ and U_- be the component of $E'_{\rho(\delta,\mathbf{o})}(\mathbf{o}) \setminus \mathcal{C}(\delta)$ whose closure contains a portion of $\partial^-\Omega$ (see Figure 3.2 with $C'_{\rho(\delta,\mathbf{o})}(\mathbf{o})$ (green) and $\mathcal{C}$ (red)).

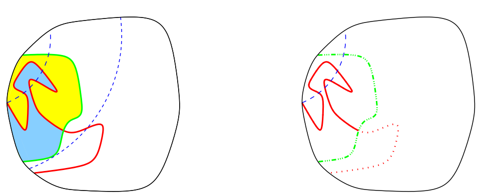

Figure 3.2: Left: U_+ (yellow), U_- (blue); Right: $\mathcal{C}(\delta)$ (red).

Since $\mathcal{C}(\delta) \subset V(\delta)$,

$$f(\mathbf{x}) \geq z(\delta) \quad \text{for } \mathbf{x} \in \mathcal{C}(\delta)$$

and, in particular, $f(\mathbf{x}(\delta)) \geq z(\delta)$. Since $|f(\mathbf{x}(\delta)) - f(\mathbf{y})| \leq l_{\rho(\delta,\mathbf{o})} < p(\delta)$ for $\mathbf{y} \in C'_{\rho(\delta,\mathbf{o})}(\mathbf{o})$, we see that

$$f \geq z(\delta) - p(\delta) \quad \text{on } C'_{\rho(\delta,\mathbf{o})}(\mathbf{o}) \cup \mathcal{C}(\delta).$$

Let q_2 denote a modulus of continuity of $-h(\widehat{r}_+)$. Then

$$f \geq z(\delta) - p(\delta) - q_2(p(\delta)) \quad \text{in } U_+ \setminus \overline{B_{r_1}(c_+)}.$$

Thus

$$\liminf_{U_+ \setminus \overline{B_{r_1}(c_+)} \ni \mathbf{x} \to \mathcal{O}} f(\mathbf{x}) \geq z_2.$$

If we set $\Omega_1 = U_+ \setminus \overline{B_{r_1}(\mathbf{c}_+)}$ and recall that $z_2 = \limsup_{\Omega \ni \mathbf{x} \to \mathcal{O}} f(\mathbf{x})$, we have

$$\lim_{\Omega_1 \ni \mathbf{x} \to \mathcal{O}} f(\mathbf{x}) = z_2. \tag{3.5}$$

(We note that Ω_1 might not be connected (see Figure 3.3) and might even have an infinite number of components but one sees that this does not affect the comparison argument which establishes (3.5).)

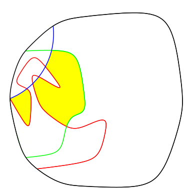

Figure 3.3: Ω and Ω_1.

In a similar manner, we see that

$$\lim_{\Omega_2 \ni \mathbf{x} \to \mathcal{O}} f(\mathbf{x}) = z_2,$$

where $\Omega_2 = U_- \setminus \overline{B_{r_1}(\mathbf{c}_-)}$. Since $\Omega_1 \cup \Omega_2 \cup \mathcal{C}(\delta) = E'_{\rho(\delta,\mathbf{o})}(\mathbf{o}) \setminus \left(\overline{B_{r_1}(\mathbf{c}_+) \cup B_{r_1}(\mathbf{c}_-)}\right)$, we see that $Rf(\theta) = z_2$ for each $\theta \in (\alpha', \beta')$. Since α' and β' are arbitrary (with $\alpha < \alpha' < \beta' < \beta$), Theorem 2.3 is proven. $\square$

Proof of Theorem 2.5. All of the claims in the theorem except those in the last sentence follow from [9, Theorem 1] and [5] (when $\beta - \alpha > \pi$) and [9, Theorem 2] and [5] (when $\beta - \alpha < \pi$). (When $\beta - \alpha = \pi$, all of the claims follow from Theorem 2.3 and [11].)

The claims follow once we prove that the results of [11] hold under the assumptions of Theorem 2.5. Let us assume

$$\limsup_{\partial^{\pm}\Omega\ni\mathbf{x}\to\mathcal{O}} \left(\Lambda(\mathbf{x}) + 2|H(\mathbf{x})|\right) < 0. \tag{3.6}$$

Suppose $\beta - \alpha > \pi$. Let $\delta_1 > 0$ be small enough that $B_{\delta_1}(\mathcal{O}) \cap \Omega \setminus \Sigma^+$ has two components. Let Ω_+ be the component whose closure contains $B_{\delta_1}(\mathcal{O}) \cap \partial^+\Omega$ (see Figure 3.4 (left)) and notice that the tangent directions to $\partial\Omega_+$ at $\mathcal{O}$ are $\alpha' = \beta - \pi$ and β and the curvature $\Lambda_+(\mathcal{O})$ of $\partial\Omega_+$ at $\mathcal{O}$ satisfies

$$\Lambda_+(\mathcal{O}) < -2H(\mathcal{O})$$

since $\Lambda_+(\mathbf{x}) = \Lambda(\mathbf{x})$ for $\mathbf{x} \in B_{\delta_1}(\mathcal{O}) \cap \partial^+\Omega$ and (3.6) implies

$$\Lambda_+(\mathcal{O}) = \limsup_{\partial^{+}\Omega\ni\mathbf{x}\to\mathcal{O}} \Lambda_+(\mathbf{x}) < -2|H(\mathcal{O})|.$$

By restricting f to Ω_+, we see that the existence of $Rf(\beta)$ follows from [11]. A similar argument implies $Rf(\alpha)$ also exists.

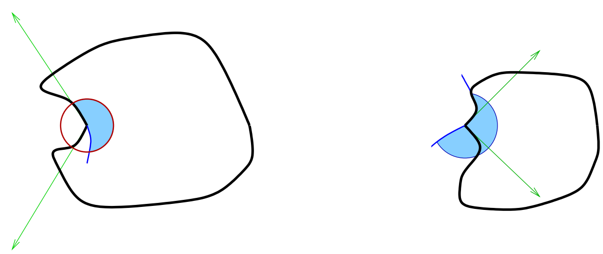

Figure 3.4: Ω_+ when $\beta - \alpha > \pi$ (left); Ω_+ when $\beta - \alpha < \pi$ (right).

Suppose $\beta - \alpha < \pi$. Then $Rf(\alpha)$ exists and equals m. Let $\delta_1 > 0$ be small enough that $B_{\delta_1}(\mathcal{O}) \setminus \Sigma^+$ has two components and let Ω_+ be the component which contains $B_{\delta_1}(\mathcal{O}) \cap \Omega$ (see Figure 3.4 (right)). Then the tangent directions to $\partial\Omega_+$ at $\mathcal{O}$ are $\alpha' = \beta - \pi$ and β and, as before, the curvature $\Lambda_+(\mathcal{O})$ of $\partial\Omega_+$ at $\mathcal{O}$ satisfies $\Lambda_+(\mathcal{O}) < -2H(\mathcal{O})$. Thus upper and lower Bernstein pairs $(U^{\pm}, \psi^{\pm})$ exist for $\Gamma = B_{\delta_2}(\mathcal{O}) \cap \partial\Omega_+$ and H when $\delta_2 \in (0, \delta_1)$ is sufficiently small and $U^{\pm} = B_{\delta_2}(\mathcal{O}) \cap \Omega_+$. We may parametrize $S_1 = S_0 \cap (B_{\delta_2}(\mathcal{O}) \times \mathbb{R})$ in isothermal coordinates

$$Y(u, v) = (a(u, v), b(u, v), c(u, v)) \in C^2(E : S_1)$$

as in [11] with the properties noted there (e.g., $a_1, \ldots, a_5$) and prove in essentially the same manner as in [11] that Y is uniformly continuous on E and so extends to a continuous function on $\overline{E}$. The existence of $Rf(\beta)$ then follows as in [11]. $\square$

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