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The Nonlinear Steepest Descent Approach for Long Time Behavior of the Two-component Coupled Sasa-Satsuma Equation with a 5×5 Lax Pair

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Abstract. Under investigation in this work is the coupled Sasa-Satsuma equation, which can describe the propagations of two optical pulse envelopes in birefringent fibers. The Riemann-Hilbert problem for the equation is formulated on the basis of the corresponding 5×5 matrix spectral problem, which allows us to present a suitable representation for the solution of the equation. Then the Deift-Zhou steepest descent method is used to analyze the long time behavior of the coupled Sasa-Satsuma equation.

1. Introduction

It is well-known that the standard nonlinear Schrödinger (NLS) equation is a key integrable system in the field of mathematical physics. There are many physical phenomenon where the NLS equation appears. For instance, the NLS equation describes slowly varying wave envelopes in dispersive media from water waves, nonlinear optics, and plasma physics. In particular, the NLS equation can be used to model the soliton propagation in optical fibers where only the self-phase modulation effects and the group velocity dispersion are discussed. However, for ultrashort pulse in optical fibers, the effects of the self steepening, the third-order dispersion, and the stimulated Raman scattering should be taken into account. Because of these effects, the dynamic behaviors of the ultrashort pulses can be described by the higher-order NLS equation (also called Sasa-Satsuma equation) [25, 26, 39, 43, 45]

$$q_T + \frac{1}{2}q_{XX} + |q|^2 q + i\epsilon\{q_{XXX} + 6|q|^2 q_X + 3q(|q|^2)_X\} = 0,$$

where $q = q(X,T)$ is a complex-valued function. In addition, to model the propagations of two optical pulse envelopes in birefringent fibers well, some coupled Sasa-Satsuma equations were proposed and discussed [23, 24, 30, 38]. In this work, we therefore focus on

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a coupled Sasa-Satsuma (CSS in brief) equation

$$\begin{aligned}
&iq_{1T} + \frac{1}{2}q_{1XX} + (|q_1|^2 + |q_2|^2)q_1 \\
&+ i\epsilon\big\{q_{1XXX} + 6(|q_1|^2 + |q_2|^2)q_{1X} + 3q_1(|q_1|^2 + |q_2|^2)_X\big\} = 0, \\
&iq_{2T} + \frac{1}{2}q_{2XX} + (|q_1|^2 + |q_2|^2)q_2 \\
&+ i\epsilon\big\{q_{2XXX} + 6(|q_1|^2 + |q_2|^2)q_{2X} + 3q_2(|q_1|^2 + |q_2|^2)_X\big\} = 0,
\end{aligned} \tag{1.1}$$

which can be rewritten in the following form [24]

$$\begin{aligned}
&u_t + \epsilon\big\{u_{xxx} + 6(|u|^2 + |v|^2)u_x + 3u(|u|^2 + |v|^2)_x\big\} = 0, \\
&v_t + \epsilon\big\{v_{xxx} + 6(|u|^2 + |v|^2)v_x + 3v(|u|^2 + |v|^2)_x\big\} = 0, \\
&u(x,0) = u_0(x), \quad v(x,0) = v_0(x),
\end{aligned} \tag{1.2}$$

by introducing the gauge, Galilean and scale transformations

$$\begin{aligned}
u(x,t) &= q_1(X,T)\exp\left[-\frac{i}{6\epsilon}\left(X - \frac{T}{18\epsilon}\right)\right], \\
v(x,t) &= q_2(X,T)\exp\left[-\frac{i}{6\epsilon}\left(X - \frac{T}{18\epsilon}\right)\right], \\
x &= X - \frac{T}{12\epsilon}, \quad t = T,
\end{aligned}$$

where (u_0, v_0) lie in the Schwartz space, ϵ is the ratio of the width of the spectra to the carrier frequency, and the last three terms in the left-hand side of (1.1) stand for the third-order dispersion, self-steepening, and stimulated Raman scattering effects, respectively. Besides, $q_1 = q_1(X,T)$ and $q_2 = q_2(X,T)$ are two complex functions of variables X, T, The CSS equation (1.2) is still completely integrable. Additionally, the CSS equation (1.2) has also been investigated via Darboux transformation, Darboux-Bäcklund transformation and Hirota method etc. Recently, we have studied the long-time behavior and rogue wave solutions of the integrable three-component coupled nonlinear Schrödinger equation [33, 35]. In this paper, we will consider the long-time asymptotics of the CSS equation (1.2) on the line. In the following, we let $\epsilon = 1$ for the convenience of the analysis.

In recent years, there are many investigations on long time asymptotics and exact solutions of nonlinear evolution equations [8–10, 16, 19–21, 27, 28, 32, 34, 36, 41]. It is also known that the Deift-Zhou steepest descent approach is a powerful approach to analyze the long time behavior for integrable nonlinear wave equations [4–7, 11–14, 17, 18, 22, 29, 31, 40, 42]. However, since (1.2) contains a 5×5 matrix spectral problem, the long time asymptotics for (1.2) is rather complicated to consider. The research in this direction, to the best of our knowledge, has not been conducted before. The main purpose of the present article is to analyze the long time asymptotics of (1.2) by utilizing the Riemann-Hilbert problem (RHP) via the Deift-Zhou steepest descent method.

The structure of this paper is given as follows. In Section 2, we derive a 5×5 matrix RHP and find that the solution of (1.2) can be given by the solution of this RHP. In Section 3, we obtain the main conclusion of this work by using the Deift-Zhou steepest descent method. Finally, the last section summarizes the main results of this article.

2. Riemann-Hilbert problem

System (1.2) is still completely integrable. Its Lax pair yields [24]

$$\begin{aligned} \psi_x(x,t;\lambda) &= i\lambda\sigma\psi(x,t;\lambda) + \mathbf{U}(x,t;\lambda)\psi(x,t;\lambda), \\ \psi_t(x,t;\lambda) &= 4i\lambda^3\sigma\psi(x,t;\lambda) + \mathbf{V}(x,t;\lambda)\psi(x,t;\lambda), \end{aligned} \tag{2.1}$$

where

$$\sigma = \begin{pmatrix} \mathcal{I}_{4\times 4} & \mathbf{0} \\ \mathbf{0} & -1 \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} \mathbf{0}_{4\times 4} & \mathcal{U} \\ -\mathcal{U}^\dagger & 0 \end{pmatrix}, \quad \mathcal{U} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u \\ \overline{u} \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v \\ \overline{v} \end{pmatrix}$$

with

$$\mathbf{V}(x,t,\lambda) = 4\lambda^2\mathbf{U} - 2i\lambda\sigma(\mathbf{U}_x - \mathbf{U}^2) + (\mathbf{U}_x\mathbf{U} - \mathbf{U}\mathbf{U}_x) - \mathbf{U}_{xx} + 2\mathbf{U}^3.$$

Here the overbar represents the complex conjugation and "†" represents Hermitian of a matrix.

In the following, introducing a new matrix function by

$$\psi(x,t;\lambda) = \mu(x,t;\lambda)e^{i(\lambda x + 4\lambda^3 t)\sigma},$$

the spectral problem (2.1) then gives

$$\begin{aligned} \mu_x(x,t;\lambda) - i\lambda[\sigma,\mu(x,t;\lambda)] &= \mathbf{U}(x,t)\mu(x,t;\lambda), \\ \mu_t(x,t;\lambda) - 4i\lambda^3[\sigma,\mu(x,t;\lambda)] &= \mathbf{V}(x,t;\lambda)\mu(x,t;\lambda). \end{aligned} \tag{2.2}$$

We next present two eigenfunctions $\mu_\pm(x,t;\lambda)$ of x-part of (2.2) by the following Volterra type integral equations

$$\mu_\pm = \mathcal{I} + \int_{\pm\infty}^{x} e^{-i\lambda(x-\xi)\widehat{\sigma}}[\mathbf{U}(\xi,t)\mu_\pm(\xi,t;\lambda)]\,d\xi, \tag{2.3}$$

where $\widehat{\sigma}$ represents the operators which act on a 5×5 matrix Ω by $\widehat{\sigma} = [\sigma,\Omega]$. Here $e^{\widehat{\sigma}} = e^{\sigma}\Omega e^{\sigma}$. Then we rewrite $\mu_\pm(x,t;\lambda)$ as

$$\mu_\pm(x,t;\lambda) = (\mu_{\pm L}(x,t;\lambda), \mu_{\pm R}(x,t;\lambda)),$$

where the first fourth columns of $\mu_\pm(x,t;\lambda)$ and fifth column are expressed by $\mu_{\pm L}(x,t;\lambda)$ and $\mu_{\pm R}(x,t;\lambda)$, respectively. From (2.3), we know that μ_{+L}, μ_{-R} and μ_{-L}, μ_{+R} are analytic in $\mathbb{C}_-$ and $\mathbb{C}_+$, respectively. Furthermore

$$\big(\mu_{+L}(x,t;\lambda),\mu_{-R}(x,t;\lambda)\big)=\mathcal{I}+O\left(\frac{1}{\lambda}\right),\quad \lambda\in\mathbb{C}_-\to\infty,$$
$$\big(\mu_{-L}(x,t;\lambda),\mu_{+R}(x,t;\lambda)\big)=\mathcal{I}+O\left(\frac{1}{\lambda}\right),\quad \lambda\in\mathbb{C}_+\to\infty.$$

The solutions of the equation of differential equation (2.2) can be related by a matrix independent of x and t. As a result

$$\mu_-(x,t;\lambda)=\mu_+(x,t;\lambda)e^{i(\lambda x+4i\lambda^3 t)\widehat{\sigma}}s(\lambda). \tag{2.4}$$

Evaluation at $t=0$ arrives at

$$s(\lambda)=\lim_{x\to+\infty}e^{-i\lambda x\widehat{\sigma}}\mu_-(x,0;\lambda), \tag{2.5}$$

i.e.,

$$s(\lambda)=\mathcal{I}+\int_{-\infty}^{+\infty}e^{-i\lambda x\widehat{\sigma}}[\mathbf{U}(x,0)\mu_-(x,0;\lambda)]\,dx. \tag{2.6}$$

The fact that $\mathrm{tr}(\mathbf{U})=0$ together with (2.3) indicates

$$\det(\mu_\pm(x,t;\lambda))=1. \tag{2.7}$$

Therefore, we obtain

$$\det(s(\lambda))=1. \tag{2.8}$$

Additionally, we know that

$$\mathbf{U}^\dagger(x,t;\overline{\lambda})=-\mathbf{U}(x,t;\lambda),\quad \overline{\mathbf{U}(x,t;-\overline{\lambda})}=\nabla\mathbf{U}(x,t;\lambda)\nabla, \tag{2.9}$$

where

$$\nabla=\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&0&0&0&1\end{pmatrix}.$$

Furthermore, it follows from (2.1) that

$$\psi_x^A(x,t;\lambda)=(i\lambda\sigma-\mathbf{U}(x,t))^T\psi^A(x,t;\lambda) \tag{2.10}$$

with $\psi^A(x,t;\lambda) = (\psi^{-1}(x,t;\lambda))^T$, where the superscript 'T' represents a matrix transpose. Consequently, we have

$$\psi^\dagger(x,t;\overline{\lambda}) = \psi^{-1}(x,t;\lambda), \quad \psi(x,t;\lambda) = \nabla\overline{\psi(x,t;-\overline{\lambda})}\nabla. \tag{2.11}$$

These relations indicate that the eigenfunctions $\mu_j(x,t;\lambda)$ meet

$$\mu^\dagger(x,t;\overline{\lambda}) = \mu^{-1}(x,t;\lambda), \quad \mu(x,t;\lambda) = \nabla\overline{\mu(x,t;-\overline{\lambda})}\nabla, \quad j = 1,2, \tag{2.12}$$

where '†' represents the Hermitian conjugate. To sum up, the matrix-valued function $s(\lambda)$ admits the following symmetries

$$s^\dagger(\overline{\lambda}) = s^{-1}(\lambda), \quad s(-\lambda) = \nabla\overline{s(\overline{\lambda})}\nabla. \tag{2.13}$$

In the following, we rewrite a 5×5 matrix $\mathbf{A}$ as a block form

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix},$$

where $\mathbf{A}_{11}$ is a 4×4 matrix and $\mathbf{A}_{22}$ is scalar. It follows from (2.4)–(2.13) that

$$\begin{aligned} s_{22}^\dagger(\overline{\lambda}) &= \det(s_{11}(\lambda)), & s_{11}(\lambda) &= \sigma_1\overline{s}_{11}(-\overline{\lambda})\sigma_1, \\ s_{12}^\dagger(\overline{\lambda}) &= -s_{21}\operatorname{adj}(s_{11}(\lambda)), & \overline{s}_{21}(-\overline{\lambda})\sigma_1 &= s_{21}(\lambda), \end{aligned}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{2.14}$$

and $\operatorname{adj}(\mathbf{B})$ represents the adjoint matrix of matrix $\mathbf{B}$. Because of the above expression (2.14), we rewrite $s(\lambda)$ as

$$s(\lambda) = \begin{pmatrix} a(\lambda) & -\operatorname{adj}(a^\dagger(\overline{\lambda}))b^\dagger(\overline{\lambda}) \\ b(\lambda) & \det(a^\dagger(\overline{\lambda})) \end{pmatrix},$$

where

$$a(\lambda) = \sigma_1\overline{a}(-\overline{\lambda})\sigma_1, \quad \overline{b}(-\overline{\lambda})\sigma_1 = b(\lambda).$$

It follows that $a(\lambda)$ and $b(\lambda)$ satisfy

$$\begin{aligned} a(\lambda) &= \mathcal{I} + \int_{-\infty}^{+\infty} \mathcal{U}(x,0)\mu_{-,21}(x,0;\lambda)\,dx, \\ b(\lambda) &= -\int_{-\infty}^{+\infty} e^{-2ix\lambda}\mathcal{U}^\dagger(x,0)\mu_{-,11}(x,0;\lambda)\,dx. \end{aligned} \tag{2.15}$$

Obviously, $a(\lambda)$ is analytic in $\mathbb{C}_+$.

Suppose that $\det(a(\lambda))$ admits $4N$ simple zeros $\lambda_1, \dots, \lambda_{4N}$ in $\mathbb{C}_+$, where $\lambda_{N+j} = -\overline{\lambda}_j$, $j = 1, 2, \dots 2N$. Define

$$M(\xi;\lambda) = \begin{cases} \left(\mu_{-L}(\lambda)a^{-1}(\lambda), \mu_{+R}(\lambda)\right), & \lambda \in \mathbb{C}_+, \\ \left(\mu_{+L}(\lambda), \mu_{-R}(\lambda)/\det a^{\dagger}(\overline{\lambda})\right), & \lambda \in \mathbb{C}_-. \end{cases} \tag{2.16}$$

Theorem 2.1. *Let $a(\lambda)$ and $b(\lambda)$ be determined by* (2.15)*. Then $M(x,t;\lambda)$ given by* (2.16) *satisfies the following matrix RHP. We find a meromorphic function $M(x,t;\lambda)$ with simple poles at $\{\lambda_j\}_1^{4N}$ and $\{\overline{\lambda}_j\}_1^{4N}$, then it admits*

$$\begin{cases} M_+(\lambda) = M_-(\lambda)J(\lambda), & \lambda \in \mathbb{R}, \\ M(\lambda) = \mathcal{I} + O\left(\frac{1}{\lambda}\right), & \lambda \to \infty, \end{cases}$$

and residue conditions

$$\operatorname*{Res}_{\lambda_j} M(\lambda) = \lim_{\lambda\to\lambda_j} M(\lambda) \begin{pmatrix} 0 & 0 \\ \frac{e^{-2i\theta(\lambda)t}b(\lambda)\operatorname{adj}(a(\lambda))}{\dot{\det}(a(\lambda))} & 0 \end{pmatrix},$$

$$\operatorname*{Res}_{\overline{\lambda}_j} M(\lambda) = \lim_{\lambda\to\overline{\lambda}_j} M(\lambda) \begin{pmatrix} 0 & -\frac{e^{2i\theta(\lambda)t}\operatorname{adj}(a^{\dagger}(\overline{\lambda}))b(\overline{\lambda})^{\dagger}}{\dot{\det}(a^{\dagger}(\overline{\lambda}))} \\ 0 & 0 \end{pmatrix},$$

where $j = 1, 2, \dots, 2N$, and

$$M_{\pm} = \lim_{\epsilon\to 0^+} M(\lambda \pm i\epsilon), \quad \gamma(\lambda) = b(\lambda)a^{-1}(\lambda), \quad \lambda \in \mathbb{R}, \tag{2.17}$$

$$J(\lambda) = \begin{pmatrix} \mathcal{I} + \gamma^{\dagger}(\overline{\lambda})\gamma(\lambda) & e^{2i\theta t}\gamma^{\dagger}(\overline{\lambda}) \\ e^{-2i\theta t}\gamma(\lambda) & 1 \end{pmatrix}, \quad \theta = \lambda\left(\frac{x}{t} + 4\lambda^2\right). \tag{2.18}$$

Here $\gamma(\lambda)$ lies in Schwartz space and satisfies

$$\gamma(\lambda) = \gamma^{\dagger}(-\overline{\lambda})\sigma_1, \quad \sup_{\lambda\in\mathbb{R}} \gamma(\lambda) < \infty.$$

Let

$$\mathcal{U}(x,t) = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = 2i \lim_{\lambda\to\infty} (\lambda M(x,t;\lambda))_{12}.$$

Then $u(x,t)$, $v(x,t)$ represent the solution of the CSS equation (1.2)*.*

3. Long-time asymptotic analysis

According to the idea of Deift and Zhou [6], we next consider the stationary points of the function θ, i.e., setting $\frac{d\theta}{d\lambda} = 0$, the stationary phase points are constructed for $x > 0$ as $\pm\lambda_0 = \pm\sqrt{\frac{x}{12t}}$, Thus, $\theta = 4\lambda(\lambda^2 - 3\lambda_0^2)$. In what follows, we mainly focus on physically interesting region $\lambda_0 \in (0, C]$, where C is a constant.

3.1. Factorization of the jump matrix

We notice that the jump matrix admits two distinct factorizations

$$J = \begin{cases} \begin{pmatrix} \mathcal{I} & e^{2i\theta}\gamma^\dagger(\bar{\lambda}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{I} & 0 \\ e^{-2i\theta}\gamma(\lambda) & 1 \end{pmatrix}, \\ \begin{pmatrix} \mathcal{I} & 0 \\ \frac{e^{-2i\theta}\gamma(\lambda)}{1+\gamma(\lambda)\gamma^\dagger(\bar{\lambda})} & 1 \end{pmatrix} \begin{pmatrix} \left(\mathcal{I}+\gamma^\dagger(\bar{\lambda})\gamma(\lambda)\right) & 0 \\ 0 & \frac{1}{1+\gamma(\lambda)\gamma^\dagger(\bar{\lambda})} \end{pmatrix} \begin{pmatrix} \mathcal{I} & \frac{e^{2it\theta}\gamma^\dagger(\bar{\lambda})}{1+\gamma(\lambda)\gamma^\dagger(\bar{\lambda})} \\ 0 & 1 \end{pmatrix}. \end{cases} \tag{3.1}$$

Next, we consider a function $\delta(\lambda)$ as the solution of the matrix problem

$$\delta_+(\lambda) = \begin{cases} (\mathcal{I}+\gamma^\dagger\gamma)\delta_-(\lambda), & |\lambda| < \lambda_0, \\ \delta_-(\lambda), & |\lambda| > \lambda_0 \end{cases} \quad \text{and} \quad \delta(\lambda) \to \mathcal{I}, \quad \lambda \to \infty. \tag{3.2}$$

As the jump matrix $(\mathcal{I}+\gamma^\dagger\gamma)$ is positive definite, the vanishing lemma gives the existence and uniqueness of the function $\delta(\lambda)$. Moreover, we have

$$\det(\delta_+(\lambda)) = \begin{cases} \left(1+|\gamma|^2\right)\det(\delta_-(\lambda)), & |\lambda| < \lambda_0, \\ \det(\delta_-(\lambda)), & |\lambda| > \lambda_0, \end{cases} \quad \text{and} \quad \det(\delta(\lambda)) \to 1, \quad \lambda \to \infty.$$

By utilizing the Plemelj formula [1], we get

$$\det(\delta(\lambda)) = \exp\left\{\frac{1}{2\pi i}\int_{-\lambda_0}^{\lambda_0} \frac{\log(1+\gamma(\xi)\gamma^\dagger(\xi))}{\xi-\lambda}\,d\xi\right\} = \left(\frac{\lambda+\lambda_0}{\lambda-\lambda_0}\right)^{i\nu} e^{\chi(\lambda)},$$

where

$$\nu = \frac{1}{2\pi}\log\left(1+\gamma(\lambda_0)\gamma^\dagger(\lambda_0)\right) > 0,$$
$$\chi(\lambda) = \frac{1}{2\pi i}\int_{-\lambda_0}^{\lambda_0} \log\left(\frac{1+\gamma(\xi)\gamma^\dagger(\xi)}{1+\gamma(\lambda_0)\gamma^\dagger(\lambda_0)}\right)\frac{d\xi}{\xi-\lambda}.$$

Then we have used the following relation

$$1+\gamma(\lambda_0)\gamma^\dagger(\lambda_0) = 1+\gamma(-\lambda_0)\gamma^\dagger(-\lambda_0),$$

which can be obtained from the second symmetry condition in (2.3).

In addition, for $|\lambda| < \lambda_0$, it follows from (3.2) that

$$\lim_{\epsilon\to 0^+}\delta(\lambda-i\epsilon) = \left(\mathcal{I}+\gamma(\lambda)^\dagger\gamma(\lambda)\right)^{-1}\lim_{\epsilon\to 0^-}\delta(\lambda+i\epsilon).$$

If we set $g(\lambda) = \left(\delta^\dagger(\bar{\lambda})\right)^{-1}$, we then get

$$g_+(\lambda) = \left(\mathcal{I}+\gamma^\dagger(\lambda)\gamma(\lambda)\right)g_-(\lambda).$$

Thus, we know

$$\left(\delta^{\dagger}(\overline{\lambda})\right)^{-1} = \delta(\lambda).$$

Similar to [6], after a direct calculation, we obtain

$$|\delta(\lambda)| \leq \text{const} < \infty, \quad |\det(\delta(\lambda))| \leq \text{const} < \infty$$

for all λ, where we define $|\mathbf{A}| = \sqrt{\left(\operatorname{tr} \mathbf{A}^{\dagger}\mathbf{A}\right)}$ for any matrix $\mathbf{A}$. Then we define

$$\Delta(\lambda) = \begin{pmatrix} \delta(\lambda)^{-1} & 0 \\ 0 & \det\left(\delta(\lambda)\right) \end{pmatrix}.$$

Introduce

$$M^{\Delta}(x,t;\lambda) = M(x,t;\lambda)\Delta(\lambda),$$

and reverse the orientation for $|\lambda| < \lambda_0$ as seen in Figure 3.1.

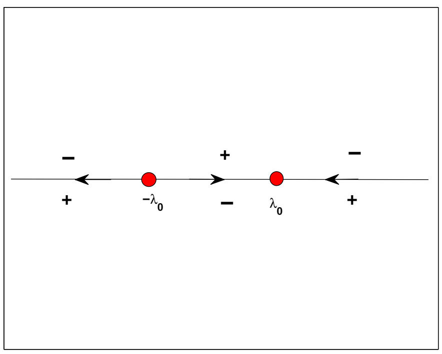

Figure 3.1: The oriented jump contour $\mathbb{R}$.

Theorem 3.1. *The M^{Δ} admits the following RHP*

$$\begin{cases} M_{+}^{\Delta}(x,t;\lambda) = M_{-}^{\Delta}(x,t;\lambda)J^{\Delta}(x,t;\lambda), & \lambda \in \mathbb{R}, \\ M^{\Delta}(x,t;\lambda) \to \mathcal{I}, & \lambda \to \infty, \end{cases}$$

where

$$J^{\Delta}(\lambda) = \begin{pmatrix} \mathcal{I} & 0 \\ \frac{e^{-2it\theta}\rho^{\dagger}(\overline{\lambda})\delta_{-}^{-1}(\lambda)}{\det \delta_{-}(\lambda)} & 1 \end{pmatrix} \begin{pmatrix} \mathcal{I} & \left(\det \delta_{+}(\lambda)\right)e^{2it\theta}\delta_{+}(\lambda)\rho(\lambda) \\ 0 & 1 \end{pmatrix},$$

and the vector-valued function

$$\rho(\lambda) = \begin{cases} \frac{\gamma^{\dagger}(\overline{\lambda})}{1+\gamma(\lambda)\gamma^{\dagger}(\overline{\lambda})}, & |\lambda| < \lambda_0, \\ -\gamma^{\dagger}(\overline{\lambda}), & |\lambda| > \lambda_0. \end{cases}$$

3.2. Analytic approximations of $\rho(\lambda)$

Our next purpose is to deform the contour, but we need to discuss the decomposition of $\rho(\lambda)$. Take

$$L : \left\{\lambda = \lambda_0 + \lambda_0 \alpha e^{\frac{3\pi i}{4}} : -\infty < \alpha \leq \sqrt{2}\right\} \cup \left\{\lambda = -\lambda_0 + \lambda_0 \alpha e^{\frac{\pi i}{4}} : -\infty < \alpha \leq \sqrt{2}\right\},$$

and

$$L_\epsilon : \left\{\lambda = \lambda_0 + \lambda_0 \alpha e^{\frac{3\pi i}{4}} : \epsilon < \alpha \leq \sqrt{2}\right\} \cup \left\{\lambda = -\lambda_0 + \lambda_0 \alpha e^{\frac{\pi i}{4}} : \epsilon < \alpha \leq \sqrt{2}\right\},$$

where $0 < \epsilon \leq \sqrt{2}$.

Lemma 3.2. [6] *As $0 < \lambda_0 \leq C$, there exists decomposition for the function $\rho(\lambda)$:*

$$\rho(\lambda) = h_1(\lambda) + h_2(\lambda) + R(\lambda), \quad \lambda \in \mathbb{R}, \tag{3.3}$$

where $R(\lambda)$ is analytic in the complex plane and $h_2(\lambda)$ is analytically and continuously extended to L. Additionally, $R(\lambda)$, $h_1(\lambda)$ and $h_2(\lambda)$ satisfy

$$\begin{cases} \left|e^{2it\theta(\lambda)} h_1(\lambda)\right| \lesssim t^{-l}, & \lambda \in \mathbb{R}, \\ \left|e^{2it\theta(\lambda)} h_2(\lambda)\right| \lesssim t^{-l}, & \lambda \in L, \\ \left|e^{2it\theta(\lambda)} R(\lambda)(\lambda)\right| \lesssim e^{-16\epsilon^2\lambda_0^3}, & \lambda \in L_\epsilon, \end{cases}$$

where positive integer l is free. It follows from the Schwartz conjugate representation of (3.3) *that*

$$\rho^\dagger(\overline{\lambda}) = h_1^\dagger(\overline{\lambda}) + h_2^\dagger(\overline{\lambda}) + R^\dagger(\overline{\lambda}),$$

we obtain the similar estimates for $e^{-2it\theta(\lambda)} h_1^\dagger(\overline{\lambda})$, $e^{-2it\theta(\lambda)} h_2^\dagger(\overline{\lambda})$ and $e^{-2it\theta(\lambda)} R^\dagger(\overline{\lambda})$ on the contour $\mathbb{R} \cup \overline{L}$.

3.3. Contour deformation

We rewrite $J^\Delta(x, t; \lambda)$ as $J^\Delta = (b_-)^{-1} b_+$, where $b_\pm = \mathcal{I} + \omega_\pm$, $\omega_\pm = \omega_\pm^o + \omega_\pm^a$,

$$\begin{aligned} b_+ &= b_+^o b_+^a = (\mathcal{I} + \omega_+^o)(\mathcal{I} + \omega_+^a) \\ &\triangleq \begin{pmatrix} \mathcal{I} & \det\delta(\lambda) e^{2i\theta} \delta(\lambda) h_1(\lambda) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{I} & \det\delta(\lambda) e^{2it\theta} \delta(\lambda)(h_2(\lambda) + R(\lambda)) \\ 0 & 1 \end{pmatrix}, \\ b_- &= b_-^o b_-^a = (\mathcal{I} - \omega_-^o)(\mathcal{I} - \omega_-^a) \\ &\triangleq \begin{pmatrix} \mathcal{I} & 0 \\ -\frac{e^{-2it\theta} h_1^\dagger(\overline{\lambda}) \delta(\lambda)}{\det\delta(\lambda)} & 1 \end{pmatrix} \begin{pmatrix} \mathcal{I} & 0 \\ -\frac{e^{-2it\theta}\left(h_2^\dagger(\overline{\lambda}) + R^\dagger(\overline{\lambda})\right)\delta^{-1}(\lambda)}{\det\delta(\lambda)} & 1 \end{pmatrix}. \end{aligned}$$

Lemma 3.3. *Take*

$$M^{\sharp}(\lambda)=\begin{cases}M^{\Delta}(\lambda), & \lambda\in\Omega_1\cup\Omega_2,\\ M^{\Delta}(\lambda)(b_-^a)^{-1}, & \lambda\in\Omega_3\cup\Omega_4\cup\Omega_5,\\ M^{\Delta}(\lambda)(b_+^a)^{-1}, & \lambda\in\Omega_6\cup\Omega_7\cup\Omega_8.\end{cases}$$

As a result, the function $M^{\sharp}(\lambda)$ admits the RHP on the contour $\Sigma=L\cup\overline{L}\cup\mathbb{R}$ displayed in Figure 3.2,

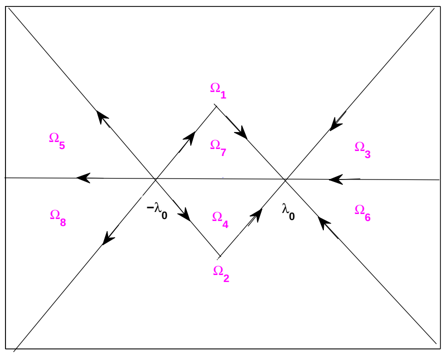

Figure 3.2: The oriented jump contour Σ.

$$\begin{cases}M_+^{\sharp}(\lambda)=M_-^{\sharp}(\lambda)J^{\sharp}(\lambda), & \lambda\in\Sigma,\\ M^{\sharp}\to\mathcal{I}, & \lambda\to\infty,\end{cases}\tag{3.4}$$

where

$$J^{\sharp}=\left(b_-^{\sharp}\right)^{-1}b_+^{\sharp}\triangleq\begin{cases}(b_-^o)^{-1}b_+^o, & \lambda\in\mathbb{R},\\ \mathcal{I}^{-1}b_+^a, & \lambda\in L,\\ (b_-^a)^{-1}\mathcal{I}, & \lambda\in\overline{L}.\end{cases}$$

The above RHP (3.4) can be obtained (see [3]). Take

$$\omega^{\sharp}=\omega_-^{\sharp}+\omega_+^{\sharp},\quad \omega_{\pm}^{\sharp}=\pm b_{\pm}^{\sharp}\mp\mathcal{I}.$$

In the following, the Cauchy operators $C_{\pm}$ for $\lambda\in\Sigma$ are denoted by

$$C_{\pm}f(\lambda)=\frac{1}{2\pi i}\int_{\Sigma}\frac{f(\xi)}{\xi-\lambda_{\pm}}\,d\xi,$$

where $f\in\mathscr{L}^2(\Sigma)$. Define

$$C_{\omega^{\sharp}}f=C_+\left(f\omega_-^{\sharp}\right)+C_-\left(f\omega_+^{\sharp}\right).\tag{3.5}$$

Theorem 3.4. [3] *Assume* $\mu^\sharp(x,t;\lambda) \in \mathscr{L}^2(\Sigma) + \mathscr{L}^\infty(\Sigma)$ *satisfies*

$$\mu^\sharp = \mathcal{I} + C_{\omega^\sharp}\mu^\sharp.$$

Thus

$$M^\sharp(\lambda) = \mathcal{I} + \frac{1}{2\pi i}\int_\Sigma \frac{\mu^\sharp(\xi)\omega^\sharp(\xi)}{\xi-\lambda}\,d\xi$$

represents the solution of the RHP (3.4)*.*

Theorem 3.5. *The solutions* $(u(x,t), v(x,t))$ *for the CSS equation* (1.2) *are expressed by*

$$\begin{aligned}\mathcal{U}(x,t) = \begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix} &= 2i\lim_{\lambda\to\infty}\left(\lambda M^\sharp(\lambda)\right)_{12}\\ &= -\frac{1}{\pi}\left(\int_\Sigma \left(\mu^\sharp(\xi)\omega^\sharp(\xi)\,d\xi\right)d\xi\right)_{12}\\ &= -\frac{1}{\pi}\left(\left(\int_\Sigma (1-C_{\omega^\sharp})^{-1}\mathcal{I}\right)(\xi)\omega^\sharp(\xi)\,d\xi\right)_{12}.\end{aligned}$$

Proof. The similar result is provided in [6]. □

3.4. Contour truncation

As seen in Figure 3.3, take $\Sigma' = \Sigma(\mathbb{R}\cup L_\epsilon \cup \overline{L}_\epsilon)$ with the orientation. We plane to replace the RHP on Σ with the truncated contour Σ' by error control.

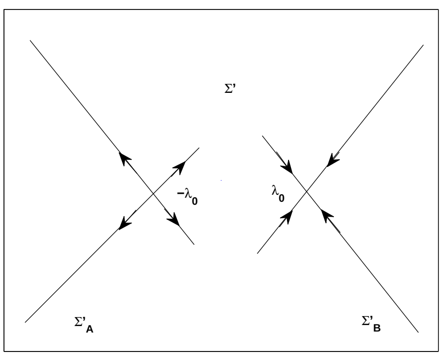

Figure 3.3: The oriented contour $\Sigma' = \Sigma'_A \cup \Sigma'_B$.

Take

$$\omega^a = \begin{cases}\omega^\sharp, & \lambda\in\mathbb{R},\\ \mathbf{0}, & \text{otherwise},\end{cases}\qquad \omega^b = \begin{cases}\begin{pmatrix}0 & \det\delta(\lambda)e^{2i\theta}\delta(\lambda)h_2(\lambda)\\ 0 & 0\end{pmatrix}, & \lambda\in L,\\ \begin{pmatrix}0 & 0\\ \frac{e^{-2it\theta}h_2^\dagger(\overline{\lambda})\delta^{-1}(\lambda)}{\det\delta(\lambda)} & 0\end{pmatrix}, & \lambda\in\overline{L},\\ \mathbf{0}, & \text{otherwise},\end{cases}$$

$$\omega^c = \begin{cases} \begin{pmatrix} 0 & e^{2i\theta} \det \delta(\lambda)\delta(\lambda)R(\lambda) \\ 0 & 0 \end{pmatrix}, & \lambda \in L_\epsilon, \\ \begin{pmatrix} 0 & 0 \\ \frac{e^{-2it\theta} R^\dagger(\overline{\lambda})\delta^{-1}(\lambda)}{\det \delta(\lambda)} & 0 \end{pmatrix}, & \lambda \in \overline{L}_\epsilon, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

Define $\omega' = \omega^\sharp - \omega^a - \omega^b - \omega^c$, so $\omega' = 0$ on the contour $\Sigma \setminus \Sigma'$. Thus, ω' is supposed on contour Σ' and related to $R(\lambda)$ and $R^\dagger(\overline{\lambda})$.

Lemma 3.6. [6] *For sufficiently small ϵ, as $t \to \infty$, we obtain*

$$\|\omega^a\|_{\mathscr{L}^\infty(\mathbb{R})\cap\mathscr{L}^1(\mathbb{R})} \lesssim t^{-l}, \quad \|\omega^b\|_{\mathscr{L}^\infty(L\cup\overline{L})\cap\mathscr{L}^1(L\cup\overline{L})} \lesssim t^{-l},$$
$$\|\omega^c\|_{\mathscr{L}^\infty(L_\epsilon\cup\overline{L}_\epsilon)\cap\mathscr{L}^1(L_\epsilon\cup\overline{L}_\epsilon)} \lesssim e^{-16\epsilon^2\tau}, \quad \|\omega'\|_{\mathscr{L}^2(\Sigma)} \lesssim \tau^{-1/4}, \quad \|\omega'\|_{\mathscr{L}^1(\Sigma)} \lesssim \tau^{-1/2},$$

where $\tau = \lambda_0^3 t$.

Lemma 3.7. [6] *In the case $0 < \lambda_0 \le C$, as $\tau \to \infty$, the inverse $(1 - C_{\omega'})^{-1} \colon \mathscr{L}^2(\Sigma) \to \mathscr{L}^2(\Sigma)$ exists, and has uniform boundedness*

$$\left\|\left(1 - C_{\omega'}\right)^{-1}\right\|_{\mathscr{L}^2(\Sigma)} \lesssim 1.$$

Besides

$$\left\|\left(1 - C_{\omega^\sharp}\right)^{-1}\right\|_{\mathscr{L}^2(\Sigma)} \lesssim 1.$$

Lemma 3.8. *The integral equation has estimate as $\tau \to \infty$*

$$\int_\Sigma \big((1 - C_{\omega^\sharp})^{-1}\mathcal{I}\big)(\xi)\omega^\sharp(\xi)\, d\xi = \int_\Sigma \big((1 - C_{\omega'})^{-1}\mathcal{I}\big)(\xi)\omega'(\xi)\, d\xi + O\left(\frac{1}{\tau^l}\right).$$

Proof. After a simple calculation, we find

$$\begin{aligned} \big((1-\omega^\sharp)^{-1}\mathcal{I}\big)\omega^\sharp = {}& \big((1 - C\omega')^{-1}\mathcal{I}\big)\omega' + \omega^e + \big((1-\omega')^{-1}(C_{\omega^e}\mathcal{I})\big)\omega^\sharp \\ &+ \big((1-\omega')^{-1}(C_{\omega'}\mathcal{I})\big)\omega^e + \big((1 - C_{\omega'})^{-1}C_{\omega^e}(1 - C_{\omega^\sharp})^{-1}\big)(C_{\omega^\sharp}\mathcal{I})\omega^\sharp. \end{aligned}$$

Then from Lemma 3.6, we have

$$\begin{aligned} \|\omega^e\|_{\mathscr{L}^1(\Sigma)} &\le \|\omega^a\|_{\mathscr{L}^1(\mathbb{R})} + \|\omega^b\|_{\mathscr{L}^1(L_\epsilon\cup\overline{L}_\epsilon)} \lesssim \tau^{-l}, \\ \left\|\big((1 - C_{\omega'})^{-1}(C_{\omega^e}\mathcal{I})\big)\omega^\sharp\right\|_{\mathscr{L}^1(\Sigma)} &\le \left\|(1 - C_{\omega'})^{-1}\right\|_{\mathscr{L}^2(\Sigma)} \left\|C_{\omega^e}\mathcal{I}\right\|_{\mathscr{L}^2(\Sigma)} \|\omega^\sharp\|_{\mathscr{L}^2(\Sigma)} \\ &\le \|\omega^e\|_{\mathscr{L}^2(\Sigma)} \|\omega^\sharp\|_{\mathscr{L}^2(\Sigma)} \lesssim t^{-l-1/4}, \\ \left\|\big((1 - C_{\omega'})^{-1}(C_{\omega'}\mathcal{I})\big)\omega^e\right\|_{\mathscr{L}^1(\Sigma)} &\le \left\|(1 - C_{\omega'})^{-1}\right\|_{\mathscr{L}^2(\Sigma)} \|C_{\omega'}\mathcal{I}\|_{\mathscr{L}^2(\Sigma)} \|\omega^e\|_{\mathscr{L}^2(\Sigma)} \\ &\le \|\omega'\|_{\mathscr{L}^2(\Sigma)} \|\omega^e\|_{\mathscr{L}^2(\Sigma)} \lesssim t^{-l-1/4}, \end{aligned}$$

$$
\begin{aligned}
&\big\|\big((1-C_{\omega'})^{-1}C_{\omega^e}(1-C_{\omega^\sharp})^{-1}\big)(C_{\omega^\sharp}\mathcal{I})\omega^\sharp\big\|_{\mathscr{L}^1(\Sigma)}\\
&=\big\|(1-C_{\omega'})^{-1}\|_{\mathscr{L}^2(\Sigma)}\|C_{\omega^e}\|_{\mathscr{L}^2(\Sigma)}\big\|(1-C_{\omega^\sharp})^{-1}\big\|_{\mathscr{L}^2(\Sigma)}\|C_{\omega^\sharp}\mathcal{I}\|_{\mathscr{L}^2(\Sigma)}\|\omega^\sharp\|_{\mathscr{L}^2(\Sigma)}\\
&\lesssim\|\omega^e\|_{\mathscr{L}^\infty(\Sigma)}\|\omega^\sharp\|^2_{\mathscr{L}^2(\Sigma)}\lesssim t^{-l-1/2}.
\end{aligned}
$$

This finishes proof of Lemma 3.8. □

Lemma 3.9. *The solution admits the following asymptotics, as $\tau\to\infty$*

$$
\mathcal{U}(x,t)=\begin{pmatrix}\mathbf{u}(x,t)\\ \mathbf{v}(x,t)\end{pmatrix}=-\frac{1}{\pi}\left(\int_{\Sigma'}\big((1-C_{\omega'})^{-1}\mathcal{I}\big)(x,t;\xi)\omega'(x,t,\xi)\,d\xi\right)_{12}+O\left(\frac{1}{\tau^l}\right).
$$

Proof. The lemma follows by Theorem 3.5 and Lemma 3.8. □

Here take $L'=L\setminus L_\epsilon$ and $\Sigma'=L'\cup\overline{L}'$. Let $\mu'=(1-C_{\omega'})^{-1}\mathcal{I}$. Then

$$
M'(\lambda)=\mathcal{I}+\frac{1}{2\pi i}\int_{\Sigma'}\frac{\mu'(\xi)\omega'(\xi)}{\xi-\lambda}\,d\xi
$$

meets

$$
\begin{cases}M'_+(\lambda)=M'_-(\lambda)J'(\lambda), & \lambda\in\Sigma',\\ M'(\lambda)\to\mathcal{I}, & \lambda\to\infty,\end{cases}
$$

where

$$
\begin{gathered}
J'=b'^{-1}_-b'_+,\quad b'_-=\mathcal{I},\quad \lambda\in L',\\
b'_+=\begin{pmatrix}\mathcal{I} & 0\\ e^{2it\theta(\lambda)}\det(\delta(\lambda))R(\lambda) & 1\end{pmatrix},\\
b'_+=\mathcal{I},\quad b'_-=\begin{pmatrix}\mathcal{I} & -\frac{e^{-2it\theta(\lambda)}\delta^{-1}(\lambda)R^\dagger(\bar\lambda)}{\det\delta(\lambda)}\\ 0 & 1\end{pmatrix},\quad \lambda\in\overline{L}'.
\end{gathered}
$$

3.5. Noninteraction of disconnected contour

Choose $\omega'=\omega'_++\omega'_-$, where $\omega'_\pm=\pm b'_\pm-\mp\mathcal{I}$. Let the contour $\Sigma'=\Sigma'_{\mathbf{A}}\cup\Sigma'_{\mathbf{B}}$ and $\omega'_\pm=\omega'_{\mathbf{A}\pm}+\omega'_{\mathbf{B}\pm}$, where $\omega'_{\mathbf{A}\pm}(\lambda)=0$ for $\lambda\in\Sigma'_{\mathbf{B}}$, $\omega'_{\mathbf{B}\pm}(\lambda)$, $\omega'_{\mathbf{B}\pm}(\lambda)=0$ for $\lambda\in\Sigma'_{\mathbf{A}}$. Give the operators $C_{\omega'_{\mathbf{A}}}$ and $C_{\omega'_{\mathbf{B}}}:\mathscr{J}^\infty(\Sigma')+\mathscr{J}^2(\Sigma')\to\mathscr{J}^2(\Sigma')$ as in (3.5).

Lemma 3.10. [6]

$$
\begin{aligned}
&\big\|C_{\omega'_{\mathbf{B}}}C_{\omega'_{\mathbf{A}}}\big\|=\big\|C_{\omega'_{\mathbf{A}}}C_{\omega'_{\mathbf{B}}}\|_{\mathscr{J}^2(\Sigma')}\lesssim\lambda_0\tau^{-1/2},\\
&\big\|C_{\omega'_{\mathbf{B}}}C_{\omega'_{\mathbf{A}}}\|_{\mathscr{J}^\infty(\Sigma')\to\mathscr{J}^2(\Sigma')},\ \big\|C_{\omega'_{\mathbf{A}}}C_{\omega'_{\mathbf{B}}}\big\|_{\mathscr{J}^\infty(\Sigma')\to\mathscr{J}^2(\Sigma')}\le\lambda_0\tau^{-3/4}.
\end{aligned}\tag{3.6}
$$

Proof. Together with Lemmas 3.6 and 3.7, we obtain (3.6). □

Lemma 3.11. *As $\tau \to \infty$,*

$$\int_{\Sigma'} \big((1 - C_{\omega'})^{-1}\mathcal{I}\big)(\xi)\omega'(\xi)\,d\xi = \int_{\Sigma'_{\mathbf{A}}} \big((1 - C_{\omega'_{\mathbf{A}}})^{-1}\mathcal{I}\big)(\xi)\omega'(\xi)\,d\xi + \int_{\Sigma'_{\mathbf{B}}} \big((1 - C_{\omega'_{\mathbf{A}}})^{-1}\mathcal{I}\big)(\xi)\omega'(\xi)\,d\xi + O\left(\frac{1}{\tau}\right).$$

Proof. From the following relation

$$\begin{aligned}&\big(1 - C_{\omega'_{\mathbf{A}}} - C_{\omega'_{\mathbf{B}}}\big)\big(1 + C_{\omega'_{\mathbf{A}}}(1 - C_{\omega'_{\mathbf{A}}})^{-1} + C_{\omega'_{\mathbf{B}}}(1 - C_{\omega'_{\mathbf{B}}})^{-1}\big)\\ &= 1 - C_{\omega'_{\mathbf{B}}}C_{\omega'_{\mathbf{A}}}(1 - C_{\omega'_{\mathbf{A}}})^{-1} - C_{\omega'_{\mathbf{A}}}C_{\omega'_{\mathbf{B}}}(1 - C_{\omega'_{\mathbf{B}}})^{-1},\end{aligned}$$

we find

$$\begin{aligned}(1 - C_{\omega'})^{-1} = (1 + C_{\omega'})^{-1} = 1 &+ C_{\omega'_{\mathbf{A}}}(1 - C_{\omega'_{\mathbf{A}}})^{-1} + C_{\omega'_{\mathbf{B}}}(1 - C_{\omega'_{\mathbf{B}}})^{-1}\\ &+ \big(1 + C_{\omega'_{\mathbf{A}}}(1 - C_{\omega'_{\mathbf{A}}})^{-1} + C_{\omega'_{\mathbf{B}}}(1 - C_{\omega'_{\mathbf{B}}})^{-1}\big)\\ &\quad\times \big(1 - C_{\omega'_{\mathbf{B}}}C_{\omega'_{\mathbf{A}}}(1 - C_{\omega'_{\mathbf{A}}})^{-1} - C_{\omega'_{\mathbf{A}}}C_{\omega'_{\mathbf{B}}}(1 - C_{\omega'_{\mathbf{B}}})^{-1}\big)^{-1}\\ &\quad\times \big(C_{\omega'_{\mathbf{B}}}C_{\omega'_{\mathbf{A}}}(1 - C_{\omega'_{\mathbf{A}}})^{-1} + C_{\omega'_{\mathbf{A}}}C_{\omega'_{\mathbf{B}}}(1 - C_{\omega'_{\mathbf{B}}})^{-1}\big).\end{aligned}$$

Together with Lemmas 3.6, 3.7 and 3.10, we obtain Lemma 3.11. □

Note. We also write the restriction $C_{\omega'_{\mathbf{A}}}|_{\mathscr{I}^2(\Sigma'_{\mathbf{A}})}$ as $C_{\omega'_{\mathbf{B}}}$.

Lemma 3.12. *As $\tau \to \infty$, we have*

$$\begin{aligned}\mathcal{U}(x,t) = \begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix} &= -\frac{1}{\pi}\left(\int_{\Sigma'_{\mathbf{A}}} \big((1 - C_{\omega'_{\mathbf{A}}})^{-1}\mathcal{I}\big)(x,t;\xi)\omega'_{\mathbf{A}}(x,t;\xi)\,d\xi\right)_{12}\\ &\quad - \frac{1}{\pi}\left(\int_{\Sigma'_{\mathbf{B}}} \big((1 - C_{\omega'_{\mathbf{B}}})^{-1}\mathcal{I}\big)(x,t;\xi)\omega'_{\mathbf{B}}(x,t;\xi)\,d\xi\right)_{12} + O\left(\frac{1}{\tau}\right).\end{aligned}$$

3.6. The model Riemann-Hilbert problem

Extend $\Sigma'_{\mathbf{A}}$ and $\Sigma'_{\mathbf{B}}$ to the following contours

$$\widehat{\Sigma}'_{\mathbf{A}} = \left\{\lambda = -\lambda_0 + \lambda_0\alpha e^{\frac{\pm i\pi}{4}} : \alpha \in \mathbb{R}\right\}, \quad \widehat{\Sigma}'_{\mathbf{B}} = \left\{\lambda = -\lambda_0 + \lambda_0\alpha e^{\frac{\pm 3i\pi}{4}} : \alpha \in \mathbb{R}\right\},$$

respectively, and give $\widehat{\omega}'_{\mathbf{A}}$, $\widehat{\omega}'_{\mathbf{B}}$ on $\widehat{\Sigma}'_{\mathbf{A}}$, $\widehat{\Sigma}'_{\mathbf{B}}$ as

$$\widehat{\omega}'_{\mathbf{A}\pm} = \begin{cases}\omega'_{A\pm}(\lambda), & \lambda \in \Sigma'_{\mathbf{A}},\\ 0, & \lambda \in \widehat{\Sigma}'_{\mathbf{A}} \setminus \Sigma'_{\mathbf{A}},\end{cases} \qquad \widehat{\omega}'_{\mathbf{B}\pm} = \begin{cases}\omega'_{\mathbf{B}\pm}(\lambda), & \lambda \in \Sigma'_{\mathbf{B}},\\ 0, & \lambda \in \widehat{\Sigma}'_{\mathbf{B}} \setminus \Sigma'_{\mathbf{B}}.\end{cases}$$

Let $\Sigma_{\mathbf{A}}$ and $\Sigma_{\mathbf{B}}$ denote the contours $\{\lambda = \lambda_0 \alpha e^{\pm\frac{i\pi}{4}} : \alpha \in \mathbb{R}\}$ shown in Figure 3.4. The scaling operators $N_{\mathbf{A}}$ and $N_{\mathbf{B}}$ is given by

$$N_{\mathbf{A}} : \mathscr{I}^2(\widehat{\Sigma}'_{\mathbf{A}}) \to \mathscr{I}^2(\Sigma^{\mathbf{A}}), \qquad f(\lambda) \mapsto (N_{\mathbf{A}} f)(\lambda) = f\left(\frac{\lambda}{4\sqrt{3\lambda_0 t}} - \lambda_0\right),$$
$$N_{\mathbf{B}} : \mathscr{I}^2(\widehat{\Sigma}'_{\mathbf{B}}) \to \mathscr{I}^2(\Sigma^{\mathbf{B}}), \qquad f(\lambda) \mapsto (N_{\mathbf{B}} f)(\lambda) = f\left(\frac{\lambda}{4\sqrt{3\lambda_0 t}} + \lambda_0\right),$$

and take

$$\omega_{\mathbf{A}} = N_{\mathbf{A}} \widehat{\omega}'_{\mathbf{A}}, \quad \omega_{\mathbf{B}} = N_{\mathbf{B}} \widehat{\omega}'_{\mathbf{B}}.$$

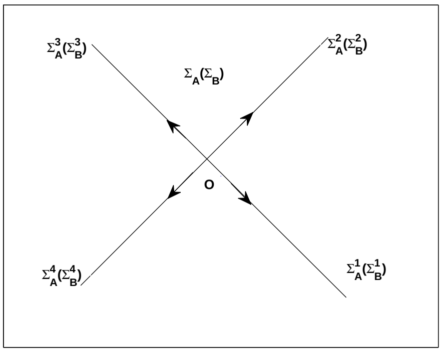

Figure 3.4: The oriented contour Σ_A or Σ_B.

A simple replacement yields

$$C_{\widehat{\omega}'_{\mathbf{A}}} = N_{\mathbf{A}}^{-1} C_{\omega_{\mathbf{A}}} N_{\mathbf{A}}, \quad C_{\widehat{\omega}'_{\mathbf{B}}} = N_{\mathbf{B}}^{-1} C_{\omega_{\mathbf{A}}} N_{\mathbf{B}},$$

where $C_{\omega_{\mathbf{A}}} : \mathscr{I}^2(\Sigma_{\mathbf{A}}) \to \mathscr{I}^2(\Sigma_{\mathbf{A}})$ is bounded, similar to $C_{\omega_{\mathbf{B}}}$.

On the other hand, we infer that

$$\omega_{\mathbf{A}} = \omega_{\mathbf{A}+} = \begin{pmatrix} 0 & (N_{\mathbf{A}} s_1)(\lambda) \\ 0 & 0 \end{pmatrix},$$

on

$$L_{\mathbf{A}} = \left\{\lambda = \alpha\lambda_0 4\sqrt{3\lambda_0 t} e^{-\frac{3\pi i}{4}} : -\epsilon < \alpha < +\infty\right\},$$

and

$$\omega_{\mathbf{A}} = \omega_{\mathbf{A}-} = \begin{pmatrix} 0 & 0 \\ (N_{\mathbf{A}} s_2)(\lambda) & 0 \end{pmatrix},$$

on $\overline{L}_{\mathbf{A}}$, where

$$s_1(\lambda) = \det\delta(\lambda) e^{2i\theta(\lambda)t} \delta(\lambda) R(\lambda), \quad s_2(\lambda) = \frac{e^{-2i\theta(\lambda)t} R^{\dagger}(\overline{\lambda}) \delta(\lambda)^{-1}}{\det\delta(\lambda)}.$$

Lemma 3.13. *As $\lambda \in L_{\mathbf{A}}$ and $t \to \infty$, we have*

$$\big|(N_{\mathbf{A}})\widetilde{\delta}(\lambda)\big| \lesssim t^{-l}, \tag{3.7}$$

where

$$\widetilde{\delta}(\lambda) = e^{2i\theta(\lambda)t}\big[\delta(\lambda) - \det\delta(\lambda)\mathcal{I}\big]R(\lambda). \tag{3.8}$$

Proof. It follows from (3.2), (3.1) and (3.8) that

$$\begin{cases} \widetilde{\delta}_+(\lambda) = e^{2it\theta}f(\lambda) + \widetilde{\delta}_-(\lambda)\big(1+|\gamma(\lambda)|^2\big), & |\lambda| < \lambda_0, \\ \widetilde{\delta}(\lambda) \to 0, & \lambda \to \infty, \end{cases}$$

where

$$f(\lambda) = \delta_-\big(\gamma^\dagger\gamma R - |\gamma|^2 R\big)(\lambda).$$

By using Plemelj formula, the solution $\widetilde{\delta}(\lambda)$ reaches to

$$\widetilde{\delta}(\lambda) = X(\lambda)\int_{-\lambda_0}^{\lambda_0} \frac{e^{2it\theta(\xi)f(\xi)}}{X_+(\xi)(\xi-\lambda)}\,d\xi, \quad X(\lambda) = e^{\frac{1}{2\pi i}\int_{-\lambda_0}^{\lambda_0}\frac{1+|\gamma(\xi)|^2}{\xi-\lambda}\,d\xi}.$$

From

$$\gamma^\dagger\gamma R - |\gamma|^2 R = \gamma^\dagger\gamma(R-\rho) - |\gamma|^2(R-\rho),$$

we obtain $f(\lambda) = O\big((\lambda^2-\lambda_0^2)^l\big)$. Then we decompose $f(\lambda)$ two parts: $f(\lambda) = f_1(\lambda)+f_2(\lambda)$, where $f_2(\lambda)$ is analytically and continuously extended to L_t and meets

$$\begin{cases} \big|e^{2it\theta(\lambda)}f_1(\lambda)\big| \lesssim \frac{1}{(1+|\lambda|^2)t^l}, & \lambda \in \mathbb{R}, \\ \big|e^{2it\theta(\lambda)}f_1(\lambda)\big| \lesssim \frac{1}{(1+|\lambda|^2)t^l}, & \lambda \in L_t, \end{cases}$$

where

$$L_t : \lambda = \left\{\lambda_0\alpha e^{3\pi i/4} : 0 \le \alpha \le \sqrt{2} - \frac{1}{\sqrt{2}t}\right\}$$
$$\cup \left\{\lambda = \lambda_0/t - \lambda_0 + \lambda_0\alpha e^{\pi i/4} : 0 \le \alpha \le \sqrt{2} - \frac{1}{\sqrt{2}t}\right\}$$

is shown in Figure 3.5.

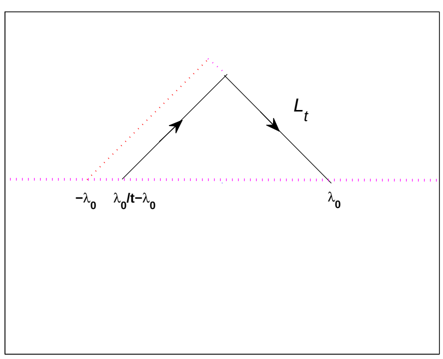

Figure 3.5: The contour L_t.

When $\lambda \in L_{\mathbf{A}}$, we have

$$\big(N_A\widetilde{\delta}\big)(\lambda) = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3,$$

with

$$|\mathcal{I}_1| \lesssim \int_{-\lambda_0}^{\lambda_0/t-\lambda_0} \frac{|f(\xi)|}{\xi+\lambda_0-\lambda/4\sqrt{3t\lambda_0}}\,d\xi \le t^{-l}\log\left|1-\frac{4\lambda_0\sqrt{3\lambda_0}}{\lambda\sqrt{t}}\right| \lesssim t^{-l-1/2},$$

$$|\mathcal{I}_2| \lesssim \int_{\lambda_0/t-\lambda_0}^{\lambda_0} \frac{|e^{2i\theta}f_1(\xi)|}{\xi+\lambda_0-\lambda/4\sqrt{3t\lambda_0}}\,d\xi \le t^{-l}\frac{\sqrt{2}t}{\lambda_0}(2\lambda_0-\lambda_0/t) \lesssim t^{-l+1},$$

with the help of Cauchy's Theorem, the original integral interval $(\lambda_0/t-\lambda_0, \lambda_0)$ in $\mathcal{I}_3$ can be replaced by contour L_t. Following the similar way, $|\mathcal{I}_3| \lesssim t^{-l+1}$, we can easily obtain Lemma 3.13. □

Similarly, we have

$$\big|(N_{\mathbf{A}}\widehat{\delta})(\lambda)\big| \lesssim t^{-l}, \quad \lambda \in \overline{L}_{\mathbf{A}}, \quad t \to \infty,$$

where

$$\widehat{\delta}(\lambda) = e^{-2i\theta(\lambda)t}R^{\dagger}(\overline{\lambda})\big[\delta(\lambda)^{-1} - [\det\delta^{-1}(\lambda)]\mathcal{I}\big].$$

Take $J^{\mathbf{A}^0} = (\mathcal{I}-\omega_{\mathbf{A}^0-})^{-1}(\mathcal{I}-\omega_{\mathbf{A}^0+})$, where

$$\omega_{\mathbf{A}^0} = \omega_{\mathbf{A}^0+} = \begin{cases} \begin{pmatrix} 0 & -(\delta_{\mathbf{A}})^2(-\lambda)^{2iv}e^{i\lambda^2/2}\gamma^{\dagger}(-\lambda_0) \\ 0 & 0 \end{pmatrix}, & \lambda \in \Sigma_{\mathbf{A}}^4, \\ \begin{pmatrix} 0 & (\delta_{\mathbf{A}})^2(-\lambda)^{2iv}e^{i\lambda^2/2}\frac{\gamma^{\dagger}(-\lambda_0)}{1+|\gamma(-\lambda_0)|^2} \\ 0 & 0 \end{pmatrix}, & \lambda \in \Sigma_{\mathbf{A}}^2, \end{cases} \tag{3.9}$$

and

$$\delta_{\mathbf{A}} = e^{\chi(-\lambda_0)-8i\tau}(192\tau)^{-iv/2}, \tag{3.10}$$

with

$$\omega_{\mathbf{A}^0} = \omega_{\mathbf{A}^0-} = \begin{cases} \begin{pmatrix} 0 & 0 \\ -(\delta_{\mathbf{A}})^{-2}(-\lambda)^{-2iv}e^{-i\lambda^2/2}\gamma(-\lambda_0) & 0 \end{pmatrix}, & \lambda \in \Sigma_{\mathbf{A}}^3, \\ \begin{pmatrix} 0 & 0 \\ (\delta_{\mathbf{A}})^2(-\lambda)^{-2iv}e^{-i\lambda^2/2}\frac{\gamma(-\lambda_0)}{1+|\gamma(-\lambda_0)|^2} & 0 \end{pmatrix}, & \lambda \in \Sigma_{\mathbf{A}}^1. \end{cases} \tag{3.11}$$

It follows from (3.7) and [6] that

$$\|\omega_{\mathbf{A}} - \omega_{\mathbf{A}^0}\|_{\mathscr{J}^{\infty}(\Sigma_{\mathbf{A}}) \cap \mathscr{J}^1(\Sigma_{\mathbf{A}}) \cap \mathscr{J}^2(\Sigma_{\mathbf{A}})} \lesssim \lambda_0 t^{1/2}\log(t).$$

Therefore

$$
\begin{aligned}
&\int_{\Sigma'_{\mathbf{A}}} \big((1-C_{\omega'_{\mathbf{A}}})^{-1}\mathcal{I}\big)(\xi)\omega'_{\mathbf{A}}(\xi)\,d\xi \\
&= \int_{\widehat{\Sigma}_{\mathbf{A}}} \big((1-C_{\widehat{\omega}'_{\mathbf{A}}})^{-1}\mathcal{I}\big)(\xi)\widehat{\omega}'_{\mathbf{A}}(\xi)\,d\xi \\
&= \int_{\widehat{\Sigma}'_{\mathbf{A}}} \big(N_{\mathbf{A}}^{-1}(1-C_{\omega_{\mathbf{A}}})^{-1}N_{\mathbf{A}}\mathcal{I}\big)(\xi)\widehat{\omega}'_{\mathbf{A}}(\xi)\,d\xi \\
&= \int_{\widehat{\Sigma}'_{\mathbf{A}}} \big((1-C_{\omega_{\mathbf{A}}})^{-1}\mathcal{I}\big)(\xi+\lambda_0)4\sqrt{3t\lambda_0}N_{\mathbf{A}}\widehat{\omega}'_{\mathbf{A}}\big((\xi+\lambda_0)4\sqrt{3t\lambda_0}\big)\,d\xi \\
&= \frac{1}{4\sqrt{3t\lambda_0}}\int_{\Sigma_{\mathbf{A}}} \big((1-C_{\omega_{\mathbf{A}}})^{-1}\mathcal{I}\big)(\xi)\omega_{\mathbf{A}}(\xi) \\
&= \frac{1}{4\sqrt{3t\lambda_0}}\int_{\Sigma_{\mathbf{A}}} \big((1-C_{\omega_{\mathbf{A}^0}})^{-1}\mathcal{I}\big)(\xi)\omega_{\mathbf{A}}(\xi) + O\left(\frac{\log t}{t}\right).
\end{aligned}
$$

Together with a similar computation for $\mathbf{B}$ yields

$$
\begin{aligned}
\mathcal{U}(x,t) = \begin{pmatrix}\mathbf{u}(x,t)\\ \mathbf{v}(x,t)\end{pmatrix} &= -\frac{1}{\pi}\frac{1}{4\sqrt{3\lambda_0 t}}\left(\int_{\Sigma_{\mathbf{A}}} \big((1-C_{\omega_{\mathbf{A}^0}})^{-1}\mathcal{I}\big)(\xi)\omega_{\mathbf{A}^0}(\xi)\,d\xi\right)_{12} \\
&\quad - \frac{1}{\pi}\frac{1}{4\sqrt{3\lambda_0 t}}\left(\int_{\Sigma_{\mathbf{B}}} \big((1-C_{\omega_{\mathbf{B}^0}})^{-1}\mathcal{I}\big)(\xi)\omega_{\mathbf{B}^0}(\xi)\,d\xi\right)_{12} + O\left(\frac{\log t}{t}\right).
\end{aligned}
$$

For $\lambda \in \mathbb{C}\setminus\Sigma_A$, let

$$
M^{A^0}(\lambda) = \mathcal{I} + \frac{1}{2\pi i}\int_{\Sigma_A} \frac{\big((1-C_{\omega_{A^0}})^{-1}\big)(\xi)\omega_{A^0}(\xi)}{\xi-\lambda}\,d\xi.
$$

Then $M^{\mathbf{A}^0}$ admits

$$
\begin{cases}
M_+^{\mathbf{A}^0}(\lambda) = M_-^{\mathbf{A}^0}(\lambda)J^{\mathbf{A}^0}(\lambda), & \lambda\in\Sigma_{\mathbf{A}},\\
M^{\mathbf{A}^0}(\lambda)\to\mathcal{I}, & \lambda\to\infty.
\end{cases}
$$

Particularly

$$
M^{\mathbf{A}^0}(\lambda) = \mathcal{I} + \frac{M_1^{\mathbf{A}^0}}{\lambda} + O\left(\frac{1}{\lambda^2}\right), \quad \lambda\to\infty,
$$

then

$$
M_1^{\mathbf{A}^0} = -\frac{1}{2\pi i}\int_{\Sigma_{\mathbf{A}}} \big((1-C_{\omega_{\mathbf{A}^0}})^{-1}\mathcal{I}\big)(\xi)\omega_{\mathbf{A}^0}(\xi)\,d\xi.
$$

A similar RHP for $\mathbf{B}^0$ on $\Sigma_{\mathbf{B}}$ reads

$$
\begin{cases}
M_+^{\mathbf{B}^0}(\lambda) = M_-^{\mathbf{B}^0}(\lambda)J^{\mathbf{B}^0}(\lambda), & \lambda\in\Sigma_{\mathbf{B}},\\
M^{\mathbf{B}^0}(\lambda)\to\mathcal{I}, & \lambda\to\infty.
\end{cases}
$$

Utilizing (3.9)–(3.11) and $\omega_{\mathbf{B}^0}$, one has

$$
J^{\mathbf{A}^0}(\lambda) = \overline{\tau}(J^{\mathbf{B}^0})(-\overline{\lambda})\tau.
$$

By uniqueness

$$M_1^{\mathbf{A}^0}(\lambda)=\overline{\tau}\big(M^{\mathbf{B}^0}\big)(-\overline{\lambda})\tau$$

and

$$M_1^{\mathbf{A}^0}=-\overline{\tau}\big(M_1^{\mathbf{B}^0}\big)\tau.$$

Consequently, we have

$$\mathcal{U}(x,t)=\begin{pmatrix}\mathbf{u}(x,t)\\ \mathbf{v}(x,t)\end{pmatrix}=\frac{i}{\sqrt{12\lambda_0 t}}\left(M_1^{\mathbf{A}^0}-\sigma_1\big(\overline{M_1^{\mathbf{A}^0}}\big)\right)_{12}+O\left(\frac{\log t}{t}\right).$$

3.7. The final transformation

To solve for $\big(M_1^{\mathbf{A}^0}\big)_{12}$ explicitly, it is worthy considering the following transformation

$$\Psi(\lambda)=H(\lambda)\left(-\frac{1}{\lambda}\right)^{iv\sigma}e^{i\lambda^2\sigma/4},\quad H(\lambda)=(\delta_{\mathbf{A}})^{-\sigma}M^{\mathbf{A}^0}(\lambda)(\delta_{\mathbf{A}})^{\sigma}.$$

Therefore

$$\Psi_+(\lambda)=\Psi_-(\lambda)\upsilon(-\lambda_0),\quad \upsilon=(-\lambda)^{iv\widehat{\sigma}}e^{-i\lambda^2\widehat{\sigma}/4}(\delta_{\mathbf{A}})^{-\widehat{\sigma}}J^{\mathbf{A}^0}.$$

Because of the jump matrix is independent of λ, we have

$$\frac{d\Psi_+(\lambda)}{d\lambda}=\frac{d\Psi_-(\lambda)}{dk}\upsilon(-\lambda_0).$$

Since $\frac{d\Psi(\lambda)}{d\lambda}$ and Ψ have the same jump matrix along any of the rays, it follows that $\frac{d\Psi(\lambda)}{d\lambda}\Psi^{-1}(\lambda)$ is holomorphic in the complex plane and admits a polynomial dependence on λ at $\lambda\to\infty$. In reality

$$\begin{aligned}\frac{d\Psi(\lambda)}{d\lambda}\Psi^{-1}(\lambda)&=\frac{i\lambda}{2}H(\lambda)\sigma H^{-1}(\lambda)-\frac{iv}{\lambda}H(\lambda)\sigma H^{-1}(\lambda)+\frac{dH(\lambda)}{d\lambda}H^{-1}(\lambda)\\&=\frac{i\lambda}{2}\sigma-\frac{i}{2}\delta_{\mathbf{A}}\sigma\big[\sigma,M_1^{\mathbf{A}^0}\big]\delta_{\mathbf{A}}^{-\sigma}+O\left(\frac{1}{\lambda}\right).\end{aligned}$$

Thus,

$$\frac{d\Psi(\lambda)}{d\lambda}=\frac{i\lambda}{2}\sigma\Psi(\lambda)+\beta\Psi(\lambda),\tag{3.12}$$

where

$$\beta=-\frac{i}{2}\delta_{\mathbf{A}}^{\sigma}\big[\sigma,M_1^{\mathbf{A}^0}\big]\delta_{\mathbf{A}}^{-\sigma}=\begin{pmatrix}0&\beta_{12}\\ \beta_{21}&0\end{pmatrix}.$$

Particularly

$$(M_1^{\mathbf{A}^0})_{12}=i(\delta_{\mathbf{A}})^2\beta_{12}.$$

Let

$$\Psi(\lambda) = \begin{pmatrix} \Psi_{11}(\lambda) & \Psi_{12}(\lambda) \\ \Psi_{21}(\lambda) & \Psi_{22}(\lambda) \end{pmatrix}.$$

It follows from (3.12) that

$$\begin{aligned}
\frac{d^2\beta_{12}\Psi_{11}(\lambda)}{d\lambda^2} &= \left(\beta_{21}\beta_{12} + 0.5i - \frac{\lambda^2}{2}\right)\beta_{21}\Psi_{11}(\lambda), \\
\frac{d^2\Psi_{22}(\lambda)}{d\lambda^2} &= \left(\beta_{21}\beta_{12} - 0.5i - \frac{\lambda^2}{4}\right)\Psi_{22}(\lambda), \\
\beta_{21}\Psi_{21}(\lambda) &= \frac{1}{\beta_{12}}\left(\frac{d\beta_{21}\Psi_{11}(\lambda)}{d\lambda} - \frac{i\lambda}{2}\beta_{21}\Psi_{11}(\lambda)\right), \\
\beta_{21}\Psi_{12}(\lambda) &= \frac{d\Psi_{22}(\lambda)}{d\lambda} + \frac{i\lambda}{2}\Psi_{22}(\lambda).
\end{aligned}$$

As is well known that

$$g(\zeta) = c_1 D_a(\zeta) + c_2 D_a(-\zeta),$$

admits the solution of Weber's equation [11, 18, 37]

$$g''(\zeta) + \left(a + \frac{1}{2} - \frac{\zeta^2}{4}\right) g(\zeta) = 0,$$

where $D_a(\zeta)$ is the standard parabolic-cylinder function, and admits the following relations

$$\begin{aligned}
D_a'(\zeta) + \frac{\zeta}{2} D_a(\zeta) - a D_{a-1}(\zeta) &= 0, \\
D_a(\pm\zeta) = \Gamma(a+1)e^{i\pi a/2} D_{-a-1}(\pm i\zeta) &+ \frac{\Gamma e^{-i\pi a/2}}{\sqrt{2\pi}} D_{-1-a}(\mp i\zeta).
\end{aligned}$$

We know that as $\zeta \to \infty$ [37]

$$D_a(\zeta) = \begin{cases} \zeta^a e^{-\zeta^2/4}\left(1 + O\left(\frac{1}{\zeta^2}\right)\right), & |\arg\zeta| < \frac{3\pi}{4}, \\ \zeta^a e^{-\zeta^2/4}\left(1 + O\left(\frac{1}{\zeta^2}\right)\right) - \frac{\sqrt{2\pi}}{\Gamma(-a)} e^{a\pi i + \zeta^2/4}\left(1 + O\left(\frac{1}{\zeta^2}\right)\right), & \frac{\pi}{4} < \arg\zeta < \frac{5\pi}{4}, \\ \zeta^a e^{-\zeta^2/4}\left(1 + O\left(\frac{1}{\zeta^2}\right)\right) - \frac{\sqrt{2\pi}}{\Gamma(-a)} e^{-a\pi i + \zeta^2/4}\left(1 + O\left(\frac{1}{\zeta^2}\right)\right), & -\frac{5\pi}{4} < \arg\zeta < -\frac{\pi}{4}, \end{cases}$$

where Γ is the Gamma function.

Let $a = -i\beta_{21}\beta_{12}$, we have

$$\begin{aligned}
\beta_{21}\Psi_{11}(\lambda) &= c_1 D_a\big(e^{3\pi i/4}\lambda\big) + c_2 D_a\big(e^{-\pi i/4}\lambda\big), \\
\Psi_{22}(\lambda) &= c_3 D_{-a}\big(e^{-3\pi i/4}\lambda\big) + c_4 D_{-a}\big(e^{\pi i/4}\lambda\big).
\end{aligned}$$

It follows from $\arg\lambda \in (\frac{3\pi}{4}, \pi) \cup (-\pi, -\frac{3\pi}{4})$ and $\lambda \to \infty$ that

$$\Psi_{11}(\lambda)(-\lambda)^{-iv} e^{-i\lambda^2/4} \to \mathcal{I}, \quad \Psi_{22}(\lambda)(-\lambda)^{-iv} e^{i\lambda^2/4} \to 1,$$

and

$$\beta_{21}\Psi_{11}(\lambda)=\beta_{21}e^{-\pi v/4}D_a(e^{3\pi i/4}\lambda),\quad v=-\beta_{21}\beta_{12},\quad \Psi_{22}(\lambda)=e^{-\pi v/4}D_{-a}(e^{-3\pi i/4}\lambda).$$

Thus

$$\Psi_{21}(\lambda)=\frac{\beta_{21}e^{-\pi v/4}}{\beta_{21}\beta_{12}}\left(D'_a(e^{3\pi i/4}\lambda)-\frac{i\lambda}{2}D_a(e^{3\pi i/4}\lambda)\right)=\beta_{21}e^{\pi(i+v)/4}D_{a-1}(e^{3\pi i/4}\lambda),$$

$$\Psi_{22}=e^{-\pi v/4}\left(D'_{-a}(e^{-3\pi i/4}\lambda)+\frac{i\lambda}{2}D_{-a}(e^{-3\pi i/4}\lambda)\right)=ae^{\pi(i-v)/4}D_{-a-1}(e^{-3\pi i/4}\lambda).$$

As $\arg\lambda\in\left(\frac{\pi}{4},\frac{3\pi}{4}\right)$ and $\lambda\to\infty$, we have

$$\Psi_{11}(\lambda)(-\lambda)^{iv}e^{-i\lambda^2/4}\to\mathcal{I},\quad \Psi_{22}(\lambda)(-\lambda)^{iv}e^{i\lambda^2/4}\to 1,$$

then

$$\beta_{21}\Psi_{11}(\lambda)=\beta_{21}e^{3\pi v/4}D_a\big(e^{-\pi i/4}\lambda\big),\quad \Psi_{22}(\lambda)=e^{-\pi v/4}D_{-a}\big(e^{-3\pi i/4}\lambda\big).$$

So that

$$\Psi_{21}(\lambda)=\frac{\beta_{21}e^{3\pi v/4}}{\beta_{21}\beta_{12}}\left(D'_a(e^{-\pi i/4}\lambda)-\frac{i\lambda}{2}D_a(e^{-\pi i/4}\lambda)\right)=\beta_{21}e^{-3\pi(i-v)/4}D_{a-1}(e^{-\pi i/4}\lambda),$$

$$\beta_{21}\Psi_{12}(\lambda)=ae^{\pi(i-v)/4}D_{-a-1}(e^{-3\pi i/4}\lambda).$$

Along the ray $\arg\lambda=3\pi/4$, we have

$$\Psi_+(\lambda)=\Psi_-(\lambda)\begin{pmatrix}\mathcal{I} & 0\\ -\gamma(-\lambda_0) & 1\end{pmatrix},$$

$$\beta_{21}e^{\pi(i-v)/4}D_{a-1}(e^{3\pi i/4}\lambda)=\beta_{21}e^{-3\pi(i-v)/4}D_{a-1}(e^{-\pi i/4}\lambda)-\gamma(-\lambda_0)e^{-\pi v/4}D_{-a}(e^{-3\pi i/4}\lambda),$$

$$D_{-a}(e^{-3\pi i/4}\lambda)=\frac{\Gamma(-a+1)e^{-\pi ia/2}}{\sqrt{2\pi}}D_{a-1}(e^{-\pi i/4}\lambda)+\frac{\Gamma(-a+1)e^{\pi ia/2}}{\sqrt{2\pi}}D_{a-1}(e^{3\pi i/4}\lambda),$$

$$\beta_{21}=\Gamma(-a+1)e^{-\pi v/2}e^{3\pi i/4}\gamma(-\lambda_0)=\frac{-v\Gamma(-iv)e^{\pi v/2}e^{-3\pi i/4}}{\sqrt{2\pi}}\gamma(-\lambda_0).$$

It is clear to see that $\Psi^{-1}(\lambda)$ and $\Psi^\dagger(\overline{\lambda})$ satisfy the same RHP. Due to the uniqueness, we get

$$\Psi^{-1}(\lambda)=\Psi^\dagger(\overline{\lambda}),$$

and thus

$$\beta_{12}=-\beta_{21}^\dagger=\frac{v\Gamma(iv)e^{\pi v/2}e^{-\pi i/4}}{\sqrt{2\pi}}\sigma_1\gamma^T(\lambda_0).$$

It follows from $\beta_{21}\beta_{12}=-v$ and $\Gamma(-iv)=\overline{\Gamma}(iv)$ that

$$\frac{v\Gamma(iv)e^{\pi v/2}}{\sqrt{2\pi}}=\frac{\sqrt{v}}{|\gamma(\lambda_0)|}.$$

Summarizing the above analysis, the following theorem holds.

Theorem 3.14. *Let (u_0, v_0) belong to the Schwartz space $\mathcal{S}(\mathbb{R})$. Then suppose $u(x,t)$, $v(x,t)$ can solve the CSS equation (1.2). As $x < 0$ and $\left|\frac{x}{t}\right|$ is bounded, the solutions $u(x,t)$, $v(x,t)$ admit the following leading asymptotics*

$$u(x,t) = \frac{u_{as}(x,t)}{\sqrt{t}} + O\left(\frac{\log t}{t}\right), \quad v(x,t) = \frac{v_{as}(x,t)}{\sqrt{t}} + O\left(\frac{\log t}{t}\right),$$

where

$$\begin{aligned}
u_{as}(x,t) &= \frac{\sqrt{v}}{\sqrt{12\lambda_0}|\gamma(\lambda_0)|}\left(|\gamma_2(\lambda_0)|e^{i(\phi+\arg\gamma_2(\lambda_0))} + |\gamma_1(\lambda_0)|e^{-i(\phi+\arg\gamma_1(\lambda_0))}\right),\\
v_{as}(x,t) &= \frac{\sqrt{v}}{\sqrt{12\lambda_0}|\gamma(\lambda_0)|}\left(|\gamma_4(\lambda_0)|e^{i(\phi+\arg\gamma_4(\lambda_0))} + |\gamma_3(\lambda_0)|e^{-i(\phi+\arg\gamma_3(\lambda_0))}\right),\\
\phi &= 16t\lambda_0^3 + \arg\Gamma(iv) + v\log(192\lambda_0^3 t) + \frac{1}{\pi}\int_{-\lambda_0}^{\lambda_0}\log\left(\frac{1+|\gamma(\xi)|^2}{1+|\gamma(\lambda_0)|^2}\right)\frac{d\xi}{\xi+\lambda_0} - \frac{5\pi}{4},\\
v &= \frac{1}{2\pi}\log(1+|\gamma(\lambda_0|^2),
\end{aligned}$$

and γ_1, γ_2, γ_3, γ_4 are the 1, 2, 3, 4*-th component of the vector function γ given by* (2.17)*, $\lambda_0 = \sqrt{-x/(12t)}$, Γ is a Gamma function.*

4. Conclusions and discussions

In this work, we have obtained a 5×5 matrix Riemann-Hilbert problem to tackle the Cauchy problem for the CSS equation (1.2) on the line, which can help us to obtain a representation for the solution of the CSS equation (1.2). We then employ the approach of the Deift-Zhou steepest descent to discuss the long-time asymptotics of the CSS equation (1.2). Similarly to [44], starting from the CSS equation (1.2), if we impose the following constraint

$$v(x,t) = \overline{u}(-x,t),$$

we then obtain

$$u_t + \left\{u_{xxx} + 6(|u|^2 + |u(-x,t)|^2)u_x + 3u(|u|^2 + |u(-x,t)|^2)_x\right\} = 0, \tag{4.1}$$

which is a nonlocal equation of reverse-time type. If we impose the solution constraint

$$v(x,t) = \overline{u}(-x,-t),$$

the coupled Sasa-Satsuma (1.2) reduces to

$$u_t + \left\{u_{xxx} + 6(|u|^2 + |u(-x,-t)|^2)u_x + 3u(|u|^2 + |u(-x,-t)|^2)_x\right\} = 0, \tag{4.2}$$

which is a nonlocal equation of reverse-space-time type. The two equations differ from the other nonlocal equations of reverse-time and reverse-space-time types [2, 15] in the nonlinear terms. Both of our nonlocal equations (4.1) and (4.2) are also integrable which have clear physical meanings.

Finally, we state that there exist several methods to derive exact solutions for the nonlinear wave equations, such as Darboux transformation, Inverse scattering transform, Riemann-Hilbert approach, Deift-Zhou steepest descent method, Hirota method, dressing method, Wronskian technique etc. Consequently, it is very worthy to consider whether the nonlocal equations (4.1) and (4.2) can be solved by using these approaches? These ideas will be left for future discussions.

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