# Global Existence and Exponential Decay of Strong Solutions to the 2D Density-dependent Nematic Liquid Crystal Flows with Vacuum

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Abstract. This paper deals with the 2D incompressible nematic liquid crystal flows with density-dependent viscosity in bounded domain. The global well-posedness of strong solutions are established in the vacuum cases, provided the assumption that $\overline{\rho} + \|\nabla d_0\|_{L^2}$ is suitably small with large velocity, which extends the recent work [Discrete Contin. Dyn. Syst. 37 (2017), 4907–4922] and [Methods Appl. Anal. 22 (2015), 201–220] to the case of variable viscosity. Furthermore, the exponential decay of the solution is also obtained.

## 1. Introduction

The motion of incompressible nematic liquid crystal flows, which are described by the following simplified version of the Ericksen-Leslie equations:

$$\text{(1.1)} \quad \begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ \rho u_t + \rho(u \cdot \nabla)u - \operatorname{div}(\mu(\rho)\nabla u) + \nabla P = -\lambda \operatorname{div}(\nabla d \odot \nabla d), \\ \operatorname{div} u = 0, \quad |d| = 1, \\ d_t + u \cdot \nabla d = \theta(\Delta d + |\nabla d|^2 d), \end{cases}$$

in $\Omega \times [0, \infty)$, where $\Omega$ is a bounded domain with smooth boundary in $\mathbb{R}^2$. Here $u \colon \Omega \times [0, \infty) \to \mathbb{R}^2$ represents the velocity field of the flow, $d \colon \Omega \times [0, \infty) \to \mathcal{S}^2$, the unit sphere in $\mathbb{R}^2$, represents the macroscopic molecular orientation of the liquid crystal material, $\rho$ and $P$ are scalar functions, respectively, denoting the density of the fluid and the pressure arising from the usual assumption of incompressibility $\operatorname{div} u = 0$. The positive constants $\lambda$ and $\theta$ represent viscosity of fluid, competition between kinetic and potential energy, and microscopic elastic relaxation time respectively. The viscosity coefficient $\mu = \mu(\rho)$ is a general function of density, which is assumed to satisfy

$$\text{(1.2)} \quad \mu \in C^1[0, \infty) \quad \text{and} \quad 0 < \underline{\mu} \leq \mu \leq \overline{\mu} < \infty \quad \text{on } [0, \infty)$$

for some positive constant $\underline{\mu}$ and $\overline{\mu}$. Without loss of generality, both $\lambda$ and $\theta$ are normalized to 1. The symbol $\nabla d \odot \nabla d$, which exhibits the property of the anisotropy of the material, denotes the $n \times n$ matrix whose $(i,j)$-th entry is given by $\partial_i d \cdot \partial_j d$, for $i,j = 1,2$.

Noticing that

$$\mathrm{div}(\nabla d \odot \nabla d) = \Delta d \cdot \nabla d + \nabla \left( \frac{|\nabla d|^2}{2} \right),$$

one can rewrite equation $(1.1)_2$ as

$$(1.3) \qquad \rho u_t + \rho(u \cdot \nabla)u - \mathrm{div}(\mu(\rho)\nabla u) + \nabla P = -\lambda \Delta d \cdot \nabla d.$$

We consider an initial boundary value problem for (1.1) with the following initial and boundary conditions:

$$(1.4) \qquad (\rho, \rho u, d)|_{t=0} = (\rho_0, \rho_0 u_0, d_0), \quad |d_0| = 1 \quad \text{in } \Omega; \quad u = 0, \quad d = d_0' \quad \text{on } \partial\Omega$$

with $d_0' \in \mathcal{S}^2$ being given with compatibility, $\mathrm{div}\, u_0 = 0$ in $\Omega$ and $d_0' \in C^1(\overline{\Omega})$ satisfying $\nabla d_0' = 0$ on the boundary $\partial\Omega$ (see e.g., [7]).

The hydrodynamics of nematic liquid crystals developed by Ericksen [2] and Leslie [12] in the 1960's, but it still retains most important mathematical structures as well as most of the essential difficulties of the original Ericksen-Leslie model. Mathematically, system (1.1) is a strongly coupled system between the nonhomogeneous incompressible Navier-Stokes equations and the transported heat flows of harmonic map, and thus, its mathematical analysis is full of challenges.

When the fluid is the homogeneous case, the systems (1.1) are the simplified model of nematic liquid crystals with constant density. Huang-Wang [9] have provided a blow-up criterion of strong solutions, and the well-posedness of (1.1) for an initial data $(u_0, d_0)$ with small $\mathrm{BMO}^{-1} \times \mathrm{BMO}$-norm, and with small $L^3_{\mathrm{uloc}}(\mathbb{R}^3)$-norm has been shown by Wang [25] and Hineman-Wang [5], and small $L^3(\mathbb{R}^3_+)$-norm has been established by Huang-Wang-Wen [8], respectively. Li [13] proved the local well-posedness of mild solutions with $L^\infty$ initial data, in particular, that the initial energy may be infinite. Most recently, Lin-Wang [20] have shown the existence of global weak solutions in dimension three under the assumption that the initial director field $d_0 \subset \mathbb{S}^2_+$.

Let's go back to the system (1.1). When viscosity $\mu$ is a constant, there is a huge literature on the studies about well-posedness of solutions. In the absence of vacuum, Li [14] and Wen-Ding [26] established the global strong solutions in dimension two. For small data, Li-Wang [19] obtained global strong solutions in three dimensional space. On the other hand, when the initial density allows to vanish, if the initial data are small (in some sense) and satisfy the additional compatibility condition, Li [15], Yu-Zhang [27] and Ding-Huang-Xia [1] obtained the global existence of strong solutions in dimension three. Li [18] considered the case on the bounded domains in $\mathbb{R}^2$. Recently, Liu-Zhang [23] and

Liu et al. [22] independently extended the result of [18] to $\mathbb{R}^2$ with large initial data, provided that the initial orientation $d_0 = (d_{01}, d_{02}, d_{03})$ satisfies a geometric condition

(1.5)  $$d_{03} \geq \epsilon_0 \quad \text{for some positive } \epsilon_0 > 0.$$

Li-Liu-Zhong [17] got the same result under small initial data without the additional geometric condition (1.5). Recently, when the viscosity coefficient is a function of the density of fluid, if the initial data satisfies the following compatibility condition

$$- \operatorname{div}(\mu(\rho_0)\nabla u_0) + \nabla P_0 + \operatorname{div}(\nabla d_0 \odot \nabla d_0) = \sqrt{\rho_0}g \quad \text{for some } (\nabla P_0, g) \in L^2,$$

Gao-Tao-Yao [3] established the local unique strong solutions to the system (1.1) in a bounded domain of $\mathbb{R}^N$ ($N = 2, 3$) with the Neumann boundary condition, and they also obtained the Serrin-type blow up criterion of the strong solutions.

Especially, Li [18] (also see [15]) studied global regularity for $d = d_0'$ on $\partial\Omega$ when initial energy is small. However, when deal with the case of variable viscosity, some new difficulties need to be overcome and the similar result is still unclear. Our main aim in this paper is to offer an answer to this open problem. Indeed, we will establish the global existence of strong solutions to the 2D incompressible nematic liquid crystal system (1.1)–(1.4), under the assumption that the initial data in some norm is small enough, and we don't need any smallness on the initial velocity.

Before stating the main results, we first explain the notations and conventions used throughout this paper. Set

$$\int f \, dx := \int_\Omega f \, dx.$$

For $1 \leq r \leq \infty$ and $k \in \mathbb{N}$, the Sobolev spaces are defined in a standard way,

$$L^r := L^r(\Omega), \quad W^{k,r} := \{f \in L^r : D^\alpha f \in L^r, |\alpha| \leq k\}, \quad H^k := W^{k,2}.$$

Moreover, $H_0^1$ and $H_{0,\sigma}^1$ represent the closure of $C_0^\infty$ and $C_{0,\sigma}^\infty := \{f \in C_0^\infty : \operatorname{div} f = 0\}$ in $H^1$, respectively.

Our main result can be stated as follows:

**Theorem 1.1.** *Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^2$ and $q \in (2, \infty)$ be a fixed constant. For any given positive numbers $M_0$, $M_1$, $M_2$, and $M_3$, suppose that the initial data $(\rho_0, \rho_0 u_0, d_0)$ satisfies the regularity conditions*

(1.6)  $0 \leq \rho_0 \leq \overline{\rho}, \quad \rho_0 \in H^1 \cap W^{1,q}, \quad \nabla\mu(\rho_0) \in L^q, \quad u_0 \in H_{0,\sigma}^1, \quad d_0 \in H^2, \quad |d_0| = 1.$

*Then for*

$$\overline{\rho} + \|\nabla d_0\|_{L^2} =: M_0, \qquad \|\sqrt{\rho_0}u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2 =: M_2,$$
$$\|\nabla\mu(\rho_0)\|_{L^q} =: M_1, \qquad \overline{\mu}\|\nabla u_0\|_{L^2}^2 + \|\Delta d_0\|_{L^2}^2 =: M_3,$$

*there exists some small positive constant $\epsilon_0$ depending on $\Omega$, $\underline{\mu}$, $\overline{\mu}$, $M_1$, $M_2$, and $M_3$ such that if*

$$M_0 \leq \epsilon_0,$$

*then the initial boundary value problem (1.1)–(1.4) admits a unique global strong solution $(\rho, u, d, P)$ satisfying that any $0 < \tau < T < \infty$ for any $r$ with $2 < r < \min\{q, 3\}$,*

$$(1.7) \quad \begin{cases} 0 \leq \rho \in C([0, T]; H^1 \cap W^{1,q}), \\ \nabla u \in L^\infty(0, T; L^2) \cap L^\infty(\tau, T; H^1) \cap L^2(\tau, T; W^{1,r}), \\ P \in L^\infty(0, T; L^2) \cap L^\infty(\tau, T; H^1) \cap L^2(\tau, T; W^{1,r}), \\ d \in L^\infty(0, T; H^2) \cap L^\infty(\tau, T; H^3) \cap L^2(\tau, T; H^4), \\ d_t \in L^\infty(\tau, T; H^1) \cap L^2(\tau, T; H^2), \\ \nabla \mu(\rho) \in C([0, T]; L^q), \quad |d| = 1 \quad in \ \Omega \times [0, T]. \end{cases}$$

*Moreover, it holds that*

$$(1.8) \quad \sup_{0 \leq t < \infty} \|\nabla \rho\|_{L^2 \cap L^q} \leq 2\|\nabla \rho_0\|_{L^2 \cap L^q}, \quad \sup_{0 \leq t < \infty} \|\nabla \mu(\rho)\|_{L^q} \leq 2\|\nabla \mu(\rho_0)\|_{L^q},$$

*and that there exists some positive constant $\sigma$ depending only on the constant of Poincaré's inequality, $\Omega$, $\overline{\rho}$, and $\underline{\mu}$ such that for all $t \geq 1$,*

$$(1.9) \quad \|\nabla d_t\|_{L^2}^2 + \|\nabla d\|_{H^2}^2 + \|\nabla u(\cdot, t)\|_{H^1}^2 + \|P(\cdot, t)\|_{H^1}^2 \leq Ce^{-\sigma t},$$

*where $C$ depends only on $\Omega$, $q$, $\underline{\mu}$, $\overline{\mu}$, $M_1$, $M_2$, and $M_3$.*

*Remark* 1.2. Theorem 1.1 extends the result of Li [18] (also see [15]) to the case of variable viscosity for arbitrarily large initial velocity with a smallness assumption only on $\overline{\rho} + \|\nabla d_0\|_{L^2}$, which is in sharp contrast to [15, 18] where they need the smallness assumption on the initial energy.

*Remark* 1.3. Compared with [15, 18], there is no need to impose the additional compatibility conditions on the initial data for the global existence of the strong solutions. Furthermore, it should be pointed out that the exponential decay of the global strong solution in (1.9) is completely new for the 2D nonhomogeneous nematic liquid crystal flows.

*Remark* 1.4. When $d$ is a constant vector and $|d| = 1$, system (1.1) turns to be the nonhomogeneous incompressible Navier-Stokes equations. Theorem 1.1 is different from the result of Huang-Wang [11], since they need the smallness assumptions on $\|\nabla \mu(\rho_0)\|_{L^q}$. Moveover, Theorem 1.1 established the exponential decay-in-time properties, since the corresponding decay-in-time rates of [11] for the strong solutions to (1.1)–(1.4) is algebraic.

We now make some comments on the analysis in this paper. To extend the local strong solutions whose existence is obtained by Lemma 2.1 globally in time, one needs to establish global a *priori* estimates on smooth solutions to (1.1)–(1.4) in suitable higher norm. The key ingredient here is to get the time-independent bounds on the $L^1(0, T; L^\infty)$-norm of $\nabla u$ and then the $L^\infty(0, T; L^q)$-norm of $\nabla \mu(\rho)$ and the $L^\infty(0, T; L^2)$-one of $\nabla \rho$. Here, some key ideas due to [11] will be used, which deals with the 2D incompressible Navier-Stokes. It should be pointed out that the crucial techniques of proof in [11] cannot be adapted directly to the situation treated here. On the one hand, as mentioned by Huang-Wang [11], the methods used in [11] depend crucially on the smallness assumptions on $\|\nabla \mu(\rho_0)\|_{L^q}$ and the following inequality

$$\|\sqrt{\rho} u\|_{L^4}^2 \leq C(\overline{\rho}, \Omega)(1 + \|\rho u\|_{L^2})\|\nabla u\|_{L^2}\sqrt{\log(2 + \|\nabla u\|_{L^2}^2)}.$$

On the other hand, compared with [11], for system (1.1)–(1.4) treated here, the strong coupling terms and strong nonlinear terms, such as $u \cdot \nabla d$, $\mathrm{div}(\nabla d \odot \nabla d)$, and $|\nabla d|^2 d$ will bring out some new difficulties. Hence, some new ideas are needed here. First, using the initial layer analysis (see [4, 6, 10]), we succeed in bounding the $L^1(0, \min\{1, T\}; L^\infty)$-norm of $\nabla u$ by $\overline{\rho}$ and $\sup_{0 \leq t \leq T} \|\nabla d\|_{L^2}$ (see (3.34)). Then, in order to estimate the $L^1(\min\{1, T\}, T; L^\infty)$-norm of $\nabla u$, we find that $\|\sqrt{\rho} u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2$ in fact decays at the rate of $e^{-\sigma t}$ ($\sigma > 0$) for large time (see (3.5)), which can be achieved by combining the standard energy equality (see (3.7)) with the following fact

$$\|\sqrt{\rho} u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 \leq \|\rho\|_{L^\infty}\|u\|_{L^2}^2 + C\|\Delta d\|_{L^2}^2 \leq C(\|\nabla u\|_{L^2}^2 + |\Delta d\|_{L^2}^2),$$

due to $(1.1)_1$, and Poincaré's inequality. With this key exponential decay-in-time rate at hand, we can obtain that both $\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2$ and $\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2$ decay at the same rate as $e^{-\sigma t}$ ($\sigma > 0$) for large time (3.10) and (3.24). In fact, all these exponential decay-in-time rates are the key to obtaining the desired uniform bound (with respect to time) on the $L^1(\min\{1, T\}, T; L^\infty)$-norm of $\nabla u$ (see (3.37)). Finally, using these a *priori* estimates and the fact that the velocity is divergent free, we establish the time-independent estimates on the $\|\nabla \mu(\rho)\|_{L^q}$, $\|\nabla u\|_{L^2}^2$, and $\|\Delta d\|_{L^2}^2$ which guarantee the extension of local strong solutions (see Proposition 3.1).

The rest of this paper is organized as follows. In Section 2, we collect some elementary facts and inequalities that will be used later. Section 3 is devoted to the a *priori* estimates. Finally, we will prove Theorem 1.1 in Section 4.

## 2. Preliminaries

In this section, we shall enumerate some auxiliary lemmas.

We start with the following local existence of strong solutions can be shown by similar strategies as in [3, 16, 24].

**Lemma 2.1.** *Assume that $(\rho_0, \rho_0 u_0, d_0)$ satisfies (1.6). Then there exists a small time $T_0 > 0$ and a unique strong solution $(\rho, u, d, P)$ to the problem (1.1)–(1.4) in $\Omega \times (0, T_0)$ satisfying (1.7).*

The following regularity results on the Stokes equations will be useful for our derivation of higher order a *priori* estimates (see [11]).

**Lemma 2.2.** *Assume that $\rho \in W^{1,q}$, $2 < q < \infty$, $0 \le \rho \le \overline{\rho}$, and $\mu(\rho)$ satisfies (1.2) on $[0, \overline{\rho}]$. Let $(u, P) \in H_{0,\sigma}^1 \times L^2$ be the unique weak solution to the following boundary value problem*

$$- \operatorname{div}(\mu(\rho)\nabla u) + \nabla P = F, \quad \operatorname{div} u = 0 \quad in \ \Omega, \quad \int P \, dx = 0.$$

*Then we have the following regularity results:*

- *If $F \in L^2$, then $(u, P) \in H^2 \times H^1$ and*

$$
\begin{aligned}
(2.1) \qquad & \|u\|_{H^2} \le C\|F\|_{L^2}(1 + \|\nabla\mu(\rho)\|_{L^q})^{q/(q-2)}, \\
& \|P\|_{H^1} \le C\|F\|_{L^2}(1 + \|\nabla\mu(\rho)\|_{L^q})^{(2q-2)/(q-2)}.
\end{aligned}
$$

- *If $F \in L^r$ for some $r \in (2, q)$, then $(u, P) \in W^{2,r} \times W^{1,r}$ and*

$$
\begin{aligned}
(2.2) \qquad & \|u\|_{W^{2,r}} \le C\|F\|_{L^r}(1 + \|\nabla\mu(\rho)\|_{L^q})^{\frac{qr}{2(q-r)}}, \\
& \|P\|_{W^{1,r}} \le C\|F\|_{L^r}(1 + \|\nabla\mu(\rho)\|_{L^q})^{1 + \frac{qr}{2(q-r)}}.
\end{aligned}
$$

*Here the constant $C$ in (2.1) and (2.2) depends on $\Omega$, $q$, $r$, $\underline{\mu}$, $\overline{\mu}$.*

## 3. A priori estimates

In this section, we will establish some necessary a *priori* bounds of local strong solutions $(\rho, u, d, P)$ to the problem (1.1)–(1.4) whose existence is guaranteed by Lemma 2.1. Thus, let $T > 0$ be a fixed time and $(\rho, u, d, P)$ be the smooth solution (1.1)–(1.4) on $\Omega \times (0, T]$ with smooth initial data $(\rho_0, u_0, d_0)$ satisfying (1.6). For simplicity, we shall the letters $C$ and $C_i$ $(i = 1, 2, \ldots)$ (except Lemma 3.6) to denote the generic constants which may be dependent on $\Omega$, $q$, $\underline{\mu}$, $\overline{\mu}$, and $M_1$, but independent of $T$, $M_2$, and $M_3$.

We aim to get the following key a *priori* estimates on $(\rho, u, d, P)$.

**Proposition 3.1.** *There exists some small positive constant $\epsilon_0$ depending only on $\Omega$, $q$, $\underline{\mu}$, $\overline{\mu}$, $M_1$, $M_2$, and $M_3$ such that if $(\rho, u, d, P)$ is a smooth solution of (1.1)–(1.4) on $\Omega \times (0, T]$ satisfying*

(3.1)
$$\sup_{0 \leq t \leq T} (\overline{\rho} + \|\nabla d\|_{L^2}) \leq M_0, \quad \sup_{0 \leq t \leq T} \|\nabla \mu(\rho)\|_{L^q} \leq 4M_1,$$
$$\sup_{0 \leq t \leq T} e^{\sigma t} \left( \underline{\mu} \|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 \right) \leq 4 \left[ 2 \left( \frac{\overline{\mu}}{\underline{\mu}} + 1 \right) M_2 + \frac{3}{2} M_3 \right],$$

*the following estimates hold*

(3.2)
$$\sup_{0 \leq t \leq T} (\overline{\rho} + \|\nabla d\|_{L^2}) \leq \frac{M_0}{2}, \quad \sup_{0 \leq t \leq T} \|\nabla \mu(\rho)\|_{L^q} \leq 2M_1,$$
$$\sup_{0 \leq t \leq T} e^{\sigma t} \left( \underline{\mu} \|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 \right) \leq 2 \left[ 2 \left( \frac{\overline{\mu}}{\underline{\mu}} + 1 \right) M_2 + \frac{3}{2} M_3 \right],$$

*provided*

$$M_0 \leq \epsilon_0,$$

*where*

$$\epsilon_0 := \min \left\{ 1, \frac{1}{\sqrt{2C_1}}, \frac{1}{(8C_2)^3}, \frac{1}{(8C_2)^2 M_3^{1/2}}, \frac{1}{4C_3 M_3}, \frac{1}{(4C_3)^2 M_3^{9/2}}, \frac{1}{12C_4 M_3^3}, \right.$$
$$\frac{1}{\left[ 12C_4 (M_3^{9/2} + 1) \right]^{3/2}}, \frac{1}{12C_4 M_3^{9/2}}, \frac{1}{(2C_5)^{1/2}}, \frac{1}{2C_6 M_3^{1/2}},$$
$$\left( \frac{\ln 2}{3C_7 M_3 (M_3^{9/2} + 1)^{3/2}} \right)^{4/3}, \left( \frac{\ln 2}{3C_7 M_3^{\frac{5r-6}{2r}} M_2} \right)^{r/(6r-6)},$$
$$\left( \frac{\ln 2}{3C_7 M_3^{7/12} (M_3^{9/4} + 1)^{7/6}} \right)^{4/3}, \frac{1}{2C_8}, \frac{1}{2C_9}, \frac{1}{12C_{10} M_3^{\frac{5r-6}{4r}} M_2},$$
$$\left. \frac{1}{12C_{10} M_3^{1/2} (M_3^{9/2} + 1)^{3/4}}, \frac{1}{12C_{10} M_3^{7/24} (M_3^{9/2} + 1)^{7/12}}, \frac{1}{4} \right\}.$$

Before proving Proposition 3.1, we establish some necessary a *priori* estimates, see Lemmas 3.2–3.5.

We begin with the nonnegativity and boundedness of density and basic energy estimates for $(\rho, u, d, P)$.

**Lemma 3.2.** *Let $(\rho, u, d, P)$ be a smooth solution to (1.1)–(1.4) on $[0, T]$. Then for*

(3.3)
$$\sigma := \min \left\{ \frac{1}{2\widetilde{C} C_p^2}, \frac{\underline{\mu}}{2C_p^2 \overline{\rho}} \right\},$$

*it holds that*

$$(3.4) \qquad 0 \leq \rho(x,t) \leq \sup_{x \in \Omega} \rho_0(x) = \overline{\rho},$$

$$(3.5) \qquad e^{\sigma t}\big(\|\sqrt{\rho}u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2\big) + \int_0^t e^{\sigma t}\big(\underline{\mu}\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2\big)\, dt$$
$$\leq M_2 \quad \text{for every } t \in [0,T]$$

*provided* $\sup_{0 \leq t \leq T}(\overline{\rho} + \|\nabla d\|_{L^2}) \leq \epsilon_1 := \min\{1, \frac{1}{\sqrt{2C_1}}\}$. *Here, $C_p$ is the constant of Poincaré's inequality.*

*Proof.* Note that (3.4) follows from the transport equation $(1.1)_1$ and making use of $(1.1)_3$ (see Lions [21, Theorem 2.1]). Next, multiplying $(1.1)_2$ and $(1.1)_4$ by $u$ and $-(\Delta d + |\nabla d|^2 d)$, respectively, summing the resulting equations up and integrating over $\Omega$, then it follows from integration by parts and using $(1.1)_1$ and $(1.1)_3$ that

$$\frac{1}{2}\frac{d}{dt}\int (\rho|u|^2 + |\nabla d|^2)\, dx + \int (\mu(\rho)|\nabla u|^2 + |\Delta d + |\nabla d|^2 d|^2)\, dx = 0,$$

from which, noticing that

$$|\Delta d + |\nabla d|^2 d|^2 = |\Delta d|^2 - |\nabla d|^4,$$

guaranteed by the condition $|d| = 1$, one arrives at

$$\frac{1}{2}\frac{d}{dt}\int (\rho|u|^2 + |\nabla d|^2)\, dx + \int (\mu(\rho)|\nabla u|^2 + |\Delta d|^2)\, dx$$
$$= \int |\nabla d|^4\, dx \leq C\|\nabla d\|_{L^2}^2\|\nabla d\|_{H^1}^2 \leq C_1\|\nabla d\|_{L^2}^2\|\Delta d\|_{L^2}^2,$$

which directly deduce that

$$(3.6) \qquad \frac{d}{dt}\int (\rho|u|^2 + |\nabla d|^2)\, dx + \int (\underline{\mu}|\nabla u|^2 + |\Delta d|^2)\, dx \leq 0,$$

provided $\sup_{0 \leq t \leq T}\|\nabla d\|_{L^2} \leq M_0 \leq \epsilon_1 := \min\{1, \frac{1}{\sqrt{2C_1}}\}$. Then, integrating (3.6) over $[0,T]$, one arrives at

$$\|\sqrt{\rho}u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 + \int_0^t \big(\underline{\mu}\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2\big)\, dt \leq M_2 \quad \text{for every } t \in [0,T].$$

Moveover, it follows from the Poincaré's inequality, and (3.4) that

$$\|\sqrt{\rho}u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 \leq \|\rho\|_{L^\infty}\|u\|_{L^2}^2 + \widetilde{C}C_p^2\|\Delta d\|_{L^2}^2$$
$$(3.7) \qquad\qquad \leq \frac{C_p^2\overline{\rho}}{\underline{\mu}}\big(\underline{\mu}\|\nabla u\|_{L^2}^2\big) + \widetilde{C}C_p^2\|\Delta d\|_{L^2}^2$$
$$\leq \sigma^{-1}\left(\frac{\underline{\mu}}{2}\|\nabla u\|_{L^2}^2 + \frac{1}{2}\|\Delta d\|_{L^2}^2\right),$$

where we have used the following inequality

$$(3.8) \qquad \|d\|_{H^2}^2 \le \widetilde{C}\|\Delta d\|_{L^2}^2.$$

Here, $\sigma$ is defined as in (3.3). Putting (3.7) into (3.6) yields

$$\frac{d}{dt}\big((\|\sqrt{\rho}u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2) + \sigma(\|\sqrt{\rho}u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2)\big) + \frac{1}{2}\big(\underline{\mu}\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2\big) \le 0,$$

that is

$$\frac{d}{dt}\big(e^{\sigma t}(\|\sqrt{\rho}u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2)\big) + e^{\sigma t}\big(\underline{\mu}\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2\big) \le 0,$$

which, integrated over $[0, t]$ gives (3.5). The proof of Lemma 3.2 is completed. $\qquad\square$

**Lemma 3.3.** *Let* $(\rho, u, d, P)$ *be a smooth solution to* (1.1)–(1.4) *satisfying* (3.1). *Then there exists a generic positive constant* $C$ *depending only on* $\Omega$, $q$, $\underline{\mu}$, $\overline{\mu}$, $M_1$, $M_2$, *and* $M_3$ *such that*

$$(3.9) \qquad \sup_{0 \le t \le \zeta(T)} (\underline{\mu}\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) + \int_0^{\zeta(T)} (\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2)\, dt \le M_3,$$

$$(3.10) \qquad \sup_{\zeta(T) \le t \le T} e^{\sigma t}(\underline{\mu}\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) + \int_{\zeta(T)}^T e^{\sigma t}(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2)\, dt \le CM_3,$$

*provided* $\sup_{0 \le t \le T}(\overline{\rho} + \|\nabla d\|_{L^2}) \le \epsilon_2 := \min\big\{\epsilon_1, \frac{1}{(8C_2)^3}, \frac{1}{(8C_2)^2 M_3^{1/2}}, \frac{1}{4C_3 M_3}, \frac{1}{(4C_3)^2 M_3^{9/2}}, \frac{1}{\left[12C_4(M_3^{9/2}+1)\right]^{3/2}}, \frac{1}{12C_4 M_3^3}, \frac{1}{12C_4 M_3^{9/2}}, \frac{1}{(2C_5)^{1/2}}\big\}$. *Here,* $\zeta(t)$ *is defined by*

$$\zeta(t) := \min\{1, t\}.$$

*Proof.* First, by $(1.1)_1$, we obtain the following equation

$$\mu(\rho)_t + u \cdot \nabla\mu(\rho) = 0.$$

Next, multiplying $(1.1)_2$ by $u_t$ in $L^2$, we obtain after using integration by parts over $\Omega$ and $(1.1)_3$ that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}&\int \mu(\rho)|\nabla u|^2\, dx + \int \rho|u_t|^2\, dx \\
&= -\int \rho u \cdot \nabla u \cdot u_t\, dx - \frac{1}{2}\int u \cdot \nabla\mu(\rho)|\nabla u|^2\, dx + \int \nabla d \odot \nabla d : \nabla u_t\, dx \\
&= \frac{d}{dt}\int \nabla d \odot \nabla d : \nabla u\, dx - \int (\nabla d \odot \nabla d)_t : \nabla u\, dx - \int \rho u \cdot \nabla u \cdot u_t\, dx \\
&\quad - \frac{1}{2}\int u \cdot \nabla\mu(\rho)|\nabla u|^2\, dx \\
&=: \frac{d}{dt}\int \nabla d \odot \nabla d : \nabla u\, dx + \sum_{i=1}^3 \mathcal{I}_{1i}.
\end{aligned}
$$

Now, we will use the Gagliardo-Nirenberg inequality, Poincaré's inequality, (3.1), (3.5), and (3.4) to estimate each term on the right hand of (3.11) as follows:

$$
\begin{aligned}
(3.12) \quad & \mathcal{I}_{11} \le C\|\nabla d_t\|_{L^2}\|\nabla d\|_{L^4}\|\nabla u\|_{L^4} \le C\|\nabla d_t\|_{L^2}\|\nabla d\|_{L^2}^{1/2}\|\nabla^2 d\|_{L^2}^{1/2}\|\nabla u\|_{H^1} \\
& \le \frac{1}{8}\|\nabla d_t\|_{L^2}^2 + C\|\nabla d\|_{L^2}\|\Delta d\|_{L^2}\|\nabla u\|_{H^1}^2 \le \frac{1}{8}\|\nabla d_t\|_{L^2}^2 + CM_3\|\nabla u\|_{H^1}^2,
\end{aligned}
$$

$$
\begin{aligned}
(3.13) \quad & \mathcal{I}_{12} \le C\bar{\rho}^{1/2}\|\sqrt{\rho}u_t\|_{L^2}\|u\|_{L^\infty}\|\nabla u\|_{L^2} \le \frac{1}{4}\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\nabla^2 u\|_{L^2}\|u\|_{L^2}\|\nabla u\|_{L^2}^2 \\
& \le \frac{1}{4}\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\nabla u\|_{H^1}^2 \le \frac{1}{4}\|\sqrt{\rho}u_t\|_{L^2}^2 + CM_3\|\nabla u\|_{H^1}^2,
\end{aligned}
$$

$$
(3.14) \quad \mathcal{I}_{13} \le C\|\nabla\mu(\rho)\|_{L^q}\|u\|_{L^{2q/(q-2)}}\|\nabla u\|_{L^4}^2 \le C\|\nabla u\|_{L^2}\|\nabla u\|_{H^1}^2 \le CM_3^{1/2}\|\nabla u\|_{H^1}^2,
$$

where we have used the following fact that

$$
(3.15) \quad \|\sqrt{\rho_0}u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2 \le \|\rho_0\|_{L^\infty}\|u_0\|_{L^2}^2 + C\|\Delta d_0\|_{L^2}^2 \le C(\|\nabla u_0\|_{L^2}^2 + \|\Delta d_0\|_{L^2}^2).
$$

Substituting (3.12)–(3.14) into (3.11), using (3.1), yields that

$$
\begin{aligned}
(3.16) \quad & \frac{1}{2}\frac{d}{dt}\int \mu(\rho)|\nabla u|^2\,dx + \int \rho|u_t|^2\,dx \\
& \le \frac{d}{dt}\int \nabla d \odot \nabla d : \nabla u\,dx + \frac{1}{8}\|\nabla d_t\|_{L^2}^2 + \frac{1}{4}\|\sqrt{\rho}u_t\|_{L^2}^2 + C(M_3 + M_3^{1/2})\|\nabla u\|_{H^1}^2.
\end{aligned}
$$

Next, multiplying $(1.1)_4$ by $\Delta d_t$ and integrating over $\Omega$, using (3.15), (3.1), Gagliardo-Nirenberg, Poincaré's, and Young's inequalities, we obtain, bearing in mind the facts that $|d| = 1$, $u = 0$ and $\nabla d_0' = 0$ on $\partial\Omega$ that

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\int |\Delta d|^2\,dx + \int |\nabla d_t|^2\,dx \\
& = \int u \cdot \nabla d \cdot \Delta d_t\,dx - \int |\nabla d|^2 d \cdot \Delta d_t\,dx \\
& \le \frac{1}{8}\|\nabla d_t\|_{L^2}^2 + C\int (|\nabla(u \cdot \nabla d)|^2 + |\nabla(|\nabla d|^2 d)|^2)\,dx \\
& \le \frac{1}{8}\|\nabla d_t\|_{L^2}^2 + C\big(\||\nabla u||\nabla d|\|_{L^2}^2 + \||u||\nabla^2 d|\|_{L^2}^2 + \|\nabla d\|_{L^6}^6 + \||\nabla d||\nabla^2 d|\|_{L^2}^2\big) \\
& \le \frac{1}{8}\|\nabla d_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\nabla d\|_{L^\infty}^2 + C\|u\|_{L^4}^2\|\nabla^2 d\|_{L^4}^2 \\
(3.17) \quad & \quad + C\|\nabla d\|_{L^\infty}^4\|\nabla d\|_{L^2}^2 + C\|\nabla d\|_{L^4}^2\|\nabla^2 d\|_{L^4}^2 \\
& \le \frac{1}{8}\|\nabla d_t\|_{L^2}^2 + C\|\nabla d\|_{L^2}\|\nabla^3 d\|_{L^2}\|\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\nabla^3 d\|_{L^2}^{3/2}\|\nabla d\|_{L^2}^{1/2} \\
& \quad + C\|\nabla^3 d\|_{L^2}^2\|\nabla d\|_{L^2}^4 + C\|\nabla d\|_{L^2}\|\nabla^2 d\|_{L^2}^2\|\nabla^3 d\|_{L^2} \\
& \le \frac{1}{8}\|\nabla d_t\|_{L^2}^2 + C(\|\nabla d\|_{L^2}^2 + \|\nabla d\|_{L^2}^{2/3} + \|\nabla d\|_{L^2}^4 + M_3^{1/2}\|\nabla d\|_{L^2})\|\nabla^3 d\|_{L^2}^2 \\
& \quad + C\|\nabla u\|_{L^2}^4 + C\|\nabla u\|_{L^2}^8
\end{aligned}
$$

$$\leq \frac{1}{8}\|\nabla d_t\|_{L^2}^2 + C(\|\nabla d\|_{L^2}^2 + \|\nabla d\|_{L^2}^{2/3} + \|\nabla d\|_{L^2}^4 + M_3^{1/2}\|\nabla d\|_{L^2})\|\nabla^3 d\|_{L^2}^2$$

$$+ C(M_3 + M_3^3)\|\nabla u\|_{H^1}^2.$$

Adding (3.16) to (3.17), it holds that

$$\frac{1}{2}\frac{d}{dt}\int \left(\mu(\rho)|\nabla u|^2 + |\Delta d|^2\right) dx + \|\sqrt{\rho} u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2$$

$$\leq \frac{d}{dt}\int \nabla d \odot \nabla d : \nabla u \, dx + \frac{1}{4}(\|\nabla d_t\|_{L^2}^2 + \|\sqrt{\rho} u_t\|_{L^2}^2) + CM_3^3\|\nabla u\|_{H^1}^2$$

$$+ C(\|\nabla d\|_{L^2}^2 + \|\nabla d\|_{L^2}^{2/3} + \|\nabla d\|_{L^2}^4 + M_3^{1/2}\|\nabla d\|_{L^2})\|\nabla^3 d\|_{L^2}^2.$$

Moreover, we estimate $\|\nabla u\|_{H^1}$ and $\|\nabla^3 d\|_{L^2}$. According to Lemma 2.2 with $F = -\rho u_t - \rho u \cdot \nabla u - \mathrm{div}(\nabla d \odot \nabla d)$, using (3.4), (3.15), and (3.1), we derive

(3.18)
$$\|\nabla u\|_{H^1} + \|P\|_{H^1}$$

$$\leq C\left[(1 + \|\nabla\mu(\rho)\|_{L^q})^{q/(q-2)} + (1 + \|\nabla\mu(\rho)\|_{L^q})^{(2q-2)/(q-2)}\right]\|F\|_{L^2}$$

$$\leq C(\overline{\rho}\|u\|_{L^\infty}\|\nabla u\|_{L^2} + \overline{\rho}^{1/2}\|\sqrt{\rho} u_t\|_{L^2} + \|\nabla d\|_{L^\infty}\|\nabla^2 d\|_{L^2})$$

$$\leq C\overline{\rho}\|\nabla^2 u\|_{L^2}^{1/2}\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^2 + C\overline{\rho}^{1/2}\|\sqrt{\rho} u_t\|_{L^2} + C\|\nabla d\|_{L^2}^{1/2}\|\nabla^3 d\|_{L^2}^{1/2}\|\nabla^2 d\|_{L^2}^2$$

$$\leq C\overline{\rho}M_3\|\nabla u\|_{H^1} + C\overline{\rho}^{1/2}\|\sqrt{\rho} u_t\|_{L^2} + C\|\nabla d\|_{L^2}^{1/2}\|\nabla^3 d\|_{L^2}\|\Delta d\|_{L^2}^{3/2}$$

$$\leq C\overline{\rho}M_3\|\nabla u\|_{H^1} + C\overline{\rho}^{1/2}\|\sqrt{\rho} u_t\|_{L^2} + CM_3^{3/4}\|\nabla d\|_{L^2}^{1/2}\|\nabla^3 d\|_{L^2}.$$

Using $H^k$ estimate of the elliptic systems, it is easy to deduce from $(1.1)_4$ that

$$\|\nabla^3 d\|_{L^2} \leq C\|\nabla d_t + \nabla(u \cdot \nabla d) - \nabla(|\nabla d|^2 d)\|_{L^2}$$

$$\leq C_2(\|\nabla d\|_{L^2} + \|\nabla d\|_{L^2}^{1/3} + \|\nabla d\|_{L^2}^2 + M_3^{1/4}\|\nabla d\|_{L^2}^{1/2})\|\nabla^3 d\|_{L^2}$$

$$+ C\|\nabla d_t\|_{L^2} + CM_3^{3/2}\|\nabla u\|_{H^1},$$

which directly leads to

(3.19)
$$\|\nabla^3 d\|_{L^2} \leq CM_3^{3/2}\|\nabla u\|_{H^1} + C\|\nabla d_t\|_{L^2},$$

provided $\sup_{0\leq t\leq T}\|\nabla d\|_{L^2} \leq M_0 \leq \epsilon_2 := \min\{\epsilon_1, \frac{1}{(8C_2)^3}, \frac{1}{(8C_2)^2 M_3^{1/2}}\}$. Inserting (3.19) into (3.18), we have

$$\|\nabla u\|_{H^1} \leq C_3(\overline{\rho}M_3 + \|\nabla d\|_{L^2}^{1/2}M_3^{9/4})\|\nabla u\|_{H^1} + \overline{\rho}^{1/2}\|\sqrt{\rho} u_t\|_{L^2} + CM_3^{3/4}\|\nabla d\|_{L^2}^{1/2}\|\nabla d_t\|_{L^2},$$

that is

(3.20)
$$\|\nabla u\|_{H^1} \leq C(\overline{\rho}^{1/2} + M_3^{3/4}\|\nabla d\|_{L^2}^{1/2})(\|\sqrt{\rho} u_t\|_{L^2} + \|\nabla d_t\|_{L^2}),$$

provided $\sup_{0\leq t\leq T}(\overline{\rho}+\|\nabla d\|_{L^2})\leq M_0\leq\epsilon_3:=\min\{\epsilon_2,\frac{1}{4C_3M_3},\frac{1}{(4C_3)^2M_3^{9/2}}\}$. Combining (3.19) and (3.20), yields that

$$
(3.21)\quad
\begin{aligned}
\|\nabla^3 d\|_{L^2} &\leq CM_3^{3/2}(\overline{\rho}^{1/2}+M_3^{3/4}\|\nabla d\|_{L^2}^{1/2})(\|\sqrt{\rho}u_t\|_{L^2}+\|\nabla d_t\|_{L^2})+C\|\nabla d_t\|_{L^2}\\
&\leq C(M_3^{9/4}+1)(\|\sqrt{\rho}u_t\|_{L^2}+\|\nabla d_t\|_{L^2}).
\end{aligned}
$$

Thus, it follows from (3.20) and (3.21) that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\left(\mu(\rho)|\nabla u|^2+|\Delta d|^2\right)dx+\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2\\
&\leq\frac{d}{dt}\int\nabla d\odot\nabla d:\nabla u\,dx+\frac{1}{4}(\|\nabla d_t\|_{L^2}^2+\|\sqrt{\rho}u_t\|_{L^2}^2)\\
&\quad+C\big[\overline{\rho}M_3^3+M_3^{9/2}\|\nabla d\|_{L^2}+(M_3^{9/2}+1)(\|\nabla d\|_{L^2}^2+\|\nabla d\|_{L^2}^{2/3}+\|\nabla d\|_{L^2}^4)\big]\\
&\qquad\times(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2)\\
&\leq\frac{d}{dt}\int\nabla d\odot\nabla d:\nabla u\,dx+\frac{1}{4}(\|\nabla d_t\|_{L^2}^2+\|\sqrt{\rho}u_t\|_{L^2}^2)\\
&\quad+C_4(\overline{\rho}M_3^3+M_3^{9/2}\|\nabla d\|_{L^2}+(M_3^{9/2}+1)\|\nabla d\|_{L^2}^{2/3})(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2),
\end{aligned}
$$

which, yields that

$$
(3.22)\quad\frac{1}{2}\frac{d}{dt}\int\left(\mu(\rho)|\nabla u|^2+|\Delta d|^2\right)dx+\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2\leq2\frac{d}{dt}\int\nabla d\odot\nabla d:\nabla u\,dx
$$

provided $\sup_{0\leq t\leq T}(\overline{\rho}+\|\nabla d\|_{L^2})\leq M_0\leq\epsilon_4:=\min\left\{\epsilon_3,\frac{1}{12C_4M_3^3},\frac{1}{\left[12C_4(M_3^{9/2}+1)\right]^{3/2}},\frac{1}{12C_4M_3^{9/2}}\right\}$. Integrating (3.22) over $[0,\zeta(T)]$, from (1.2) and (3.8), gives

$$
\begin{aligned}
&\sup_{0\leq t\leq\zeta(T)}\left(\underline{\mu}\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\right)+\int_0^{\zeta(T)}\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2\right)dt\\
&\leq\frac{\overline{\mu}}{2}\sup_{0\leq t\leq\zeta(T)}\|\nabla u\|_{L^2}^2+C_5\sup_{0\leq t\leq\zeta(T)}\left[\|\nabla d\|_{L^2}^2\|\Delta d\|_{L^2}^2\right]+M_3.
\end{aligned}
$$

From the above inequality, we get

$$
\sup_{0\leq t\leq\zeta(T)}\left(\underline{\mu}\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\right)+\int_0^{\zeta(T)}\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2\right)dt\leq M_3,
$$

provided $\sup_{0\leq t\leq T}\|\nabla d\|_{L^2}\leq M_0\leq\epsilon_5:=\min\{\epsilon_4,\frac{1}{(2C_5)^{1/2}}\}$.

Finally, it remains to estimate (3.10). In fact, multiplying (3.22) by $e^{\sigma t}$ and integrating the result inequality over $[\zeta(T),T]$, we obtain

$$
\begin{aligned}
&\sup_{\zeta(T)\leq t\leq T}e^{\sigma t}\left(\underline{\mu}\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\right)+\int_{\zeta(T)}^T e^{\sigma t}\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2\right)dt\\
&\leq C\sup_{\zeta(T)\leq t\leq T}e^{\sigma t}\left[\|\nabla d\|_{L^2}^2\|\Delta d\|_{L^2}^2\right]+\int_{\zeta(T)}^T e^{\sigma t}\left(\overline{\mu}\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\right)dt
\end{aligned}
$$

$$+ \sup_{\zeta(T)\leq t\leq T} \frac{\mu}{2}\big[e^{\sigma t}\|\nabla u\|_{L^2}^2\big] + C\int_{\zeta(T)}^T e^{\sigma t}\|\nabla d\|_{L^4}^2\|\nabla u\|_{L^2}\,dt + e^{\sigma\zeta(T)}M_3$$

$$\leq C_5 \sup_{\zeta(T)\leq t\leq T} e^{\sigma t}\big[\|\nabla d\|_{L^2}^2\|\Delta d\|_{L^2}^2)\big] + \int_{\zeta(T)}^T e^{\sigma t}\big(\overline{\mu}\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2\big)\,dt + e^{\sigma\zeta(T)}M_3$$

$$+ \sup_{\zeta(T)\leq t\leq T} \frac{\mu}{2}\big[e^{\sigma t}\|\nabla u\|_{L^2}^2\big] + C\sup_{\zeta(T)\leq t\leq T}\|\nabla d\|_{L^2}\int_{\zeta(T)}^T e^{\sigma t}\big(\underline{\mu}\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2\big)\,dt$$

which directly deduce from (3.8), (3.15), and (3.5) that

$$\sup_{\zeta(T)\leq t\leq T} e^{\sigma t}\big(\underline{\mu}\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2\big) + \int_{\zeta(T)}^T e^{\sigma t}\big(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2\big)\,dt \leq CM_3,$$

provided $\sup_{0\leq t\leq T}\|\nabla d\|_{L^2} \leq M_0 \leq \epsilon_5$. The proof of Lemma 3.3 is finished. $\qquad\square$

**Lemma 3.4.** *Let $(\rho, u, d, P)$ be a smooth solution to $(1.1)$–$(1.4)$ satisfying $(3.1)$. Then there exists a generic positive constant $C$ depending only on $\Omega$, $q$, $\underline{\mu}$, $\overline{\mu}$, $M_1$, $M_2$, and $M_3$ such that*

$$(3.23) \quad \sup_{0\leq t\leq \zeta(T)} \big[t(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2)\big] + \int_0^{\zeta(T)} t\big(\underline{\mu}\|\nabla u_t\|_{L^2}^2 + \|\Delta d_t\|_{L^2}^2\big)\,dt \leq 2M_3,$$

$$(3.24) \quad \sup_{\zeta(T)\leq t\leq T} \big[e^{\sigma t}(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2)\big] + \int_{\zeta(T)}^T e^{\sigma t}\big(\underline{\mu}\|\nabla u_t\|_{L^2}^2 + \|\Delta d_t\|_{L^2}^2\big)\,dt$$
$$\leq CM_3^2(M_3^{9/2}+1)^3,$$

$$(3.25) \quad \sup_{\zeta(T)\leq t\leq T} e^{\sigma t}(\|\nabla u\|_{H^1}^2 + \|\nabla^3 d\|_{L^2}^2) \leq CM_3^2(M_3^{9/2}+1)^4,$$

*provided $\sup_{0\leq t\leq T}\|\nabla d\|_{L^2} \leq M_0 \leq \epsilon_6 := \min\{\epsilon_5, \frac{1}{2C_6 M_3^{1/2}}\}.$*

*Proof.* Differentiating $(1.1)_2$ with respect to time variable $t$ gives

$$(3.26) \quad \rho u_{tt} + \rho u\cdot\nabla u_t - \mathrm{div}(\mu(\rho)\nabla u)_t = -\rho_t(u_t + u\cdot\nabla u) - \rho u_t\cdot\nabla u - \nabla P_t - \mathrm{div}(\nabla d\odot\nabla d)_t.$$

Multiplying (3.26) by $u_t$ and integrating the resulting equality by parts over $\Omega$, we obtain after applying $(1.1)_1$ and the divergence free condition $(1.1)_3$ that

$$(3.27)$$
$$\frac{1}{2}\frac{d}{dt}\int \rho|u_t|^2\,dx + \int \mu(\rho)|\nabla u_t|^2\,dx$$
$$= \int \big[(\nabla d\odot\nabla d)_t : \nabla u_t - \rho(u_t\cdot\nabla)u\cdot u_t + \mathrm{div}(\rho u)(u_t + u\cdot\nabla u)\cdot u_t - \mu(\rho)_t\nabla u\cdot\nabla u_t\big]\,dx$$
$$= \int \big\{(\nabla d\odot\nabla d)_t : \nabla u_t - \rho(u_t\cdot\nabla)u\cdot u_t - \rho u\cdot\nabla(|u_t|^2) + \rho u\cdot\nabla(u\cdot\nabla u\cdot u_t)$$
$$- \mu(\rho)_t\nabla u\cdot\nabla u_t\big\}\,dx$$

$$\leq C \int \rho|u||u_t|(|\nabla u|^2 + |u||\nabla^2 u|)\,dx + C\int \rho|u|^2|\nabla u||\nabla u_t|\,dx + C\int \rho|u||u_t||\nabla u_t|\,dx$$

$$+ C\int \rho|u_t|^2|\nabla u|\,dx + C\int |\nabla d||\nabla d_t||\nabla u_t|\,dx + C\int |u||\nabla\mu(\rho)||\nabla u||\nabla u_t|\,dx$$

$$=: \sum_{i=1}^{6} \mathcal{I}_{2i}.$$

Now, we use (3.4), (3.20), (3.21), the Gagliardo-Nirenberg, Poincaré's and Hölder's inequalities to estimate each term on the right-hand side of (3.27) as follows:

$$\mathcal{I}_{21} \leq C\overline{\rho}\|u_t\|_{L^4}\|u\|_{L^4}\|\nabla u\|_{L^4}^2 + C\overline{\rho}\|u_t\|_{L^2}\|u\|_{L^\infty}^2\|\nabla^2 u\|_{L^2}$$
$$\leq C\overline{\rho}\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^2}\|\nabla u\|_{H^1}^2 \leq \delta\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C\overline{\rho}^2\|\nabla u\|_{L^2}^2\|\nabla u\|_{H^1}^4,$$

$$\mathcal{I}_{22} \leq C\overline{\rho}\|u\|_{L^\infty}^2\|\nabla u\|_{L^2}\|\nabla u_t\|_{L^2} \leq C\overline{\rho}\|u\|_{L^2}\|\nabla u\|_{H^1}\|\nabla u\|_{L^2}\|\nabla u_t\|_{L^2}$$
$$\leq \delta\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C\overline{\rho}^2\|\nabla u\|_{L^2}^2\|\nabla u\|_{H^1}^4,$$

$$\mathcal{I}_{23} \leq C\overline{\rho}^{1/2}\|u\|_{L^\infty}\|\nabla u_t\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2} \leq C\overline{\rho}^{1/2}\|u\|_{L^2}^{1/2}\|\nabla u\|_{H^1}^{1/2}\|\nabla u_t\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}$$
$$\leq \delta\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C\overline{\rho}\|\nabla u\|_{H^1}^2\|\sqrt{\rho}u_t\|_{L^2}^2,$$

$$\mathcal{I}_{24} \leq C\overline{\rho}^{1/2}\|\sqrt{\rho}u_t\|_{L^2}\|u_t\|_{L^4}\|\nabla u\|_{L^4} \leq C\overline{\rho}^{1/2}\|\nabla u_t\|_{L^2}\|\nabla u\|_{H^1}\|\sqrt{\rho}u_t\|_{L^2}$$
$$\leq \delta\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C\overline{\rho}\|\nabla u\|_{H^1}^2\|\sqrt{\rho}u_t\|_{L^2}^2,$$

$$\mathcal{I}_{25} \leq C\|\nabla d\|_{L^4}\|\nabla d_t\|_{L^4}\|\nabla u_t\|_{L^2} \leq C\|\nabla d\|_{L^2}^{1/2}\|\Delta d\|_{L^2}^{1/2}\|\Delta d_t\|_{L^2}\|\nabla u_t\|_{L^2}$$
$$\leq \delta\underline{\mu}\|\nabla u_t\|_{L^2}^2 + CM_3^{1/2}\|\nabla d\|_{L^2}\|\Delta d_t\|_{L^2}^2,$$

$$\mathcal{I}_{26} \leq C\|u\|_{L^\infty}\|\nabla\mu(\rho)\|_{L^q}\|\nabla u\|_{L^{2q/(q-2)}}\|\nabla u_t\|_{L^2}$$
$$\leq C\|u\|_{L^2}^{1/2}\|\nabla u\|_{H^1}^{1/2}\|\nabla\mu(\rho)\|_{L^q}\|\nabla u\|_{H^1}\|\nabla u_t\|_{L^2} \leq \delta\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C\|\nabla u\|_{H^1}^2\|\nabla u\|_{H^1}^2.$$

Substituting the estimates of $\mathcal{I}_{2i}$ $(i=1,2,\ldots,6)$ into (3.27), and then using (3.1), we obtain

$$\frac{1}{2}\frac{d}{dt}\int \rho|u_t|^2\,dx + \int \mu(\rho)|\nabla u_t|^2\,dx$$
(3.28)
$$\leq \delta\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C(M_3+1)\|\nabla u\|_{H^1}^2(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla u\|_{H^1}^2)$$
$$+ CM_3^{1/2}\|\nabla d\|_{L^2}\|\Delta d_t\|_{L^2}^2$$
$$\leq \delta\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C(M_3^{3/2}+1)^2\|\nabla u\|_{H^1}^2(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2)$$
$$+ CM_3^{1/2}\|\nabla d\|_{L^2}\|\Delta d_t\|_{L^2}^2.$$

Now, differentiating $(1.1)_4$ with respect to time variable $t$ yields

(3.29)
$$d_{tt} - \Delta d_t = -(u\cdot\nabla d)_t + (|\nabla d|^2 d)_t.$$

Multiplying (3.29) by $\Delta d_t$, and integrating the resulting equality over $\Omega$, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\nabla d_t|^2\,dx + \int |\Delta d_t|^2\,dx \\
&= \int u_t \cdot \nabla d \cdot \Delta d_t\,dx + \int u \cdot \nabla d_t \cdot \Delta d_t\,dx - \int |\nabla d|^2 d_t \cdot \Delta d_t\,dx \\
&\quad - 2\int (\nabla d : \nabla d_t) d \cdot \Delta d_t\,dx \\
&\leq C\int |\nabla u_t||\nabla d||\nabla d_t|\,dx + C\int |u_t||\nabla^2 d||\nabla d_t|\,dx + C\int |\nabla d|^2|d_t||\Delta d_t|\,dx \\
&\quad + C\int |\nabla d||\nabla d_t||\Delta d_t|\,dx + C\int |u||\nabla d_t||\Delta d_t|\,dx \\
&=: \sum_{i=1}^{5} \mathcal{I}_{3i}.
\end{aligned}
$$

(3.30)

By virtue of Hölder's, Poincaré's and Gagliardo-Nirenberg inequalities, (3.8), and (3.1), we have

$$
\begin{aligned}
\mathcal{I}_{31} &\leq C\|\nabla u_t\|_{L^2}\|\nabla d\|_{L^4}\|\nabla d_t\|_{L^4} \leq \delta\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C\|\nabla d\|_{L^4}^2\|\nabla d_t\|_{L^4}^2 \\
&\leq \delta\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C\|\nabla d\|_{L^2}\|\Delta d\|_{L^2}\|\Delta d_t\|_{L^2}^2 \leq \delta\underline{\mu}\|\nabla u_t\|_{L^2}^2 + CM_3^{1/2}\|\nabla d\|_{L^2}\|\Delta d_t\|_{L^2}^2, \\
\mathcal{I}_{32} &\leq C\|u_t\|_{L^2}\|\nabla^2 d\|_{L^4}\|\nabla d_t\|_{L^4} \leq C\|\nabla u_t\|_{L^2}\|\Delta d\|_{L^2}^{1/2}\|\nabla^3 d\|_{L^2}^{1/2}\|\nabla d_t\|_{L^2}^{1/2}\|\Delta d_t\|_{L^2}^{1/2} \\
&\leq \delta\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C\|\Delta d\|_{L^2}\|\nabla^3 d\|_{L^2}\|\nabla d_t\|_{L^2}\|\Delta d_t\|_{L^2} \\
&\leq \delta\underline{\mu}\|\nabla u_t\|_{L^2}^2 + \kappa\|\Delta d_t\|_{L^2}^2 + C\|\Delta d\|_{L^2}^2\|\nabla^3 d\|_{L^2}^2\|\nabla d_t\|_{L^2}^2 \\
&\leq \delta\underline{\mu}\|\nabla u_t\|_{L^2}^2 + \kappa\|\Delta d_t\|_{L^2}^2 + CM_3\|\nabla^3 d\|_{L^2}^2\|\nabla d_t\|_{L^2}^2, \\
\mathcal{I}_{33} &\leq C\|\Delta d_t\|_{L^2}\|\nabla d\|_{L^\infty}^2\|d_t\|_{L^2} \leq C\|\Delta d_t\|_{L^2}\|\nabla d\|_{L^2}\|\nabla^3 d\|_{L^2}\|\nabla d_t\|_{L^2} \\
&\leq \delta\|\Delta d_t\|_{L^2}^2 + C\|\nabla^3 d\|_{L^2}^2\|\nabla d_t\|_{L^2}^2, \\
\mathcal{I}_{34} &\leq C\|\Delta d_t\|_{L^2}\|\nabla d\|_{L^4}\|\nabla d_t\|_{L^4} \leq \delta\|\Delta d_t\|_{L^2}^2 + C\|\nabla d\|_{L^2}^2\|\nabla^2 d\|_{L^2}^2\|\nabla d_t\|_{L^2}^2 \\
&\leq \delta\|\Delta d_t\|_{L^2}^2 + C\|\nabla^3 d\|_{L^2}^2\|\nabla d_t\|_{L^2}^2, \\
\mathcal{I}_{35} &\leq C\|u\|_{L^\infty}\|\nabla d_t\|_{L^2}\|\Delta d_t\|_{L^2} \leq \delta\|\Delta d_t\|_{L^2}^2 + C\|\nabla u\|_{H^1}^2\|\nabla d_t\|_{L^2}^2.
\end{aligned}
$$

Thus, inserting the estimates of $\mathcal{I}_{3i}$ $(i = 1, 2, \ldots, 5)$ into (3.30), we arrive at

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla d_t\|_{L^2}^2 + \|\Delta d_t\|_{L^2}^2 &\leq \delta\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C(\|\nabla u\|_{H^1}^2 + (M_3+1)\|\nabla^3 d\|_{L^2}^2)\|\nabla d_t\|_{L^2}^2 \\
&\quad + (\delta+\kappa)\|\Delta d_t\|_{L^2}^2 + CM_3^{1/2}\|\nabla d\|_{L^2}\|\Delta d_t\|_{L^2}^2.
\end{aligned}
$$

(3.31)

Adding the resulting inequality with (3.31) and (3.28), taking $\delta$ and $\kappa$ suitable small, we

get after using (3.20) and (3.21) that

(3.32)
$$\frac{1}{2}\frac{d}{dt}\big(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2\big) + \underline{\mu}\|\nabla u_t\|_{L^2}^2 + \|\Delta d_t\|_{L^2}^2$$
$$\le C(M_3^{3/2}+1)^2(\|\nabla u\|_{H^1}^2 + \|\nabla^3 d\|_{L^2}^2)(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) + CM_3^{1/2}\|\nabla d\|_{L^2}\|\Delta d_t\|_{L^2}^2$$
$$\le C(M_3^{9/2}+1)^3(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2)(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) + CM_3^{1/2}\|\nabla d\|_{L^2}\|\Delta d_t\|_{L^2}^2.$$

Next, multiplying the above inequality by $t$, we get

$$\frac{1}{2}\frac{d}{dt}\big[t(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2)\big] + t\big(\underline{\mu}\|\nabla u_t\|_{L^2}^2 + \|\Delta d_t\|_{L^2}^2\big)$$
$$\le C_6(M_3^{9/2}+1)^3(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2)\big[t(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2)\big]$$
$$+ C_6 M_3^{1/2}\|\nabla d\|_{L^2}\big[t\|\Delta d_t\|_{L^2}^2\big] + \|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2,$$

which, integrating over $[0, \zeta(T)]$, using Gronwall's inequality and (3.9) lead to

$$(3.33) \quad \sup_{0\le t\le \zeta(T)}\big[t(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2)\big] + \int_0^{\zeta(T)} t\big(\underline{\mu}\|\nabla u_t\|_{L^2}^2 + \|\Delta d_t\|_{L^2}^2\big)\,dt \le 2M_3,$$

provided $\sup_{0\le t\le T}\|\nabla d\|_{L^2} \le M_0 \le \epsilon_6 := \min\{\epsilon_5, \frac{1}{2C_6 M_3^{1/2}}\}$.

Similarly, multiplying (3.32) by $e^{\sigma t}$ and integrating over $[\zeta(T), T]$, by Gronwall's inequality, (3.33), and (3.10), we get

$$\sup_{\zeta(T)\le t\le T}\big[e^{\sigma t}(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2)\big] + \int_{\zeta(T)}^{T} e^{\sigma t}\big(\underline{\mu}\|\nabla u_t\|_{L^2}^2 + \|\Delta d_t\|_{L^2}^2\big)\,dt$$
$$\le C(M_3^{9/4}+1)^2\exp\left\{e^{-\zeta(T)}\int_{\zeta(T)}^{T}e^{\sigma t}\big(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2\big)\,dt\right\}$$
$$\times\left\{e^{\sigma(t)}\big(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2\big)\Big|_{t=\zeta(T)} + \int_{\zeta(T)}^{T}e^{\sigma t}\big(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2\big)\,dt\right\}$$
$$\le CM_3^2(M_3^{9/2}+1)^3,$$

provided $\sup_{0\le t\le T}\|\nabla d\|_{L^2} \le M_0 \le \epsilon_6 := \min\{\epsilon_5, \frac{1}{2C_6 M_3^{1/2}}\}$.

Finally, it follows from (3.20), (3.21), (3.10), and (3.24) that (3.25) holds. The proof of Lemma 3.4 is completed. $\qquad\square$

We will use Lemmas 3.2–3.4 to prove the following time-independent bound on the $L^1(0,T;L^\infty)$-norm of $\nabla u$ which is important for obtaining the uniform one (with respect to time) on the $L^\infty(0,T;L^q)$-norm of the gradient of $\mu(\rho)$.

**Lemma 3.5.** *Let $(\rho, u, d, P)$ be a smooth solution to $(1.1)$–$(1.4)$ satisfying $(3.1)$. Then there exists a generic positive constant $C$ depending only on $\Omega$, $q$, $\underline{\mu}$, $\overline{\mu}$, $M_1$, $M_2$, and $M_3$ such that*

$$(3.34) \qquad \int_0^T \|\nabla u\|_{L^\infty} \le C\overline{\rho}^{3/4} M_3 (M_3^{9/2} + 1)^{3/2} + C\overline{\rho}^{(6r-6)/r} M_3^{\frac{5r-6}{2r}} M_2$$

$$+ C \sup_{0 \le t \le T} \|\nabla d\|_{L^2}^{5/6} M_3^{7/12} (M_3^{9/2} + 1)^{7/6}.$$

*Proof.* First, it follows from Lemma 2.2, $(3.1)$, $(3.4)$, Gagliardo-Nirenberg and Hölder's inequalities that for $r \in (2, \min\{q, 3\})$,

$$\|\nabla u\|_{W^{1,r}} \le C\|F\|_{L^r}(1 + \|\nabla \mu(\rho)\|_{L^q})^{\frac{qr}{2(q-r)}}$$

$$\le C\|\rho u_t\|_{L^3} + C\|\rho u \cdot \nabla u\|_{L^r} + C\|\,|\nabla d||\nabla^2 d|\,\|_{L^3}$$

$$\le C\|\rho u_t\|_{L^2}^{1/2}\|\rho u_t\|_{L^6}^{1/2} + C\overline{\rho}\|u\|_{L^3}\|\nabla u\|_{L^{3r/(3-r)}} + C\|\nabla d\|_{L^\infty}\|\nabla^2 d\|_{L^3}$$

$$\le C\overline{\rho}^{3/4}\|\sqrt{\rho}u_t\|_{L^2}^{1/2}\|\nabla u_t\|_{L^2}^{1/2} + C\overline{\rho}\|\nabla u\|_{L^2}^{\frac{7r-6}{6r-6}}\|\nabla u\|_{W^{1,r}}^{\frac{5r-6}{6r-6}}$$

$$+ C\|\nabla d\|_{L^\infty}\|\nabla^3 d\|_{L^2}^{2/3}\|\nabla d\|_{L^2}^{1/3}$$

$$\le C\overline{\rho}^{3/4}\|\sqrt{\rho}u_t\|_{L^2}^{1/2}\|\nabla u_t\|_{L^2}^{1/2} + C\overline{\rho}^{(6r-6)/r}\|\nabla u\|_{L^2}^{(7r-6)/r}$$

$$+ \frac{1}{2}\|\nabla u\|_{W^{1,r}} + C\|\nabla d\|_{L^2}^{5/6}\|\nabla^3 d\|_{L^2}^{7/6},$$

which directly deduces that

(3.35)
$$\|\nabla u\|_{W^{1,r}} \le C\overline{\rho}^{3/4}\|\sqrt{\rho}u_t\|_{L^2}^{1/2}\|\nabla u_t\|_{L^2}^{1/2} + C\overline{\rho}^{(6r-6)/r}\|\nabla u\|_{L^2}^{(7r-6)/r} + C\|\nabla d\|_{L^2}^{5/6}\|\nabla^3 d\|_{L^2}^{7/6}.$$

On the one hand, it follows from $(3.23)$, $(3.5)$, $(3.21)$, and $(3.9)$ that for $t \in [0, \zeta(T)]$,

$$\int_0^{\zeta(T)} \|\nabla u\|_{L^\infty}\, dt$$

$$\le C\overline{\rho}^{3/4}\left[\int_0^{\zeta(T)} t^{-1/2}\|\sqrt{\rho}u_t\|_{L^2}^{2/3}\, dt\right]^{3/4} \cdot \left[\int_0^{\zeta(T)} t^{3/2}\|\nabla u_t\|_{L^2}^2\, dt\right]^{1/4}$$

$$+ C(M_3^{9/2} + 1)^{7/6} \sup_{0 \le t \le \zeta(T)} \|\nabla d\|_{L^2}^{5/6}$$

$$\times \sup_{0 \le t \le \zeta(T)} \left(t\|\sqrt{\rho}u_t\|_{L^2}^2 + t\|\nabla d_t\|_{L^2}^2\right)^{7/12} \int_0^{\zeta(T)} t^{-1+5/7}\, dt$$

$$(3.36) \qquad + C\overline{\rho}^{(6r-6)/r} \sup_{0 \le t \le \zeta(T)} \left(\|\nabla u\|_{L^2}^2\right)^{\frac{5r-6}{2r}} \int_0^{\zeta(T)} \|\nabla u\|_{L^2}^2\, dt$$

$$\le C\overline{\rho}^{3/4} M_3^{1/4}\left[\int_0^{\zeta(T)} t^{-1/2} \cdot t^{-\frac{1}{2} \cdot \frac{2}{3}}\, dt\right]^{3/4} \cdot \left[\int_0^{\zeta(T)} t^{3/2}\|\nabla u_t\|_{L^2}^2\, dt\right]^{1/4}$$

$$+ C\overline{\rho}^{(6r-6)/r} M_3^{\frac{5r-6}{2r}} M_2 + C \sup_{0 \le t \le \zeta(T)} \|\nabla d\|_{L^2}^{5/6} M_3^{7/12} (M_3^{9/2} + 1)^{7/6}$$

$$\le C\overline{\rho}^{3/4} M_3^{1/2} + C\overline{\rho}^{(6r-6)/r} M_3^{\frac{5r-6}{2r}} M_2 + C \sup_{0 \le t \le \zeta(T)} \|\nabla d\|_{L^2}^{5/6} M_3^{7/12} (M_3^{9/2} + 1)^{7/6}.$$

On the other hand, using (3.10), (3.24), (3.21) and (3.5), we obtain that for $t \in [\zeta(T), T]$,

(3.37)
$$\int_{\zeta(T)}^T \|\nabla u\|_{L^\infty} \, dt \le C\overline{\rho}^{(6r-6)/r} \sup_{\zeta(T) \le t \le T} \left( e^{\sigma t} \|\nabla u\|_{L^2}^2 \right)^{\frac{5r-6}{2r}} e^{-\frac{(5r-6)\sigma\zeta(T)}{2r}} \int_{\zeta(T)}^T \|\nabla u\|_{L^2}^2 \, dt$$

$$+ C \sup_{0 \le t \le \zeta(T)} \|\nabla d\|_{L^2}^{5/6} \left[ \int_{\zeta(T)}^T e^{\sigma t} \|\nabla^3 d\|_{L^2}^2 \, dt \right]^{7/12} \left[ \int_{\zeta(T)}^T e^{-\frac{7}{5}\sigma t} \, dt \right]^{5/12}$$

$$+ C\overline{\rho}^{3/4} \int_{\zeta(T)}^T (\|\sqrt{\rho} u_t\|_{L^2} + \|\nabla u_t\|_{L^2}) \, dt$$

$$\le C\overline{\rho}^{3/4} \sup_{\zeta(T) \le t \le T} (e^{\sigma t} \|\sqrt{\rho} u_t\|_{L^2}^2)^{1/2} \int_{\zeta(T)}^T e^{-\frac{\sigma t}{2}} \, dt + C\overline{\rho}^{(6r-6)/r} M_3^{\frac{5r-6}{2r}} M_2$$

$$+ C\overline{\rho}^{3/4} \left( \int_{\zeta(T)}^T e^{-\sigma t} \, dt \right)^{1/2} \left( \int_{\zeta(T)}^T e^{\sigma t} \|\nabla u_t\|_{L^2}^2 \, dt \right)^{1/2}$$

$$+ C \sup_{\zeta(T) \le t \le T} \|\nabla d\|_{L^2}^{5/6} M_3^{7/12} (M_3^{9/2} + 1)^{7/6}$$

$$\le C\overline{\rho}^{3/4} M_3 (M_3^{9/2} + 1)^{3/2} + C\overline{\rho}^{(6r-6)/r} M_3^{\frac{5r-6}{2r}} M_2$$

$$+ C \sup_{\zeta(T) \le t \le T} \|\nabla d\|_{L^2}^{5/6} M_3^{7/12} (M_3^{9/2} + 1)^{7/6}.$$

Combining this with (3.36) gives (3.34) and finishes the proof of Lemma 3.5. □

With Lemmas 3.2–3.5 at hand, we are in a position to prove Proposition 3.1.

*Proof of Proposition* 3.1. First, it follows from $(1.1)_1$ that

(3.38)
$$\frac{d}{dt} \|\nabla \mu(\rho)\|_{L^q} \le C \|\nabla u\|_{L^\infty} \|\nabla \mu(\rho)\|_{L^q},$$

which together with Gronwall's inequality and (3.34) yields

$$\sup_{0 \le t \le T} \|\nabla \mu(\rho)\|_{L^q}$$

$$\le \|\nabla \mu(\rho_0)\|_{L^q} \exp \left\{ C \int_0^T \|\nabla u\|_{L^\infty} \, dt \right\}$$

(3.39)
$$\le \|\nabla \mu(\rho_0)\|_{L^q} \exp \left\{ C_7 \overline{\rho}^{3/4} M_3 (M_3^{9/2} + 1)^{3/2} + C_7 \overline{\rho}^{(6r-6)/r} M_3^{\frac{5r-6}{2r}} M_2 \right.$$

$$\left. + C_7 \sup_{0 \le t \le T} \|\nabla d\|_{L^2}^{5/6} M_3^{7/12} (M_3^{9/2} + 1)^{7/6} \right\}$$

$$\le 2M_1,$$

provided $\sup_{0\leq t\leq T}(\overline{\rho} + \|\nabla d\|_{L^2}) \leq M_0 \leq \epsilon_7 := \min\big\{\epsilon_6, \big(\frac{\ln 2}{3C_7 M_3(M_3^{9/2}+1)^{3/2}}\big)^{4/3},$

$\big(\frac{\ln 2}{3C_7 M_3^{(5r-6)/(2r)} M_2}\big)^{r/(6r-6)}, \big(\frac{\ln 2}{3C_7 M_3^{7/12}(M_3^{9/2}+1)^{7/6}}\big)^{4/3}\big\}.$

Next, we deduce from (3.22) and (3.5) that

$$\sup_{0\leq t\leq T} e^{\sigma t}\big(\underline{\mu}\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2\big) + \int_0^T e^{\sigma t}\big(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2\big)\,dt$$

$$\leq C\sup_{0\leq t\leq T} e^{\sigma t}\big[\|\nabla d\|_{L^2}^2\|\Delta d\|_{L^2}^2\big] + \int_0^T e^{\sigma t}\big(\overline{\mu}\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2\big)\,dt + M_3$$

$$+ \sup_{0\leq t\leq T} \frac{\underline{\mu}}{2}\big[e^{\sigma t}\|\nabla u\|_{L^2}^2\big] + C\sup_{0\leq t\leq T}\|\nabla d\|_{L^2}\int_0^T e^{\sigma t}\big(\|\Delta d\|_{L^2}^2 + \underline{\mu}\|\nabla u\|_{L^2}^2\big)\,dt$$

(3.40) $$\leq C\sup_{0\leq t\leq T} e^{\sigma t}\big[\|\nabla d\|_{L^3}^2\|\Delta d\|_{L^2}^2\big] + 2\Big(\frac{\overline{\mu}}{\underline{\mu}}+1\Big)M_2 + M_3$$

$$+ \sup_{0\leq t\leq T}\frac{\underline{\mu}}{2}\big[e^{\sigma t}\|\nabla u\|_{L^2}^2\big] + C\sup_{0\leq t\leq T}\|\nabla d\|_{L^2}\big(\|\sqrt{\rho_0}u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2\big)$$

$$\leq C_8\sup_{0\leq t\leq T} e^{\sigma t}\big[\|\nabla d\|_{L^2}^2\|\Delta d\|_{L^2}^2\big] + 2\Big(\frac{\overline{\mu}}{\underline{\mu}}+1\Big)M_2 + M_3$$

$$+ \sup_{0\leq t\leq T}\frac{\underline{\mu}}{2}\big[e^{\sigma t}\|\nabla u\|_{L^2}^2\big] + C_8\sup_{0\leq t\leq T}\|\nabla d\|_{L^2}M_3$$

which leads to

$$\sup_{0\leq t\leq T} e^{\sigma t}\big(\underline{\mu}\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2\big) \leq 2\Big[2\Big(\frac{\overline{\mu}}{\underline{\mu}}+1\Big)M_2 + \frac{3}{2}M_3\Big],$$

provided $\sup_{0\leq t\leq T}\|\nabla d\|_{L^2} \leq M_0 \leq \epsilon_8 := \min\{\epsilon_7, \frac{1}{2C_8}\}.$

Moreover, it remains ro prove $\sup_{0\leq t\leq T}\|\nabla d\|_{L^2} \leq M_0/4$. In fact, multiplying $(1.1)_4$ by $-\Delta d$ and integrating by parts over $\Omega$, it follows that

$$\frac{1}{2}\frac{d}{dt}\|\nabla d\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 \leq C\int |\nabla u||\nabla d||\nabla d|\,dx + C\int |\nabla d|^2|\nabla^2 d|\,dx$$

$$\leq C\|\nabla u\|_{L^\infty}\|\nabla d\|_{L^2}^2 + C\|\nabla d\|_{L^4}^2\|\Delta d\|_{L^2}$$

$$\leq C\|\nabla u\|_{L^\infty}\|\nabla d\|_{L^2}^2 + C_9\|\nabla d\|_{L^2}\|\Delta d\|_{L^2}^2,$$

that is

(3.41) $$\frac{d}{dt}\|\nabla d\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 \leq C\|\nabla u\|_{L^\infty}\|\nabla d\|_{L^2}^2,$$

provided $\sup_{0\leq t\leq T}\|\nabla d\|_{L^2} \leq M_0 \leq \epsilon_9 = \min\{\epsilon_8, \frac{1}{2C_9}\}$. Then, (3.41) together with (3.34) and Gronwall's inequality leads to

(3.42)
$$\sup_{0\leq t\leq T}\|\nabla d\|_{L^2} \leq C_{10}\big\{M_3^{1/2}(M_3^{9/2}+1)^{3/4} + M_3^{\frac{5r-6}{4r}}M_2 + M_3^{7/24}(M_3^{9/2}+1)^{7/12}\big\}\|\nabla d_0\|_{L^2}$$

$$\leq \frac{M_0}{4},$$

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provided $M_0 \le \epsilon_{10} := \min\left\{\epsilon_9, \frac{1}{12C_{10}M_3^{1/2}(M_3^{9/2}+1)^{3/4}}, \frac{1}{12C_{10}M_3^{(5r-6)/(4r)}M_2}, \right.$
$\left. \frac{1}{12C_{10}M_3^{7/24}(M_3^{9/2}+1)^{7/12}}\right\}$.

Finally, choosing $\epsilon_0 := \min\{\epsilon_{10}, \epsilon/4\}$, we directly obtain (3.2) from (3.4), (3.39), and (3.42). The proof of Proposition 3.1 is finished. $\qquad\square$

The following Lemma 3.6 is necessary for further estimates on the higher-order derivatives of the strong solution $(\rho, u, d, P)$.

**Lemma 3.6.** *Let $(\rho, u, d, P)$ be a smooth solution to $(1.1)$–$(1.4)$ satisfying $(3.1)$. Then there exists a generic positive constant $C$ depending only on $\Omega$, $q$, $\underline{\mu}$, $\overline{\mu}$, $M_1$, $M_2$, and $M_3$ such that for $r \in (2, \min\{q, 3\})$,*

(3.43)
$$\sup_{0 \le t \le T} \|\rho\|_{H^1 \cap W^{1,q}} + \sup_{0 \le t \le T} e^{\sigma t}\big(\zeta\|\nabla u\|_{H^1}^2 + \zeta\|P\|_{H^1}^2 + \zeta\|\nabla d_t\|_{L^2}^2 + \zeta\|\nabla^3 d\|_{L^2}^2\big)$$
$$+ \int_0^T e^{\sigma t}\big(\|\nabla u\|_{H^1}^2 + \|P\|_{H^1}^2 + \|\nabla^3 d\|_{L^2}^2 + \zeta^{7/6}\|\nabla u\|_{W^{1,r}}^2 + \zeta^{7/6}\|P\|_{W^{1,r}}^2 + \zeta^2\|\nabla^4 d\|_{L^2}^2\big)\, dt$$
$$+ \int_0^T \zeta e^{\sigma t}\big(\|\nabla u_t\|_{L^2}^2 + \|\Delta d_t\|_{L^2}^2\big)\, dt \le C.$$

*Proof.* First, similar to (3.38) and (3.39), we have

(3.44)
$$\|\nabla \rho\|_{L^2 \cap L^q} \le 2\|\nabla \rho_0\|_{L^2 \cap L^q}.$$

Next, multiplying (3.32) by $\zeta e^{\sigma t}$, we get after using Gronwall's inequality, (3.9), (3.10), (3.23), and (3.24) that

(3.45)
$$\sup_{0 \le t \le T} \zeta e^{\sigma t}\big(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2\big) + \int_0^T \zeta e^{\sigma t}\big(\|\nabla u_t\|_{L^2}^2 + \|\Delta d_t\|_{L^2}^2\big)\, dt \le C.$$

Combining this, (3.20), (3.21), and (3.40) gives

(3.46)
$$\sup_{0 \le t \le T} e^{\sigma t}\big(\zeta\|\nabla u\|_{H^1}^2 + \zeta\|P\|_{H^1}^2 + \zeta\|\nabla d_t\|_{L^2}^2 + \zeta\|\nabla^3 d\|_{L^2}^2\big)$$
$$+ \int_0^T e^{\sigma t}\big(\|\nabla u\|_{H^1}^2 + \|P\|_{H^1}^2 + \|\nabla^3 d\|_{L^2}^2\big)\, dt \le C.$$

Moreover, by $H^4$-estimate of $(1.1)_4$, using Poincaré's inequality, we have

$$\|\nabla^4 d\|_{L^2} \le C\|d_t\|_{H^2} + C\|u \cdot \nabla d\|_{H^2} + C\||\nabla d|^2 d\|_{H^2}$$
$$\le C\|d_t\|_{H^2} + C\||u|(|\nabla d| + |\nabla^2 d| + |\nabla^3 d|)\|_{L^2} + C\||\nabla u|(|\nabla d| + |\nabla^2 d|)\|_{L^2}$$
$$+ C\||\nabla^2 u||\nabla d|\|_{L^2} + C\||\nabla d|^2\|_{L^2} + C\||\nabla d||\nabla^2 d|\|_{L^2} + C\||\nabla^2 d|^2\|_{L^2}$$
$$+ C\||\nabla d||\nabla^3 d|\|_{L^2}$$

$$\leq C\|d_t\|_{H^2} + C\|u\|_{L^\infty}\|\nabla d\|_{H^2} + C\|\nabla d\|_{L^\infty}\|\nabla u\|_{H^1} + C\|\nabla u\|_{H^1}\|\nabla^3 d\|_{L^2}$$
$$+ C\|\nabla d\|_{L^\infty}\|\nabla d\|_{H^2} + C\|\nabla^2 d\|_{H^1}^2$$
$$\leq C\|\Delta d_t\|_{L^2} + C(\|\nabla^3 d\|_{L^2} + \|\nabla u\|_{H^1})\|\nabla^3 d\|_{L^2} + C\|\nabla^3 d\|_{L^2}^2$$

which along with (3.45), and (3.46) leads to

$$\int_0^T \zeta^2 e^{\sigma t} \|\nabla^4 d\|_{L^2}^2 \, dt \leq C.$$

Finally, it follows from (3.20), (3.21), (3.35), (3.40), and (3.4) that for $r \in (2, \min\{q, 3\})$,

$$\|\nabla u\|_{H^1 \cap W^{1,r}} + \|P\|_{H^1 \cap W^{1,r}} \leq C\|\nabla u_t\|_{L^2} + C\|\nabla u\|_{L^2} + C\|\nabla^3 d\|_{L^2}^{7/6}$$
$$\leq C\|\nabla u_t\|_{L^2} + C\|\nabla u\|_{L^2} + C(\|\sqrt{\rho} u_t\|_{L^2} + \|\nabla d_t\|_{L^2})^{7/6},$$

which together with (3.45) and (3.5) implies

$$\int_0^T \zeta^{7/6} e^{\sigma t} \big( \|\nabla u\|_{H^1 \cap W^{1,r}}^2 + \|P\|_{H^1 \cap W^{1,r}}^2 \big) \, dt \leq C.$$

This combined with (3.45)–(3.46) gives (3.43) and completed the proof of Lemma 3.6.  □

## 4.  Proof of Theorem 1.1

With all the a *priori* estimates obtained in Section 3 at hand, we are now in a position to prove Theorem 1.1.

*Proof of Theorem* 1.1. First, by Lemma 2.1, there exists a $T_* > 0$ such that the initial boundary value problem (1.1)–(1.4) has a unique local strong solution $(\rho, u, d, P)$ on $\Omega \times (0, T_*]$. It follows from (1.6) that there exists a $T_1 \in (0, T_*]$ such that (3.1) holds for $T = T_1$.

Next, set

(4.1)     $T^* := \sup\{T \mid (\rho, u, d, P) \text{ is a strong solution on } \Omega \times (0, T] \text{ and } (3.1) \text{ holds}\}.$

First, for any $0 < \tau < T \leq T^*$, one deduces from (3.2), (3.5), and (3.43) that for any $q > 2$,

(4.2)                     $\nabla u, \ \nabla d, \ \Delta d \in C([\tau, T]; L^2 \cap L^q),$

where one has used the standard embedding theory

$$L^\infty(\tau, T; H^1) \cap H^1(\tau, T; H^{-1}) \hookrightarrow C([\tau, T]; L^{q_0}) \quad \text{for any } q_0 \in [2, \infty).$$

Moreover, it follows from (3.1), (3.4), (3.44), and [21, Lemma 2.3] that

(4.3)                    $\rho \in C([0, T]; H^1 \cap W^{1,q}), \quad \nabla \mu(\rho) \in C([0, T]; L^q).$

Now, we claim that

(4.4)                                        $T^* = \infty.$

Otherwise $T^* < \infty$. Proposition 3.1 implies that (3.2) holds at $T = T^*$. It follows from (3.2), (3.5), (4.2), and (4.3) that

$$(\rho^*, u^*, d^*)(x) := (\rho, u, d)(x, T^*) = \lim_{t \to T^*} (\rho, u, d)(x, t)$$

satisfies

$$\rho^* \in H^1 \cap W^{1,q}, \quad u^* \in H^1_{0,\sigma}, \quad d^* \in H^2.$$

Therefore, one can take $(\rho^*, \rho^* u^*, d^*)$ as the initial data and apply Lemma 2.1 to extend the local strong solution beyond $T^*$. This contradicts the assumption of $T^*$ in (4.1). Hence, (4.4) holds. We thus finish the proof of Theorem 1.1 since (1.8) and (1.9) follow directly from (3.44), and (3.43), respectively.                                                   $\square$

## Acknowledgments

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