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Unified Approach to Spectral Properties of Multipliers

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Abstract. Let $\mathbb{B}_n$ be the open unit ball in $\mathbb{C}^n$. We characterize the spectra of pointwise multipliers M_u acting on Banach spaces of analytic functions on $\mathbb{B}_n$ satisfying some general conditions. These spaces include Bergman-Sobolev spaces $A^p_{\alpha,\beta}$, Bloch-type spaces $\mathcal{B}_\alpha$, weighted Hardy spaces H^p_w with Muckenhoupt weights and Hardy-Sobolev Hilbert spaces H^2_β. Moreover, we describe the essential spectra of multipliers in most of the aforementioned spaces, in particular, in those spaces for which the set of multipliers is a subset of the ball algebra.

1. Introduction and preliminaries

In a very recent article [10], Cao, He, and Zhu considered the multiplication operator M_u acting on the Hardy-Sobolev Hilbert space and characterized the spectrum and essential spectrum of M_u. In the present work, we extend and generalize the results obtained there from Hardy-Sobolev Hilbert space to the Bergman-Sobolev and Bloch-type spaces of the open unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$ and weighted Hardy spaces of the open unit disk $\mathbb{D}$ with Muckenhoupt weights. In particular, our main focus is to allow the multiplier space $M(X(\mathbb{B}_n))$ to be contained in the ball algebra, which holds for example for certain Bergman-Sobolev spaces and Bloch-type spaces. We formulate our results on spectral properties of M_u acting on a Banach space $X(\mathbb{B}_n)$ of analytic functions in $\mathbb{B}_n$, where $X(\mathbb{B}_n)$ satisfies very general and natural properties regarding its multiplier space and the norm topology. Consequently, we approach the spectral properties of multipliers in a unified manner and key examples of such spaces include the aforementioned spaces. Aside from obtaining a description of the spectrum for all spaces satisfying the mentioned properties, we also have to develop some new techniques to determine the essential spectrum of M_u regarding the non-Hilbert space case. Other previous work regarding spectral and related properties of multiplication operators on analytic function spaces includes [3, 5, 8, 9, 14, 18, 19].

The article is organised as follows. In Section 2, we introduce general Banach spaces $X(\mathbb{B}_n)$ of analytic functions on $\mathbb{B}_n$ and give central concrete examples of them. Section 3

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focuses on the spectrum of M_u by first establishing a characterization of invertibility of M_u and then obtaining the spectrum of M_u and giving admissible examples of spaces on which M_u can be defined. In Section 4, we begin with a characterization of the essential spectrum in the high-dimensional case $n > 1$. Then we consider the case $n = 1$ by first establishing a characterization of the Fredholmness of M_u when conditions (I), (IV) and $M(X(\mathbb{D})) = H^\infty(\mathbb{D})$ hold. Examples of spaces satisfying the previous conditions are also given. Next, we consider the difficult case when $M(X(\mathbb{D})) \subset A(\mathbb{D})$ (or $u \in M(X(\mathbb{D})) \cap A(\mathbb{D})$) and starting off with the space $X(\mathbb{D}) = \mathcal{B}_\alpha(\mathbb{D})$ for $0 < \alpha \leq 1$ and showing that the condition, earlier observed to be sufficient for the Fredholmness of M_u, is also necessary. Finally, we show the necessity of the condition in the case of those Bergman-Sobolev spaces $A^p_{\alpha,\beta}(\mathbb{D})$ for which $M(A^p_{\alpha,\beta}(\mathbb{D})) \subset A(\mathbb{D})$. From these two cases we obtain the essential spectrum of M_u for several scales of spaces $\mathcal{B}_\alpha(\mathbb{D})$ and $A^p_{\alpha,\beta}(\mathbb{D})$ as the main result of Section 4.

To conclude, our main result regarding the spectra of multiplication operators acting on $X(\mathbb{B}_n)$ is Theorem 3.2. The essential spectra of operators M_u acting on certain spaces $X(\mathbb{D})$ having their multiplier spaces $M(X(\mathbb{D}))$ contained in the disk algebra are described in Theorem 4.13. In the case of general $X(\mathbb{D})$ with $M(X(\mathbb{D})) = H^\infty(\mathbb{D})$, the essential spectra of operators M_u are characterized in Theorem 4.5. In Theorem 4.1, we present the high-dimensional case $n > 1$ concerning the essential spectra of operators M_u acting on general spaces $X(\mathbb{B}_n)$.

Now we introduce some definitions and notations. Throughout this article, let $\mathbb{Z}_{\geq a} = \{n \in \mathbb{Z} : n \geq a\}$ and $\mathbb{Z}_{>a} = \{n \in \mathbb{Z} : n > a\}$, where $a \in \mathbb{R}$. Furthermore, let $\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}$, $n \in \mathbb{Z}_{\geq 1}$, be the open unit ball in $\mathbb{C}^n$ and $\mathbb{D} = \mathbb{B}_1$. Moreover, let $\mathcal{H}(\mathbb{B}_n)$ be the space of all analytic functions $f\colon \mathbb{B}_n \to \mathbb{C}$ and $\mathcal{P}(\mathbb{B}_n)$ be the set of all analytic polynomials $p\colon \mathbb{B}_n \to \mathbb{C}$ such that $p(z) = \sum_{k \in J} c_k z^k$, where $J \subset \mathbb{Z}^n_{\geq 0}$ is a finite set, $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n_{\geq 0}$, $|k| = k_1 + \cdots + k_n$, $z^k = z_1^{k_1} \cdots z_n^{k_n}$ and $c_k \in \mathbb{C}$ for $k \in J$.

We also recall that a bounded linear operator T acting on a Banach space is Fredholm if it has closed range and both kernel and cokernel of T are finite dimensional. The essential spectrum $\sigma_e(T)$ of an operator T is defined as $\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\}$, where I is the identity operator, and the reader may observe that $\sigma_e(T)$ is a subset of the spectrum $\sigma(T)$. See [1] for more details on Fredholm properties of bounded operators.

For any $f \in \mathcal{H}(\mathbb{B}_n)$, the gradient of f is given by

$$\nabla f(z) = \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \right)$$

and will be denoted $Df(z)$ in the case $n = 1$.

Let $\beta \in \mathbb{R}$ and $f \in \mathcal{H}(\mathbb{B}_n)$. The fractional radial derivative R^β is given by

$$R^\beta f(z) = \sum_{k=1}^{\infty} k^\beta f_k(z),$$

where $f(z) = \sum_{k=0}^{\infty} f_k(z)$ is the homogeneous expansion of $f \in \mathcal{H}(\mathbb{B}_n)$. Let $I\colon \mathcal{H}(\mathbb{B}_n) \to \mathcal{H}(\mathbb{B}_n)$ be the identity operator. The operator $(I+R)^\beta$ will also be used and is naturally defined by

$$(I+R)^\beta f(z) = \sum_{k=0}^{\infty} (1+k)^\beta f_k(z).$$

For expressing asymptotic behaviour, the notation $a_k \sim b_k$ as $k \to \infty$ means $\lim_{k\to\infty} \frac{a_k}{b_k} = 1$. Moreover, by $a(x) \gtrsim b(x)$ (or $a(x) \lesssim b(x)$) we indicate the existence of a constant $C > 0$ independent of x such that $a(x) \geq Cb(x)$ (or $a(x) \leq Cb(x)$) for all x in some implicit set. If both $a(x) \gtrsim b(x)$ and $a(x) \lesssim b(x)$ hold, we write $a(x) \asymp b(x)$. When two Banach spaces X_1 and X_2 are isomorphic, we use the notation $X_1 \simeq X_2$.

2. Conditions and examples

We deal with a vector space $X(\mathbb{B}_n)$ of analytic functions on $\mathbb{B}_n$ and a norm $\|\cdot\|_X$ on it, that renders $X(\mathbb{B}_n)$ a Banach space. As usual, for each $z \in \mathbb{B}_n$, the evaluation functional δ_z is defined by $\delta_z(f) = f(z)$ for all $f \in X(\mathbb{B}_n)$. We assume that $X(\mathbb{B}_n)$ contains the constant functions, so then all δ_z are non-zero. Furthermore, we associate to $X(\mathbb{B}_n)$ another Banach space $Y(\mathbb{B}_n) \subset \mathcal{H}(\mathbb{B}_n)$ containing the constant functions and equipped with the norm $\|\cdot\|_Y$ as will be explained below.

The Banach spaces $X(\mathbb{B}_n)$ and $Y(\mathbb{B}_n)$ are often assumed to satisfy the first three conditions below:

(I) *The topologies induced by* $\|\cdot\|_X$ *and* $\|\cdot\|_Y$ *are both finer than the compact-open topology* τ_0. In particular, for every $z \in \mathbb{B}_n$, δ_z is a bounded linear functional on both $X(\mathbb{B}_n)$ and $Y(\mathbb{B}_n)$.

Let

$$M(X(\mathbb{B}_n)) = \{u \in \mathcal{H}(\mathbb{B}_n) : uf \in X(\mathbb{B}_n) \text{ for all } f \in X(\mathbb{B}_n)\}.$$

Using condition (I) and the closed graph theorem, it follows that every $u \in M(X(\mathbb{B}_n))$ induces a bounded linear operator $M_u\colon X(\mathbb{B}_n) \to X(\mathbb{B}_n)$.

(II) *For some* $N \in \mathbb{Z}_{\geq 1}$ *it holds that* $\|f\|_X \asymp |f(0)| + \|R^N f\|_Y$ *for all* $f \in \mathcal{H}(\mathbb{B}_n)$.

Condition (II) describes a relationship between the Banach spaces $X(\mathbb{B}_n)$ and $Y(\mathbb{B}_n)$ such that Lemma 3.1 holds. Since the lemma is trivial for spaces $X(\mathbb{B}_n)$ with $M(X(\mathbb{B}_n)) = H^\infty(\mathbb{B}_n)$, this condition may be omitted when such spaces are considered. For these spaces we have $Y(\mathbb{B}_n) = X(\mathbb{B}_n)$.

(III) $H^\infty(\mathbb{B}_n) \subset M(Y(\mathbb{B}_n))$.

By condition (I) it is well-known that $\sup_{z\in\mathbb{B}_n} |u(z)| \leq \|M_u\|$ for all $u \in M(X(\mathbb{B}_n))$, so $M(X(\mathbb{B}_n)) \subset H^\infty(\mathbb{B}_n)$ and $M(Y(\mathbb{B}_n)) = H^\infty(\mathbb{B}_n)$, where also condition (III) is used in the second statement. Since $u \mapsto M_u$ is bounded according to the bounded inverse theorem, it follows from the boundedness of M_u, that there exists a constant $C > 0$ such that $\|M_u g\|_Y \le C\|u\|_\infty \|g\|_Y$ for all $g \in Y(\mathbb{B}_n)$ and $u \in M(Y(\mathbb{B}_n))$.

When considering the case $n = 1$, we will need the following condition to determine the essential spectra of the multiplication operator generated by $u \in M(X(\mathbb{D}))$.

(IV) *If $f \in X(\mathbb{D})$ has a zero at $z_0 \in \mathbb{D}$, then $\frac{f(z)}{z-z_0} \in X(\mathbb{D})$.*

Lemma 2.1. *Let $f \in \mathcal{H}(\mathbb{D})$ and $v\colon \mathbb{D} \to [0,\infty)$ be a bounded function such that $v(z) = v(|z|)$ for all $z \in \mathbb{D}$. Moreover, let $N \in \mathbb{Z}_{\ge 0}$ be such that*

$$\sup_{z\in\mathbb{D}} v(z)|D^N f(z)| < \infty.$$

If $z_0 \in \mathbb{D}$ is a zero of f, then

$$\sup_{z\in\mathbb{D}} v(z)\left|D^N \frac{f(z)}{z-z_0}\right| < \infty.$$

Proof. Let $g(z) = \frac{f(z)}{z-z_0}$. Since f is analytic with a zero at z_0 we have $D^N g \in \mathcal{H}(\mathbb{D})$. Thus, $h(z) = v(z)|D^N g(z)|$ is bounded on $\mathbb{D}$ if and only if h is bounded near the boundary. For $z \in T = \left\{z \in \mathbb{D} : |z| > \frac{1+|z_0|}{2}\right\}$, we have the following estimate

$$|D^N g(z)| = \left|\sum_{j=0}^{N} \binom{N}{j} D^j f(z) D^{N-j}(z-z_0)^{-1}\right| \le \sum_{j=0}^{N} \frac{N!|D^j f(z)|}{(|z|-|z_0|)^{N-j+1}}. \tag{2.1}$$

Furthermore, for $k \ge 0$ we have

$$\begin{aligned}
|D^k f(z)| &\le \left|\int_{C_z} D^{k+1} f(w)\, dw\right| + |D^k f(0)| \\
&\le |z| \sup_{|w|=|z|} |D^{k+1} f(w)| + |D^k f(0)| \\
&\le \sup_{|w|=|z|} |D^{k+1} f(w)| + |D^k f(0)|,
\end{aligned}$$

where C_z is the line from 0 to z in $\mathbb{D}$.

By induction, it can be shown that

$$|D^k f(z)| \le \sup_{|y|=|z|} |D^N f(y)| + \sum_{j=0}^{N-k-1} |D^{k+j} f(0)|$$

for $0 \le k \le N$. Moreover, from the fact that $\sup_{z\in\mathbb{D}} \sup_{|w|=|z|}$ is interchangable with

$\sup_{w\in\mathbb{D}}$ and $v(z)=v(|z|)$ we now obtain

$$\begin{aligned} v(z)|D^k f(z)| &\le \sup_{|y|=|z|} v(z)|D^N f(y)| + v(z)\sum_{j=0}^{N-k-1}|D^{k+j}f(0)| \\ &= \sup_{|y|=|z|} v(y)|D^N f(y)| + v(z)\sum_{j=0}^{N-k-1}|D^{k+j}f(0)| \\ &\le \sup_{z\in\mathbb{D}} v(z)|D^N f(z)| + \sup_{z\in\mathbb{D}} v(z)\sum_{j=0}^{N-1}|D^j f(0)| = M_{f,N,v} < \infty \end{aligned}$$

for all $z\in\mathbb{D}$. Especially for $z\in T$, using (2.1), we have

$$v(z)|D^N g(z)| \le \sum_{k=0}^{N}\frac{N!v(z)|D^k f(z)|}{(|z|-|z_0|)^{N-k+1}} \le M_{f,N,v}\sum_{k=0}^{N}\frac{N!2^{N-k+1}}{(1-|z_0|)^{N-k+1}} < \infty,$$

which proves the lemma. $\square$

Next, we list a number of spaces satisfying the above conditions (I)–(IV). However, in Example 2.4 we consider spaces $X(\mathbb{B}_n)$ for which $M(X(\mathbb{B}_n)) = H^\infty(\mathbb{B}_n)$, implying that condition (II) is irrelevant.

Example 2.2. For $\alpha>0$ the Bloch-type space $X(\mathbb{B}_n)=\mathcal{B}_\alpha(\mathbb{B}_n)$ is the space of all $f\in\mathcal{H}(\mathbb{B}_n)$ satisfying $\|f\|_{\mathcal{B}_\alpha}=|f(0)|+\sup_{z\in\mathbb{B}_n}(1-|z|^2)^\alpha|\nabla f(z)|<\infty$, see [21]. To these spaces correspond

$$Y(\mathbb{B}_n)=H^\infty_\alpha(\mathbb{B}_n)=\left\{f\in\mathcal{H}(\mathbb{B}_n):\|f\|_{H^\infty_\alpha}=\sup_{z\in\mathbb{B}_n}(1-|z|^2)^\alpha|f(z)|<\infty\right\},$$

see [6]. The little Bloch-type space $\mathcal{B}_{0,\alpha}(\mathbb{B}_n)$ is the subspace of $\mathcal{B}_\alpha(\mathbb{B}_n)$ satisfying $\lim_{|z|\to1}(1-|z|^2)^\alpha|\nabla f(z)|=0$. It is well-known that these spaces obey (I). Let $\|f\|_{\mathcal{B}R_\alpha}=|f(0)|+\sup_{z\in\mathbb{B}_n}(1-|z|^2)^\alpha|Rf(z)|$. According to Theorem 7.1 in [21], it holds that

$$\{f\in\mathcal{H}(\mathbb{B}_n):\|f\|_{\mathcal{B}_\alpha}<\infty\}=\{f\in\mathcal{H}(\mathbb{B}_n):\|f\|_{\mathcal{B}R_\alpha}<\infty\}.$$

Therefore it follows from the bounded inverse theorem that $\|\cdot\|_{\mathcal{B}_\alpha}\asymp\|\cdot\|_{\mathcal{B}R_\alpha}$ and, hence, both of these spaces satisfy condition (II). Condition (III) holds by definition. We will consider the space $(\mathcal{B}_\alpha(\mathbb{D}),\|\cdot\|_{\mathcal{B}_\alpha})$ in the one-dimensional case.

By Theorem 2.1(i) in [17], for $0<\alpha<1$, $u\in M(\mathcal{B}_\alpha(\mathbb{D}))$ if and only if $u\in\mathcal{B}_\alpha(\mathbb{D})\cap H^\infty(\mathbb{D})=\mathcal{B}_\alpha(\mathbb{D})\subset A(\mathbb{D})$, where the inclusion is found in Theorem 7.9 in [21]. When $\alpha=1$, we get from Theorem 2.1(ii) in [17] that $u\in M(\mathcal{B}_\alpha(\mathbb{D}))$ if and only if

$$\sup_{z\in\mathbb{D}}|u'(z)|(1-|z|^2)\log\left(\frac{e}{1-|z|^2}\right)<\infty \quad\text{and}\quad u\in H^\infty(\mathbb{D}).$$

Therefore $u \in \mathcal{B}_{0,1}(\mathbb{D})$. Finally by Theorem 2.1(iii) in [17] we have for $\alpha > 1$ that $u \in M(\mathcal{B}_\alpha(\mathbb{D}))$ if and only if $u \in \mathcal{B}_1(\mathbb{D}) \cap H^\infty(\mathbb{D}) = H^\infty(\mathbb{D})$. According to Lemma 2.1 a function belonging to $\mathcal{B}_\alpha(\mathbb{D})$ will remain in $\mathcal{B}_\alpha(\mathbb{D})$ after removing a finite number of zeros z_0 through division by $z - z_0$, which proves (IV). The notations $\mathcal{B}(\mathbb{D})$ and $\mathcal{B}_0(\mathbb{D})$ stand for $\mathcal{B}_1(\mathbb{D})$ and $\mathcal{B}_{0,1}(\mathbb{D})$ respectively.

Example 2.3. Let $\beta \geq 0$, $\alpha \geq -1$ and $1 \leq p < \infty$. The holomorphic Sobolev space $A^p_{\alpha,\beta}(\mathbb{B}_n)$ is defined by

$$A^p_{\alpha,\beta}(\mathbb{B}_n) = \{f \in \mathcal{H}(\mathbb{B}_n) : \|f\|_{A^p_{\alpha,\beta}} < \infty\},$$

where the norm is defined by

$$\|f\|_{A^p_{\alpha,\beta}} = \|(I+R)^\beta f\|_{A^p_\alpha} = \left(\int_{\mathbb{B}_n} |(I+R)^\beta f(z)|^p \, dA_\alpha(z)\right)^{1/p}$$

for $\alpha > -1$ and

$$\|f\|_{A^p_{-1,\beta}} = \|(I+R)^\beta f\|_{H^p} = \left(\int_{\partial\mathbb{B}_n} |(I+R)^\beta f(z)|^p \, dS(z)\right)^{1/p}.$$

Furthermore, $dA_\alpha(z) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}(1-|z|^2)^\alpha \, dA(z)$, where $dA(z)$ is the $2n$-dimensional Lebesgue measure normalized so that $\int_{\mathbb{B}_n} dA(z) = 1$, and hence, $\int_{\mathbb{B}_n} dA_\alpha(z) = 1$ for every $\alpha > -1$. The notation $dS(z)$ stands for the surface measure satisfying $\int_{\partial\mathbb{B}_n} dS(z) = 1$. The holomorphic Sobolev spaces can be partitioned into the Bergman-Sobolev spaces, $\alpha > -1$, and the Hardy-Sobolev spaces, $H^p_\beta(\mathbb{B}_n) = A^p_{-1,\beta}(\mathbb{B}_n)$. In case of $\beta = 0$, these spaces are called the weighted Bergman spaces $A^p_\alpha(\mathbb{B}_n) = A^p_{\alpha,0}(\mathbb{B}_n)$ with $\alpha > -1$ and the Hardy spaces $H^p(\mathbb{B}_n) = A^p_{-1,0}(\mathbb{B}_n)$.

For $p \geq 1$, $\alpha_j > -1$, $\beta_j \geq 0$ $(j = 1, 2)$, with $\alpha_1 - \alpha_2 = p(\beta_1 - \beta_2)$, the following equivalence holds by Theorem 5.12 in [4] (see also [11]):

$$A^p_{\alpha_1,\beta_1}(\mathbb{B}_n) \simeq A^p_{\alpha_2,\beta_2}(\mathbb{B}_n), \tag{2.2}$$

where the isomorphism is given by the identity operator, and hence, the spaces have equivalent norms. By the same theorem, one also obtains the statement (2.2) for $\alpha_1 = -1$ and $p = 2$. From this it follows that for $\beta_1 < \frac{1+\alpha_1}{p}$, where equality may be used in the case of $p = 2$, we have $A^p_{\alpha_1,\beta_1}(\mathbb{B}_n) \simeq A^p_{\alpha_1-\beta_1 p,0}(\mathbb{B}_n)$. The right-hand side is a weighted Bergman space or $H^2(\mathbb{B}_n)$, hence, $M(A^p_{\alpha,\beta}(\mathbb{B}_n)) = H^\infty(\mathbb{B}_n)$ for $\beta < \frac{1+\alpha}{p}$, where equality may be used in the case of $p = 2$. Regarding the case $n = 1$, if $\beta > \frac{2+\alpha}{p}$, then $A^p_{\alpha,\beta}(\mathbb{D})$ is an algebra and $M(A^p_{\alpha,\beta}(\mathbb{D})) = A^p_{\alpha,\beta}(\mathbb{D})$, see [5]. In this setting, there is a $b < \beta$ satisfying $0 < b - \frac{2+\alpha}{p} < 1$, so that

$$A^p_{\alpha,\beta}(\mathbb{D}) \subset A^p_{\alpha,b}(\mathbb{D}) \subset \Lambda_{b-\frac{2+\alpha}{p}}(\mathbb{D}) \subset A(\mathbb{D}),$$

where $\Lambda_{b-\frac{2+\alpha}{p}}$ is a Lipschitz space, see [21]. The first inclusion follows from (2.2), the second inclusion can be found in Theorem 5.5 in [4] and the last one is given by Theorem 7.9 in [21]. Furthermore, by Proposition 2.2 in [11], we have for $p \geq 1$, $\alpha \geq -1$, and every positive integer N that

$$\|f\|_{A^p_{\alpha,N}} \asymp \sum_{j=0}^{N-1} |D^j f(0)| + \|D^N f\|_{A^p_\alpha} \tag{2.3}$$

for $f \in \mathcal{H}(\mathbb{D})$. Next, we check the conditions (I)–(IV).

The topology generated by $\|\cdot\|_{A^p_{\alpha,\beta}}$ is finer than the compact-open topology τ_0, so condition (I) holds. Indeed, the statement follows from Lemma 5.6 in [4] with the use of supremum over an arbitrary compact subset of $\mathbb{B}_n$. Hereafter, we will assume that $\beta \geq \frac{1+\alpha}{p}$. For smaller β it was mentioned that the multiplier space is $H^\infty(\mathbb{B}_n)$ which is considered in Example 2.4, where the the space $A^p_{\alpha,\beta}(\mathbb{B}_n)$ can be viewed as a weighted Bergman space.

In the case $N > \beta - \frac{\alpha+1}{p} \geq 0$, an application of (2.2) gives that

$$f \in A^p_{\alpha,\beta}(\mathbb{B}_n) \text{ if and only if } f \in A^p_{(N-\beta)p+\alpha,N}(\mathbb{B}_n) \text{ if and only if } R^N f \in A^p_{(N-\beta)p+\alpha}(\mathbb{B}_n).$$

Therefore, let $X(\mathbb{B}_n) = A^p_{\alpha,\beta}(\mathbb{B}_n)$ and $Y(\mathbb{B}_n) = A^p_{(N-\beta)p+\alpha}(\mathbb{B}_n)$, where $N = \inf\big\{\widehat{N} \in \mathbb{Z}_{\geq 1} : \widehat{N} > \beta - \frac{\alpha+1/2}{p}\big\}$. Moreover, for $f \in \mathcal{H}(\mathbb{B}_n)$ we have

$$\|(I+R)^\beta f\|_{A^p_\alpha} \asymp |f(0)| + \|R^\beta f\|_{A^p_\alpha}, \tag{2.4}$$

according to Lemma 2.5. Condition (II) follows by first using the equivalence of the norms $\|\cdot\|_{A^p_{\alpha,\beta}}$ and $\|\cdot\|_{A^p_{(N-\beta)p+\alpha,N}}$ by (2.2), and then applying (2.4) to the latter norm. Furthermore, it holds that $M(A^p_\alpha(\mathbb{B}_n)) = H^\infty(\mathbb{B}_n)$, which shows that condition (III) is satisfied.

Let us check the condition (IV) for $A^p_{\alpha,\beta}(\mathbb{D})$. We assume that $f \in A^p_{\alpha,\beta}(\mathbb{D})$ has a zero at $z = z_0$. Let us show that $\frac{f}{z-z_0} \in A^p_{\alpha,\beta}(\mathbb{D})$ by establishing that $R^N\big(\frac{f}{z-z_0}\big) \in A^p_{(N-\beta)p+\alpha}(\mathbb{D})$. Let us take $|z_0| < r < 1$. We may assume that $|z| \geq r$, since $R^N\big(\frac{f}{z-z_0}\big) \in H(\mathbb{D})$ is bounded on $r\mathbb{D}$. We will utilize the following formula given in Proposition 6 in [10]:

$$R^N\left(\frac{f(z)}{z-z_0}\right) = \frac{(-1)^N}{(z-z_0)^{N+1}} \sum_{k=0}^{N} (-1)^k \binom{N+1}{k} (z-z_0)^k R^N((z-z_0)^{N-k} f),$$

where $r \leq |z| < 1$. It suffices to show that $R^N((z-z_0)f) \in A^p_{(N-\beta)p+\alpha}(\mathbb{D})$, which implies that $R^N((z-z_0)^{N-k}f) \in A^p_{(N-\beta)p+\alpha}(\mathbb{D})$ for $k = 0, 1, \ldots, N$. Using the general Leibniz rule we obtain

$$R^N((z-z_0)f) = \sum_{k=0}^{N} \binom{N}{k} R^{N-k}(z-z_0) R^k(f) = \sum_{k=0}^{N} \binom{N}{k} z R^k(f).$$

We observe that $\|zR^kf\|_{A^p_{(N-\beta)p+\alpha}} \leq \|R^kf\|_{A^p_{(N-\beta)p+\alpha}}$ and $R^kf \in A^p_{(N-\beta)p+\alpha}(\mathbb{D})$ if and only if

$$(1-|z|^2)^{N-k}R^{(N-k)}R^kf = (1-|z|^2)^{N-k}R^Nf \in A^p_{(N-\beta)p+\alpha}(\mathbb{D}),$$

see [21, p. 75]. The last statement holds, since

$$\|(1-|z|^2)^{N-k}R^Nf\|_{A^p_{(N-\beta)p+\alpha}} \leq \|R^Nf\|_{A^p_{(N-\beta)p+\alpha}} < \infty,$$

where we used the fact $R^Nf \in A^p_{(N-\beta)p+\alpha}(\mathbb{D})$. So we have that $R^kf \in A^p_{(N-\beta)p+\alpha}(\mathbb{D})$ for $k = 0, 1, \ldots, N$ and consequently $R^N((z-z_0)f) \in A^p_{(N-\beta)p+\alpha}(\mathbb{D})$. Therefore $R^N\big(\frac{f}{z-z_0}\big) \in A^p_{(N-\beta)p+\alpha}(\mathbb{D})$.

It should also be mentioned that the spaces $A^p_{\alpha,\beta}(\mathbb{B}_n)$ are reflexive for $p > 1$, see Proposition 5.7(iv) in [4].

Example 2.4. We consider all spaces $X(\mathbb{B}_n)$ that satisfies (I) and $M(X(\mathbb{B}_n)) = H^\infty(\mathbb{B}_n)$. Furthermore, condition (IV) is also assumed to hold if $n = 1$. Letting $Y(\mathbb{B}_n) = X(\mathbb{B}_n)$ condition (III) is also satisfied and condition (II) is irrelevant, see the remark after condition (II). These spaces include growth spaces $H^\infty_\alpha(\mathbb{D})$, $\alpha > 0$, and weighted Hardy spaces $H^p_w(\mathbb{D})$, $p > 1$, where $w \in (A^p)$, that is, w satisfies the Muckenhoupt (A^p)-condition, see details in [7]. Considering the weighted Hardy spaces, condition (I) follows from the proof of Lemma 2.1 in [7]. Notice that if $w \in (A^p)$, $p > 1$, then the critical exponent $q_w < p$. For $f \in \mathcal{H}(\mathbb{D})$ we have $\|f\|_{H^p_w} < \infty$ if and only if

$$\lim_{r\to 1^-} \int_{-\pi}^{\pi} \big|f(re^{i\theta})\big|^p w(\theta)\, d\theta < \infty.$$

Since, for every $z_0 \in \mathbb{D}$, there exists $r < 1$ such that $\frac{1}{z-z_0}$ is bounded on $\mathbb{D} \setminus r\mathbb{D}$, condition (IV) follows. Condition (IV) is proved by similar arguments for many spaces, for example, weighted Bergman spaces, growth spaces and Hardy spaces.

Lemma 2.5. *Let $\beta \geq 0$. If either $\alpha > -1$ and $p \geq 1$, or $\alpha = -1$ and $p = 2$, it holds that*

$$\|f\|_{A^p_{\alpha,\beta}} \asymp |f(0)| + \|R^\beta f\|_{A^p_\alpha}, \quad f \in \mathcal{H}(\mathbb{B}_n).$$

Moreover, the space $A^p_{\alpha,\beta}(\mathbb{B}_n)$ endowed with the norm defined as $\|f\|_{p,\alpha,\beta} = |f(0)| + \|R^\beta f\|_{A^p_\alpha}$ is a Banach space.

Proof. Let N be the smallest integer in the set $\mathbb{Z}_{>\beta+1}$ and $\gamma = p(N-\beta)+\alpha > -1$. By (2.2) and Lemma 1 in [10] we have $\|f\|_{A^p_{\alpha,\beta}} \asymp \|f\|_{A^p_{\gamma,N}}$ and $\|f\|_{p,\alpha,\beta} \asymp \|f\|_{p,\gamma,N}$ respectively. The norm equivalences are also well known to experts in the case $\alpha = -1$ and $p = 2$. Therefore

$$\big(A^p_{\alpha,\beta}(\mathbb{B}_n), \|\cdot\|_{p,\alpha,\beta}\big)$$

is a Banach space, since this is true for

$$\left(A^p_{\alpha,\beta}(\mathbb{B}_n), \|\cdot\|_{p,\gamma',N}\right)$$

for all $\gamma' > -1$, see [20]. It now suffices to show that $\|f\|_{A^p_{\gamma,N}} \asymp \|f\|_{p,\gamma,N}$. Using Jensen's inequality we have

$$\begin{aligned}
|(I+R)^N f(z)| = \left| \sum_{j=0}^{N} \binom{N}{j} R^j f(z) \right| &\leq 2^N \sum_{j=0}^{N} \frac{1}{2^N} \binom{N}{j} |R^j f(z)| \\
&\leq 2^N \left(\sum_{j=0}^{N} \frac{1}{2^N} \binom{N}{j} |R^j f(z)|^p \right)^{1/p}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\|f\|_{A^p_{\gamma,N}} &= \left(\int_{\mathbb{B}_n} |(I+R)^N f(z)|^p \, dA_\gamma(z) \right)^{1/p} \\
&\leq \left(2^{N(p-1)} \sum_{j=0}^{N} \binom{N}{j} \int_{\mathbb{B}_n} |R^j f(z)|^p \, dA_\gamma(z) \right)^{1/p} \\
&\asymp \left(\sum_{j=0}^{N} \int_{\mathbb{B}_n} |R^N f(z)|^p \, dA_{p(N-j)+\gamma}(z) \right)^{1/p} \\
&\lesssim \left(\int_{\mathbb{B}_n} |R^N f(z)|^p \, dA_\gamma(z) \right)^{1/p} + |f(0)| = \|f\|_{p,\gamma,N}.
\end{aligned}$$

Our approach to prove the converse is very similar. It holds that

$$\begin{aligned}
\int_{\mathbb{B}_n} |R^N f(z)|^p \, dA_\gamma(z) &= \int_{\mathbb{B}_n} |(I+R-I)^N f(z)|^p \, dA_\gamma(z) \\
&\leq 2^{N(p-1)} \sum_{j=0}^{N} \binom{N}{j} \int_{\mathbb{B}_n} |(I+R)^j f(z)|^p \, dA_\gamma(z) \\
&\asymp \sum_{j=0}^{N} \int_{\mathbb{B}_n} |(I+R)^N f(z)|^p \, dA_{p(N-j)+\gamma}(z) \\
&\lesssim \int_{\mathbb{B}_n} |(I+R)^N f(z)|^p \, dA_\gamma(z).
\end{aligned}$$

From this and Lemma 5.6 in [4], it follows that there exists a constant $M = M(n, N, p, \gamma) > 0$ such that

$$\|R^N f\|_{A^p_\gamma} + |f(0)| \leq M \|(I+R)^N f\|_{A^p_\gamma} + |f(0)| \leq 2M \|(I+R)^N f\|_{A^p_\gamma},$$

which finishes the proof. $\square$

3. The spectrum of M_u

Next, we will characterize the spectra of multiplication operators acting on $X(\mathbb{B}_n)$ in the case that there exists a space $Y(\mathbb{B}_n)$ such that conditions (I)–(III) are satisfied. Condition (II) is crucial in the following lemma. The corresponding results for the Hardy-Sobolev Hilbert spaces were obtained in [10].

Lemma 3.1. *Assume that* (I)–(III) *are satisfied and let* $u \in M(X(\mathbb{B}_n))$. *The following statements are equivalent:*

(a) $1/u \in M(X(\mathbb{B}_n))$,

(b) $1/u \in H^\infty(\mathbb{B}_n)$,

(c) M_u *is invertible.*

Proof. Assuming $1/u \in M(X(\mathbb{B}_n))$ we obtain immediately, by the remark after condition (III), that $1/u \in H^\infty(\mathbb{B}_n)$. To prove the converse implication we will use the formula

$$R^N\left(\frac{f}{u}\right) = \frac{(-1)^N}{u^{N+1}} \sum_{k=0}^{N} (-1)^k \binom{N+1}{k} u^k R^N\left(u^{N-k} f\right), \quad f \in \mathcal{H}(\mathbb{B}_n), \tag{3.1}$$

which can be found in Corollary 5 in [10]. The proof of the formula uses the derivative D, but the formula remains valid for all linear operators S that admit the law $S(fg) = fSg + gSf$, $f, g \in \mathcal{H}(\mathbb{D})$, and for which the formula is valid for $N = 1$. Moreover, the dimension n is irrelevant for the proof, and therefore we may replace D with R and also consider the formula in higher dimensions. Notice that (3.1) is invalid for $N = 0$.

If $1/u \in H^\infty(\mathbb{B}_n)$, then u is uniformly bounded from below, that is, there exists a $0 < c < 1$ such that $\inf_{z\in\mathbb{B}_n} |u(z)| \ge c$ and hence formula (3.1) is applicable. By condition (II) we have that $f \in X(\mathbb{B}_n)$ if and only if $R^N f \in Y(\mathbb{B}_n)$.

One should also notice that $u^k \in M(X(\mathbb{B}_n))$ and $(1/u)^k \in H^\infty(\mathbb{B}_n)$ for all $k \in \mathbb{Z}_{\ge 0}$. For $f \in X(\mathbb{B}_n)$ we obtain that $u^k f \in X(\mathbb{B}_n)$ for all $k \in \mathbb{Z}_{\ge 0}$, and therefore,

$$\begin{aligned}
\left\|\frac{f}{u}\right\|_X &\asymp \left\|R^N\left(\frac{f}{u}\right)\right\|_Y + \left|\frac{f(0)}{u(0)}\right| \\
&\le \sum_{k=0}^{N} \binom{N+1}{k} \left\| \left(\frac{1}{u}\right)^{N+1-k} R^N\left(u^{N-k} f\right)\right\|_Y + \left|\frac{f(0)}{u(0)}\right| \\
&\lesssim \sum_{k=0}^{N} \left\| \left(\frac{1}{u}\right)^{N+1-k} \right\|_\infty \left\|R^N\left(u^{N-k} f\right)\right\|_Y + \left|\frac{f(0)}{u(0)}\right| \\
&\le \left(\frac{1}{c}\right)^{N+1} \sum_{k=0}^{N} \|R^N(u^{N-k} f)\|_Y + \left|\frac{f(0)}{u(0)}\right| \lesssim \sum_{k=0}^{N} \|u^{N-k} f\|_X + \left|\frac{f(0)}{u(0)}\right| < \infty,
\end{aligned}$$

where the remark after condition (III) gives the second inequality. Hence, we have shown that the two statements (a) and (b) are equivalent. If $1/u \in M(X(\mathbb{B}_n))$, then clearly $f \mapsto \frac{f}{u}$ is the inverse of M_u. Conversely, if M_u is invertible, then $M_{1/u}$ must be the unique bounded inverse, so $1/u \in M(X(\mathbb{B}_n))$. $\square$

Theorem 3.2. *Assume that* (I)–(III) *are satisfied and let* $M_u\colon X(\mathbb{B}_n) \to X(\mathbb{B}_n)$ *be a multiplication operator generated by* $u \in M(X(\mathbb{B}_n))$. *The spectrum of* M_u *is given by* $\sigma(M_u) = \overline{u(\mathbb{B}_n)}$.

Proof. Let $\lambda \in \mathbb{C}$. Clearly $u - \lambda \in M(X(\mathbb{B}_n))$. If $\lambda \in \overline{u(\mathbb{B}_n)}$, then $|u(z) - \lambda|$ is not bounded from below so $M_u - \lambda I = M_{u-\lambda}$ is not invertible by Lemma 3.1. Using again Lemma 3.1, it follows that for any $\lambda \in \mathbb{C} \setminus \overline{u(\mathbb{B}_n)}$ the operator $M_u - \lambda I$ is invertible since $|u(z) - \lambda|$, in this case, is bounded from below. Hence, the spectrum is given by $\sigma(M_u) = \overline{u(\mathbb{B}_n)}$. $\square$

Remark 3.3. The above result implies that $r(M_u) = \|u\|_\infty \leq \|M_u\|$. Moreover, since the spectrum $\sigma(M_u) = \overline{u(\mathbb{B}_n)}$ is connected, when u is continuous, any nonzero spectral radius would imply an uncountable number of points in the spectrum, from which it follows that the operator is not compact. Consequently, M_u is never compact if $u \neq 0$.

Corollary 3.4. *Let* $X(\mathbb{B}_n)$ *be any of the following spaces*

(a) $A^p_{\alpha,\beta}(\mathbb{B}_n)$, $p \geq 1$, $\beta \geq 0$, *and* $\alpha > -1$;

(b) $\mathcal{B}_\alpha(\mathbb{B}_n)$, $\alpha > 0$;

(c) $H^2_\beta(\mathbb{B}_n)$, $\beta \geq 0$;

(d) $H^p_w(\mathbb{D})$, $p > 1$, $w \in (A^p)$.

Then the spectrum of a multiplication operator $M_u\colon X(\mathbb{B}_n) \to X(\mathbb{B}_n)$ *is given by* $\sigma(M_u) = \overline{u(\mathbb{B}_n)}$.

4. The essential spectrum of M_u

Examining the essential spectrum of a multiplication operator when the domain is $\mathbb{B}_n$, $n > 1$, the result concerning H^2_β, obtained by Cao, He and Zhu, can be made quite general, see Theorem 4.1. In the case $n = 1$, we have obtained a sufficient condition for Fredholmness in Lemma 4.4, where all four conditions (I)–(IV) were assumed. For the spaces mentioned in our main result, namely Theorem 4.13, this condition is also necessary for Fredholmness, see Lemmas 4.9 and 4.12, but for this to be proved, space-specific properties were used. An asymptotic approximation for the behaviour of the norm of the peak functions is necessary for the result concerning Bergman-Sobolev spaces. The estimate given in Lemma 11 in [10] is insufficient for our purposes, not only because it only considers $p = 2$, but also because it is not a sharp lower bound. The necessity of an asymptotic approximation instead of a non-sharp lower bound of the behaviour is clear when an arbitrary $p \in (1, \infty)$ is considered in Lemma 4.12.

Theorem 4.1. *Assume that condition* (I) *is satisfied and* $n > 1$. *Furthermore, let* $u \in M(X(\mathbb{B}_n))$ *and* $P_j\colon \mathbb{B}_n \to \mathbb{C}$, $P_j(z) = z_j$ *for every* $j = 1, \ldots, n$. *Suppose that* $P_j \in M(X(\mathbb{B}_n))$ *for every* j. *Then* $\sigma_e(M_u) = \bigcap_{0<r<1} \overline{u(\mathbb{B}_n \setminus r\mathbb{B}_n)} = \overline{u(\mathbb{B}_n)} = \sigma(M_u)$.

Proof. Let $\lambda \in u(\mathbb{B}_n)$. Since $n > 1$, the function $u(z) - \lambda$ has infinitely many distinct zeros, and therefore, there must exist an infinite subset $\{\alpha_k\}_{k=1}^{\infty}$, $\alpha_k = (\alpha_{k,1}, \ldots, \alpha_{k,n})$, of these zeros such that for some $j = 1, \ldots, n$ we have $\alpha_{k,j} \neq \alpha_{l,j}$ whenever $k \neq l$. We first show, by induction, that $(\delta_{\alpha_k})_{k=1}^{\infty}$ are linearly independent in $\operatorname{Ker} M_{u-\lambda}^*$. Clearly all $\delta_{\alpha_k} \in \operatorname{Ker} M_{u-\lambda}^*$. Suppose that

$$\sum_{k=1}^{m} c_k \delta_{\alpha_k} = 0$$

for some $m \in \mathbb{Z}_{\geq 1}$. If $m = 1$, it follows that $c_1 = 0$. Assume that $m \geq 2$. For arbitrary $f \in X(\mathbb{B}_n)$ we have by assumption that $P_j f \in X(\mathbb{B}_n)$, so

$$\sum_{k=1}^{m} c_k \alpha_{k,j} \delta_{\alpha_k}(f) = 0 \quad \text{and} \quad \sum_{k=1}^{m} c_k \delta_{\alpha_k}(f) = 0.$$

Hence

$$\sum_{k=2}^{m} c_k(\alpha_{k,j} - \alpha_{1,j})\delta_{\alpha_k}(f) = \sum_{k=1}^{m} c_k(\alpha_{k,j} - \alpha_{1,j})\delta_{\alpha_k}(f) = 0 \quad \text{for all } f \in X(\mathbb{B}_n),$$

and therefore, by the induction hypothesis, $c_k(\alpha_{k,j} - \alpha_{1,j}) = 0$ for all $k = 2, \ldots, m$. This implies that $c_k = 0$ for $k = 2, \ldots, m$, and consequently $c_1 = 0$. Then $\operatorname{Ker} M_{u-\lambda}^*$ is infinite dimensional so that $M_{u-\lambda}^*$, and equivalently $M_{u-\lambda}$, is not Fredholm. It follows that $u(\mathbb{B}_n) \subset \sigma_e(M_u)$ and, moreover, that

$$\bigcap_{0<r<1} \overline{u(\mathbb{B}_n \setminus r\mathbb{B}_n)} \subset \overline{u(\mathbb{B}_n)} \subset \sigma_e(M_u) \subset \sigma(M_u).$$

For the converse conclusion, let $\lambda \notin \bigcap_{0<r<1} \overline{u(\mathbb{B}_n \setminus r\mathbb{B}_n)}$. Hence, there are $r \in (0,1)$ and $\delta > 0$ such that $|\lambda - u(z)| \geq \delta$ for all $r < |z| < 1$. Then $v(z) = (u(z) - \lambda)^{-1}$ is holomorphic and bounded on $\mathbb{B}_n \setminus r\overline{\mathbb{B}}_n$. As in [10], using Hartogs' extension theorem and the identity theorem, we can extend v to a function $\widetilde{v} \in \mathcal{H}(\mathbb{B}_n)$ such that $\widetilde{v}(z) = (u(z) - \lambda)^{-1}$ for all $z \in \mathbb{B}_n$, and therefore $\widetilde{v} \in H^{\infty}(\mathbb{B}_n)$. Now $M_{u-\lambda}$ is invertible by Lemma 3.1, so $\lambda \notin \sigma(M_u)$. $\square$

Remark 4.2. Following the proof of Theorem 4.1 it is clear that $(\delta_{\alpha_k})_{k=1}^{\infty}$ are linearly independent when $n = 1$.

Now we proceed to the case $n = 1$.

The following result is based on ideas due to Axler [3] that was carried on in [8]. It holds for all spaces $X(\mathbb{D})$ such that $M(X(\mathbb{D})) = H^\infty(\mathbb{D})$.

Lemma 4.3. *Assume that condition* (I) *is satisfied and let* $u \in M(X(\mathbb{D})) = H^\infty(\mathbb{D})$*. If* $M_u\colon X(\mathbb{D}) \to X(\mathbb{D})$ *is Fredholm, then there are* $r \in (0,1)$ *and* $\delta > 0$ *such that* $|u(z)| \geq \delta$ *for all* $r \leq |z| < 1$*.*

Proof. Assume we can find a sequence $(z_n)_{n=1}^{\infty} \subset \mathbb{D}$ with $|z_n| \to 1$ and $|u(z_n)| \to 0$ when $n \to \infty$. Then we can assume that $(z_n)_n$ is an interpolating sequence in $H^\infty(\mathbb{D})$ by going to a subsequence if necessary. Therefore, (see, e.g., [2, Ch. 7.3]) there is a constant $M > 0$ such that for each $N \in \mathbb{N}$ there is a function $u_N \in H^\infty(\mathbb{D})$ with

$$u_N(z_n) = \begin{cases} u(z_n) & \text{if } n \geq N, \\ 0 & \text{if } n < N \end{cases}$$

and $\|u_N\|_\infty \leq M \sup_{n \geq N} |u(z_n)|$. Let

$$Z_N = \{f \in X(\mathbb{D}) : \delta_{z_n}(f) = 0 \text{ for all } n \geq N\},$$

which is a closed subspace of $X(\mathbb{D})$. From Remark 4.2 we know that the $\delta_{z_n} \in X(\mathbb{D})^*$ are linearly independent, which implies that $Z_N^\perp$ is infinite-dimensional. Since $\delta_{z_n}(u - u_N) = 0$ for all $n \geq N$, we get $M_{u-u_N}(X(\mathbb{D})) \subset Z_N$. Now $(X(\mathbb{D})/Z_N)^* = Z_N^\perp$, so $X(\mathbb{D})/Z_N$ is infinite-dimensional. Hence $X(\mathbb{D})/M_{u-u_N}(X(\mathbb{D}))$ is also infinite-dimensional, and $M_{u-u_N}\colon X(\mathbb{D}) \to X(\mathbb{D})$ is not Fredholm. As $M(X(\mathbb{D})) = H^\infty(\mathbb{D})$ and the set of non-Fredholm operators is closed, it follows from

$$\|M_{u-u_N} - M_u\| = \|M_{u_N}\| \leq C\|u_N\|_\infty \leq CM \sup_{n \geq N} |u(z_n)| \to 0 \quad \text{as } N \to \infty,$$

that M_u is not Fredholm. □

Lemma 4.4. *Assume that* (I)–(IV) *are satisfied and let* $u \in M(X(\mathbb{D}))$*. If there are* $r \in (0,1)$ *and* $\delta > 0$ *such that* $|u(z)| \geq \delta$ *for all* $r \leq |z| < 1$*, then* $M_u\colon X(\mathbb{D}) \to X(\mathbb{D})$ *is Fredholm.*

Proof. By assumption we have that u can have only finitely many zeros $\alpha_1, \ldots, \alpha_n$ inside $\mathbb{D}$ with multiplicities $m_1, \ldots, m_n$ respectively. Then for all $z \in \mathbb{D}$,

$$u(z) = v(z)(z - \alpha_1)^{m_1} \cdots (z - \alpha_n)^{m_n} = v(z)p(z),$$

where $v \in \mathcal{H}(\mathbb{D})$ and $1/v \in H^\infty(\mathbb{D})$.

Let us now define the point evaluation maps for derivatives by $\delta_z^{(k)}(f) = f^{(k)}(z)$ for all $z \in \mathbb{D}$ and all $k \in \mathbb{Z}_{\geq 0}$. By assumption (I), it holds that $\delta_z^{(k)} \in X(\mathbb{D})^*$ for all k and z. Clearly,

$$M_u(X(\mathbb{D})) \subset \bigcap_{i=1}^{n} \bigcap_{k=0}^{m_i-1} \operatorname{Ker} \delta_{\alpha_i}^{(k)}.$$

Let $f \in \bigcap_{i=1}^{n} \bigcap_{k=0}^{m_i-1} \operatorname{Ker} \delta_{\alpha_i}^{(k)}$, so $f^{(k)}(\alpha_i) = 0$ for all $i = 1, \ldots, n$ and all $k = 0, \ldots, m_i - 1$. Then $\frac{f}{u} \in \mathcal{H}(\mathbb{D})$. Now assumption (IV) implies that $v \in M(X(\mathbb{D}))$. Indeed, if $g \in X(\mathbb{D})$, then $ug \in X(\mathbb{D})$ and by assumption (IV) it follows that $vg = \frac{ug}{p} \in X(\mathbb{D})$. Therefore, $1/v \in M(X(\mathbb{D}))$ by Lemma 3.1, so that $\frac{f}{u} = \frac{f/p}{v} \in X(\mathbb{D})$ by assumption (IV). As a result, $f = u\frac{f}{u} \in M_u(X(\mathbb{D}))$, and thus

$$M_u(X(\mathbb{D})) = \bigcap_{i=1}^{n} \bigcap_{k=0}^{m_i-1} \operatorname{Ker} \delta_{\alpha_i}^{(k)}.$$

Consequently, M_u has closed range, and since $M_u\colon X(\mathbb{D}) \to X(\mathbb{D})$ is always injective, the dimension of the kernel of M_u is finite. Since ${}^\perp(\operatorname{span}\{\delta_{\alpha_i}^{(k)}\}) = \operatorname{Ker} \delta_{\alpha_i}^{(k)}$, it follows that the w^*-closed one-dimensional space $\operatorname{span}\{\delta_{\alpha_i}^{(k)}\} = (\operatorname{Ker} \delta_{\alpha_i}^{(k)})^\perp$, see [16, Theorem 11 on p. 341]. Therefore, by [16, Theorem 13 on p. 342], we have

$$M_u(X(\mathbb{D}))^\perp = \sum_{i=1}^{n} \sum_{k=0}^{m_i-1} (\operatorname{Ker} \delta_{\alpha_i}^{(k)})^\perp = \sum_{i=1}^{n} \sum_{k=0}^{m_i-1} \operatorname{span}\{\delta_{\alpha_i}^{(k)}\},$$

and hence, the dimension of the co-kernel of M_u is finite, and M_u is Fredholm. □

Theorem 4.5. *Assume that* (I), (III) *and* (IV) *are satisfied and* $M(X(\mathbb{D})) = H^\infty(\mathbb{D})$. *Let* $M_u\colon X(\mathbb{D}) \to X(\mathbb{D})$ *be a multiplication operator generated by* $u \in M(X(\mathbb{D}))$. *The essential spectrum of* M_u *is given by* $\sigma_e(M_u) = \bigcap_{0<r<1} \overline{u(\mathbb{D} \setminus r\mathbb{D})}$.

Proof. We have that $\lambda \in \overline{u(\mathbb{D} \setminus r\mathbb{D})}$ for all $r \in (0,1)$ if and only if for all $r \in (0,1)$ there is a sequence $(z_n)_{n=1}^\infty \subset \mathbb{D}$ such that $|z_n| \geq r$ for all $n \in \mathbb{N}$ and $|u(z_n) - \lambda| \to 0$ when $n \to \infty$. Since $M_u - \lambda I = M_{u-\lambda}$, we can now apply Lemmas 4.3 and 4.4 to conclude that the last statement equivalently means that $M_u - \lambda I$ is not Fredholm, that is $\lambda \in \sigma_e(M_u)$. The use of Lemma 4.4 is justified by the remark after condition (II). □

In Example 2.2–2.4 they were stated that the multiplier spaces for $A^p_{\alpha,\beta}(\mathbb{D})$ with $p \geq 1$, $\alpha > -1$, $\beta < \frac{1+\alpha}{p}$; $H^p_w(\mathbb{D})$ with $p > 1$, $w \in (A^p)$ and $\mathcal{B}_\alpha(\mathbb{D})$ with $\alpha > 1$ are $H^\infty(\mathbb{D})$. Thus, we obtain the following results.

Corollary 4.6. *In each of the following three cases:*

(a) $p \geq 1$, $\alpha > -1$ *and* $\beta < \frac{1+\alpha}{p}$ *with* $u \in M(A^p_{\alpha,\beta}(\mathbb{D}))$*;*

(b) $\alpha > 1$ *with* $u \in M(\mathcal{B}_\alpha(\mathbb{D}))$*;*

(c) $p > 1$, $w \in (A^p)$ *with* $u \in M(H^p_w(\mathbb{D}))$*,*

the essential spectrum of M_u *is given by*

$$\sigma_e(M_u) = \bigcap_{0<r<1} \overline{u(\mathbb{D} \setminus r\mathbb{D})}.$$

It was shown in Theorem 4.1 that in higher dimensions, $n > 1$, the essential spectra of multiplication operators coincide with their spectra for many spaces. This is seldom true for $n = 1$. In Corollaries 3.4 and 4.6 and Theorem 4.13, we list some spaces, on which multiplier operators have the spectrum given by $\overline{u(\mathbb{B}_n)}$ and the essential spectrum given by $\bigcap_{0<r<1} \overline{u(\mathbb{B}_n \setminus r\mathbb{B}_n)}$. Although the sets may differ, their spectral and essential spectral radii coincide according to the following remark.

Remark 4.7. (a) Let $n \in \mathbb{Z}_{\geq 1}$. Since the decreasing sequence $\left(\overline{u(\mathbb{B}_n \setminus (1-\frac{1}{k})\mathbb{B}_n)}\right)_{k=2}^\infty$ consists of compact and connected sets, the intersection $\bigcap_{0<r<1} \overline{u(\mathbb{B}_n \setminus r\mathbb{B}_n)}$ is compact and connected.

(b) For $n \in \mathbb{Z}_{\geq 1}$ and $u \in H^\infty(\mathbb{B}_n)$ we have

$$\sup\left\{ |\lambda| : \lambda \in \bigcap_{0<r<1} \overline{u(\mathbb{B}_n \setminus r\mathbb{B}_n)} \right\} = \|u\|_\infty = \sup_{\lambda \in \overline{u(\mathbb{B}_n)}} |\lambda|.$$

Moreover, both suprema are attained. Clearly

$$\|u\|_\infty = \sup_{z \in \mathbb{B}_n} |u(z)| = \sup_{\lambda \in u(\mathbb{B}_n)} |\lambda| = \sup_{\lambda \in \overline{u(\mathbb{B}_n)}} |\lambda| \geq \sup\left\{ |\lambda| : \lambda \in \bigcap_{0<r<1} \overline{u(\mathbb{B}_n \setminus r\mathbb{B}_n)} \right\}.$$

Furthermore, since $u \in H^\infty$ there is a sequence $(z_j)_{j=1}^\infty$ such that $z_j \in \mathbb{B}_n \setminus r_j\mathbb{B}_n$ and $\lim_{j\to\infty} |u(z_j)| = \|u\|_\infty$, where $r_j = 1 - j^{-1}$. The sequence $(u(z_j))_{j=1}^\infty$ is bounded, and therefore, by Bolzano-Weierstrass theorem, there is a convergent subsequence $(\lambda_k)_{k=1}^\infty$, where $\lambda_k = u(z_{j_k}) \in \overline{u(\mathbb{B}_n \setminus r_{j_k}\mathbb{B}_n)}$. Since the sets $U_k = \overline{u(\mathbb{B}_n \setminus r_{j_k}\mathbb{B}_n)}$ are compact and $U_{k+1} \subset U_k$, $k = 1, 2, \ldots$, it holds that $\lim_{k\to\infty} \lambda_k = \lambda \in U_j$ for every j, and hence, we have $\lambda \in \bigcap_{0<r<1} \overline{u(\mathbb{B}_n \setminus r\mathbb{B}_n)}$ and $|\lambda| = \|u\|_\infty$.

For $\xi \in \partial\mathbb{D}$ and $k \in \mathbb{Z}_{\geq 1}$, let $f_{\xi,k} \colon \mathbb{D} \to \mathbb{D}$ be a peak function defined by

$$f_{\xi,k}(z) = \left(\frac{1 + \overline{\xi} z}{2}\right)^k.$$

For $\alpha > 0$ it is well-known that $\mathcal{B}_{0,\alpha}(\mathbb{D})^* \simeq A^1_0(\mathbb{D})$ and $A^1_0(\mathbb{D})^* \simeq \mathcal{B}_\alpha(\mathbb{D})$ via an integral pairing, see [21].

Lemma 4.8. *Let $0 < \alpha \leq 1$, $\xi \in \mathbb{D}$, and $g_{\xi,k}(z) = \big(\frac{1+\bar{\xi}z}{2}\big)^k \big\|\big(\frac{1+\bar{\xi}z}{2}\big)^k\big\|_{\mathcal{B}_\alpha}^{-1}$ be the normalized peak function. Then we have $g_{\xi,k}^{(m)} \to 0$, $m \in \mathbb{Z}_{\geq 0}$ uniformly on every set $A_\delta = \{z \in \mathbb{D} : |z - \xi| \geq \delta\}$, $\delta > 0$, and $g_{\xi,k} \to 0$ weakly in $\mathcal{B}_\alpha(\mathbb{D})$ as $k \to \infty$.*

Proof. For the Bloch-type spaces $\mathcal{B}_\alpha(\mathbb{D})$, it can be shown that

$$\left\|\left(\frac{1+\bar{\xi}z}{2}\right)^k\right\|_{\mathcal{B}_\alpha} \asymp k^{1-\alpha}.$$

The property $g_{\xi,k}^{(m)} \to 0$, $m \in \mathbb{Z}_{\geq 0}$, uniformly on the sets A_δ as $k \to \infty$ is a consequence of the definition of $g_{\xi,k}$. Moreover, the sequence $(g_{\xi,k})_{k=1}^\infty$ is a weak* null sequence by using Lemma 3.1 in [12]. Since $(g_{\xi,k})_k \subset \mathcal{P}(\mathbb{D}) \subset \mathcal{B}_{0,\alpha}(\mathbb{D})$, we conclude that $g_{\xi,k} \to 0$ weakly when $k \to \infty$. □

Lemma 4.9. *Let us assume that either $u \in M(\mathcal{B}(\mathbb{D})) \cap A(\mathbb{D})$ or $u \in M(\mathcal{B}_\alpha(\mathbb{D})) = \mathcal{B}_\alpha(\mathbb{D})$ with $0 < \alpha < 1$. If $M_u : \mathcal{B}_\alpha(\mathbb{D}) \to \mathcal{B}_\alpha(\mathbb{D})$ is Fredholm, then there are $r \in (0,1)$ and $\delta > 0$ such that $|u(z)| \geq \delta$ for all $r \leq |z| < 1$.*

Proof. Suppose there is a sequence $(z_k)_{k=1}^\infty \subset \mathbb{D}$ such that $|z_k| \to 1$ and $|u(z_k)| \to 0$ when $k \to \infty$. Then, by going to a subsequence if necessary, we can assume that $z_k \to \xi \in \partial\mathbb{D}$ when $k \to \infty$. Since u is continuous up to the boundary in both cases, $u(\xi) = 0$. Now by Lemma 4.8 it holds that $g_{\xi,k} \to 0$, $g'_{\xi,k} \to 0$ uniformly on every set $A_\delta = \{z \in \mathbb{D} : |z - \xi| \geq \delta\}$, $\delta > 0$, and $g_{\xi,k} \to 0$ weakly as $k \to \infty$. We consider the two cases: (i) when $\alpha = 1$ and (ii) when $0 < \alpha < 1$.

(i) It holds that $\sup_{k \in \mathbb{Z}_{\geq 1}} \|g_{\xi,k}\|_\infty < \infty$. Since $u \in M(\mathcal{B}(\mathbb{D}))$, we know that $u \in \mathcal{B}_0(\mathbb{D}) \cap H^\infty(\mathbb{D})$. Let $B_\delta = \{z \in \mathbb{D} : |z - \xi| < \delta\}$, so $\mathbb{D} = A_\delta \cup B_\delta$. Let $\varepsilon > 0$ be given, and choose $\delta > 0$ such that $|u(z)| < \varepsilon$ and $|u'(z)|(1 - |z|^2) < \varepsilon$ for $z \in B_\delta$. The following estimates hold,

$$\|M_u(g_{\xi,k})\|_{\mathcal{B}} \leq I_{k,A_\delta} + II_{k,B_\delta} + |g_{\xi,k}(0)u(0)|,$$

where

$$I_{k,A_\delta} = \sup_{z \in A_\delta} |u(z)||g'_{\xi,k}(z)|(1 - |z|^2) + \sup_{z \in A_\delta} |u'(z)||g_{\xi,k}(z)|(1 - |z|^2),$$
$$II_{k,B_\delta} = \sup_{z \in B_\delta} |u(z)||g'_{\xi,k}(z)|(1 - |z|^2) + \sup_{z \in B_\delta} |u'(z)||g_{\xi,k}(z)|(1 - |z|^2).$$

Consequently, we get that $\lim_{k\to\infty} I_{k,A_\delta} = 0$ and $\lim_{k\to\infty} II_{k,B_\delta} \leq 2\varepsilon$. We also have $|g_{\xi,k}(0)| \asymp 2^{-k}$. Thus $\|M_u(g_{\xi,k})\|_{\mathcal{B}} \to 0$ when $k \to \infty$, which means by Lemma 4.3.15 in [13] that $0 \in \sigma_e(M_u)$. Therefore M_u is not Fredholm.

(ii) The result follows similarily from showing that $\|ug_{\xi,k}\|_{\mathcal{B}_\alpha} \to 0$ as $k \to \infty$. Take $\varepsilon > 0$ and choose $\delta > 0$ such that $|u(z)| < \varepsilon$ on B_δ. It is clear that $I_{k,A_\delta} \to 0$ as $k \to \infty$.

From the definition of $g_{\xi,k}$ we have $\|g_{\xi,k}\|_\infty \asymp k^{\alpha-1}$, so $\|g_{\xi,k}\|_\infty \to 0$ as $k \to \infty$, hence, $II_{k,B_\delta} < 2\varepsilon$ for k large enough. □

Let us now consider the space $X(\mathbb{D}) = A^p_{\alpha,\beta}(\mathbb{D})$ with $1 < p < \infty$. The following lemma will be used to obtain an estimate for the Bergman-Sobolev norm of the peak function.

Lemma 4.10. *Let $L, M \geq 0$. Then*

$$\frac{\Gamma(K+L)}{\Gamma(K)} \sim K^L \quad \textit{and} \quad \frac{\Gamma(2K+L)}{\Gamma(K+L)\Gamma(K+M)} \sim \frac{2^{2K+L-1}}{\sqrt{\pi}} K^{1/2-M}$$

as $K \to \infty$.

Proof. According to Stirling's approximation, $\Gamma(x) \sim \sqrt{\frac{2\pi}{x}}\left(\frac{x}{e}\right)^x$ as $x \to \infty$, we have

$$\begin{aligned}\frac{\Gamma(K+L)}{K^L\Gamma(K)} &\sim \sqrt{\frac{K}{K+L}} e^{K-(K+L)} \frac{(K+L)^{K+L}}{K^L K^K} \\ &= \left(1+\frac{L}{K}\right)^{-1/2} e^{-L} \left(1+\frac{L}{K}\right)^K \left(1+\frac{L}{K}\right)^L \to 1\end{aligned}$$

as $K \to \infty$. Moreover,

$$\begin{aligned}&\frac{\sqrt{\pi}K^{M-1/2}\Gamma(2K+L)}{2^{L+2K-1}\Gamma(K+L)\Gamma(K+M)} \\ &\sim \frac{\sqrt{\pi}K^{M-1/2}}{2^{L+2K-1}} \sqrt{\frac{(K+L)(K+M)}{2\pi(2K+L)}} \frac{e^M(2K+L)^{2K+L}}{(K+L)^{K+L}(K+M)^{K+M}} \\ &= e^M \sqrt{\frac{\left(1+\frac{L}{K}\right)\left(1+\frac{M}{K}\right)}{\left(1+\frac{L}{2K}\right)}} \frac{\left(1+\frac{L}{2K}\right)^{2K+L}}{\left(1+\frac{L}{K}\right)^{K+L}\left(1+\frac{M}{K}\right)^{K+M}} \to 1\end{aligned}$$

as $K \to \infty$. □

In the following important lemma a fairly good approximation of the behaviour of the Bergman-Sobolev norm of the peak functions is obtained. The proof also gives an exact asymptotic formula for $\|D^j f_{\xi,k}\|_{A^p_\alpha}$ as $k \to \infty$ in the case of $p \in \mathbb{Z}_{\geq 1}$, namely,

$$\|D^j f_{\xi,k}\|^p_{A^p_\alpha} \sim \frac{\Gamma(\alpha+2)2^{2\alpha+5/2-jp}}{\sqrt{\pi}p^{\alpha+3/2}}(k+1)^{jp-(\alpha+3/2)}.$$

Furthermore, some properties for the normalized peak function are given in order to prove Lemma 4.12, from which a part of the main result follows. Observe that, as already mentioned in the beginning of Section 4, the following lemma is a necessary refinement of Lemma 11 in [10] and, as a sharp estimate, it is also of independent interest.

Lemma 4.11. *Let $p \geq 1$, $\alpha > -1$ or $p = 2$, $\alpha = -1$. If $\beta \geq 0$, then*

$$\|f_{\xi,k}\|^p_{A^p_{\alpha,\beta}} \asymp (k+1)^{-\alpha+\beta p-3/2}$$

for $k \in \mathbb{Z}$ large enough. Consequently, if $\beta > \frac{2+\alpha}{p}$ and $\xi \in \partial\mathbb{D}$, then the functions $g_{\xi,k} = f_{\xi,k}/\|f_{\xi,k}\|_{A^p_{\alpha,\beta}} \in \mathcal{P}(\mathbb{D})$ have the properties that $\|g_{\xi,k}\|_{A^p_{\alpha,\beta}} = 1$; $g_{\xi,k} \to 0$; $R^m g_{\xi,k} \to 0$, $m \in \mathbb{Z}_{\geq 1}$, uniformly on every set $A_\delta = \{z \in \mathbb{D} : |z-\xi| \geq \delta\}$, $\delta > 0$, and for $p > 1$ it also holds that $g_{\xi,k} \to 0$ weakly in $A^p_{\alpha,\beta}$ as $k \to \infty$.

Proof. Let N be a positive integer satisfying $N > \beta - \frac{1/2+\alpha}{p}$. By (2.2) and (2.3) we have that

$$\begin{aligned}\|f_{\xi,k}\|_{A^p_{\alpha,\beta}} &\asymp \|f_{\xi,k}\|_{A^p_{(N-\beta)p+\alpha,N}} \asymp \sum_{l=0}^{N-1} |D^l f_{\xi,k}(0)| + \|D^N f_{\xi,k}\|_{A^p_{(N-\beta)p+\alpha}} \\ &\asymp \|D^N f_{\xi,k}\|_{A^p_{(N-\beta)p+\alpha}}.\end{aligned}$$

The last equivalence follows from

$$0 \leq |D^l f_{\xi,k}(0)| \leq |D^N f_{\xi,k}(0)| \leq \|D^N f_{\xi,k}\|_{A^p_{(N-\beta)p+\alpha}}$$

for $l \leq N$. To finish the proof, it will be shown that for $\gamma > -1$ and $j \in \mathbb{Z}_{\geq 0}$ we have

$$\|D^j f_{\xi,k}\|^p_{A^p_\gamma} \asymp (k+1)^{jp-(\gamma+3/2)},$$

from which the lemma follows by letting $\gamma = (N-\beta)p + \alpha$ and $j = N$.

Let q be the smallest integer greater than or equal to p and $k \geq j$. We have

$$\begin{aligned}\frac{\|D^j f_{\xi,k}\|^p_{A^p_\gamma}}{\gamma+1} &= \int_{\mathbb{D}} |D^j f_{\xi,k}(z)|^p (1-|z|^2)^\gamma \, dA(z) \\ &= \left(\frac{k!}{(k-j)!}\right)^p \int_{\mathbb{D}} \left(\frac{\left|1+\bar{\xi}z\right|^{(k-j)}}{2^k}\right)^p (1-|z|^2)^\gamma \, dA(z) \\ &\overset{(*)}{\geq} \left(\frac{k!}{(k-j)!}\right)^p \int_{\mathbb{D}} \left(\frac{\left|1+\bar{\xi}z\right|^{(k-j)}}{2^k}\right)^q (1-|z|^2)^\gamma \, dA(z) \\ &\overset{(**)}{\geq} \left(\frac{k!}{(k-j)!}\right)^p \int_{\mathbb{D}} \frac{1}{2^{jq}} \frac{\left|1+\bar{\xi}z\right|^{2K}}{2^{2K}} (1-|z|^2)^\gamma \, dA(z) = U_{k,\xi,j,\gamma}.\end{aligned}$$

The $(*)$ indicates that choosing q to be the greatest integer smaller than p we similarly obtain the opposite strict inequality. The function $K\colon \mathbb{Z}_{\geq j} \to \mathbb{Z}_{\geq 0}$ is defined as $K = K_j(k) = \frac{(k-j)q}{2}$ if $k-j$ is even. In this case $\overset{(**)}{\geq}$ is an equality. If $k-j$ is odd, then K is defined by $K = \frac{(k+1-j)q}{2}$ or $K = \frac{(k-1-j)q}{2}$ depending on which inequality we want to obtain. In the latter case $\overset{(**)}{\geq}$ is replaced by $\leq$.

We continue the proof by evaluating the integral with respect to the angle. It is enough to examine the expression for $\xi = 1$. Now consider the functions $g_r \in L^2([0, 2\pi))$, $g_r(t) = (1 + re^{it})^K = \sum_{n=0}^{K} \binom{K}{n} r^n e^{itn}$ for $r \geq 0$. From Parseval's equality we obtain

$$\int_0^{2\pi} |1 + re^{it}|^{2K}\, dt = 2\pi \sum_{n=0}^{K} \binom{K}{n}^2 r^{2n},$$

for every $0 \leq r < 1$ and thus,

$$\begin{aligned}
U_{k,\xi,j,\gamma} &= \left(\frac{k!}{(k-j)!}\right)^p \frac{2}{2^{(jq+2K)}} \int_0^1 \sum_{n=0}^{K} \binom{K}{n}^2 r^{2n}(1-r^2)^\gamma r\, dr \\
&= \left(\frac{k!}{(k-j)!}\right)^p \frac{1}{2^{(jq+2K)}} \sum_{n=0}^{K} \binom{K}{n}^2 \int_0^1 r^{2n}(1-r^2)^\gamma 2r\, dr \\
&= \left(\frac{k!}{(k-j)!}\right)^p \frac{1}{2^{(jq+2K)}} \sum_{n=0}^{K} \binom{K}{n}^2 \int_0^1 r^{n}(1-r)^\gamma\, dr.
\end{aligned}$$

Moreover,

$$\begin{aligned}
U_{k,\xi,j,\gamma} &= \left(\frac{k!}{(k-j)!}\right)^p \frac{1}{2^{(jq+2K)}} \sum_{n=0}^{K} \binom{K}{n}^2 \beta(n+1, \gamma+1) \\
&= \left(\frac{k!}{(k-j)!}\right)^p \frac{1}{2^{(jq+2K)}} \sum_{n=0}^{K} \binom{K}{n}^2 \frac{\Gamma(\gamma+1)\Gamma(n+1)}{\Gamma(n+\gamma+2)} \\
&= \left(\frac{\Gamma(k-j+1+j)}{\Gamma(k-j+1)}\right)^p \frac{1}{2^{(jq+2K)}} \frac{\Gamma(\gamma+1)}{\Gamma(K+\gamma+2)} \frac{\Gamma(2K+\gamma+2)}{\Gamma(K+\gamma+2)} \\
&\sim k^{jp} \frac{\Gamma(\gamma+1)}{2^{(jq+2K)}} K^{-\gamma-3/2} \frac{2^{\gamma+1+2K}}{\sqrt{\pi}} \\
&\sim \Gamma(\gamma+1) k^{jp} (kq)^{-\gamma-3/2} \frac{2^{2\gamma+5/2-jq}}{\sqrt{\pi}} \\
&= \frac{\Gamma(\gamma+1) 2^{2\gamma+5/2-jq}}{\sqrt{\pi} q^{\gamma+3/2}} k^{jp-(\gamma+3/2)}
\end{aligned}$$

as $k \to \infty$, where the first asymptotic approximation is given by Lemma 4.10 and $(k-c)^a \sim k^a$ as $k \to \infty$ for every $c \in \mathbb{R}$. The third equality follows from the Chu-Vandermonde identity, see [15, p. 32] with the parameters $n = K$, $b = -K$ and $c = \gamma + 2$.

To prove that $(g_{\xi,k})_{k=1}^\infty$ is a weak null sequence, let $B_{A^p_{\alpha,\beta}(\mathbb{D})}$ denote the closed unit ball of the Bergman-Sobolev space $A^p_{\alpha,\beta}(\mathbb{D})$, $p > 1$. Let τ_p denote the topology of pointwise convergence. Notice that $(B_{A^p_{\alpha,\beta}(\mathbb{D})}, \tau_p)$ is a Hausdorff space and that $B_{A^p_{\alpha,\beta}(\mathbb{D})}$ is weakly compact, since the space is reflexive. Since $\delta_z \in A^p_{\alpha,\beta}(\mathbb{D})^*$ by condition (I), the identity map

$$\mathrm{id}\colon (B_{A^p_{\alpha,\beta}(\mathbb{D})}, w) \to (B_{A^p_{\alpha,\beta}(\mathbb{D})}, \tau_p)$$

is continuous, and hence, it represents a homeomorphism between the spaces $(B_{A^p_{\alpha,\beta}(\mathbb{D})}, w)$ and $(B_{A^p_{\alpha,\beta}(\mathbb{D})}, \tau_p)$. Since $\mathrm{id}^{-1}\colon (B_{A^p_{\alpha,\beta}(\mathbb{D})}, \tau_p) \to (B_{A^p_{\alpha,\beta}(\mathbb{D})}, w)$ is continuous, we conclude that $g_{\xi,k} \to 0$ weakly, when $k \to \infty$. $\square$

Lemma 4.12. *Let $\alpha > -1$, $p > 1$ or $\alpha = -1$, $p = 2$ and assume $\beta > \frac{2+\alpha}{p}$. If $M_u\colon A^p_{\alpha,\beta} \to A^p_{\alpha,\beta}$ is Fredholm, then there exist $\delta > 0$ and $r \in (0,1)$ such that $|u(z)| \geq \delta$ for all $r \leq |z| < 1$.*

Proof. The proof will be carried out by contraposition. Since u belongs to the disk algebra it is continuous up to the boundary of $\mathbb{D}$. Assume there is a point $\xi \in \partial\mathbb{D}$ such that $u(\xi) = 0$. This assumption is equivalent to u not being bounded from below arbitrarily close to the boundary, since u is continuous. It will be shown that

$$\|ug_{\xi,k}\|_{A^p_{p(N-\beta)+\alpha,N}} \to 0 \quad \text{as } k \to \infty,$$

which by (2.2) implies that

$$\|ug_{\xi,k}\|_{A^p_{\alpha,\beta}} \to 0 \quad \text{as } k \to \infty, \tag{4.1}$$

where N is the positive integer satisfying

$$0 < N - \beta + \frac{1/2 + \alpha}{p} \leq 1,$$

and $g_{\xi,k}$ is the function defined in Lemma 4.11. The lemma follows from Lemma 4.11, (4.1) and Lemma 4.3.15 in [13].

To prove the null sequence statement, we will make use of (2.4). First, notice that by Lemma 4.11 we obtain

$$|u(0)g_{\xi,k}(0)| \lesssim \frac{|u(0)f_{\xi,k}(0)|}{(k+1)^{-\frac{\alpha}{p}+\beta-\frac{3}{2p}}} \to 0$$

as $k \to \infty$. Using the general Leibniz formula we have

$$R^N(ug_{\xi,k}) = \sum_{j=0}^{N} \binom{N}{j} R^j u R^{N-j} g_{\xi,k},$$

from which it follows that

$$\|R^N(ug_{\xi,k})\|_{A^p_{p(N-\beta)+\alpha}} \leq \sum_{j=0}^{N} \binom{N}{j} \|R^j u R^{N-j} g_{\xi,k}\|_{A^p_{p(N-\beta)+\alpha}}.$$

Therefore, it suffices to show that

$$I_{k,j} = \int_{\mathbb{D}} |R^j u R^{N-j} g_{\xi,k}|^p \, dA_{p(N-\beta)+\alpha}$$

approaches zero for $j = 0, 1, \dots, N$ as k tends to infinity. To prove the assertion for the case $j = 0$, we take $\varepsilon > 0$ and choose $\delta > 0$ such that $|u(z)|^p < \varepsilon$ for all

$$z \in B_\delta = \{z \in \mathbb{D} : |z - \xi| < \delta\}.$$

We can now choose a $K > 0$ such that

$$\int_{A_\delta} |R^N g_{\xi,k}(z)|^p \, dA_{p(N-\beta)+\alpha}(z) < \varepsilon,$$

which implies

$$\int_{A_\delta} |u(z) R^N g_{\xi,k}(z)|^p \, dA_{p(N-\beta)+\alpha}(z) < \|u\|_\infty^p \varepsilon$$

for $k > K$, where Lemma 4.11 has been used and $A_\delta = \{z \in \mathbb{D} : |z - \xi| \geq \delta\}$. Thus, for $k > K$

$$I_{k,0} < \big(\|u\|_\infty^p + \|R^N g_{\xi,k}\|^p_{A^p_{p(N-\beta)+\alpha}}\big)\varepsilon \leq \big(\|u\|_\infty^p + M\|g_{\xi,k}\|^p_{A^p_{\alpha,\beta}}\big)\varepsilon$$

where (2.2) gives the second inequality for some $M > 0$. Since $u \in A^p_{\alpha,\beta}(\mathbb{D}) \subset H^\infty(\mathbb{D})$ and $\|g_{\xi,k}\|_{A^p_{\alpha,\beta}} = 1$ for every k, the result follows. To assure the result in the case $j \geq 1$, we will use the following approximation:

$$I_{k,j} \leq \frac{(k+1)^{p(N-j)}}{\|f_{\xi,k}\|^p_{A^p_{\alpha,\beta}}} \int_{\mathbb{D}} |R^j u(z)|^p \left|\frac{1+\bar{\xi}z}{2}\right|^{(k-(N-j))p} dA_{p(N-\beta)+\alpha}(z).$$

From Lemma 4.11 it follows that

$$\begin{aligned} I_{k,j} &\lesssim \frac{(k+1)^{p(N-j)}}{(k+1)^{-\alpha+\beta p-3/2}} \int_{\mathbb{D}} |R^j u(z)|^p \left|\frac{1+\bar{\xi}z}{2}\right|^{(k-(N-j))p} (1-|z|^2)^{p(N-\beta)+\alpha} \, dA(z) \\ &= (k+1)^{p\left(N-\beta+\frac{\alpha+3/2}{p}-j\right)} \int_{\mathbb{D}} |R^j u(z)|^p \left|\frac{1+\bar{\xi}z}{2}\right|^{(k-(N-j))p} (1-|z|^2)^{p(N-\beta)+\alpha} \, dA(z). \end{aligned}$$

For integers $j \in [2, N]$ the result $I_{k,j} \to 0$ as $k \to \infty$ is obtained from the following three facts:

$$u \in A^p_{\alpha,\beta} \simeq A^p_{p(N-\beta)+\alpha,N} \subset A^p_{p(N-\beta)+\alpha,j},$$
$$\|f_{\xi,k}\|_\infty \leq 1 \quad \forall\, k \in \mathbb{Z}_{\geq 1},$$
$$p\left(N - \beta + \frac{3/2+\alpha}{p} - j\right) \leq 1 + p - pj < 0.$$

For $j = 1$ we make an additional partition. We will, once at a time, assume that $N - \beta + \frac{3/2+\alpha}{p} - 1$ is strictly less than zero, equal to zero or strictly larger than zero. In the first case we can apply the procedure used for $j \geq 2$. In the second case we may utilize the Lebesgue dominated convergence theorem to functions

$$\left|\frac{1+\bar{\xi}z}{2}\right|^{(k-(N-j))p} \leq 1$$

for all $z \in \mathbb{D}$ and $k \in \mathbb{Z}_{\geq N}$ to obtain the result.

The only thing that remains to show is that $I_{k,1} \to 0$ as $k \to \infty$ when $N-\beta+\frac{3/2+\alpha}{p}-1 > 0$. This condition implies that

$$N > \beta - \frac{3/2+\alpha}{p} + 1 > 1,$$

so that $N \geq 2$.

To prove that $(I_{k,1})_{k=1}^{\infty}$ is a null sequence we will use Lemma 5.4 in [4] and Lemma 4.11. Lemma 5.4 in [4] gives us three different approximations for the behaviour of $|Du(z)|$, depending on values of some parameters. Hence, it suffices to prove the null convergence for all of these approximations, one at a time. Notice that $q = \alpha + 1$ when comparing notations with [4]. First, assume $\beta < \frac{2+\alpha}{p} + 1$. Then we have

$$\begin{aligned} I_{k,1} &\lesssim \|u\|^p_{A^p_{\alpha,\beta}} \int_{\mathbb{D}} |R^{N-1} g_{\xi,k}|^p (1-|z|^2)^{-p(\frac{2+\alpha}{p}+1-\beta)}\, dA_{p(N-\beta)+\alpha}(z) \\ &\lesssim \frac{\|u\|^p_{A^p_{\alpha,\beta}}}{\|f_{\xi,k}\|^p_{A^p_{\alpha,\beta}}} \int_{\mathbb{D}} |R^{N-1} f_{\xi,k}|^p\, dA_{p(N-1)-2}(z) \\ &= \frac{\|u\|^p_{A^p_{\alpha,\beta}}}{\|f_{\xi,k}\|^p_{A^p_{\alpha,\beta}}} \|f_{\xi,k}\|^p_{A^p_{p(N-1)-2,N-1}} \asymp \|u\|^p_{A^p_{\alpha,\beta}} (k+1)^{2+\alpha-\beta p}, \end{aligned}$$

therefore $(I_{k,1})_k$ is a null sequence in this case. If $\beta \geq \frac{2+\alpha}{p} + 1$, then a worse upper bound than the one stated in Lemma 5.4 is given by $C\|u\|^p_{A^p_{\alpha,\beta}} \frac{1}{(1-|z|^2)^r}$ for some positive constant C and any $r > 0$. In this case we have, for $0 < r < 1/2$, that

$$\begin{aligned} I_{k,1} &\lesssim \|u\|^p_{A^p_{\alpha,\beta}} \int_{\mathbb{D}} |R^{N-1} g_{\xi,k}|^p (1-|z|^2)^{-r}\, dA_{p(N-\beta)+\alpha}(z) \\ &\lesssim \frac{\|u\|^p_{A^p_{\alpha,\beta}}}{\|f_{\xi,k}\|^p_{A^p_{\alpha,\beta}}} \int_{\mathbb{D}} |R^{N-1} f_{\xi,k}|^p\, dA_{p(N-\beta)+\alpha-r}(z) \\ &= \frac{\|u\|^p_{A^p_{\alpha,\beta}}}{\|f_{\xi,k}\|^p_{A^p_{\alpha,\beta}}} \|f_{\xi,k}\|^p_{A^p_{p(N-\beta)+\alpha-r,N-1}} \asymp \|u\|^p_{A^p_{\alpha,\beta}} (k+1)^{r-p}, \end{aligned}$$

which completes the proof. □

We are now ready to present the main result.

Theorem 4.13. *Let $X(\mathbb{D})$ be any of the following spaces:*

(a) $\mathcal{B}_\alpha(\mathbb{D})$, $0 < \alpha < 1$, *with* $u \in M(\mathcal{B}_\alpha(\mathbb{D})) = \mathcal{B}_\alpha(\mathbb{D}) \subset A(\mathbb{D})$;

(b) $\mathcal{B}(\mathbb{D})$ *with* $u \in M(\mathcal{B}(\mathbb{D})) \cap A(\mathbb{D})$;

(c) $A^p_{\alpha,\beta}(\mathbb{D})$ *with* $u \in M(A^p_{\alpha,\beta}(\mathbb{D})) = A^p_{\alpha,\beta}(\mathbb{D}) \subset A(\mathbb{D})$, *where* $p > 1$, $\alpha > -1$ *and* $\beta > \frac{2+\alpha}{p}$;

(d) $H^2_\beta(\mathbb{D})$ *with* $u \in M(H^2_\beta(\mathbb{D})) = H^2_\beta(\mathbb{D}) \subset A(\mathbb{D})$, *where* $\beta > 1/2$.

Then the essential spectrum of $M_u\colon X(\mathbb{D}) \to X(\mathbb{D})$ *is given by*

$$\sigma_e(M_u) = \bigcap_{0<r<1} \overline{u(\mathbb{D} \setminus r\mathbb{D})} = u(\partial\mathbb{D}).$$

Proof. As in the proof of Theorem 4.5, now using Lemmas 4.4, 4.9 and 4.12, we obtain $\sigma_e(M_u) = \bigcap_{0<r<1} \overline{u(\mathbb{D} \setminus r\mathbb{D})}$ whenever $u \in M(X(\mathbb{D})) \cap A(\mathbb{D})$ and $X(\mathbb{D})$ is any of the spaces listed above. To prove the last equality, we utilize the continuity of u on $\overline{\mathbb{D}}$, which implies the first equality below

$$\bigcap_{0<r<1} \overline{u(\mathbb{D} \setminus r\mathbb{D})} = \bigcap_{0<r<1} u(\overline{\mathbb{D} \setminus r\mathbb{D}}) = \bigcap_{0<r<1} u(\overline{\mathbb{D}} \setminus r\mathbb{D}) \supset u(\partial\mathbb{D}).$$

To show the opposite inclusion, take $z \in \bigcap_{0<r<1} u(\overline{\mathbb{D}} \setminus r\mathbb{D})$. Now there is a sequence $(y_n)_{n=1}^\infty$, $1 - \frac{1}{n} \leq |y_n| \leq 1$ such that $u(y_n) = z$. Since $(y_n)_{n=1}^\infty$ is bounded there is a convergent subsequence $(y_{n_k})_{k=1}^\infty$ such that $y_{n_k} \to y \in \partial\mathbb{D}$ as $k \to \infty$. Since u is continuous on $\overline{\mathbb{D}}$ we have

$$z = \lim_{k\to\infty} u(y_{n_k}) = u(y),$$

so $z \in u(\partial\mathbb{D})$, which proves the theorem. □

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