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The Monochromatic Connectivity of Graphs

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Abstract. In 2011, Caro et al. introduced the *monochromatic connection* of graphs. An edge-coloring of a connected graph G is called a *monochromatically connecting* (MC-coloring, for short) if there is a monochromatic path joining any two vertices. The *monochromatic connection number* $\mathrm{mc}(G)$ of a graph G is the maximum integer k such that there is a k-edge-coloring, which is an MC-coloring of G. Clearly, a monochromatic spanning tree can monochromatically connect any two vertices. So for a graph G of order n and size m, $\mathrm{mc}(G) \geq m-n+2$. Caro et al. proved that both triangle-free graphs and graphs of diameter at least three meet the lower bound.

In this paper, we consider the monochromatic connectivity of graphs containing triangles which meet the lower bound too. Also, in order to study the graphs of diameter two, we present the formula for the monochromatic connectivity of join graphs. This will be helpful to solve the problem for graphs of diameter two.

1. Introduction

An edge-colored connected graph G is *rainbow connected* if any two vertices of G are connected by a rainbow path, i.e., a path whose edges have distinct colors. The *rainbow connection number* of a graph G, denoted by $\mathrm{rc}(G)$, is the minimum number such that there is a rainbow connected coloring of G. The concept of rainbow connection was introduced by Chartrand et al. [4] in 2008. Since then, the rainbow connection numbers were well studied and the number for several special graph classes has been determined or characterized. More details about the rainbow connection numbers of graphs can be found in [6].

In 2011, Caro et al. [3] introduced a natural opposite question of the rainbow connection, which is called the *monochromatic connection*. An edge-coloring of a connected graph G is called a *monochromatically connecting* (MC-coloring, for short) if there is a monochromatic path joining any two vertices. The *monochromatic connection number* of a graph G, denoted by $\mathrm{mc}(G)$, is the maximum number such that there is a monochromatic connection coloring of G. Caro et al. [3] gave some upper and lower bounds for $\mathrm{mc}(G)$

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characterized by other graph parameters. A straightforward lower bound for $\mathrm{mc}(G)$ is $|E(G)| - |V(G)| + 2$, which can be verified by coloring the edges of a spanning tree with one color, and coloring the remaining edges by new distinct colors. In particular, Caro et al. [3] showed that both triangle-free graphs and graphs of diameter at least three meet the lower bound. For more results about the monochromatic connectivity of graphs, see [2, 5, 7]. In Section 3, we consider the problem for graphs containing triangles which meets the lower bound too. Another open problem is about the monochromatic connectivity of graphs of diameter two. In order to study this graph class, in Section 4, we will consider the problem for the join graphs.

Now we present some definitions and notations necessary. Given a graph G and $D \subseteq V(G)$, let $G[D]$ be the subgraph of G induced by D. Denote by $\overline{G}$ the complement of graph G. Here we always denote by m and n the edge number and vertex number of the graph G, respectively. As proved in [3], an important property of an extremal MC-coloring (a coloring that uses $\mathrm{mc}(G)$ colors) is that each color forms a tree. For a color c, let T_c be the tree consisting of the edges colored c. The color c is *nontrivial* if T_c has at least two edges. Otherwise, c is trivial.

2. Preliminaries and lemmas

An extremal coloring is *simple* if any two nontrivial color trees have at most one common vertex. For the existence of the extremal MC-coloring of connected graphs, Caro et al. [3] gave the following lemma.

Lemma 2.1. [3] *Every connected graph has a simple extremal MC-coloring.*

Moreover, we have the following properties of the simple extremal MC-coloring.

Lemma 2.2. *Let f be a simple extremal MC-coloring of a graph G. Then*

(1) *Any two nontrivial color trees have at most one common vertex.*

(2) *If a vertex u lies in only one nontrivial color tree T, then u is adjacent to all the vertices $V(G) \setminus V(T)$ in G.*

Proof. The first statement is clear. We consider the second statement. Let u lie in only one nontrivial color tree T in f. For any vertex $v \in V(G)\setminus V(T)$, there is a monochromatic u–v path, say P. Clearly, $V(P) \nsubseteq V(T)$. If $V(P) \neq \{u, v\}$, then the path P lies in a nontrivial color tree different from T. This implies that the vertex u lies in at least two nontrivial color trees, a contradiction. □

The following lemma is obvious.

Lemma 2.3. *For any integers $a, b \geq 2$, $\binom{a}{2} + \binom{b}{2} \leq \binom{a+b-2}{2} + 1$.*

The graph G is called H-free if G does not contain any subgraph isomorphic to H. Now we consider K_4^--free graphs and present the following Turán type result.

Lemma 2.4. [1] *The size of a K_4^--free graph of order n is at most $\lfloor n^2/3 \rfloor$.*

The *hourglass graph* is defined to be the union of two triangles with exactly one common point. Next, we consider the hourglass-free graphs. That is to say, any two triangles are either vertex disjoint or having a common edge.

By the known results about K_3-free graphs, we have the following Turán type result.

Lemma 2.5. [1] *The size of a K_3-free graph of order n is at most $\lfloor n^2/4 \rfloor$.*

According to Lemma 2.5, we have the following two lemmas.

Lemma 2.6. *If G is a hourglass-free graph, then $m \leq \lfloor n^2/4 \rfloor + n/2$.*

Proof. Since G has no two triangles that have exactly one common point, for any two distinct triangles C and C', either $C \cap C' = \emptyset$ or $G[C \cup C'] = K_4$ or K_4^-. Assume that G has s independent triangles that have no any common point with other triangles and assume that G has t copies of K_4 or K_4^-. Then $3s + 4t \leq n$. It is easy to see that, when deleting at most $s + 2t$ edges from G, we will get a triangle-free subgraph of G. It follows from Lemma 2.5 that $m - s - 2t \leq \lfloor n^2/4 \rfloor$. Since $s + 2t \leq (n - s)/2$, we have that $m \leq \lfloor n^2/4 \rfloor + (n - s)/2 \leq \lfloor n^2/4 \rfloor + n/2$. So $m \leq \lfloor n^2/4 \rfloor + n/2$. □

A graph is called $2K_3$-free if it does not contain independent triangles.

Lemma 2.7. *Let G be a $2K_3$-free graph of order $n \geq 7$. Then $m \leq \lfloor n^2/4 \rfloor + n/2$.*

Proof. By Lemma 2.5, if G has at most one triangle, then the result holds obviously. Let C and C' be two distinct triangles in G. Since G without two independent triangles, we have that either $|C \cap C'| = 1$ or $G[C \cup C'] = K_4$ or K_4^-.

Assume that $|C \cap C'| = 1$. Let $C = uvw$ and $C' = uxy$, where $v, w \neq x, y$. Let $S = \{x, y, v, w\}$. Since G has no $2K_3$, we have that $G - \{u, v, w, x, y\}$ is K_3-free. Let $E_1 = E(G - \{u, v, w, x, y\})$ and so $|E_1| \leq \lfloor (n-5)^2/4 \rfloor$. Since G has no $2K_3$, each vertex of $G - \{u, v, w, x, y\}$ has at most two neighbour in S. So the set E_2 of the edges between S and $G - \{u\} - S$ is of size at most $2(n - 5)$, i.e., $|E_2| \leq 2(n - 5)$.

If $|E(G[S])| \leq 4$, then

$$\begin{aligned} m &= |E_1| + |E_2| + d(u) + |E(G[S])| \\ &\leq \left\lfloor \frac{(n-5)^2}{4} \right\rfloor + 2(n-5) + (n-1) + 4 \leq \left\lfloor \frac{n^2}{4} \right\rfloor + \frac{n}{2}. \end{aligned}$$

If $|E(G[S])| = 5$, then $G[S]$ has a triangle. So $d(u) < n-1$ or $E_1 = \emptyset$, i.e., $|E_1|+d(u) \leq \lfloor (n-5)^2/4 \rfloor + n - 2$. Hence

$$\begin{aligned} m &= |E_1| + |E_2| + d(u) + |E(G[S])| \\ &\leq \left\lfloor \frac{(n-5)^2}{4} \right\rfloor + n - 2 + 2(n-5) + 5 \leq \left\lfloor \frac{n^2}{4} \right\rfloor + \frac{n}{2}. \end{aligned}$$

If $|E(G[S])| = 6$, then $G[S] = K_4$ and each vertex of $G - \{u\} \cup S$ is adjacent to at most one vertex of $S \cup \{u\}$, i.e., $|E_2| \leq n-5$. Hence

$$\begin{aligned} m &= |E_1| + |E_2| + d(u) + |E(G[S])| \\ &\leq \left\lfloor \frac{(n-5)^2}{4} \right\rfloor + n - 1 + (n-5) + 6 \leq \left\lfloor \frac{n^2}{4} \right\rfloor + \frac{n}{2}. \end{aligned}$$

Let $G[C \cup C'] = K_4$ or K_4^-. Moreover, we can assume that any two triangles have two common vertices. Then each vertex of $V(G) \setminus V(C \cup C')$ has at most two neighbours in $V(C \cup C')$. It is easy to see that $G - V(C \cup C')$ is triangle free and so $|E(G - V(C \cup C'))| \leq \lfloor (n-4)^2/4 \rfloor$. Then $m \leq \lfloor (n-4)^2/4 \rfloor + 2(n-4) + 6 \leq \lfloor n^2/4 \rfloor + n/2$. This completes the proof. $\square$

3. Results concerning triangles

For convenience, the *waste* of a exactly k-edge-colored graph G is defined to be $|E(G)| - k$. We have the following result.

Theorem 3.1. *Let G be a connected graph of order $n \geq 7$ and $|E(G)| = m$. If G is K_4^--free, then* $\mathrm{mc}(G) = m - n + 2$.

Proof. It is clear that $\mathrm{mc}(G) \geq m - n + 2$ and here we only need to show that $\mathrm{mc}(G) \leq m - n + 2$, i.e., any MC-coloring of G contains at most $m - n + 2$ colors, i.e., the waste of G is at least $n - 2$.

Let f be a simple extremal MC-coloring of G. Since G is K_4^--free ($G \neq K_n$), f contains at least one nontrivial color tree. Denote by $T_1, T_2, \ldots, T_k$, $k \geq 1$, all the nontrivial color trees in G. Let $|T_i| = t_i$. Clearly, the waste of the tree T_i is $t_i - 2$. Also, the waste of G is $\sum_{i=1}^{k}(t_i - 2)$. This means that below we only need to show $\sum_{i=1}^{k}(t_i - 2) \geq n - 2$. If $t_1 = n$, then we are done. So let $t_1 < n$.

Claim 1. *Each vertex must lie in a nontrivial color tree.*

Proof of Claim 1. Otherwise, there is a vertex, say x, which does not appear in any nontrivial colore tree. Then x is adjacent to all the vertices in G. It follows from $k \geq 1$ and $t_1 \geq 3$ that G must contain a subgraph isomorphic to K_4^-, a contradiction. $\square$

Claim 2. *Each vertex appears in at least two distinct nontrivial color trees.*

Proof of Claim 2. On the contrary, assume that there are vertices that appear in unique nontrivial color trees.

Assume that there are two vertices u and v which appears in distinct unique nontrivial color trees, say T_1 and T_2, respectively.

If $V(T_1) \cap V(T_2) = \emptyset$, then $G[\{u\} \cup V(\text{a path of order three in } T_2)]$ contains a K_4^-, a contradiction. So let $V(T_1) \cap V(T_2) = \{w\}$. Clearly, $w \notin \{u, v\}$.

Suppose that neither $N_{T_1}(u) = \{w\}$ nor $N_{T_2}(v) = \{w\}$. Then there exist $u_0 \in V(T_1) \setminus \{w\}$ and $v_0 \in V(T_2) \setminus \{w\}$ with $uu_0 \in E(T_1)$ and $vv_0 \in E(T_2)$, respectively. From Lemma 2.2, $G[u, v, u_0, v_0]$ contains a K_4^-, a contradiction. So either $N_{T_1}(u) = \{w\}$ or $N_{T_2}(v) = \{w\}$. Without loss of generality, let $N_{T_1}(u) = \{w\}$. Since $t_2 \geq 3$, take a path P of order three through w in T_2. From Lemma 2.2, $G[V(P) \cup \{u\}]$ contains a K_4^-, a contradiction.

Thus the set, denoted by S, of all the vertices that appear in the unique nontrivial color tree must lie in the same nontrivial color tree, say T_1.

Suppose that $k = 1$. Then since each vertex appears in at least one nontrivial color tree, we have that $t_1 = n$, contradicting to the assumption $t_1 < n$. So let $k \geq 2$. Suppose that $S = V(T_1)$. Then $V(T_2) \cap V(T_1) = \emptyset$. Take a vertex $u \in S$ and a path P of order three in T_2. From Lemma 2.2, the vertex u is adjacent to all vertices of P, i.e., $G[V(P) \cup \{u\}]$ contains a K_4^-, a contradiction. So $V(T_1) \setminus S \neq \emptyset$.

Clearly, each vertex in $V(T_1) \setminus S$ appears in at least two nontrivial color trees. Let $u \in S$ and $v \in V(T_1) \setminus S$ with $uv \in E(T_1)$. Let $v \in V(T_2)$. From $t_2 \geq 3$, we have that T_2 contains a path P of order three through v. From Lemma 2.2, $G[V(P) \cup \{u\}]$ contains a K_4^-, a contradiction. This completes the proof of the claim. □

From Claim 2, $\sum_{i=1}^{k} t_i \geq 2n$. If $k \leq n/2+1$, then $\sum_{i=1}^{k}(t_i - 2) \geq 2n - 2k \geq n - 2$, and we are done. So let $k > n/2+1$. From Lemma 2.4, $m \leq n^2/3$. Then $|E(\overline{G})| \geq \binom{n}{2} - n^2/3$. On the other hand, each nontrivial color tree T_i can monochromatically connect at most $\binom{t_i}{2} - (t_i - 1) = \binom{t_i - 1}{2}$ pairs of non-neighbors in G. Notice that the two end vertices of each edge in $\overline{G}$ lie in a nontrivial tree of G. So we have that $\sum_{i=1}^{k} \binom{t_i - 1}{2} \geq |E(\overline{G})| \geq \binom{n}{2} - n^2/3$.

Assume that $\sum_{i=1}^{k}(t_i - 2) < n - 2$. Then $\sum_{i=1}^{k}(t_i - 1) \leq n - 3 + k$ and $k \leq n - 3$ because $t_i \geq 3$. Since $t_i \geq 3$, it follows from Lemma 2.3 that $\sum_{i=1}^{k} \binom{t_i - 1}{2} \leq k - 1 + \binom{n-k-1}{2}$. Note that $g(k) = k - 1 + \binom{n-k-1}{2}$ is a decreasing function of k for $k \leq n-3$. From $k > n/2+1$, by the convex function property we have that $\sum_{i=1}^{k} \binom{t_i - 1}{2} \leq k - 1 + \binom{n-k-1}{2} \leq n^2/8 - n + 39/8$.

So we get $\binom{n}{2} - n^2/3 \leq |E(\overline{G})| \leq n^2/8 - n + 39/8$, i.e., $n^2 + 12n \leq 108$. This contradicts to $n \geq 7$. Hence $\sum_{i=1}^{k}(t_i - 2) \geq n - 2$. □

Theorem 3.2. *Let G be a connected graph of order $n \geq 7$. If G is hourglass-free and $G \neq K_n - E(K_{n-2})$, then* $\mathrm{mc}(G) = m - n + 2$.

Proof. As mentioned in the proof of Theorem 3.1, we only need to show that the waste of an MC-coloring of G is at least $n - 2$.

Let f be a simple extremal MC-coloring of G. Since G is hourglass-free ($G \neq K_n$), f contains at least one nontrivial color tree. Denote by $T_1, T_2, \ldots, T_k$, $k \geq 1$, all the nontrivial color trees in G. Let $|T_i| = t_i$. Clearly, waste of the tree T_i is $t_i - 2$. Also, the waste of G is $\sum_{i=1}^{k}(t_i - 2)$. This means that below we only need to show $\sum_{i=1}^{k}(t_i - 2) \geq n - 2$. We may assume that $t_i < n$ for $1 \leq i \leq k$.

Suppose that there is a vertex, say u, which is not in any nontrivial color tree. Then u is adjacent to all the other vertices in G and $G - u$ is connected, since each pair of $V(G - u)$ are monochromatically connected in $G - u$. Since G is hourglass-free, $G - u$ has no two independent edges, i.e., $G - u = K_{1,n-2}$. This implies that $G = K_n - E(K_{n-2})$, a contradiction. So we assume that each vertex lies in at least one nontrivial color tree.

Claim 3. *Each vertex appears in at least two distinct nontrivial color trees.*

Proof of Claim 3. On the contrary, assume that there are vertices that appear in unique nontrivial color trees. Denote by S the vertices that appear in the unique nontrivial color tree.

Observation. *All the vertices in S lie in the same nontrivial color tree.*

Proof of Observation. Assume that there exist $u, v \in S$ appearing in the distinct nontrivial color trees, say T_1 and T_2, respectively.

Assume that $V(T_1) \cap V(T_2) = \emptyset$. If T_1 is not a star with the center u, then from Lemma 2.2, we can easily find that $G[V(T_1) \cup V(T_2)]$ contains a hourglass subgraph, a contradiction. So $T_1 = K_{1,t_1-1}$ and $T_2 = K_{1,t_2-1}$ with the center u and v, respectively. Let $ux \in E(T_1)$. If x also appears only in the nontrivial color tree T_1, then from Lemma 2.2, we can easily find that $G[V(T_1) \cup V(T_2)]$ contains a hourglass subgraph, a contradiction. So we can assume that $x \in V(T_3)$. From $t_i \geq 3$ and Lemma 2.2, we can take two vertices y and z with $xy \in E(T_3)$ and $vz \in E(T_2)$. From Lemma 2.2, $G[\{x, y, u, v, z\}]$ contains a hourglass subgraph, a contradiction.

So let $V(T_1) \cap V(T_2) = \{w\}$. Clearly, $w \notin \{u, v\}$. Also, both $G[V(T_1 - w)]$ and $G[V(T_2 - w)]$ does not have independent edges, since otherwise from Lemma 2.2 there will be a hourglass subgraph.

Suppose that there is a vertex $u_0 \in V(T_1) \setminus \{w\}$ with $uu_0 \in E(T_1)$. We claim that u_0 also lies only in the nontrivial color tree T_1. Otherwise, without loss of generality, assume that $u_0 \in V(T_1) \cap V(T_3)$. Clearly, there is a path P of order three through u_0 in T_3. If $P = u_0xy$, then from Lemma 2.2, uu_0vu and $uxyu$ are two triangles in G which form a hourglass subgraph in G, a contradiction. So $P = xu_0y$. From Lemma 2.2, v is adjacent to at least one of x, y, say y. Then from Lemma 2.2, yu_0vy and uxu_0u are two triangles in G which form a hourglass subgraph in G, a contradiction. So u_0 only lies in the nontrivial color tree T_1, i.e., $u_0 \in S \cap V(T_1)$. This implies that each neighbour of u in $T_1 - w$ belongs to $S \cap V(T_1)$.

If there is a vertex $u_1 \in V(T_1) \setminus \{w, u_0\}$ with $uu_1 \in E(G)$, then from Lemma 2.2, $G[\{u, u_0, u_1, v, x\}]$ contains a hourglass subgraph, where $x \in V(T_2) \setminus \{v, w\}$ and $vx \in E(G)$, a contradiction. So each vertex of $V(T_1) \setminus \{w, u_0\}$ is not adjacent to u in G. Hence the vertices u, u_0 are not adjacent to any vertex of $T_1 - \{w, u, u_0\}$. If there are two vertices x, y of $T_1 - \{w, u, u_0\}$ are adjacent in G, then from Lemma 2.2, we have that both $vxyv$ and vuu_0v are triangles which form a hourglass subgraph in G, a contradiction. So any two vertices of $T_1 - \{w, u, u_0\}$ are non-adjacent in G, i.e., each component of $T_1 - \{w, u, u_0\}$ is an isolated vertex. Without loss of generality, let $uw \in E(T_1)$.

Now we plan to show that T_1 is just the path wuu_0. On the contrary, assume that there is a vertex $z \neq w, u$ with $zw \in E(T_1)$. Let $vv' \in E(T_2)$. If $v' \neq w$, then by the analysis above, we have that v' also lies only in the nontrivial color tree T_2, i.e., $v'z \in E(G)$ from Lemma 2.2. From Lemma 2.2, both uu_0vu and $vv'zv$ are two triangles with exactly one common vertex, i.e., G contains a hourglass subgraph, a contradiction. So $v' = w$. Then from Lemma 2.2, both $vwzv$ and vuu_0v are triangles in G, a contradiction. Hence, we have $T_1 = wuu_0$.

Suppose that there is v_0 with $vv_0 \in E(T_2 - w)$. By the same analysis we have $T_2 = wvv_0$ and v_0 also lies only in the nontrivial color tree T_2. From Lemma 2.2, $G[\{u, u_0, w, v_0, v\}]$ contains a hourglass subgraph, a contradiction. So v is only adjacent to w in T_2. Take a vertex $x \in V(T_2) \setminus \{v, w\}$ so that $x \in N_{T_2}(w)$. From Lemma 2.2, $uwxu$ and uu_0vu are two triangles with exactly one common vertex, i.e., G contains a hourglass subgraph, a contradiction. So $N_{T_1}(u) = \{w\}$, and for the same reason we have $N_{T_2}(v) = \{w\}$. Take two vertices $x \in N_{T_1}(w) \setminus \{u\}$ and $y \in N_{T_2}(w) \setminus \{v\}$. From Lemma 2.2, both $uwyu$ and $vwxv$ are triangles in G, again a contradiction. This completes the proof of the observation. $\square$

Let $S \subseteq V(T_1)$. Suppose that $k = 1$. Then since each vertex appears in at least one nontrivial color tree, we have that $t_1 = n$, contradicting to the assumption $t_1 < n$. So let $k \geq 2$. Suppose that $S = V(T_1)$. Take $u, v \in S$ in T_1 and a path xyz in T_2. From Lemma 2.2, $G[\{u, v, x, y, z\}]$ contains a hourglass subgraph, a contradiction. So $V(T_1) \setminus S \neq \emptyset$.

Clearly, each vertex in $V(T_1) \setminus S$ appears in at least two nontrivial color trees. Let $u \in S$ and $v \in V(T_1) \setminus S$ with $uv \in E(T_1)$. Let $v \in V(T_2)$ and P be a path of order three through v in T_2. Notice that, from Lemma 2.2, the vertex u is adjacent to all the vertices of $V(T_2)$.

Suppose that $G[V(T_2)]$ contains a path P_4 of order four. Since the vertex u is adjacent to all the four vertices of this path P_4, the graph $G[u, V(P_4)]$ contains a hourglass subgraph, a contradiction. So $G[V(T_2)]$ does not contain any path of order four, i.e., $G[V(T_2)] = K_3$ or K_{1,t_2-1}. In particular, T_2 is a star.

If there is a vertex $u' \in S \setminus \{u\}$ with $u'v \in E(G)$, then from Lemma 2.2, $G[\{u, u'\} \cup V(P)]$ contains a hourglass subgraph, a contradiction. So $N_G(v) \cap S = \{u\}$.

Suppose that the graph $G[S]$ is not empty. By the choice of u and v, we can assume that there is a vertex $u' \in S$ with $uu' \in E(G)$. Then from Lemma 2.2, we have that $G[u', u, V(P)]$ contains a hourglass subgraph, a contradiction. So $E(G[S]) = \emptyset$.

Suppose that there is a vertex $v' \in N_G(u) \cap V(T_1)$ with $v \neq v'$. From $E(G[S]) = \emptyset$, we have that $v' \in V(T_1) \setminus S$. From Lemma 2.2, we have that $v' \in V(T_1) \cap V(T_i)$ for some $i \geq 3$. Take a path Q of order three through v' in T_i. From Lemma 2.2, $G[\{u\} \cup V(P) \cup V(Q)]$ contains a hourglass subgraph, a contradiction. So $N_G(u) \cap V(T_1) = \{v\}$. Together with $N_G(v) \cap S = \{u\}$, we have $k \geq 3$.

Now we consider the nontrivial color tree T_i, $i \geq 3$. If $V(T_i) \cap V(T_1) = \emptyset$ $(i \geq 3)$, then the vertex u is adjacent to all the vertices of T_i in G. Take a path Q of order three in T_i. Hence we have that u is adjacent all the vertices of P and Q. This implies that $G[\{u\} \cup V(P \cup Q)]$ contains a hourglass subgraph, a contradiction. So $V(T_i) \cap V(T_1) \neq \emptyset$ for each $i \geq 3$. Notice that any two nontrivial color trees have at most one common vertex. Let $v_i \in V(T_i) \cap V(T_1)$ for $i \geq 3$. Clearly, $v_i \in V(T_1) \setminus S$.

If $G[V(T_i - v_i)]$ contains an edge, say xy, for some $i \geq 3$. From Lemma 2.2, $x, y \notin V(T_1)$ and u is adjacent to both x and y in G. Notice that u is adjacent to all vertices of P and at most one of x and y belongs to $V(P)$. Hence, if x or y is not the center vertex of P, we have that $G[\{u, x, y\} \cup V(P)]$ contains a hourglass subgraph, a contradiction. If x or y is not the center vertex of P, say x, then it is clear that $x \neq v_1$. Consider the monochromatic path from y to v_1 and let y' be the neighbour of y on it. Then either $y' = v_1$ or $y' \notin V(T_1)$ and no matter what happens, we have $uy' \in E(G)$. Hence $G[\{u, x, y, y'\} \cup V(P)]$ contains a hourglass subgraph, a contradiction. So $G[V(T_i - v_i)]$ is an empty graph for each $i \geq 3$. This implies that $G[V(T_i)]$ is a star with the center $v_i \in V(T_1 \setminus S)$ for each $i \geq 3$.

Since $t_1 \geq 3$ and $N_G(u) \cap V(T_1) = \{v\}$, there is a vertex $w \neq u$ with $vw \in E(T_1)$. Let $xv \in E(P)$. If $wx \in E(G)$, then $G[\{u, w\} \cup V(P)]$ contains a hourglass subgraph, a contradiction. So $wx \notin E(G)$. Let T_3 be the nontrivial color tree monochromatically connecting x and w. Remember that T_3 is a star with the center $v_3 \in V(T_1) \setminus S$. Hence $v_3 \neq w$ and so $|V(T_1) \cap V(T_3)| \geq 2$, a contradiction. This completes the proof of Claim 3. $\square$

From Claim 3, $\sum_{i=1}^{k} t_i \geq 2n$. If $k \leq n/2 + 1$, then $\sum_{i=1}^{k}(t_i - 2) \geq 2n - 2k \geq n - 2$, and we are done. So let $k > n/2 + 1$. From Lemma 2.6, $m \leq n^2/4 + n/2$ and then $|E(\overline{G})| \geq \binom{n}{2} - n^2/4 - n/2$. On the other hand, each nontrivial color tree T_i can monochromatically connect at most $\binom{t_i}{2} - (t_i - 1) = \binom{t_i - 1}{2}$ pairs of non-neighbors in G, we have that $\sum_{i=1}^{k} \binom{t_i-1}{2} \geq |E(\overline{G})| \geq \binom{n}{2} - n^2/4 - n/2$.

Assume that $\sum_{i=1}^{k}(t_i - 2) < n - 2$. Then $\sum_{i=1}^{k}(t_i - 1) \leq n - 3 + k$ and $k \leq n - 3$. Since $t_i \geq 3$, it follows from Lemma 2.3 that $\sum_{i=1}^{k} \binom{t_i-1}{2} \leq k - 1 + \binom{n-k-1}{2}$. Note that $g(k) = k-1+\binom{n-k-1}{2}$ is a decreasing function of k for $k \leq n-3$. From $k > n/2+1$, by the convex function property we have that $\sum_{i=1}^{k} \binom{t_i-1}{2} \leq k - 1 + \binom{n-k-1}{2} \leq n^2/8 - n + 39/8$.

So $\binom{n}{2} - n^2/4 - n/2 \leq n^2/8 - n + 39/8$, a contradiction for $n \geq 7$. Hence $\sum_{i=1}^{k}(t_i - 2) \geq n - 2$. This completes the proof of Theorem 3.2. $\square$

Finally, we present the following result.

Theorem 3.3. *Let G be a connected $2K_3$-free graph of order $n \geq 7$. If for any vertex u of degree $n-1$, $G - u$ is disconnected, then* $\mathrm{mc}(G) = m - n + 2$.

Remark 3.4. The assumption that $G - u$ is disconnected for any vertex u of degree $n - 1$ is necessary. Otherwise, we can monochromatically color the edges of a spanning tree of $G - u$ and color the other edges of G by distinct new colors. Then we get an MC-coloring of G which contains $m - n + 3$ colors, i.e., $\mathrm{mc}(G) \geq m - n + 3$.

Proof of Theorem 3.3. As mentioned in the proof of Theorem 3.1, we only need to show that the waste of an MC-coloring of G is at least $n - 2$.

Let f be a simple extremal MC-coloring of G. Since G is $2K_3$-free ($G \neq K_n$), f contains at least one nontrivial color tree. Denote by $T_1, T_2, \ldots, T_k$, $k \geq 1$, all the nontrivial color trees in G. Let $|T_i| = t_i$. Clearly, the waste of the tree T_i is $t_i - 2$. Also, the waste of G is $\sum_{i=1}^{k}(t_i - 2)$. This means that below we only need to show $\sum_{i=1}^{k}(t_i - 2) \geq n - 2$. Without loss of generality, let $t_i < n$ for $1 \leq i \leq k$.

Suppose that there is a vertex, say u, which is not in any nontrivial color tree. Then u is adjacent to all the other vertices in G and, since each pair of $V(G-u)$ are monochromatically connected in $G - u$, $G - u$ is connected, a contradiction. So each vertex lies in at least one nontrivial color tree. Denote by S the vertices that appear in the unique nontrivial color tree.

Claim 4. *All the vertices of S lie in the same nontrivial color tree.*

Proof of Claim 4. On the contrary, assume that there exist $u, v \in S$ appearing in the distinct nontrivial color trees, say T_1 and T_2, respectively.

Claim 4.1. *There are not two disjoint paths of order three through u and v in $G[V(T_1) \setminus V(T_2)]$ and $G[V(T_2) \setminus V(T_1)]$, respectively.*

Proof of Claim 4.1. On the contrary, assume that there are two disjoint paths, say P and Q, of order three through u and v in $G[V(T_1) \setminus V(T_2)]$ and $G[V(T_2) \setminus V(T_1)]$, respectively.

It is impossible that both $P-u$ and $Q-v$ have edges, since otherwise it follows from from Lemma 2.2 that G contains two disjoint triangles, a contradiction. So either P or Q is a star with the center u or v, respectively. Without loss of generality, let T_2 is a star with the center v. Let $Q = xvy$ and $V(P) = \{u, u_1, u_2\}$.

Case 1: $P = uu_1u_2$.

Now we consider the color trees monochromatically connecting $\{x, y\}$ and $\{u_1, u_2\}$. From Lemma 2.2, we have $uv, ux, uy, vu_1, vu_2 \in E(G)$. If x is adjacent to both u_1 and u_2, then G contains a $2K_3$, a contradiction. So let x be not adjacent to a vertex $u' \in \{u_1, u_2\}$. Clearly, there is a monochromatic x–u' path $P_{xu'}$ in G and let w be the neighbour vertex of x on $P_{xu'}$. Clearly, $P_{xu'}$ contains at least three vertex and lies in a nontrivial color tree. From Lemma 2.2, $w \notin V(T_1) \cup V(T_2)$ and $uw \in E(G)$. So $uxwu$ and vu_1u_2v are two disjoint triangles in G, a contradiction.

Case 2: $P = u_1uu_2$.

Since $u, v \in S$, it follows from Lemma 2.2 that $ux, uy, vu_1, vu_2 \in E(G)$. Notice that $uu_1, uu_2, vx, vy \in E(G)$. Since G does not contain $2K_3$, we can easily find that $G[\{u_1, u_2, x, y\}]$ does not contain two independent edges. This implies that each component of $G[\{u_1, u_2, x, y\}]$ is either an isolated vertex or a star. Moreover, at most one component of $G[\{u_1, u_2, x, y\}]$ is a star. Hence there is a vertex $p \in \{x, y, u_1, u_2\}$ such that $G[\{u_1, u_2, x, y\}] - p$ does not contain any edges.

Here we let $p = u_1$ and the other cases can be verified in the same way whose details are omitted here. So $xu_2, yu_2 \notin E(G)$. So x and u_2 are monochromatically connected in a nontrivial color tree, say T_3. Let $P_1 = xz \cdots u_2$ be the x–u_2 path in T_3. From Lemma 2.2, $y \notin V(T_3)$.

Since $yu_2 \notin E(G)$, we have that y and u_2 are monochromatically connected in a nontrivial color tree and from Lemma 2.2, this tree is different from T_1, T_2, T_3. Let $y, u_2 \in V(T_4)$. Let $P_2 = u_2z' \cdots y$ be the y–u_2 path in T_4. Clearly, $x, z, u, v \neq z'$. From Lemma 2.2, both $vxzv$ and $uu_2z'u$ are triangles in G, i.e., G contains a $2K_3$, a contradiction. This completes the proof of Claim 4.1. □

In particular, it follows from Claim 4.1 that $V(T_1) \cap V(T_2) \neq \emptyset$. So let $V(T_1) \cap V(T_2) = \{w\}$. Clearly, $w \notin \{u, v\}$.

Claim 4.2. *There is not any path of order four in $G[V(T_i)]$ for $i \geq 3$. In particular, $G[V(T_i)] = K_{1,t_i-1} = T_i$ for $i \geq 3$.*

Proof of Claim 4.2. On the contrary, suppose that there is a path $P = x_1x_2x_3x_4$ of order four in $G[V(T_3)]$.

Assume that $w \notin V(P)$. If $|V(P) \cap V(T_1 \cup T_2)| \leq 1$, without loss of generality, let $|V(P) \cap V(T_1)| = x_1$ or $|V(P) \cap V(T_1)| = x_2$, then from Lemma 2.2, we can easily find both

vx_1x_2v and ux_3x_4u are triangles in G, a contradiction. So let $|V(P) \cap V(T_1 \cup T_2)| \geq 2$. More precisely, we have that $|V(P) \cap V(T_1)| = 1$ and $|V(P) \cap V(T_2)| = 1$. If $V(P) \cap V(T_1 \cup T_2) \neq \{x_1, x_2\}$ or $\{x_3, x_4\}$, then from Lemma 2.2, we can easily find a $2K_3$ in G, a contradiction. By the symmetry, let $V(P) \cap V(T_1 \cup T_2) = \{x_1, x_2\}$, say $x_1 \in V(P) \cap V(T_1)$ and $x_2 \in V(P) \cap V(T_2)$. Clearly, $ux_3, vx_3, ux_4, vx_4, vx_1, ux_2 \in E(G)$. Then $vx_2 \notin E(G)$, since otherwise both ux_3x_4u and vx_1x_2v are triangles, a contradiction. For the same reason, $ux_1 \notin E(G)$.

Let x_5 be a neighbour of x_1 in T_1. Clearly, $x_5 \neq u$. If $x_5x_2 \in E(G)$, then both $x_1x_2x_5x_1$ and vx_3x_4v are triangles, a contradiction. So $x_5x_2 \notin E(G)$. If $x_5 \neq w$, then x_5 and x_2 is monochromatically connected in a nontrivial color tree different from T_1, T_2, T_3. Denote the monochromatic x_2–x_5 path by $x_5 \cdots x_6x_2$. Clearly, $x_6 \notin V(T_i)$, $i = 1, 2, 3$. From Lemma 2.2, $ux_6 \in E(G)$. Then both ux_6x_2u and vx_3x_4v are triangles, a contradiction. So $x_5 = w$.

If $x_2w \in E(G)$, then both ux_3x_4u and wx_1x_2w are triangles, a contradiction. Then $x_2w \notin E(G)$. Denote by x_7 a neighbour of x_2 in T_2 and clearly, $x_7 \neq w$. If $x_1x_7 \in E(G)$, then we get a contradiction easily. Let x_8 be the neighbour of x_1 on the monochromatic x_1–x_7 path. Clearly, $x_8 \notin V(T_i)$, $i = 1, 2, 3$. Now both ux_3x_4u and vx_1x_8v are triangles, a contradiction.

So $w \in V(P)$. Moreover, this implies that each path of order four in $G[V(T_i)]$, $i \geq 3$, must pass through the vertex w. Assume that there is a nontrivial color tree T_i for some $i \geq 4$ such that $G[V(T_i)]$ contains a path Q of order four. Clearly, $V(T_3) \cap V(T_i) = \{w\}$. Denote by x_1x_2 and x_3x_4 the edges of $P - w$ and $Q - w$, respectively. From Lemma 2.2, both u and v are adjacent to all the vertices $\{x_1, x_2, x_3, x_4\}$ in G. Hence both ux_1x_2u and vx_3x_4v are triangles in G, a contradiction. So we have that each $G[V(T_i)]$, $i \geq 4$, does not contain any path of order four. That is to say, each $G[V(T_i)]$, $i \geq 4$, is a star.

Let $V(P) = \{w, x, y, z\}$. Clearly, $x, y, z \notin V(T_1) \cup V(T_2)$. It follows from Lemma 2.2 that $ux, xy, uz, vx, vy, vz \in E(G)$. Without loss of generality, let $xy, wz \in E(P)$. If $wv \in E(G)$, then both $vwzv$ and $uxyu$ are triangles, a contradiction. So $wv \notin E(G)$. Take a vertex $a \in V(T_2) \setminus \{w, v\}$ such that $wa \in E(G)$. If $za \in E(G)$, then both $awza$ and $uxyu$ are triangles, a contradiction. So $za \notin E(G)$. Let zb be an edge on the monochromatic path connecting z and a. Clearly, $b \notin V(T_2)$. From Lemma 2.2, $vb \in E(G)$. Then both $uxyu$ and $vzbv$ are triangles in G, a contradiction. This completes the proof of Claim 4.2. □

Claim 4.3. *There do not exist two disjoint paths of order three through u and v in $G[V(T_1)]$ and $G[V(T_2)]$, respectively.*

Proof of Claim 4.3. On the contrary, assume that there are two disjoint paths P and Q

of order three through u and v in $G[V(T_1)]$ and $G[V(T_2)]$, respectively. From Claim 4.1, we have that one of P and Q contains the vertex w. Without loss of generality, let $V(P) = \{u, x, y\}$ and $V(Q) = \{w, v, z\}$. Since $u, v \in S$ and $w \in V(T_1) \cap V(T_2)$, it follows from Lemma 2.2 that $uv', vu' \in E(G)$ for any $u' \in V(T_1 - w)$, $v' \in V(T_2 - w)$.

Case 1: $P = uxy$ and $Q = wvz$.

Assume that $xz \notin E(G)$. Take an edge zz' on the monochromatic path connecting z and x. Clearly, $z' \notin V(T_i)$, $i = 1, 2$. From Lemma 2.2, both $uzz'u$ and $vxyv$ are triangles in G, a contradiction. So $xz \in E(G)$. If there is a vertex $w' \notin V(T_2) \cup \{u, x\}$ such that $ww' \in E(G)$, then from Lemma 2.2 we have that $vw' \in E(G)$. Hence both $uxzu$ and $vww'v$ are triangles in G, a contradiction. So $N_G(w) \subseteq V(T_2) \cup \{u, x\}$. In particular, $N_G(w) \cap V(T_1) \subseteq \{u, x\}$. Clearly, from Lemma 2.2, any two nontrivial color trees have at most one common vertex. Since w is monochromatically connected to all the vertices of G, we have that $V(G) = V(T_1) \cup V(T_2)$. This implies that $k = 2$. From Lemma 2.2, each vertex of $T_1 - w$ is adjacent to all vertices of $T_2 - w$.

Then $G[V(T_1 - w)]$ is a star with the center vertex x. Otherwise, $G[V(T_1 - w)]$ contains two independent edges whose vertices are adjacent to both v and z and hence we can find two disjoint triangles easily, a contradiction.

Clearly, $N_G(w) \cap V(T_1) \subseteq \{u, x\}$. If $wx \in E(G)$, then we have that $d(x) = n - 1$. But now $G - x$ is connected, a contradiction. So $wu \in E(G)$. Then both $uvwu$ and $xyzx$ are triangles, a contradiction.

Case 2: $P = xuy$ and $Q = wvz$.

Suppose that w is adjacent to a vertex $w' \in V(G - u) \setminus V(T_2)$ in G. Notice that $w' \neq x$ or $w' \neq y$. Without loss of generality, let $w' \neq x$. If $xz \in E(G)$, then both $ww'vw$ and $uxzu$ are triangles in G, a contradiction. So $xz \notin E(G)$. From Claim 4.2, $xw'', w''z \in E(G)$ for some vertex $w'' \notin V(T_1) \cup V(T_2)$. From Lemma 2.2, $uw'' \in E(G)$. Hence both $ww'vw$ and $uxw''u$ are triangles in G, a contradiction. So $N_G(w) \cap V(G) \setminus V(T_2) = \{u\}$. Then in order to avoid falling into Case 1 above, we can assume that T_1 is a star with the center u.

Clearly, from Lemma 2.2, any two nontrivial color trees have at most one common vertex. Since w is monochromatically connected to all the vertices of G, we have that $V(G) = V(T_1) \cup V(T_2)$. Then $d(u) = n-1$ and however, $G-u$ is connected, a contradiction.

Case 3: $P = uxy$ and $Q = wzv$.

Suppose that there is a vertex $w' \in V(G - x) \setminus V(T_2)$ with $ww' \in E(G)$. Let $a \in \{u, y\} \setminus \{w'\}$. If $w'z \in E(G)$, then both $ww'zw$ and $vxav$ are triangles in G, a contradiction. So $w'z \notin E(G)$. From Claim 4.2, w' and z are monochromatically connected in a nontrivial color tree, say T_3, which is a star with the center $w'' \notin V(T_1) \cup V(T_2)$. If $ww'' \in E(G)$, then both $ww''zw$ and $vxav$ are triangles, a contradiction. So $ww'' \notin E(G)$. Again from

Claim 4.2, there is a vertex $w''' \notin V(T_1) \cup V(T_2) \cup V(T_3)$ such that $ww''', w''w''' \in E(G)$. Hence from Lemma 2.2, both $uw''w'''u$ and $vxyv$ are triangles, a contradiction. Thus $N_G(w) \cap V(G) \setminus V(T_2) = \{x\}$.

Clearly, from Lemma 2.2, any two nontrivial color trees have at most one common vertex. Since w is monochromatically connected to all the vertices of G, we have that $V(G) = V(T_1) \cup V(T_2)$. Then from Claim 4.2 and Lemma 2.2, we have that $k = 2$ and hence $z \in S$. By the choice of u, v, we can replace v by z and then we fall into Case 1 above. So we are done.

Case 4: $P = xuy$ and $Q = wzv$.

By choice of u, v and in order to avoiding to fall into Cases above, we can assume that $x, y, z \notin S$. From Lemma 2.2, at least one of $\{x, y, z\}$ is adjacent to a vertex $w' \notin V(T_1) \cup V(T_2)$ in a nontrivial color tree, say T_3.

Assume that $xw' \in E(T_3)$ or $yw' \in E(T_3)$. By the symmetry of x and y, let $xw' \in E(T_3)$. Then $uw', vw' \in E(G)$. If there is a vertex $y' \in V(G - w') \setminus (V(T_1) \cup V(T_2))$ with $yy' \in E(G)$, then $vy' \in E(G)$. Hence both $uxw'u$ and $vyy'v$ are triangles, a contradiction. Thus we have that $N_G(y) \setminus (V(T_1) \cup V(T_2)) \subseteq \{w'\}$. If there is a vertex $x' \in V(G - w') \setminus (V(T_1) \cup V(T_2))$ with $xx' \in E(G)$, then $vx' \in E(G)$. Hence both $uxw'u$ and $vxx'v$ are triangles, a contradiction. Thus we have that $N_G(x) \setminus (V(T_1) \cup V(T_2)) \subseteq \{w'\}$. If there is a vertex $z' \in V(G - w') \setminus (V(T_1) \cup V(T_2))$ with $zz' \in E(G)$, then $vz' \in E(G)$. Hence both $uxw'u$ and $vzz'v$ are triangles, a contradiction. Thus we have that $N_G(z) \setminus (V(T_1) \cup V(T_2)) \subseteq \{w'\}$.

If $yz \in E(G)$, then both $uxw'u$ and $vzyv$ are triangles, a contradiction. So $yz \notin E(G)$. Then from Claim 4.2, it is only possible that y and z are monochromatically connected by the monochromatic path $yw'z$. From Lemma 2.2, $yw'z$ does not lie in T_3. Now consider the vertices x, z. It is only possible that $xz \in E(G)$. Hence both $uxzu$ and $vyw'v$ are triangles, a contradiction.

So $zw' \in E(T_3)$. Notice that at least one of x, y is not in T_3. Without loss of generality, let $x \notin V(T_3)$. If $xw' \in E(G)$, then both $uxw'u$ and $vyzv$ are triangles, a contradiction. Hence x and w' are monochromatically connected by a monochromatic path $xw''w'$ with $w'' \notin V(T_1) \cup \{z\}$. Then both $uxw''u$ and $vzw'v$ are triangles, a contradiction.

Case 5: $P = xuy$ and $Q = vwz$.

First we show that T_1 is a star with the center u. Otherwise, let $ab \in E(T_1 - u)$. Suppose that $\{x, y\} = \{a, b\}$ and without loss of generality, let $x = a$ and $y = b$. From Lemma 2.2, $vx, vy, uz \in E(G)$. If $uw \in E(G)$, then both $G[\{u, w, z\}]$ and $G[\{v, x, y\}]$ are triangles, a contradiction. So we can take a vertex $w' \in N_{T_1}(w)$ and it is clear that $w' \neq u$ and $vw' \in E(G)$. Hence $G([\{v, w, w'\}])$ is a triangle. Without loss of generality, let $y \neq w'$. If $yz \in E(G)$, then both $G[\{u, y, z\}]$ and $G[\{v, w, w'\}]$ are triangles, a contradiction. So

$yz \notin E(G)$. Then there is a monochromatic path from y to z and take the neighbour vertex, say y', of y on it. Clearly, $y' \notin V(T_1)$. From Lemma 2.2, $uy' \in E(G)$. Hence both $G[\{u, y, y'\}]$ and $G[\{v, w, w'\}]$ are triangles, a contradiction. Thus, at least one of x, y does not belong to $\{a, b\}$.

By the symmetry of x and y, we can let $x \notin \{a, b\}$. If $xz \in E(G)$, then both $uxzu$ and $vabv$ are triangles, a contradiction. So $xz \notin E(G)$. From Claim 4.2, there is a vertex $w' \notin V(T_1) \cup V(T_2)$ such that $xw'z$ is a monochromatic path. From Lemma 2.2, $uw' \in E(G)$. Hence both $uxw'u$ and $vabv$ are triangles, a contradiction. Hence T_1 is a star with the center u.

Hence $V(T_1) \cup V(T_2) \setminus \{u\} \subseteq N_G(u)$. From Lemma 2.2, $V(G) \setminus (V(T_1) \cup V(T_2)) \subseteq N_G(u)$, $V(G) \setminus (V(T_1) \cup V(T_2)) \subseteq N_G(v)$, and $V(T_1) \subseteq N_G(v)$. This implies that $d(u) = n - 1$ and $G - u$ is connected, a contradiction.

Case 6: $P = uxy$ and $Q = vwz$.

Here we can assume that $uy \notin E(G)$, since otherwise we can fall into Case 5 above. If $uw \in E(G)$, then both $uwzu$ and $vxyv$ are triangles, a contradiction. Assume that there is a vertex $w' \in V(T_1) \setminus \{u, x\}$ such that $ww' \in E(G)$. Consider the monochromatic path between x and z. If $xz \in E(G)$, then both $uxzu$ and $vww'v$ are triangles, a contradiction. So from Claim 4.2, there is a vertex $w'' \notin V(T_1) \cup V(T_2)$ such that $xw''z$ is a monochromatic path. Hence from Lemma 2.2, $uw'' \in E(G)$. Then both $uxw''u$ and $vww'v$ are triangles, a contradiction. Hence $N_G(w) \cap V(T_1) = \{x\}$.

Now consider the neighbour of u. Assume that there is a vertex $u' \in V(T_1) \setminus \{u, x, y\}$ with $uu' \in E(G)$. If $u'z \in E(G)$, then both $uu'zu$ and $vwxv$ are triangles, a contradiction. So $u'z \notin E(G)$. From Claim 4.2, there is a vertex $w'' \notin V(T_1) \cup V(T_2)$ such that $u'w''z$ is a monochromatic path. Hence from Lemma 2.2, $uw'' \in E(G)$. Then both $uu'w''u$ and $vwxv$ are triangles, a contradiction. Hence $N_G(u) \cap V(T_1) = \{x\}$.

Now consider the neighbour of y. Assume that there is a vertex $y' \in V(T_1) \setminus \{u, y, x\}$ with $yy' \in E(G)$. If $xz \in E(G)$, then both $uxzu$ and $vyy'v$ are triangles, a contradiction. So $xz \notin E(G)$. From Claim 4.2, there is a vertex $w' \notin V(T_1) \cup V(T_2)$ such that $xw'z$ is a monochromatic path. Hence from Lemma 2.2, $uw' \in E(G)$. Then both $uxw'u$ and $vwxv$ are triangles, a contradiction. Hence $N_G(y) \cap V(T_1) = \{x\}$.

Assume that there is an edge ab in $T_1 - x$. Clearly, $a, b \neq u, y$. If $xz \in E(G)$, then both $uxzu$ and $vabv$ are triangles, a contradiction. So $xz \notin E(G)$. From Claim 4.2, there is a vertex $w' \notin V(T_1) \cup V(T_2)$ such that $xw'z$ is a monochromatic path. Hence from Lemma 2.2, $uw' \in E(G)$. Then both $uxw'u$ and $vabv$ are triangles, a contradiction.

Thus we have that T_1 is a star with the center x. Now we show that $V(T_2) \subseteq N_G(x)$. Otherwise, there is a vertex $x' \in V(T_2) \setminus N_G(x)$. Clearly, $x' \neq v, w$. From Claim 4.2, there is a vertex $w' \notin V(T_1) \cup V(T_2)$ such that $xw'x'$ is a monochromatic path. Hence from Lemma 2.2, $uw' \in E(G)$. Then both $ux'w'u$ and $vxwv$ are triangles, a contradiction. Hence $V(T_2) \subseteq N_G(x)$. If $d(x) \neq n-1$, then there is a vertex $x' \notin V(T_1) \cup V(T_2)$ with $xx' \notin E(G)$. From Claim 4.2, there is a vertex x'' such that $xx''x'$ is a monochromatic path. Clearly, $x'' \notin V(T_1)$. From Lemma 2.2, $ux'', ux' \in E(G)$. Hence both $ux''x'u$ and $vwxv$ are triangles, a contradiction. Thus $d(x) = n-1$. However, it is obviously that $G - x$ is connected, a contradiction. This completes the proof of Claim 4.3. □

Immediately, from Claim 4.3 above, we have the following observation.

Observation. *Each path of order three through u and v in $G[V(T_1)]$ and $G[V(T_2)]$, respectively, contains the vertex w.*

Let P and Q be a path of order three through u and v in $G[V(T_1)]$ and $G[V(T_2)]$, respectively. Let $V(P) = \{w, u, x\}$ and $V(Q) = \{w, v, y\}$. From Lemma 2.2, $uv, uy, vx \in E(G)$. By the symmetry, we distinguish the following cases.

Case a: $P = uxw \subseteq T_1$ and $Q = vyw \subseteq T_2$.

First we show that $xy \in E(G)$. Otherwise, from Claim 4.2, there is a vertex $w' \notin V(T_1) \cup V(T_2)$ such that $xw'y$ is a monochromatic path. If $ww' \in E(G)$, then from Lemma 2.2 both $uvxu$ and $yww'y$ are triangles in G, a contradiction. So $ww' \notin E(G)$. From Claim 4.2, there is a vertex $w'' \notin V(T_1) \cup V(T_2) \cup \{w'\}$ such that $ww''w'$ is a monochromatic path. Notice that from Lemma 2.2, $uw', uw'', vw', vw'' \in E(G)$.

Now we consider the monochromatic y–w'' path. If $yw'' \in E(G)$, then both $wyw''w$ and $uvxu$ are triangles in G, a contradiction. So $yw'' \notin E(G)$. From Claim 4.2 and Lemma 2.2, there is a vertex $w''' \neq u, v, x, y, w, w', w''$ such that $yw'''w''$ is a monochromatic path. From Lemma 2.2, both $uxw'u$ and $yvw'''y$ are triangles in G, a contradiction. Hence $xy \in E(G)$.

Assume that there is a vertex $w' \neq x, y, u, v$ with $ww' \in E(G)$. Without loss of generality, let $w' \notin V(T_2)$. If $yw' \in E(G)$, then both $uvxu$ and $yww'y$ are triangles in G, a contradiction. So $yw' \notin E(G)$. From Claim 4.2 and Lemma 2.2, there is a vertex $w'' \neq u, v, x, y, w, w'$ such that $yw''w'$ is a monochromatic path. Hence from Lemma 2.2, both $uxyu$ and $vw'w''v$ form a $2K_3$ in G, a contradiction.

So $N_G(w) \subseteq \{x, y, u, v\}$. Assume that there is a vertex $w' \notin V(T_1) \cup V(T_2)$. Then $ww' \notin E(G)$, and from Claim 4.2 there is a vertex w'' such that $ww''w'$ is a monochromatic path. From Lemma 2.2, $w'' \notin V(T_1) \cup V(T_2)$. However, $w'' \in N_G(w) \subseteq \{x, y, u, v\} \subseteq V(T_1) \cup V(T_2)$, a contradiction. Hence $V(G) = V(T_1) \cup V(T_2)$.

From $n \geq 7$, there is a vertex $w^* \neq w, u, v, x, y$ in G. Since $N_G(w) \subseteq \{x, y, u, v\}$, w^* is adjacent to one of $\{x, y, u, v\}$ in T_1 or T_2. Thus we can get a path of order three through u or v in $T_1 - w$ or $T_2 - w$, a contradiction.

Case b: $P = uxw$ and $Q = yvw$.

From Observation, $N_{T_2}(v) = \{y, w\}$, $N_{T_2}(y) = \{v\}$, $N_{T_1}(u) \subseteq \{x, w\}$ and $N_{T_1}(x) = \{u, w\}$.

First we show that $xy \in E(G)$. Otherwise, from Claim 4.2, there is a vertex $w' \notin V(T_1) \cup V(T_2)$ such that $xw'y$ is a monochromatic path. From Lemma 2.2, $uw', vw' \in E(G)$. Then both $uw'yu$ and $vwxv$ are triangles in G, a contradiction. Hence $xy \in E(G)$.

Suppose that there is a vertex $w' \notin V(T_1) \cup V(T_2)$. If $ww' \in E(G)$, then both $vww'v$ and $uxyu$ are triangles, a contradiction. So $ww' \notin E(G)$. From Claim 4.2 and Lemma 2.2, there is a vertex $w'' \notin V(T_1) \cup V(T_2)$ such that $ww''w'$ is a monochromatic path. From Lemma 2.2, $vw'' \in E(G)$. Then both $vww''v$ and $uxyu$ are triangles, a contradiction. Hence $V(G) = V(T_1) \cup V(T_2)$. From Claim 4.2 and Lemma 2.2, we have that $k = 2$. Hence $u, v, x, y \in S$ and we fall into Case a above.

Case c: $P = xuw$ and $Q = yvw$.

From two cases above, we can let $x, y \notin S$. Assume that $xy \notin E(G)$. From Claim 4.2, there is a vertex $w' \notin V(T_1) \cup V(T_2)$ such that $xw'y$ is a monochromatic path. Let $xw'y \subseteq T_3$. From Claim 4.2, T_3 is a star with the center w'.

Assume that there is a vertex $y' \in V(T_2)\backslash\{w, v, y\}$. If $xy' \in E(G)$, then both $uxy'u$ and $vyw'v$ are triangles in G, a contradiction. So $xy' \notin E(G)$. We have that $x \in V(T_1 \cup T_3)$ and $y' \in V(T_2)$. From Claim 4.2 and Lemma 2.2, there is a vertex $y'' \notin V(T_1) \cup V(T_2) \cup V(T_3)$ such that $xy''y'$ is a monochromatic path. Then it follows from Lemma 2.2 that both $uxy''u$ and $vyw'v$ are triangles in G, a contradiction. So $V(T_2) = \{w, v, y\}$. By the symmetry, we have that $V(T_1) = \{w, u, x\}$.

Assume that there is a vertex $x' \in N_G(x) \setminus (V(T_1) \cup V(T_2) \cup \{w'\})$. From Lemma 2.2, we have that both $uxx'u$ and $vyw'v$ are triangles, a contradiction. So $N_G(x) \subseteq V(T_1) \cup V(T_2) \cup \{w'\}$. By the symmetry of x and y, we have that $N_G(y) \subseteq V(T_1) \cup V(T_2) \cup \{w'\}$.

Notice that $n \geq 7$ and then there is a vertex $z \in V(G)\backslash\{u, v, x, y, w, w'\}$. Since $u, v \in S$ and $N_G(y) \cup N_G(x) \subseteq V(T_1) \cup V(T_2) \cup \{w'\}$, from Lemma 2.2 and Claim 4.2, $xw'z$ is a monochromatic path. Since $xw' \in E(T_3)$, this implies that $V(G) = V(T_1) \cup V(T_2) \cup V(T_3)$. Notice that from Lemma 2.2, $w \notin V(T_3)$. If $wz \notin E(G)$, then from Claim 4.2, both w and z are monochromatically connected in a nontrivial color tree, say T_4, by a path $wzz'z$ for some vertex z'. Clearly, $z' \in V(T_i)$ for some $i \in \{1, 2, 3\}$. Hence $|V(T_4) \cap V(T_i)| \geq 2$, a contradiction to the statement of Lemma 2.2. Thus it must hold that $wz \in E(G)$. Hence from Lemma 2.2, both $uwzu$ and $vyw'v$ are triangles, a contradiction. Thus $xy \in E(G)$.

Since $x \notin S$, there is a vertex $z \notin V(T_1) \cup V(T_2)$ such that $xz \in E(G)$. If $wz \in E(G)$, then both $uxyu$ and $vwzv$ are triangles, a contradiction. So $wz \notin E(G)$. From Claim 4.2 and Lemma 2.2, there is a vertex $w' \notin V(T_1) \cup V(T_2)$ such that $ww'z$ is a monochromatic path. From Lemma 2.2, both $uxyu$ and $vww'v$ are triangles, a contradiction.

Case d: $P = uxw$ and $Q = ywv$.

We can assume $N_{T_2}(v) = \{w\}$, otherwise, we will fall into Case b above. If there is a vertex $z \in N_{T_2}(y) \setminus \{w\}$, then from Lemma 2.2, both $uyzu$ and $vwxv$ are triangles, a contradiction. So $N_{T_2}(y) \subseteq \{v, w\}$. Hence T_2 is a star with the center w.

Assume that $xy \notin E(G)$. From Claim 4.2, there is a vertex $w' \notin V(T_1 \cup T_2)$ such that $xw'y$ is a monochromatic path. From Lemma 2.2, both $vxwv$ and $uyw'u$ are triangles in G, a contradiction. Thus $xy \in E(G)$.

Suppose that there is a vertex $w'' \notin N_G(w) \setminus (V(T_2) \cup \{x, u\})$ such that $ww'' \in E(G)$. From Lemma 2.2, both $vww''v$ and $uxyu$ are triangles in G, a contradiction. So $N_G(w) \subseteq V(T_2) \cup \{x, u\}$. From Observation, $N_G(u) \cap V(T_1) \subseteq \{x, w\}$ and $N_G(x) \cap V(T_1) = \{u, w\}$, and hence $T_1 = uxw$.

Since w is monochromatically connected to each vertex in G, we have that $V(G) = V(T_1) \cup V(T_2)$. From Lemma 2.2, $k = 2$. Hence $x \in S$. From Lemma 2.2, $d(x) = n - 1$ and $G - x$ is connected, a contradiction.

Case e: $P = xuw$ and $Q = ywv$.

In order to avoid falling to Case d above, let $x \notin S$. Without loss of generality, let $x \in V(T_3)$ and $xz \in E(T_3)$. Clearly, $z \notin V(T_1)$. If $z \notin V(T_2)$, then from Lemma 2.2, both $vxzv$ and $uwyu$ are triangles in G, a contradiction. So $z \in V(T_2)$. From Claim 4.2, T_3 is a star and hence there is a vertex $z' \notin V(T_1) \cup V(T_2)$ with $zz' \in E(T_3)$. If $wz' \in E(G)$, then both $uxzu$ and $vwz'v$ are triangles, a contradiction. So $wz' \notin E(G)$. From Claim 4.2 and Lemma 2.2, there is a vertex $z'' \notin V(T_1) \cup V(T_2)$ such that $wz''z'$ is a monochromatic path. From Lemma 2.2, both $uxzu$ and $vwz''v$ are triangles, a contradiction.

Case f: $P = xwu$ and $Q = ywv$.

In order to avoid falling into cases above, we can assume that $ux, vy \notin E(G)$, $N_G(u) \cap V(T_1) = N_G(v) \cap V(T_2) = \{w\}$. Suppose that there is a vertex $z \in N_G(x) \setminus V(T_2)$, then from Lemma 2.2, both $vxzv$ and $uwyu$ are triangles, a contradiction. So $N_G(x) \subseteq V(T_2)$. By the symmetry, we have that $N_G(a) \subseteq V(T_2)$ for each vertex $a \in N_G(w) \cap V(T_1)$. By the symmetry, we have that $N_G(b) \subseteq V(T_1)$ for each vertex $b \in N_G(w) \cap V(T_2)$. This implies that both T_1 and T_2 are stars with the center w. Moreover, this forces $V(G) = V(T_1) \cup V(T_2)$. Hence $d_G(w) = n - 1$ and $k = 2$. From Lemma 2.2, $G - w$ is connected, a contradiction. This completes the proof of Claim 4. □

Claim 5. $|S| \leq 1$.

Proof of Claim 5. On the contrary, let $|S| \geq 2$. From Claim 4, let $S \subseteq V(T_1)$. Suppose that $k = 1$. Then since each vertex appears in at least one nontrivial color tree, we have that $t_1 = n$, a contradiction. So let $k \geq 2$.

Suppose that $S = V(T_1)$. Take a path uvw in T_1 and a path xyz in T_2. From Lemma 2.2, $G[u, v, w, x, y, z]$ contains a $2K_3$, a contradiction. So $V(T_1) \setminus S \neq \emptyset$.

Take $u, v \in S$ such that at least one of $\{u, v\}$ is adjacent to a vertex of $V(T_1) \setminus S$ in T_1. We can choose u, v in G with one of the following cases.

Case A: There are two different vertices $x, y \in V(T_1) \setminus \{u, v\}$ with $x \notin S$ and $ux, vy \in E(G)$.

Suppose that $y \in S$. Since $x \notin S$, there is a nontrivial color tree T_i $(i \geq 2)$ containing x. Take $x', x'' \in V(T_i - x)$ with $xx' \in E(G)$. Then from Lemma 2.2, both $uxx'u$ and $vyx''v$ are triangles in G, a contradiction. So $y \notin S$.

From Lemma 2.2, any nontrivial color tree other than T_1 cannot contain both x and y. Since $x, y \notin S$, we let $x \in V(T_2)$ and $y \in V(T_3)$. Let $xx' \in E(T_2)$ and $yy' \in E(T_3)$. If $x' \neq y'$, then both $uxx'u$ and $vyy'v$ are triangles, a contradiction. So $x' = y'$, i.e., $V(T_2) \cap V(T_3) = \{x'\}$. This implies that $N_{T_2}(x) = \{x'\}$ and $N_{T_3}(y) = \{x'\}$. Hence there are vertices $x'' \in V(T_2 - x)$ and $y'' \in V(T_3 - y)$ such that $x'x'' \in E(T_2)$ and $x'y'' \in E(T_3)$. Clearly, $x', x'', y'' \notin V(T_1)$.

Suppose that $yx'' \notin E(G)$. Then y and x'' are monochromatically connected in a nontrivial color tree, say T_4. Clearly, $V(T_4) \cap V(T_i) \subseteq \{y, x''\}$, $i = 1, 2, 3$. So we can take a vertex $w \in V(T_4)$, $w \neq x'$, with $x''w \in E(T_4)$. From Lemma 2.2, both $ux''wu$ and $vy''x'v$ are triangles in G, a contradiction. So $yx'' \in E(G)$. From Lemma 2.2, both $uxx'u$ and $vyx''v$ are two disjoint triangles in G, a contradiction.

Case B: There is a vertex $x \in V(T_1) \setminus S$ with $ux, vx \in E(G)$.

Here we can assume that $N_{T_1}(u) = N_{T_1}(v) = \{x\}$.

Let $x \in V(T_2)$. Suppose that T_2 contains a path of order four, say $x_1x_2x_3x_4$. Since $ux, vx \in E(G)$, from Lemma 2.2, we have $ux_i, vx_i \in E(G)$ for $i = 1, 2, 3, 4$. Hence both ux_1x_2u and vx_3x_4v are triangles, a contradiction. So T_2 must be a star.

Let $wx \in E(T_2)$ and take a vertex $y \in V(T_2) \setminus \{w, x\}$ with either $xy \in E(G)$ or $wy \in E(G)$. Clearly, $w, y \notin V(T_1) \cup S$. So there is a non-$T_2$ nontrivial color tree, say T_3, containing y. Suppose that there is an edge $ab \in E(T_3 - V(T_1))$. Clearly, $a, b \notin V(T_1) \cup \{w\}$. Then both $uxwu$ and $vabv$ are triangles, a contradiction. So $T_3 - V(T_1)$ cannot contain any edges, i.e., T_3 is a star with the center $z \in V(T_1)$. Since $x, y \in V(T_2)$ and $y \in V(T_3)$, we have $z \neq x$ and hence $z \in V(T_1) \setminus \{u, v, x\}$.

For the same reason, we have that each non-T_2 nontrivial color tree containing y is a star with the center vertex in $V(T_1) \setminus \{u, v, x\}$.

*Case B.*1: $wy \in E(G)$.

Let $y' \in V(T_3) \setminus \{y, z\}$ and $zy' \in E(T_3)$. If $xy' \in E(G)$, from Lemma 2.2, both $uywu$ and $vxy'v$ are disjoint triangles in G, a contradiction. So $xy' \notin E(G)$. Clearly, x and y' are not monochromatically connected in T_i, $i = 1, 2, 3$. Let w' be the neighbour of x on the monochromatic x–y' path. Clearly, $w' \notin V(T_1 \cup T_2 \cup T_3)$. From Lemma 2.2, both $uywu$ and $vxw'v$ are disjoint triangles in G, a contradiction.

*Case B.*2: $xy \in E(G)$ and $wy \notin E(G)$.

Then we can assume that $N_{T_2}(w) = \{x\}$. If there is a vertex $w' \in N_G(w) \setminus V(T_1)$, then from Lemma 2.2, both $uww'u$ and $vxyv$ are triangles, a contradiction. So $N_G(w) \subseteq V(T_1)$. Suppose that there is an edge $ab \in E(G - V(T_1))$. Clearly, $w \neq a, b$. From Lemma 2.2, both $uabu$ and $vxwv$ are triangles, a contradiction. Hence $E(G - V(T_1)) = \emptyset$. Hence for any $a \in V(T_1)$ and $b \notin V(T_1)$, it must hold that $ab \in E(G)$.

Notice that $N_{T_1}(u) = N_{T_1}(v) = \{x\}$. Suppose that there are two vertices $a, b \in V(T_1 - x)$ with $ab \in E(G)$. Then both $uxwu$ and $yaby$ are triangles, a contradiction. Hence T_1 is a star with the center x. Thus we have that $d(x) = n - 1$ and $G - x$ is connected, a contradiction.

Case C: There is a vertex $x \in V(T_1) \setminus (\{u, v\} \cup S)$ with $vx, uv \in E(G)$.

Then we can assume that $N_{T_1}(u) = \{v\}$. Let $x \in V(T_2)$. From Lemma 2.2, we can easily find that T_2 does not contain a path of order four. So T_2 is a star. Let $wx \in E(T_2)$ and take a vertex $y \in V(T_2) \setminus \{x, w\}$ such that either $xy \in E(G)$ or $wy \in E(G)$. Clearly, $y \notin V(T_1) \cup S$. So there is a non-T_2 nontrivial color tree, say T_3, containing y.

If there is an edge $ab \in E(T_3 - V(T_1))$, then from Lemma 2.2, both $uabu$ and $vxwv$ are triangles, a contradiction. Hence T_3 is a star with the center $z \in V(T_1 - x)$. Let $y' \in V(T_3) \setminus \{y, z\}$.

Case C.1: $wy \in E(G)$.

Clearly, there is not any monochromatic x–y' path in T_i for $i = 1, 2, 3$. Take a monochromatic x–y' path and let w' be the neighbour vertex of x on it. From Lemma 2.2, $w' \notin V(T_1) \cup V(T_2)$. Then from Lemma 2.2, both $uwyu$ and $vxw'v$ are triangles in G, a contradiction.

Case C.2: $xy \in E(G)$.

Then we can assume that $N_{T_2}(w) = \{x\}$. Suppose that there is an edge $ab \in E(G[V(T_2 - x)])$. Then it is clear that $a, b \neq w$. From Lemma 2.2, both $uabu$ and $vxwv$ are triangles, a contradiction. Hence $G[V(T_2)]$ is a star with the center x.

Suppose that there is an edge $ab \in E(G - V(T_1))$. From Lemma 2.2, both $uabu$ and $vxwv$ are triangles, a contradiction. So $E(G - V(T_1)) = \emptyset$. Hence for any $a \in V(T_1)$ and $b \notin V(T_1)$, it must hold that $ab \in E(G)$.

Suppose that there is an edge $ab \in G[V(T_1 - v)]$. Then it is clear that $a, b \neq u, v$. Hence both $aby'a$ and $uvwu$ are triangles, a contradiction. So $G[V(T_1)]$ is a star with the center v. Hence $d(v) = n - 1$ and $G - v$ is connected, a contradiction. This completes the proof of Claim 5. $\square$

By Claim 5, $\sum_{i=1}^{k} t_i \geq 2n - 1$. If $k \leq (n+1)/2$, then $\sum_{i=1}^{k}(t_i - 2) \geq 2n - 1 - 2k \geq n - 2$, and we are done. So let $k > (n+1)/2$. From Lemma 2.7, $m \leq n^2/4 + n/2$, i.e., $|E(\overline{G})| \geq \binom{n}{2} - n^2/4 - n/2$. On the other hand, each nontrivial color tree T_i can monochromatically connect at most $\binom{t_i}{2} - (t_i - 1) = \binom{t_i - 1}{2}$ pairs of non-neighbors in G, we have that $\sum_{i=1}^{k} \binom{t_i - 1}{2} \geq |E(\overline{G})| \geq \binom{n}{2} - n^2/4 - n/2$.

Assume that $\sum_{i=1}^{k}(t_i-2)<n-2$. Then $\sum_{i=1}^{k}(t_i-1)\le n-3+k$ and $k\le n-3$. Since $t_i\ge 3$, it follows from Lemma 2.3 that $\sum_{i=1}^{k}\binom{t_i-1}{2}\le k-1+\binom{n-k-1}{2}$. Let $g(k)=k-1+\binom{n-k-1}{2}$ is a decreasing function of k for $k\le n-3$. From $k>(n+1)/2$, by the convex function property we have that that $\sum_{i=1}^{k}\binom{t_i-1}{2}\le k-1+\binom{n-k-1}{2}\le \max\{n^2/8-3n/4+3, n^2/8-n+39/8\}$.

So $\binom{n}{2}-n^2/4-n/2\le \max\{n^2/8-3n/4+3, n^2/8-n+39/8\}$, a contradiction for $n\ge 7$. Hence $\sum_{i=1}^{k}(t_i-2)\ge n-2$. This completes the proof of Theorem 3.3. □

4. Join of graphs

First we consider the join graph of two connected graphs. The join of two disjoint graphs G and H, denoted by $G+H$, is defined to be the graph $\overline{\overline{G}\cup\overline{H}}$.

Let G_1 and G_2 be two disjoint graphs and $G=G_1+G_2$. Given an MC-coloring of the graph G. A nontrivial color tree T intersecting both G_1 and G_2 is called a *star-type* color tree if $|V(T)\cap V(G_1)|=1$ or $|V(T)\cap V(G_2)|=1$, otherwise we call it *non-star-type*.

A nontrivial color tree T with $|V(T)\cap V(G_1)|\ge 2$ and $|V(T)\cap V(G_2)|\ge 2$ is called a *double-star-type* color tree, if there are $u\in V(T)\cap V(G_1)$ and $v\in V(T)\cap V(G_2)$, $uv\in E(T)$, such that both $T-V(G_1-u)$ and $T-V(G_2-v)$ are connected. Otherwise we call it *non-double-star-type*. Also, the vertices u and v are called the *centers* of the double-star-type color tree T.

Lemma 4.1. *Let f be an extremal MC-coloring of G. Then any two nontrivial color trees have at most two common vertices.*

Proof. If f is simple, then we are done. So let f be not simple and then there are two nontrivial color trees T_1 and T_2 in f with $|V(T_1)\cap V(T_2)|\ge 1$. If $|V(T_1)\cap V(T_2)|\le 2$, then we are done. Let $|V(T_1)\cap V(T_2)|\ge 3$. Since both T_1 and T_2 are trees, we have that $T_1\cup T_2$ contains at least two edges e_1 and e_2 such that $F_1\cup F_2-\{e_1,e_2\}$ is connected. So we can recolor $F_1\cup F_2-\{e_1,e_2\}$, e_1 and e_2 by distinct new colors. Hence we get another edge-coloring of G, where G is still monochromatically connected. But the new edge-coloring contains more colors than f. This contradicts to the fact that f is an extremal MC-coloring of G. □

Lemma 4.2. *Let G_1 and G_2 be two disjoint graphs and $G=G_1+G_2$. There is a simple extremal MC-coloring of G such that each nontrivial color tree T intersecting both G_1 and G_2 is either of the star-type or the double-star-type. Moreover, for any nontrivial color tree T intersecting both G_1 and G_2, $T-V(G_{3-i})$ is disconnected or $|V(T)\cap V(G_i)|=1$ for $i=1,2$.*

Proof. Let f be a simple extremal MC-coloring of G. Let T_c be a nontrivial color tree with $V(T_c) \cap V(G_i) \neq \emptyset$, $i = 1, 2$. Suppose that T_c is neither of star-type nor double-star-type. We choose f with the minimum number of this kind of nontrivial color trees.

Let $\{B_{1,i} \mid i = 1, 2, \ldots, t_1\}$ denote the set of components of $T_c - V(G_2)$ and let $\{B_{2,i} \mid i = 1, 2, \ldots, t_2\}$ denote the set of components of $T_c - V(G_1)$. Let $E_1 = E(T_c) \setminus \bigcup_{i,j} E(B_{i,j})$. Clearly, $|E_1| = t_1 + t_2 - 1$.

Let $v_{i,j} \in B_{i,j}$, $1 \leq i \leq 2$ and $1 \leq j \leq t_i$, and $S = \{v_{1,1}v_{2,j} \mid j = 1, 2, \ldots, t_2\} \cup \{v_{2,1}v_{1,j} \mid j = 1, 2, \ldots, t_1\}$.

Clearly, $|S| = t_1 + t_2 - 1$ and $|S \setminus E_1| = |E_1 \setminus S|$. Let $T_c^* = T_c - E_1 + S$. It is easy to see that T_c^* is still a tree with the vertex set $V(T_c)$. Also, each edge of $S \cap E_1$ is of the color c. By the definition of f^*, each edge of $S \setminus E_1$ is a trivial color tree in f. We exchange the colors of $S \setminus E_1$ and $E_1 \setminus S$ one-to-one. Then we get another edge-coloring f^* of G. From Lemma 2.2, each edge of $E_1 \setminus S$ is a trivial color tree in f^*. For each color tree T in f^* which is not an edge of $E_1 \setminus S$, we have that either $T = T_c^*$ or T is also a color tree in f. So we have that the edge-coloring f^* is a simple extremal MC-coloring of G too. However, f^* contains fewer non-star-type and non-double-star-type nontrivial color trees, a contradiction to the choice of f. Hence each nontrivial color tree T with $V(T) \cap V(G_1) \neq \emptyset$ and $V(T) \cap V(G_2) \neq \emptyset$ is either of star-type or double-star-type.

Take a nontrivial color tree T with $V(T) \cap V(G_i) \neq \emptyset$, $i = 1, 2$. Observe that if $t_1 = 1$ and $|V(B_{1,1})| = 1$, then T_c is of star-type. If $t_1 = 1$ and $|V(B_{1,1})| > 1$, recolor the edges of $B_{1,1}$ by a new color. Then we obtain another edge-coloring f^* of the graph G and f^* has more colors than f. Notice that each monochromatic path connecting two vertices in f also belongs to f^*, unless both the endpoints of it lie in $B_{1,1}$ and $T_c - V(B_{1,1})$, respectively. On the other hand, we know that each vertex of $V(B_{1,1})$ is adjacent to each vertex of $T_c - V(B_{1,1})$ in G. Hence, each two vertices are still monochromatically connected in f^*, i.e., f^* is an MC-coloring of G too. That is to say, we find an MC-coloring using colors more than the extremal MC-coloring f of G, a contradiction. □

Remark 4.3. As we do in the proof of Lemma 4.2, for any simple extremal MC-coloring f of $G = G_1 + G_2$, we can normalize each non-double-star-type nontrivial color tree T_c in f to a double-star-type nontrivial color tree T_c^* by exchanging colors of some edges of T_c and some trivial color trees. Also, $T_c - V(G_i) = T_c^* - V(G_i)$ for $i = 1, 2$. Repeating such a procedure for all non-double-star-type color trees in f, finally we get a simple extremal MC-coloring, which we call the *normalized MC-coloring* of f and is denoted by $n(f)$. Clearly, we have the following lemma.

Lemma 4.4. *For each non-double-star-type nontrivial color tree T_c of the color c in f, the nontrivial color tree T_c^* of the color c in $n(f)$ is of the double-star-type and $T_c - V(G_i) = T_c^* - V(G_i)$ for $i = 1, 2$.*

Lemma 4.5. *Let G_1 and G_2 be two disjoint graphs and $G = G_1+G_2$. Let G_2 be connected. Let f be a simple extremal MC-coloring of G. Then there is not any nontrivial color tree T_c intersecting both G_1 and G_2 such that $V(G_2) \subseteq V(T_c)$.*

Proof. From Lemma 4.4, without loss of generality, we assume that f is a normalized MC-coloring of G. Let T_c be a nontrivial color tree intersecting both G_1 and G_2. Let $\{B_{1,i} \mid i = 1, 2, \ldots, t_1\}$ denote the set of components of $T_c - V(G_2)$ and let $\{B_{2,i} \mid i = 1, 2, \ldots, t_2\}$ denote the set of components of $T_c - V(G_1)$. Choose $v_{i,j} \in B_{i,j}$ so that $v_{1,1}$ and $v_{2,1}$ are the centers of T_c, where $1 \leq i \leq 2$ and $1 \leq j \leq t_i$. Let $E_1 = E(T_c) \setminus \bigcup_{i,j} E(B_{i,j})$, where

$$E_1 = \{v_{1,1}v_{2,j} \mid j = 1, 2, \ldots, t_2\} \cup \{v_{2,1}v_{1,j} \mid j = 1, 2, \ldots, t_1\}.$$

Assume that $V(G_2) \subseteq V(T_c)$. Clearly, $t_2 \geq 2$. Then from Lemma 2.2, each edge of $E(G_2) \setminus E(T_c)$ is a trivial color tree and any other nontrivial color tree contains at most one vertex of G_2 in f. Take a spanning tree T of G_2. We recolor $G_2 \cup T_c$ as follows. Recolor T by a new color and recolor the edges $(E(G_2) \setminus E(T)) \cup \{v_{1,1}v_{2,i} \mid i = 2, 3, \ldots, t_2\}$ by other distinct new colors. Denote by f^* the obtained edge-coloring of G. Clearly, f^* contains more colors than f. We now show that f^* is also an MC-coloring of G.

Consider any two vertices, say $x, y \in V(G)$. If $\{x, y\} \nsubseteq V(T_c)$ or $\{x, y\} \subseteq V(T_c) \setminus V(G_2)$, then f^* contains the same monochromatic x–y path as f. If $\{x, y\} \subseteq V(G_2)$, then x and y are clearly monochromatic connected. If x, y belongs to $V(T_c) \setminus V(G_1)$ and $V(T_c) \setminus V(G_2)$ respectively, then $xy \in E(G)$ is a trivial color tree in f^*. So each two vertices of G are monochromatically connected in f^*. Hence f^* is also an MC-coloring of G, which contains more colors than f, a contradiction to the assumption that f is an extremal MC-coloring of G. □

Lemma 4.6. *Let G_1 and G_2 be two disjoint graphs and $G = G_1+G_2$. Let G_2 be connected. Then there is a simple extremal MC-coloring of G such that, for any nontrivial color tree T intersecting both G_1 and G_2, it holds that $|V(T) \cap V(G_2)| = 1$.*

Remark 4.7. The statement $|V(T) \cap V(G_2)| = 1$ implies that the graph G_2 itself is monochromatically connected under the current simple extremal MC-coloring of G.

Proof of Lemma 4.5. From Lemma 4.4, without loss of generality, we assume that f is a normalized MC-coloring of G. We take f with the minimum number of nontrivial color trees intersecting both G_1 and G_2 which contains at least two vertices of G_2. Let T be a nontrivial color tree intersecting both G_1 and G_2. Suppose that $|V(T) \cap V(G_2)| \geq 2$.

Let $\{B_{1,i} \mid i = 1, 2, \ldots, t_1\}$ denote the set of components of $T - V(G_2)$ and let $\{B_{2,i} \mid i = 1, 2, \ldots, t_2\}$ denote the set of components of $T - V(G_1)$. Let $v_{i,j} \in B_{i,j}$, $1 \leq i \leq 2$ and $1 \leq j \leq t_i$, and $E_1 = E(T) \setminus \bigcup_{i,j} E(B_{i,j})$, where

$$E_1 = \{v_{1,1}v_{2,j} \mid j = 1, 2, \ldots, t_2\} \cup \{v_{2,1}v_{1,j} \mid j = 1, 2, \ldots, t_1\}.$$

Clearly, $t_2 \geq 2$.

From Lemma 4.5, there is a vertex $u \in V(G_2) \setminus V(T)$. Now we consider the color tree, say T_j, monochromatically connecting u and $v_{2,j}$ in f for $1 \leq j \leq t_2$. Denote by j the color of T_j for $1 \leq j \leq t_2$ in f. Clearly, T is not of the color $1, 2, \ldots, t_2$. Recolor all the edges of $T' = \bigcup_{i=1}^{t_2}(B_{2,i} \cup T_i)$ by the color 1 and recolor the edge $v_{1,1}v_{2,j}$ by the color j for $j = 2, 3, \ldots, t_2$.

Denote by f' the resulting edge-coloring of G. Clearly, f' contains the same number of color as f and any two vertices of G are still monochromatically connected. That is to say, f' is also an extremal MC-coloring of G. In particular, f' contains fewer nontrivial color trees than f, which intersects both G_1 and G_2 and contains at least two vertices of G_2.

If f' is simple, then we get a contradiction to the choice of f. So f' is not simple. It is obviously that any two nontrivial color trees other than T' contain at most one common vertex. Hence there is a nontrivial color tree F which has two common vertices with T'. Take an edge $e \in E(F)$ such that $T'' = T' \cup F - e$ is a tree. Recolor all the edges of $F - e$ by the color of T'. Then we get another extremal MC-coloring f'' of G. Clearly, f'' contains nontrivial color trees no more than f', which intersects both G_1 and G_2 and contains at least two vertices of G_2.

Notice that T'' is a nontrivial color tree in f'' which contains more vertices of G_2 than T'. Clearly, any two nontrivial color trees other than T'' contain at most one common vertex in f''. If f'' is not simple, we repeat the same process above. After finite times, we get a simple extremal MC-coloring f^* of G with nontrivial color trees no more than f', which intersects both G_1 and G_2 and contains at least two vertices of G_2. Hence f^* contains nontrivial color trees fewer than f, which intersects both G_1 and G_2 and contains at least two vertices of G_2. This leads to a contradiction to the choice of f. This completes the proof of the lemma. □

4.1. Two connected graphs

Based on the simple extremal MC-coloring illustrated in Lemma 4.6, we have the following lemma.

Lemma 4.8. *Let G_1 and G_2 be two disjoint connected graphs and $G = G_1 + G_2$. Then there is a simple extremal MC-coloring of G such that for any nontrivial color tree T, either $T \subseteq G_1$ or $T \subseteq G_2$.*

Proof. From Lemma 4.6, we assume that f is a normalized MC-coloring of G such that any nontrivial color tree intersecting both G_1 and G_2 contains exactly one vertex of G_2. We take f with the minimum number of nontrivial color trees intersecting both G_1 and G_2 which contains at least two vertices of G_1.

Let T be a nontrivial color tree with $V(T) \cap V(G_1) \neq \emptyset$ and $V(T) \cap V(G_2) = \{v_{2,1}\}$. Suppose that $|V(T) \cap V(G_1)| \geq 2$. Let $\{B_{1,i} \mid i = 1, 2, \ldots, t\}$ denote the set of components of $T - v_{2,1}$. Let $v_{1,j} \in B_{1,j}$, $1 \leq j \leq t$, and $E_1 = E(T) \setminus \cup_j^t E(B_{1,j})$, where $E_1 = \{v_{2,1}v_{1,j} \mid j = 1, 2, \ldots, t\}$. Clearly, $t \geq 2$.

From Lemma 4.5, there is a vertex $u \in V(G_1) \setminus V(T)$. Now we consider the color tree, say T_j, monochromatically connecting u and $v_{1,j}$ in f for $1 \leq j \leq t$. Denote by j the color of T_j for $1 \leq j \leq t$ in f. Clearly, T is not of the color $1, 2, \ldots, t$. Recolor all the edges of $T' = \bigcup_{i=1}^t (B_{1,i} \cup T_i)$ by the color 1 and recolor the edge $v_{2,1}v_{1,j}$ by the color j for $j = 2, 3, \ldots, t$. Denote by f' the resulting edge-coloring of G. Since each nontrivial color tree in f intersecting both G_1 and G_2 contains exactly one vertex of G_2, we have that

(1′) if T' contains vertices of G_2, then each component of $T' - V(G_1)$ contains only one vertex belonging to G_2;

(2′) any other nontrivial color tree other than T' in f' intersecting both G_1 and G_2 contains exactly one vertex of G_2;

(3′) $f(xy) = f'(xy)$ for any edge $xy \in E(G_2)$;

(4′) the edge-coloring f' contains fewer nontrivial color trees than f, which intersects both G_1 and G_2 and contains at least two vertices of G_1.

Clearly, f' contains the same number of color as f and any two vertices of G are still monochromatically connected. That is to say, f' is also an extremal MC-coloring of G. If f' is simple, then we get a contradiction to the choice of f. So f' is not simple.

It is obviously that any two nontrivial color trees other than T' in f' contain at most one common vertex. Hence there is a nontrivial color tree F which has two common vertices with T'. Take an edge $e \in E(F)$ such that $T'' = T' \cup F - e$ is a tree. Recolor all the edges of $F - e$ by the color of T'. Then we get another extremal MC-coloring f'' of G.

Since each nontrivial color tree in f intersecting both G_1 and G_2 contains exactly one vertex of G_2, we have that

(1″) the edge-coloring f'' contains nontrivial color trees no more than f', which intersects both G_1 and G_2 and contains at least two vertices of G_1;

(2″) if T'' contains vertices of G_2, then each component of $T'' - V(G_1)$ contains only one vertex belonging to G_2;

(3″) any other nontrivial color tree other than T'' in f'' intersecting both G_1 and G_2 contains exactly one vertex of G_2;

(4″) $f(xy) = f''(xy)$ for any edge $xy \in E(G_2)$.

Notice that T'' is a nontrivial color tree in f'' which contains more vertices of G_1 than T'. Clearly, any two nontrivial color trees other than T'' in f'' contain at most one common vertex. If f'' is not simple, we repeat the same process above. After finite times, we get a simple extremal MC-coloring f^* of G. We have that

(1*) the edge-coloring f^* contains nontrivial color trees no more than f' (hence, fewer than f), which intersects both G_1 and G_2 and contains at least two vertices of G_1;

(2*) $f(xy) = f''(xy)$ for any edge $xy \in E(G_2)$.

From the recoloring procedure we have that there is at most one nontrivial color tree T^* in f^* intersecting both G_1 and G_2, which contains at least two vertices from both G_1 and G_2. Moreover, we have that

(3*) each component of $T^* - V(G_1)$ is a vertex belonging to G_2;

(4*) any other nontrivial color tree other than T^* in f^* intersecting both G_1 and G_2 contains exactly one vertex of G_2.

Now we start to normalize the edge-coloring f^* such that each nontrivial color tree intersecting both G_1 and G_2 is of the star-type. Clearly, we only need to consider the tree T^* if it exists.

Let $v \in T^* \cap V(G_2)$. Since f^* is simple, from Lemma 2.2, for any edge $xy \in E(T^*)$, where $x \in T^* \cap V(G_2)$ and y lies in a component of $T^* - x$ which does not contain the vertex v, the edge vy is a trivial color tree in f^*. Exchange the colors of xy and vy and then finally we get another edge-coloring f^{**} of G. Since $f(xy) = f^*(xy) = f^{**}(xy)$ for any edge $xy \in E(G_2)$, from the choice of f we have that G_2 itself is monochromatically connected under the edge-coloring f and f^{**}. Hence, f^{**} is also a simple extremal MC-coloring of G.

Clearly, f^{**} contains nontrivial color trees fewer than f, which intersects both G_1 and G_2 and contains at least two vertices of G_1, and each nontrivial color tree intersecting both G_1 and G_2 contains at most one vertex of G_2. This leads to a contradiction to the choice of f. This completes the proof of Lemma 4.8. $\square$

From Lemma 4.8, we have the following main theorem.

Theorem 4.9. *Let G_1 and G_2 be two disjoint connected graphs and $G = G_1 + G_2$. Then* $\mathrm{mc}(G) = \mathrm{mc}(G_1) + \mathrm{mc}(G_2) + |V(G_1)||V(G_2)|$.

Proof. From Lemma 4.8, there is a simple extremal MC-coloring f of G such that each edge between G_1 and G_2 is a trivial color tree. Hence both G_1 and G_2 are monochromatically connected itself under the edge-coloring f. Thus we have that G_i contains at most $\mathrm{mc}(G_i)$

colors for $i = 1, 2$. Hence $\mathrm{mc}(G) \le \mathrm{mc}(G_1) + \mathrm{mc}(G_2) + |V(G_1)||V(G_2)|$. On the other hand, it is obviously that $\mathrm{mc}(G) \ge \mathrm{mc}(G_1) + \mathrm{mc}(G_2) + |V(G_1)||V(G_2)|$. This completes the proof of the theorem. □

4.2. A connected graph and a disconnected graph

By the similar proof of Lemma 4.8, we have the following lemma.

Lemma 4.10. *Let G_1 and G_2 be two disjoint graphs and $G = G_1 + G_2$. If G_1 is disconnected and G_2 is connected, then there is a simple extremal MC-coloring of G such that there is only one nontrivial color tree T intersecting both G_1 and G_2. Moreover, $V(G_1) \subseteq V(T)$ and $|V(T) \cap V(G_2)| = 1$.*

Proof. From Lemma 4.6, we assume that f is a normalized MC-coloring of G such that any nontrivial color tree T intersecting both G_1 and G_2 contains exactly one vertex of G_2. We take f with the minimum number of nontrivial color trees intersecting both G_1 and G_2 which contains at least two vertices of G_1.

Let T be a nontrivial color tree with $V(T) \cap V(G_1) \ne \emptyset$ and $V(T) \cap V(G_2) = \{v_{2,1}\}$. Suppose that $|V(T) \cap V(G_1)| \ge 2$. Let $\{B_{1,i} \mid i = 1, 2, \ldots, t\}$ denote the set of components of $T - v_{2,1}$. Let $v_{1,j} \in B_{1,j}$, $1 \le j \le t$, and $E_1 = E(T) \setminus \cup_j^t E(B_{1,j})$, where $E_1 = \{v_{2,1}v_{1,j} \mid j = 1, 2, \ldots, t\}$. Clearly, $t \ge 2$.

If $V(G_1) \subseteq V(T)$, then we are done. So let $u \in V(G_1) \setminus V(T)$. Now we consider the color tree, say T_j, monochromatically connecting u and $v_{1,j}$ in f for $1 \le j \le t$. Denote by j the color of T_j for $1 \le j \le t$ in f. Clearly, T is not of the color $1, 2, \ldots, t$. Recolor all the edges of $T' = \bigcup_{i=1}^t (B_{1,i} \cup T_i)$ by the color 1 and recolor the edge $v_{2,1}v_{1,j}$ by the color j for $j = 2, 3, \ldots, t$. Denote by f' the resulting edge-coloring of G. Since each nontrivial color tree in f intersecting both G_1 and G_2 contains exactly one vertex of G_2, we have that

(1′) if T' contains vertices of G_2, then each component of $T' - V(G_1)$ contains only one vertex belonging to G_2;

(2′) any other nontrivial color tree other than T' in f' intersecting both G_1 and G_2 contains exactly one vertex of G_2;

(3′) $f(xy) = f'(xy)$ for any edge $xy \in E(G_2)$;

(4′) the edge-coloring f' contains fewer nontrivial color trees than f, which intersects both G_1 and G_2 and contains at least two vertices of G_1.

Clearly, f' contains the same number of color as f and any two vertices of G are still monochromatically connected. That is to say, f' is also an extremal MC-coloring of G. If f' is simple, then we get a contradiction to the choice of f. So f' is not simple.

It is obviously that any two nontrivial color trees other than T' in f' contain at most one common vertex. Hence there is a nontrivial color tree F which has two common vertices with T'. Take an edge $e \in E(F)$ such that $T'' = T' \cup F - e$ is a tree. Recolor all the edges of $F - e$ by the color of T'. Then we get another extremal MC-coloring f'' of G.

Since each nontrivial color tree in f intersecting both G_1 and G_2 contains exactly one vertex of G_2, we have that

($1''$) the edge-coloring f'' contains nontrivial color trees no more than f', which intersects both G_1 and G_2 and contains at least two vertices of G_1;

($2''$) if T'' contains vertices of G_2, then each component of $T'' - V(G_1)$ only contains one vertex belonging to G_2;

($3''$) any nontrivial color tree other than T'' in f'' intersecting both G_1 and G_2 contains exactly one vertex of G_2;

($4''$) $f(xy) = f''(xy)$ for any edge $xy \in E(G_2)$.

Notice that T'' is a nontrivial color tree in f'' which contains more vertices of G_1 than T'. Clearly, any two nontrivial color trees other than T'' in f'' contain at most one common vertex. If f'' is not simple, we repeat the same process above. After finite times, we get a simple extremal MC-coloring f^* of G. We have that

(1^*) the edge-coloring f^* contains nontrivial color trees no more than f' (hence, fewer than f), which intersects both G_1 and G_2 and contains at least two vertices of G_1;

(2^*) $f(xy) = f''(xy)$ for any edge $xy \in E(G_2)$.

From the recoloring procedure we have that there is at most one nontrivial color tree T^* in f^* intersecting both G_1 and G_2, which contains at least two vertices from both G_1 and G_2. Moreover, we have that

(3^*) each component of $T^* - V(G_1)$ is a vertex belonging to G_2;

(4^*) any other nontrivial color tree other than T^* in f^* intersecting both G_1 and G_2 contains exactly one vertex of G_2.

Now we start to normalize the edge-coloring f^* such that each nontrivial color tree intersecting both G_1 and G_2 is of the star-type. Clearly, we only need to consider the tree T^* if it exists.

Let $v \in T^* \cap V(G_2)$. Since f^* is simple, from Lemma 2.2, for any edge $xy \in E(T^*)$, where $x \in T^* \cap V(G_2)$ and y lies in a component of $T^* - x$ which does not contain the vertex v, the edge vy is a trivial color tree in f^*. Exchange the colors of xy and vy and then finally we get another edge-coloring f^{**} of G. Since $f(xy) = f^*(xy) = f^{**}(xy)$ for any edge $xy \in E(G_2)$, from the choice of f we have that G_2 itself is monochromatically connected under the edge-coloring f and f^{**}. Hence, f^{**} is also a simple extremal MC-coloring of G.

Clearly, f^{**} contains nontrivial color trees fewer than f, which intersects both G_1 and G_2 and contains at least two vertices of G_1, and each nontrivial color tree intersecting both G_1 and G_2 contains at most one vertex of G_2. This leads to a contradiction to the choice of f. This completes the proof of Lemma 4.10. □

We have the following theorem.

Theorem 4.11. *Let G_1 and G_2 be two disjoint graphs and $G = G_1+G_2$. If G_1 is connected and G_2 is disconnected, then* $\mathrm{mc}(G) = \mathrm{mc}(G_1) + |E(G_2)| + |V(G_1)||V(G_2)| - |V(G_2)| + 1$.

Proof. From Lemma 4.10, there is a simple extremal MC-coloring f of G such that G_1 itself is monochromatically connected and there is nontrivial color tree of the star-type containing all the vertices of G_2. Hence $\mathrm{mc}(G) \le \mathrm{mc}(G_1) + |E(G_2)| + |V(G_1)||V(G_2)| - |V(G_2)| + 1$. On the other hand, it is obviously that $\mathrm{mc}(G) \ge \mathrm{mc}(G_1) + |E(G_2)| + |V(G_1)||V(G_2)| - |V(G_2)| + 1$. This completes the proof of the theorem. □

4.3. Two disconnected graphs

Lemma 4.12. *Let G_1 and G_2 be two disjoint graphs and $G = G_1 + G_2$. If both G_1 and G_2 are disconnected, then there is a simple extremal MC-coloring of G such that there is a nontrivial color tree T with $V(G_1) \subseteq V(T)$.*

Proof. From Lemma 4.4, we assume that f is a normalized MC-coloring of G such that any nontrivial color tree T intersecting both G_1 and G_2 is of the star-type or the double-star-type. Since G_1 is disconnected, there are nontrivial color trees intersecting both G_1 and G_2 which contain at least two vertices of G_1. We take f with the minimum number of nontrivial color trees intersecting both G_1 and G_2 which contains at least two vertices of G_1. Let T be a nontrivial color tree intersecting both G_1 and G_2 which contains at least two vertices of G_1.

If $V(G_1) \subseteq V(T)$, then we are done. So let $u \in V(G_1) \setminus V(T)$. Let $\{B_{1,i} \mid i = 1, 2, \ldots, t\}$ denote the set of components of $T - V(G_2)$. From Lemma 4.2, $t \ge 2$. Let $v_{2,1} \in V(T) \cap V(G_2)$ and $E_1 = \{v_{2,1}v_{1,j} \mid j = 1, 2, \ldots, t\} \subseteq E(T)$, where $v_{1,j} \in B_{1,j}$ for $1 \le j \le t$.

Now we consider the color tree, say T_j, monochromatically connecting u and $v_{1,j}$ in f for $1 \le j \le t$. Denote by j the color of T_j for $1 \le j \le t$ in f. Clearly, T is not of the color $1, 2, \ldots, t$. Recolor all the edges of $T' = \bigcup_{i=1}^{t}(B_{1,i} \cup T_i)$ by the color 1 and recolor the edge $v_{2,1}v_{1,j}$ by the color j for $j = 2, 3, \ldots, t$. Denote by f' the resulting edge-coloring of G. We have the following statements.

(1′) The edge-coloring f' contains fewer nontrivial color trees than f, which intersects both G_1 and G_2 and contains at least two vertices of G_1.

(2′) The edge-coloring f' contains the same number of color as f and any two vertices of G are still monochromatically connected. That is to say, f' is also an extremal MC-coloring of G.

If f' is simple, then we get a contradiction to the choice of f. So f' is not simple. It is obviously that any two nontrivial color trees other than T' in f' contain at most one common vertex. Hence there is a nontrivial color tree F which has two common vertices with T'. Take an edge $e \in E(F)$ such that $T'' = T' \cup F - e$ is a tree. Recolor all the edges of $F - e$ by the color of T'. Then we get another extremal MC-coloring f'' of G. We have the following statements.

(1″) The edge-coloring f'' contains fewer nontrivial color trees than f, which intersects both G_1 and G_2 and contains at least two vertices of G_1.

(2″) The edge-coloring f'' is also an extremal MC-coloring of G.

Notice that T'' is a nontrivial color tree in f'' which contains more vertices of G_1 than T'. Clearly, any two nontrivial color trees other than T'' in f'' contain at most one common vertex. If f'' is not simple, we repeat the same process above. After finite times, we get a simple extremal MC-coloring f^* of G. We have that

(1*) The edge-coloring f^* contains fewer nontrivial color trees than f, which intersects both G_1 and G_2 and contains at least two vertices of G_1.

(2*) The edge-coloring f^* is also an extremal MC-coloring of G.

This leads to a contradiction to the choice of f. This completes the proof of the lemma. □

Furthermore, we have the following lemma.

Lemma 4.13. *Let G_1 and G_2 be two disjoint graphs and $G = G_1 + G_2$. If both G_1 and G_2 are disconnected, then there is a simple extremal MC-coloring of G such that there is a nontrivial color tree T with $V(G_1) \cup V(G_2) \subseteq V(T)$.*

Proof. From Lemma 4.12, we can assume that f is a simple extremal MC-coloring of G and T is a nontrivial color tree in f with $V(G_1) \subseteq V(T)$. Suppose that $V(G_2) \nsubseteq V(T)$. Since G_2 is disconnected, there are two vertices x, y from different components of G_2 such that $x \in V(T) \cap V(G_2)$ and $y \in V(G_2) \setminus V(T)$. Then any monochromatic path between x and y must contain a vertex of G_1 and hence, it contains at least two vertices of the nontrivial color tree T, a contradiction to the fact that f is a simple extremal MC-coloring of G. Hence $V(G_1) \cup V(G_2) \subseteq V(T)$. This completes the proof of the lemma. □

It follows from Lemma 4.13 that, if G is a join graph of two disjoint disconnected graphs, there is a simple extremal MC-coloring of G which contains a monochromatic spanning tree. Hence we have the following main result.

Theorem 4.14. *Let G_1 and G_2 be two disjoint disconnected graphs and $G = G_1 + G_2$. Then* $\mathrm{mc}(G) = |E(G)| - |V(G_1)| - |V(G_2)| + 2$.

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