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High Spatial Accuracy Analysis of Linear Triangular Finite Element for Distributed Order Diffusion Equations

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Abstract. In this paper, an effective numerical fully discrete finite element scheme for the distributed order time fractional diffusion equations is developed. By use of the composite trapezoid formula and the well-known $L1$ formula approximation to the distributed order derivative and linear triangular finite element approach for the spatial discretization, we construct a fully discrete finite element scheme. Based on the superclose estimate between the interpolation operator and the Ritz projection operator and the interpolation post-processing technique, the superclose approximation of the finite element numerical solution and the global superconvergence are proved rigorously, respectively. Finally, a numerical example is presented to support the theoretical results.

1. Introduction

In this paper, we are concerned with the following distributed order fractional diffusion equations:

$$\begin{cases} \mathcal{D}_t^w u(x,y,t) = \Delta u(x,y,t) + g(x,y,t) & \text{if } (x,y) \in \Omega,\ t \in (0,T], \\ u(x,y,0) = u^0(x,y) & \text{if } (x,y) \in \Omega, \\ u(x,y,t)|_{\partial\Omega} = 0 & \text{if } t \in [0,T], \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded, convex domain with Lipschitz boundary $\partial\Omega$ and $[0,T]$ is the time interval. The symbol Δ denotes the Laplacian operator. The given functions $g(x,y,t)$ and $u^0(x,y)$ are assumed to be smooth. The distributed order fractional derivative under

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consideration is defined by [10] as follows:

$$\begin{aligned} &\mathcal{D}_t^w u(x,y,t) = \int_0^1 w(\alpha)\, {}_0^C D_t^\alpha u(x,y,t)\, d\alpha, \\ &w(\alpha) \geq 0, \quad \int_0^1 w(\alpha)\, d\alpha = \omega_0 > 0, \\ &{}_0^C D_t^\alpha u(x,y,t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial u(x,y,s)}{\partial s}\, ds & \text{if } 0 \leq \alpha < 1, \\ \frac{\partial u(x,y,t)}{\partial t} & \text{if } \alpha = 1. \end{cases} \end{aligned} \tag{1.2}$$

Equation (1.1) was obtained to improve the modeling accuracy of the single-term model for describing problems in mathematical physics and engineering [1, 2, 10, 15, 22]. Due to their extensive development of fractional partial differential equations (PDEs) in engineering and science, there has been significant interest in constructing some numerical schemes for their solutions. Liao et al. [8] obtained a Du Fort-Frankel type explicit scheme for solving the distributed order subdiffusion equation by combining the $L1$ formula of Riemann-Liouville derivative with the midpoint quadrature of the weighted integral. Using the weighted and shifted Grünwald formula proposed in [21], Deng et al. established some effective difference schemes for solving one-dimensional and two-dimensional distributed order fractional PDEs. Gao et al. [4] investigated the unconditional stability and convergence of the obtained schemes for distributed order fractional PDEs with the energy method. In recent work [5], a rigorous numerical analysis of two fully discrete schemes for the distributed-order time fractional diffusion equation with nonsmooth initial data was presented. An implicit difference scheme for the time distributed-order and Riesz space fractional diffusions in bounded domains was discussed, and its stability and convergence was analyzed in [23]. Chen et al. [3] studied a fully discrete spectral method for the distributed order time fractional reaction-diffusion equation on an unbounded domain. Recently, Ren et al. [12] considered an efficient algorithm for the evaluation of the Caputo fractional derivative and the superconvergence property of fully discrete finite element approximation for the time fractional subdiffusion equation. Later, they presented a fully discrete scheme for the diffusion-wave equations in [11]. The unconditional stability and superconvergence error estimates of the obtained schemes are investigated using the integral identities and postprocessing techniques. The optimal time accuracy $\mathcal{O}(\tau^{3-\alpha})$ $(1 < \alpha < 2)$ is obtained.

Recently, Shi et al. [17–19] considered some finite element methods for solving the fractional PDEs. As we all know, the superconvergence analysis is widely applied to the classical PDEs for different finite element schemes, to the best of our knowledge, it seems that there are few studies focusing on the superconvergence error estimates of the triangular finite element for the fractional PDEs with distributed order. This gap in the research literature is the motivation for our work. Based on the relationship between Ritz projection and the interpolated operator of linear triangular element, the unconditional stability priori estimate and the global superconvergence estimate of the FEM scheme for (1.1) are proved rigorously.

The plan of this paper is as follows. In Section 2, some notations and auxiliary lemmas are presented. In Sections 3 and 4, a fully discrete scheme for the distributed order time fractional subdiffusion equations is developed and the unconditional stability as well as the superconvergence of the scheme are proved, respectively. In Section 5, some numerical results are provided to verify our theoretical analysis. Throughout, the notation C denotes a generic constant, which may not be the same at different occurrences, but it is always independent of the mesh size h, the time step size τ and $\Delta\alpha$.

2. Preliminaries

In this section, some useful notations, lemmas and formulae will be prepared for the forthcoming work.

Let $\Im_h = \{K\}$ be a be a family of uniform triangular meshes, $h_K = \mathrm{diam}\{K\}$ and $h = \max_{K\in\Im_h} h_K$. Then over the triangulation $\Im_h$, we define a continuous piecewise linear finite element space V_h by

$$V_h = \{v_h \in H_0^1(\Omega), v_h|_K \text{ is a linear function}, \forall\, K \in \Im_h\}.$$

Moreover, let $R_h\colon H_0^1(\Omega) \to V_h$ be the Ritz projection operator defined by

$$(\nabla(u - R_h u), \nabla v_h) = 0, \quad \forall\, v_h \in V_h.$$

Then, from the projection error estimates, there holds

$$\|u - R_h u\| + h\|\nabla(u - R_h u)\| \le Ch^2\|u\|_2, \quad \forall\, u \in H^2(\Omega) \cap H_0^1(\Omega). \tag{2.1}$$

For the finite difference discretization of the fractional derivative, let $0 = t_0 < t_1 < \cdots < t_N = T$ be the equidistant partition of time interval $[0, T]$ with step size $\tau = T/N$ for some positive integer N. We divide the interval $[0, 1]$ into $2J$-subintervals with $\Delta\alpha = 1/(2J)$ and $\alpha_l = l\Delta\alpha$, $l = 0, 1, 2, \dots, 2J$. Let u^n denote the solution $u(x, y, t)$ at $t = t_n$. The $L1$ approximation of the Caputo fractional derivative is approximated by [20]:

$$\begin{aligned} {}_0^C D_t^\alpha u(x, y, t_k) &\approx \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \frac{u^{j+1} - u^j}{\tau} \frac{1}{(t_n - s)^\alpha}\, ds \\ &= \frac{1}{\mu^{(\alpha)}} \Big[a_0^{(\alpha)} u^k - \sum_{j=1}^{k-1} \big(a_{k-j-1}^{(\alpha)} - a_{k-j}^{(\alpha)}\big) u^j - a_{k-1}^{(\alpha)} u^0 \Big] \\ &\equiv D_\tau^\alpha u^k, \quad 1 \le k \le N, \end{aligned} \tag{2.2}$$

where $\mu^{(\alpha)} = \tau^\alpha \Gamma(2-\alpha)$, $a_0^{(\alpha)} = 1$, $a_k^{(\alpha)} = (k+1)^{1-\alpha} - k^{1-\alpha}$. The local truncation error of (2.2) is of the order $\mathcal{O}(\tau^{2-\alpha})$ when the function u is twice continuously differentiable.

To describe the numerical approximation, the following lemma is useful.

Lemma 2.1. *Let $s(\alpha) \in C^2[0,1]$, then we have*

$$\int_0^1 s(\alpha)\,d\alpha = \Delta\alpha \sum_{l=0}^{2l} c_l s(\alpha_l) - \frac{(\Delta\alpha)^2}{12} s''(\xi), \quad \xi \in (0,1),$$

where $c_0 = c_{2J} = 1/2$, $c_l = 1$, $l = 1, 2, \ldots, 2J-1$.

3. Stability of the fully discrete finite element scheme

This section is devoted to the study of the stability of the fully discrete finite element scheme.

Suppose that $w(\alpha) \in C^2[0,1]$ and ${}_0^C D_t^\alpha u(x,y,t) \in C^2[0,1]$, then by Lemma 2.1, (1.2) and (2.2), we have

$$\mathcal{D}_t^w u(x,y,t_k) = \Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) D_\tau^{\alpha_l} u^k(x,y) + O(\tau + (\Delta\alpha)^2). \tag{3.1}$$

We construct the fully discrete finite element for the problem (1.1) as: Find $u_h^k \in V_h$ such that

$$\begin{cases} \Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l)(D_\tau^{\alpha_l} u_h^k, v_h) + (\nabla u_h^k, \nabla v_h) = (g^k, v_h), \quad \forall\, v_h \in V_h,\ 1 \le k \le N, \\ u_h^0 = R_h u^0. \end{cases} \tag{3.2}$$

Now, we focus on the stability analysis of the fully discrete scheme (3.2). Using the similar arguments of [13, 14, 24], we can can obtain the result of Theorem 3.1. Since the following estimates will play a key role in the error analysis of the FEM approximations, we give some hints of the proof.

Theorem 3.1. *The fully discrete finite element scheme* (3.2) *is unconditionally stable with respect to the initial value u_0 and the inhomogeneous term g, i.e.,*

$$\|\nabla u_h^k\|^2 \le \|\nabla u_h^0\|^2 + \frac{1}{4} \max_{0\le\alpha\le1} \{\Gamma(1-\alpha)T^\alpha\} \max_{1\le k\le N} \|g^k\|^2, \quad 1 \le k \le N.$$

Proof. Choosing $v_h = \Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) D_\tau^{\alpha_l} u_h^k$ in (3.2), it follows that

$$\left(\Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) D_\tau^{\alpha_l} u_h^k, \Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) D_\tau^{\alpha_l} u_h^k\right) + \left(\nabla u_h^k, \Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) D_\tau^{\alpha_l} \nabla u_h^k\right)$$
$$= \left(g^k, \Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) D_\tau^{\alpha_l} u_h^k\right), \quad 1 \le k \le N.$$

Noticing that $a_{k-1}^{(\alpha_l)}$ and $\big(a_{k-j-1}^{(\alpha_l)} - a_{k-j}^{(\alpha_l)}\big)$ are positive, we have for all $1 \le k \le N$,

$$\begin{aligned}
&\left\|\Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) D_\tau^{\alpha_l} u_h^k\right\|^2 + \Delta\alpha \sum_{l=0}^{2J} \frac{c_l w(\alpha_l)}{\mu^{\alpha_l}} \|\nabla u_h^k\|^2 \\
&= \Delta\alpha \sum_{l=0}^{2J} \frac{c_l w(\alpha_l)}{\mu^{\alpha_l}} \sum_{j=1}^{k-1} \big(a_{k-j-1}^{(\alpha_l)} - a_{k-j}^{(\alpha_l)}\big)(\nabla u_h^j, \nabla u_h^k) + \Delta\alpha \sum_{l=0}^{2J} \frac{c_l w(\alpha_l)}{\mu^{\alpha_l}} a_{k-1}^{(\alpha_l)} (\nabla u_h^0, \nabla u_h^k) \\
&\quad + \left(g^k, \Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) D_\tau^{\alpha_l} u_h^k\right) \\
&\le \Delta\alpha \sum_{l=0}^{2J} \frac{c_l w(\alpha_l)}{\mu^{\alpha_l}} \sum_{j=1}^{k-1} \big(a_{k-j-1}^{(\alpha_l)} - a_{k-j}^{(\alpha_l)}\big) \frac{\|\nabla u_h^j\|^2 + \|\nabla u_h^k\|^2}{2} \\
&\quad + \Delta\alpha \sum_{l=0}^{2J} \frac{c_l w(\alpha_l)}{\mu^{\alpha_l}} a_{k-1}^{(\alpha_l)} \frac{\|\nabla u_h^0\|^2 + \|\nabla u_h^k\|^2}{2} + \frac{1}{4}\|g^k\|^2 + \left\|\Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) D_\tau^{\alpha_l} u_h^k\right\|^2 .
\end{aligned}$$

Noticing (1.2), we get

$$\begin{aligned}
\Delta\alpha \sum_{l=0}^{2J} \frac{c_l w(\alpha_l)}{\mu^{\alpha_l}} \|\nabla u_h^k\|^2 &\le \Delta\alpha \sum_{l=0}^{2J} \frac{c_l w(\alpha_l)}{\mu^{\alpha_l}} \sum_{j=1}^{k-1} \big(a_{k-j-1}^{(\alpha_l)} - a_{k-j}^{(\alpha_l)}\big) \|\nabla u_h^j\|^2 \\
&\quad + \Delta\alpha \sum_{l=0}^{2J} \frac{c_l w(\alpha_l)}{\mu^{\alpha_l}} a_{k-1}^{(\alpha_l)} B, \quad 1 \le k \le N,
\end{aligned} \tag{3.3}$$

where

$$B = \|\nabla u_h^0\|^2 + \frac{1}{4} \max_{0\le\alpha\le1} \{\Gamma(1-\alpha)T^\alpha\} \max_{1\le k\le N} \|g^k\|^2.$$

Then, using mathematical induction, we have

$$\|\nabla u_h^k\|^2 \le B, \quad 1 \le k \le N. \tag{3.4}$$

According to (3.3), (3.4) is obviously true for $k = 1$. Assume that (3.4) is valid for $k = 2, \ldots, l-1$, then we obtain

$$\begin{aligned}
&\Delta\alpha \sum_{l=0}^{2J} \frac{c_l w(\alpha_l)}{\mu^{\alpha_l}} \|\nabla u_h^k\|^2 \\
&\le \Delta\alpha \sum_{l=0}^{2J} \frac{c_l w(\alpha_l)}{\mu^{\alpha_l}} \sum_{j=1}^{l-1} \big(a_{l-j-1}^{(\alpha_l)} - a_{l-j}^{(\alpha_l)}\big) \|\nabla u_h^j\|^2 + \Delta\alpha \sum_{l=0}^{2J} \frac{c_l w(\alpha_l)}{\mu^{\alpha_l}} a_{l-1}^{(\alpha_l)} B \\
&\le \Delta\alpha \sum_{l=0}^{2J} \frac{c_l w(\alpha_l)}{\mu^{\alpha_l}} \sum_{j=1}^{l-1} \big(a_{l-j-1}^{(\alpha_l)} - a_{l-j}^{(\alpha_l)}\big) B + \Delta\alpha \sum_{l=0}^{2J} \frac{c_l w(\alpha_l)}{\mu^{\alpha_l}} a_{l-1}^{(\alpha_l)} B \\
&= \Delta\alpha \sum_{l=0}^{2J} \frac{c_l w(\alpha_l)}{\mu^{\alpha_l}} B.
\end{aligned}$$

Therefore, we get the desired result. □

4. Superconvergent error estimate for the fully discrete finite element scheme

In this section, the superclose estimate of $\|\nabla(R_h u^k - u_h^k)\|$ is deduced unconditionally through the relationship between the interpolated operator I_h and the Ritz projection operator R_h, then the superconvergent error estimate for the scheme is derived.

For convenience, we employ the following splitting of the error

$$e^k = (u^k - R_h u^k) + (R_h u^k - u_h^k) \equiv \rho^k + \theta^k.$$

Then subtracting (1.1) from the problem (3.2), we get the error equation for $1 \le k \le N$ that

$$\begin{aligned}&\left(\Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) D_\tau^{\alpha_l}\theta^k, v_h\right) + (\nabla\theta^k, \nabla v_h)\\ &= -\left(\Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) D_\tau^{\alpha_l}\rho^k, v_h\right) + \left(\Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) D_\tau^{\alpha_l} u^k - D_t^w u^k, v_h\right), \quad \forall\, v_h \in V_h.\end{aligned} \tag{4.1}$$

Theorem 4.1. *Suppose that ${}_0^C\mathcal{D}_t^\alpha u \in C^2([0,1]; H^2(\Omega) \cap H_0^1(\Omega))$, u and u_h^k are the solutions of* (1.1) *and* (3.2)*, respectively. Then we have the following superclose estimate for $1 \le k \le N$,*

$$\|R_h u^k - u_h^k\|_1 \le C\big(\tau + (\Delta\alpha)^2 + h^2\big)\left(1 + \max_{\substack{0\le s\le T\\ 0\le\alpha\le 1}} \|{}_0^C\mathcal{D}_t^\alpha u(s)\|_2\right).$$

Proof. Taking $v_h = \Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) D_\tau^{\alpha_l}\theta^k$ in (4.1) yields

$$\begin{aligned}&\left\|\Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) D_\tau^{\alpha_l}\theta^k\right\|^2 + \Delta\alpha \sum_{l=0}^{2J} \frac{c_l w(\alpha_l)}{\mu^{\alpha_l}} \|\nabla\theta^k\|^2\\ &= \Delta\alpha \sum_{l=0}^{2J} \frac{c_l w(\alpha_l)}{\mu^{\alpha_l}} \sum_{j=1}^{k-1} \big(a_{k-j-1}^{(\alpha_l)} - a_{k-j}^{(\alpha_l)}\big)(\nabla\theta^j, \nabla\theta^k)\\ &\quad + \Delta\alpha \sum_{l=0}^{2J} \frac{c_l w(\alpha_l)}{\mu^{\alpha_l}} a_{k-1}^{(\alpha_l)}(\nabla\theta^0, \nabla\theta^k)\\ &\quad - \left(\Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) D_\tau^{\alpha_l}\rho^k, \Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) D_\tau^{\alpha_l}\theta^k\right)\\ &\quad + \left(\Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) D_\tau^{\alpha_l} u^k - D_t^w u^k, \Delta\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) D_\tau^{\alpha_l}\theta^k\right).\end{aligned} \tag{4.2}$$

Using (2.2), we have

$$\begin{aligned}
&\left\|\Delta\alpha\sum_{l=0}^{2J}c_l w(\alpha_l)D_\tau^{\alpha_l}u^k\right\|_2^2\\
&\le\left(\Delta\alpha\sum_{l=0}^{2J}c_l w(\alpha_l)\right)^2\max_{0\le\alpha\le1}\|D_\tau^{\alpha}u^k\|_2^2\\
&\le\left(\Delta\alpha\sum_{l=0}^{2J}c_l w(\alpha_l)\right)^2\max_{0\le\alpha\le1}\left(\|D_\tau^{\alpha}u^k-{}_0^C\mathcal{D}_t^{\alpha}u^k\|_2^2+\|{}_0^C\mathcal{D}_t^{\alpha}u^k\|_2^2\right)\\
&\le\left(\Delta\alpha\sum_{l=0}^{2J}c_l w(\alpha_l)\right)^2\max_{0\le\alpha\le1}\left(\tau^{2-\alpha}\max_{0\le s\le T}\|u_{tt}(s)\|_2^2+\max_{0\le s\le T}\|{}_0^C\mathcal{D}_t^{\alpha}u(s)\|_2^2\right)\\
&\le(\omega_0+1)^2\left(1+\max_{\substack{0\le s\le T\\0\le\alpha\le1}}\|{}_0^C\mathcal{D}_t^{\alpha}u(s)\|_2\right)^2,
\end{aligned}\tag{4.3}$$

where in the last inequality we have used Lemma 2.1.

For the third term on the right-hand side of (4.2), we have by (2.1) and (4.3) that

$$\begin{aligned}
&\left|-\left(\Delta\alpha\sum_{l=0}^{2J}c_l w(\alpha_l)D_\tau^{\alpha_l}\rho^k,\Delta\alpha\sum_{l=0}^{2J}c_l w(\alpha_l)D_\tau^{\alpha_l}\theta^k\right)\right|\\
&\le\frac{3}{4}\left\|\Delta\alpha\sum_{l=0}^{2J}c_l w(\alpha_l)D_\tau^{\alpha_l}\rho^k\right\|^2+\frac{1}{3}\left\|\Delta\alpha\sum_{l=0}^{2J}c_l w(\alpha_l)D_\tau^{\alpha_l}\theta^k\right\|^2\\
&\le Ch^4\left\|\Delta\alpha\sum_{l=0}^{2J}c_l w(\alpha_l)D_\tau^{\alpha_l}u^k\right\|_2^2+\frac{1}{3}\left\|\Delta\alpha\sum_{l=0}^{2J}c_l w(\alpha_l)D_\tau^{\alpha_l}\theta^k\right\|^2\\
&\le Ch^4\left(1+\max_{\substack{0\le s\le T\\0\le\alpha\le1}}\|{}_0^C\mathcal{D}_t^{\alpha}u(s)\|_2\right)^2+\frac{1}{3}\left\|\Delta\alpha\sum_{l=0}^{2J}c_l w(\alpha_l)D_\tau^{\alpha_l}\theta^k\right\|^2.
\end{aligned}\tag{4.4}$$

For the last term on the right-hand side of (4.2), using (3.1) to get

$$\begin{aligned}
&\left|\left(\Delta\alpha\sum_{l=0}^{2J}c_l w(\alpha_l)D_\tau^{\alpha_l}u^k-D_t^w u^k,\Delta\alpha\sum_{l=0}^{2J}c_l w(\alpha_l)D_\tau^{\alpha_l}\theta^k\right)\right|\\
&\le C\big(\tau+(\Delta\alpha)^2\big)\left\|\Delta\alpha\sum_{l=0}^{2J}c_l w(\alpha_l)D_\tau^{\alpha_l}\theta^k\right\|\\
&\le C\big(\tau+(\Delta\alpha)^2\big)^2+\frac{1}{3}\left\|\Delta\alpha\sum_{l=0}^{2J}c_l w(\alpha_l)D_\tau^{\alpha_l}\theta^k\right\|.
\end{aligned}\tag{4.5}$$

Then, substituting (4.4) and (4.5) into (4.2), and using the analytical method of Theorem 3.1, we obtain the desired result. □

In what follows, we will give the estimate of $\|I_h u^k - u_h^k\|_1$ with the idea of [16], where I_h is the associated interpolation operator over V_h. It has been proved in [9] that for all $u \in H^3(\Omega)$,

$$|(\nabla(u - I_h u), \nabla v_h)| \le Ch^2 \|u\|_3 \|\nabla v_h\|, \quad \forall\, v_h \in V_h. \tag{4.6}$$

With the help of Theorem 4.1 and (4.6), we can derive the following superclose result easily.

Theorem 4.2. *Suppose that u and u_h^k are solutions of* (1.1) *and* (3.2)*, respectively, $u \in C^2([0,1]; H^3(\Omega) \cap H_0^1(\Omega))$, then we have the superclose estimate for $1 \le k \le N$ that*

$$\|I_h u^k - u_h^k\|_1 \le C\big(\tau + (\Delta\alpha)^2 + h^2\big) \left(1 + \max_{\substack{0 \le s \le T \\ 0 \le \alpha \le 1}} \|{}_0^C\mathcal{D}_t^\alpha u(s)\|_2 + \|u\|_3 \right).$$

Now we consider a coarser decomposition T_{2h} of Ω into patches $\widetilde{K}$ such that $\overline{\Omega} = \bigcup_{\widetilde{K} \in T_{2h}} \widetilde{K}$ and each patch $\widetilde{K}$ consists of a fixed number of element K. The decomposition T_h can be generated from T_{2h} by a regular refinement, i.e., a patch consists of four elements (see Figure 4.1).

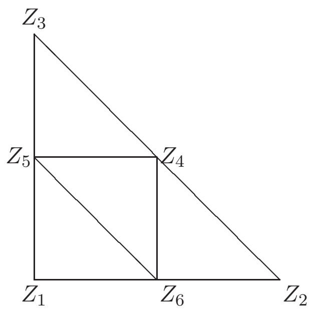

Figure 4.1: The element $\widetilde{K}$.

Next we construct the postprocessing operator $\Pi_{2h} \colon C(\widetilde{K}) \to P_2(\widetilde{K})$ on $\widetilde{K}$ as follows:

$$\Pi_{2h} u(Z_i) = u(Z_i), \quad i = 1, 2, \ldots, 6,$$

where $P_2(\widetilde{K})$ is the space of polynomials of degrees no more than 2 on $\widetilde{K}$. One can check that the interpolation postprocessing operator defined above is well-posed and has the following properties (cf. [9]):

$$\begin{cases} \Pi_{2h} I_h u = \Pi_{2h} u, & \forall\, u \in H^2(\Omega), \\ \|\Pi_{2h} u - u\|_1 \le Ch^r |u|_{r+1}, & \forall\, u \in H^{r+1}(\Omega),\ 0 \le r \le 2, \\ \|\Pi_{2h} v_h\|_1 \le C\|v_h\|_1, & \forall\, v_h \in V_h. \end{cases}$$

Thus we can obtain the following superconvergent result.

Theorem 4.3. *Under the assumptions of Theorem* 4.2, *we have the superconvergent result for* $1 \le k \le N$ *that*

$$\|u^k - \Pi_{2h} u_h^k\|_1 \le C\big(\tau + (\Delta\alpha)^2 + h^2\big) \left(1 + \max_{\substack{0\le s\le T\\ 0\le\alpha\le 1}} \|{}_0^C\mathcal{D}_t^\alpha u(s)\|_2 + \|u\|_3\right).$$

Remark 4.4. The main contribution of this paper is to obtain thee superconvergence error estimates for linear triangular finite element with the relationship between I_h and R_h. Here, it should be pointed out that if we choose $I_h u$ instead of $R_h u$ in Theorem 4.1, then the regularity of solution ${}_0^C\mathcal{D}_t^\alpha u \in C^2([0,1]; H^2(\Omega) \cap H_0^1(\Omega))$ must be replaced by $u \in C^2([0,T]; H^3(\Omega) \cap H_0^1(\Omega))$.

Remark 4.5. For the distributed-order fractional diffusion equations, we noticed that there are some discussions on the regularity of distributed-order fractional diffusion equations, such as Theorem 2.1 in [5], Theorems 2.1–2.2 in [7] and Theorems 1.1–1.3 in [6], but they are not adopted to our convergence analysis as we may need some explicit bounds on the time derivatives of the solution. We also believe that the solution of distributed-order fractional diffusion equations may exists the weak singularity in the time direction, it would be constructive and challenging to derive some more general regularities with application to the analysis of some standard numerical schemes. We will go into further investigation of this point in our future study.

Remark 4.6. The proposed method of this paper can be applied to other equations, such as the time fractional wave equation, nonlinear Schrödinger equation.

Remark 4.7. Due to distributed order derivative is based on the composite trapezoid formula and $L1$-type formula, whereas the temporal direction convergence order is only one. Thus, it is of practical interest to develop time higher-order schemes and provide rigorous error analysis.

5. Numerical experiment

In this section, we will present a numerical example to confirm the theoretical analysis.

Let $\Omega = [0,1] \times [0,1]$, $T = 0.5$, $w(\alpha) = \Gamma(4-\alpha)$ and initial condition $u^0(x,y) = 0$ in the problem (1.1), then the exact solution of the example is

$$u(x,y,t) = 8t^3 \sin(\pi x)\sin(\pi y).$$

Then $g(x,y,t)$ is chosen corresponding to the exact solution.

In the computations, the domain $\Omega = [0,1] \times [0,1]$ is triangulated as Figure 5.1. It is the first divided into $M_x \times M_y$ small rectangles (M_x and M_y are positive integers), and then divide each rectangle into two triangles.

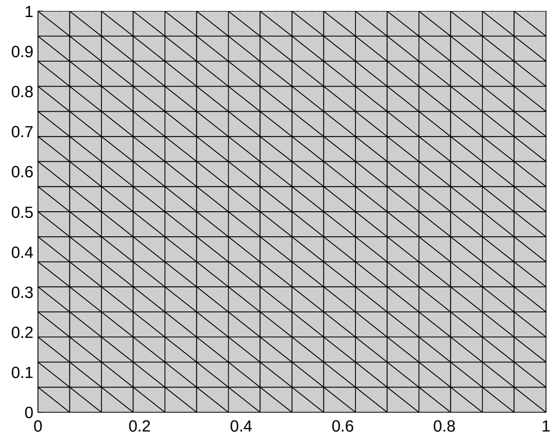

Figure 5.1: The triangulation of domain Ω.

First, to investigate the numerical convergence rate of the proposed scheme in time, we choose $M_x = M_y = 300$ and $\Delta\alpha = 1/200$, which are large enough to render negligible the error caused by spatial discretization. The numerical errors and convergence orders in H^1 norm are given in Table 5.1. From Table 5.1, the first-order accuracy of scheme (3.2) in time is verified by the example.

τ	$\|u^n - u_h^n\|_1$	rate
1/2	7.8128e-2	1.00
1/4	3.9162e-2	1.07
1/8	1.8593e-2	1.09
1/16	8.7343e-3	1.10
1/32	4.0747e-3	

Table 5.1: Numerical errors and convergence orders in temporal direction with $M_x = M_y = 300$ and $\Delta\alpha = 1/200$ at $T = 0.5$.

Secondly, the numerical accuracies of the fully finite element scheme in space is verified by the example. Taking the fixed and sufficiently small temporal stepsizes and $\Delta\alpha$, the $\|u^n - u_h^n\|$, $\|u^n - u_h^n\|_1$ and $\|u^n - \Pi_{2h}u_h^n\|_1$ norm errors and spatial convergence orders of the scheme are illustrated in Table 5.2, from which, the second-order supercloseness and superconvergence of the scheme (3.2) is apparent, indicating the sharpness of our estimate in Theorem 4.3.

Finally, we would like to investigate the numerical accuracy of the scheme (3.2) in distributed-order variable $\Delta\alpha$. The H^1-norm errors decrease as the step sizes in distribute order are reduced. The optimal second-order convergence of $\Delta\alpha$ is observed in Table 5.3, which is consistent with our theoretical analysis.

$M_x \times M_y$	$\|u^n - u_h^n\|$	rate	$\|u^n - u_h^n\|_1$	rate	$\|u^n - \Pi_{2h} u_h^n\|_1$	rate
4×4	4.3339e-2	1.87	2.9753e-1	0.94	5.4440e-2	1.63
8×8	1.1863e-2	1.97	1.5471e-1	0.99	1.7581e-2	1.90
16×16	3.0358e-3	2.00	7.8128e-2	1.00	4.6980e-3	1.99
32×32	7.6026e-4	2.02	3.9162e-2	1.00	1.1846e-3	2.05
64×64	1.8699e-4		1.9593e-2		2.8659e-4	

Table 5.2: Numerical errors and convergence orders in spatial direction with $\tau = 1/200$ and $\Delta\alpha = 1/200$ at $T = 0.5$.

$\Delta\alpha$	$\|u^n - u_h^n\|_1$	rate
1/2	1.3117e-1	1.60
1/4	4.3335e-2	1.87
1/8	1.1858e-2	1.97
1/16	3.0300e-3	2.01
1/32	7.5444e-4	

Table 5.3: Numerical errors and convergence orders in distributed order with $M_x = M_y = 300$ and $\tau = 1/200$ at $T = 0.5$.

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References

[1] M. Caputo, *Distributed order differential equations modelling dielectric induction and diffusion*, Fract. Calc. Appl. Anal. **4** (2001), no. 4, 421–442.

[2] ______, *Diffusion with space memory modelled with distributed order space fractional differential equations*, Ann. Geophys. **46** (2003), no. 2, 223–234.

[3] H. Chen, S. Lü and W. Chen, *Finite difference/spectral approximations for the distributed order time fractional reaction-diffusion equation on an unbounded domain*, J. Comput. Phys. **315** (2016), 84–97.

[4] G.-h. Gao, H.-w. Sun and Z.-z. Sun, *Some high-order difference schemes for the distributed-order differential equations*, J. Comput. Phys. **298** (2015), 337–359.

[5] B. Jin, R. Lazarov, D. Sheen and Z. Zhou, *Error estimates for approximations of distributed order time fractional diffusion with nonsmooth data*, Fract. Calc. Appl. Anal. **19** (2016), no. 1, 69–93.

[6] Z. Li, Y. Kian and E. Soccorsi, *Initial-boundary value problem for distributed order time-fractional diffusion equations*, arXiv:1709.06823.

[7] Z. Li, Y. Luchko and M. Yamamoto, *Asymptotic estimates of solutions to initial-boundary-value problems for distributed order time-fractional diffusion equations*, Fract. Calc. Appl. Anal. **17** (2014), no. 4, 1114–1136.

[8] H.-l. Liao, Y.-n. Zhang, Y. Zhao and H.-s. Shi, *Stability and convergence of modified Du Fort–Frankel schemes for solving time-fractional subdiffusion equations*, J. Sci. Comput. **61** (2014), no. 3, 629–648.

[9] Q. Lin and N. Yan, *The Construction and Analysis of High Efficient Elements*, Hebei University Press, China, 1996.

[10] I. Podlubny, *Fractional Differential Equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Mathematics in Science and Engineering **198**, Academic Press, San Diego, CA, 1999.

[11] J. Ren, X. Long, S. Mao and J. Zhang, *Superconvergence of finite element approximations for the fractional diffusion-wave equation*, J. Sci. Comput. **72** (2017), no. 3, 917–935.

[12] J. Ren, S. Mao and J. Zhang, *Fast evaluation and high accuracy finite element approximation for the time fractional subdiffusion equation*, Numer. Methods Partial Differential Equations **34** (2018), no. 2, 705–730.

[13] J. Ren and Z.-z. Sun, *Efficient and stable numerical methods for multi-term time fractional sub-diffusion equations*, East Asian J. Appl. Math. **4** (2014), no. 3, 242–266.

[14] J. Ren, Z.-z. Sun and X. Zhao, *Compact difference scheme for the fractional sub-diffusion equation with Neumann boundary conditions*, J. Comput. Phys. **232** (2013), 456–467.

[15] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and applications*, Gordon and Breach, London, 1993.

[16] D. Shi, P. Wang and Y. Zhao, *Superconvergence analysis of anisotropic linear triangular finite element for nonlinear Schrödinger equation*, Appl. Math. Lett. **38** (2014), 129–134.

[17] D. Shi and H. Yang, *Superconvergence analysis of finite element method for time-fractional Thermistor problem*, Appl. Math. Comput. **323** (2018), 31–42.

[18] ______, *A new approach of superconvergence analysis for two-dimensional time fractional diffusion equation*, Comput. Math. Appl. **75** (2018), no. 8, 3012–3023.

[19] ______, *Superconvergence analysis of a new low order nonconforming MFEM for time-fractional diffusion equation*, Appl. Numer. Math. **137** (2018), 109–122.

[20] Z.-z. Sun and X. Wu, *A fully discrete difference scheme for a diffusion-wave system*, Appl. Numer. Math. **56** (2006), no. 2, 193–209.

[21] W. Tian, H. Zhou and W. Deng, *A class of second order difference approximations for solving space fractional diffusion equations*, Math. Comp. **84** (2015), no. 294, 1703–1727.

[22] C. Yao, *The solvability of coupling the thermal effect and magnetohydrodynamics field with turbulent convection zone and the flow field*, J. Math. Anal. Appl. **476** (2019), no. 2, 495–521.

[23] H. Ye, F. Liu, V. Anh and I. Turner, *Numerical analysis for the time distributed-order and Riesz space fractional diffusions on bounded domains*, IMA J. Appl. Math. **80** (2015), no. 3, 825–838.

[24] X. Zhao and Z.-z. Sun, *A box-type scheme for fractional sub-diffusion equation with Neumann boundary conditions*, J. Comput. Phys. **230** (2011), no. 15, 6061–6074.

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