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On the Bogomolov-Miyaoka-Yau Inequality for Stacky Surfaces

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Abstract. We discuss a generalization of the Bogomolov-Miyaoka-Yau inequality to Deligne-Mumford surfaces of general type.

1. Introduction

We work over $\mathbb{C}$.

For a smooth complex projective surface S of general type, the Bogomolov-Miyaoka-Yau inequality for S reads (see [9])

$$3c_2(T_S) \geq c_1(T_S)^2. \tag{1.1}$$

Together with Noether's inequality, this puts constraints on the topology of surfaces of general types. Generalizations of (1.1) to singular surfaces and surface pairs have been found, see for example [6, 7, 10].

In this paper we discuss a generalization of (1.1) to Deligne-Mumford stacks. Let $\mathcal{X}$ be a smooth proper Deligne-Mumford $\mathbb{C}$-stack of dimension 2. Let $\pi\colon \mathcal{X} \to X$ be the natural map to the coarse moduli space. We assume that X is a projective variety. Since $\mathcal{X}$ is assumed to be smooth, it has a tangent bundle $T_{\mathcal{X}}$. A good theory of Chern classes is available for Deligne-Mumford stacks, see for example [5, 15].

Theorem 1.1. *Let $\mathcal{X}$ be as above. Assume that the canonical bundle $K_{\mathcal{X}} := \wedge^2 T_{\mathcal{X}}^\vee$ is numerically effective, then*

$$3c_2(T_{\mathcal{X}}) \geq c_1(T_{\mathcal{X}})^2. \tag{1.2}$$

Certainly (1.2) takes the same shape as (1.1). A proof of (1.2), along the lines of Miyaoka's original proof of (1.1) in [9], is given in Section 2. Section 3 contains examples of (1.2). In Section 3.2 we consider (1.2) for a class of stacks $\mathcal{X}$ with stack structures in codimension 1, recovering [6, Corollary 0.2]. In Section 3.3 we consider (1.2) for Gorenstein stacks $\mathcal{X}$ with isolated stack points, recovering [10, Corollary 1.3].

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Generalizations of the Bogomolov-Miyaoka-Yau inequality to varieties with quotient singularities (i.e., orbifolds) certainly have been studied before by many authors using various approaches. References to these can be found in e.g., [6, 7]. In this paper we work in the context of Deligne-Mumford stacks. This viewpoint has the advantage that (1.2) can be proven by following Miyaoka's original arguments in [9]. Also, as discussed in Section 3, (1.2) specializes to some generalizations of the original (1.1) by straightforward and elementary means.

2. Proof of (1.2)

In this section we give a proof of (1.2). Our proof is adapted from Miyaoka's original proof in [9].

Let $\mathcal{X}$ be a smooth proper Deligne-Mumford stack of dimension 2. If $\mathcal{X}$ has non-trivial stack structures at generic points, then $\mathcal{X}$ is an étale gerbe over a stack with trivial generic stack structure, see for example [2, Proposition 4.6]. More precisely, there is a finite group G, a stack $\mathcal{X}'$ with trivial generic stabilizers, and a morphism $f\colon \mathcal{X} \to \mathcal{X}'$ realizing $\mathcal{X}$ as a G-gerbe over $\mathcal{X}'$. Since $T_{\mathcal{X}} = f^*T_{\mathcal{X}'}$, we see that (1.2) for $\mathcal{X}$ is equivalent to (1.2) for $\mathcal{X}'$. Therefore it suffices to consider only those $\mathcal{X}$ with stack structures in codimension ≥ 1. For the rest of this section we assume this.

Let $\mathcal{F}$ be a locally free sheaf of rank 2 on $\mathcal{X}$. Let $\mathcal{V} := \mathbb{P}(\mathcal{F})$ be the projectivization, with natural projection $p\colon \mathcal{V} \to \mathcal{X}$. Let $\mathcal{H}$ be the divisor associated to the tautological sheaf on $\mathcal{V}$.

Lemma 2.1. *Assume that $\mathcal{W} \subset \mathcal{V}$ is linearly equivalent to $\mathcal{H} - p^*\mathcal{D}$, where $\mathcal{D} \subset \mathcal{X}$ is a divisor on $\mathcal{X}$. Then we have*

$$\mathcal{D} \cdot \det \mathcal{F} \leq c_2(\mathcal{F}) + \mathcal{D}^2.$$

Proof. We closely follow Miyaoka's original proof [9]. Let $i\colon \mathcal{W} \subset \mathcal{V}$ be the inclusion morphism. Note that the composition $p \circ i\colon \mathcal{W} \to \mathcal{X}$ is birational by our assumption on the linear equivalence class of $\mathcal{W}$. Since resolutions can be chosen such that they are compatible with étale base change, there is a sequence of blow-ups

$$\mu\colon \mathcal{V}_s \xrightarrow{\mu_s} \mathcal{V}_{s-1} \to \cdots \to \mathcal{V}_1 \xrightarrow{\mu_1} \mathcal{V}_0 = \mathcal{V}$$

such that the proper transform $\mathcal{W}'$ of $\mathcal{W}$ is a smooth Deligne-Mumford stack in $\mathcal{V}_s$. Let $i'\colon \mathcal{W}' \subset \mathcal{V}_s$ and $\rho\colon \mathcal{W}' \to \mathcal{X}$ be the natural maps.

Let $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_s$ be the exceptional divisors on $\mathcal{V}_s$. The divisor $\mathcal{W}'$ is linearly equivalent to $\mu^*(\mathcal{H} - p^*\mathcal{D}) - \sum a_i\mathcal{E}_i$. It can be seen[1] that the canonical bundle $K_{\mathcal{W}'}$ satisfies $K_{\mathcal{W}'} =$

[1] The argument is similar to that of [9, Lemma 7] and is omitted.

$\rho^* K_{\mathcal{X}} + \sum \mathcal{C}_i$ where C_i is a curve and $\rho(\mathcal{C}_i) =$ point. By the Hodge index theorem (for a stacky version see [8, Theorem 3.1.3]), it follows that $(K_{\mathcal{W}'} - \rho^* K_{\mathcal{X}} + \sum c_i i'^* \mathcal{E}_i)^2 \leq 0$ for any $c_i \in \mathbb{R}$.

Write $K_{\mathcal{V}_s} = \mu^*(-2\mathcal{H} + p^* K_{\mathcal{X}} + p^*(\det \mathcal{F})) + \sum b_i \mathcal{E}_i$. The adjunction formula implies that

$$K_{\mathcal{W}'} = i'^* \left[\mu^*(-\mathcal{H}) + (p \circ \mu)^*(K_{\mathcal{X}} + \det \mathcal{F} - \mathcal{D}) + \sum (b_i - a_i) \mathcal{E}_i \right].$$

Thus $i'^* \mu^*(-\mathcal{H} + p^*(\det \mathcal{F} - \mathcal{D}))^2 \leq 0$. Set $k := i'^* \mu^*(-\mathcal{H} + p^*(\det \mathcal{F} - \mathcal{D}))^2$. We can also compute this self-intersection number k in another way:

$$\begin{aligned} k &= \mu^*(-\mathcal{H} + p^*(\det \mathcal{F}) - p^*\mathcal{D})^2 \left(\mu^*\mathcal{H} - (p \circ \mu)^*\mathcal{D} - \sum a_i \mathcal{E}_i \right) \\ &= \mu^*(-\mathcal{H} + p^*(\det \mathcal{F}) - p^*\mathcal{D})^2 (\mu^*\mathcal{H} - (p \circ \mu)^*\mathcal{D}) \qquad \text{(since } \mathcal{E}_i \text{ is exceptional)} \\ &= \mathcal{H}^3 - \mathcal{H}^2 \cdot p^*(\mathcal{D} - 2\det \mathcal{F}) + \mathcal{H} \cdot (p^*(\det \mathcal{F})^2 - (p^*\mathcal{D})^2). \end{aligned}$$

Using the standard relations for the intersection numbers on the projectivization of a rank 2 vector bundle, we calculate that

$$k = c_1^2(\det(\mathcal{F})) - c_2(\mathcal{F}) - (\det \mathcal{F})^2 + \det \mathcal{F} \cdot \mathcal{D} - \mathcal{D}^2 = -c_2(\mathcal{F}) + \det \mathcal{F} \cdot \mathcal{D} - \mathcal{D}^2.$$

The result follows. □

Let $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ be a subsheaf of $\Omega^1_{\mathcal{X}}$. One key observation used in Miyaoka's original proof is that the Iitaka dimension of $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ is at most 1.

Theorem 2.2. *If $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ is a subsheaf of $\Omega^1_{\mathcal{X}}$ of a projective Deligne-Mumford stack $\mathcal{X}$, then $h^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(n\mathcal{D})) \leq cn$ for some positive constant c and $n \gg 0$.*

The proof of Theorem 2.2 is very similar to that of [9, Theorem 2"]. Two main ingredients are needed in the proof of [9, Theorem 2"]: (1) the Riemann-Roch formula, and (2) a lemma due to de Franchis. The de Franchis lemma states that any global holomorphic differential form on a Kähler manifold or a surface is d-closed [9, Lemma 9]. This lemma follows essentially from Stoke's theorem. The argument still works for smooth Kähler Deligne-Mumford stacks (Kähler orbifolds) or smooth surface Deligne-Mumford stacks. The Riemann-Roch for stacks is proved in [12].

One can prove the following result using Theorem 2.2.

Proposition 2.3. *Let $\mathcal{F} \subset \Omega^1_{\mathcal{X}}$ be a locally free sheaf of rank 2 and assume that $\det(\mathcal{F})^{\otimes n}$ is generated by global sections for some $n > 0$. If $\mathcal{F} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D})$ has a non-trivial section, then*

$$\mathcal{D} \cdot \det(\mathcal{F}) \leq \max\{c_2(\mathcal{F}), 0\}.$$

Proof. Consider $p\colon \mathcal{V} = \mathbb{P}(\mathcal{F}) \to \mathcal{X}$. The canonical isomorphism gives us

$$H^0(\mathcal{X}, \mathcal{F} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D})) = H^0(\mathcal{V}, \mathcal{O}_{\mathcal{V}}(\mathcal{H} - p^*\mathcal{D})).$$

If $\mathcal{F}\otimes\mathcal{O}_{\mathcal{X}}(-\mathcal{D})$ has a non-trivial section, then $|\mathcal{H}-p^*\mathcal{D}|$ is non-empty. Pick $\mathcal{W} \in |\mathcal{H}-p^*\mathcal{D}|$. Decompose $\mathcal{W}$ as $\mathcal{W} = \mathcal{W}_0 + p^*\mathcal{D}'$ where $\mathcal{W}_0$ is effective and irreducible which is linearly equivalent to $\mathcal{H} - p^*(\mathcal{D} + \mathcal{D}')$ and $\mathcal{D}'$ is effective. Note that $(\det \mathcal{F})^{\otimes n}$ is generated by global sections, so the intersection number $\mathcal{D}' \cdot \det(\mathcal{F}) \geq 0$. It follows that $\mathcal{D} \cdot \det(\mathcal{F}) \leq (\mathcal{D}+\mathcal{D}')\cdot\det(\mathcal{F})$ and it suffices to prove $(\mathcal{D}+\mathcal{D}')\cdot\det(\mathcal{F}) \leq \max\{c_2(\mathcal{F}), 0\}$. Set $\mathcal{D}'' = \mathcal{D}+\mathcal{D}'$ to simplify notation. By Lemma 2.1, $\mathcal{D}'' \cdot \det(\mathcal{F}) \leq c_2(\mathcal{F}) + \mathcal{D}'' \cdot \mathcal{D}''$. Observe that $\mathcal{O}_{\mathcal{X}}(\mathcal{D}'')$ is a subsheaf of $\Omega^1_{\mathcal{X}}$. Indeed, the effectiveness of $\mathcal{W}_0$ ensures the existence of a non-trivial section of $\mathcal{F} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D}'')$, i.e., an injection $\mathcal{O}_{\mathcal{X}} \hookrightarrow \mathcal{F} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D}'')$. Twisting by $\mathcal{O}_{\mathcal{X}}(-\mathcal{D}'')$, embeds $\mathcal{O}_{\mathcal{X}}(-\mathcal{D}'')$ into $\mathcal{F} \subset \Omega^1_{\mathcal{X}}$. By Theorem 2.2, $\mathcal{D}''$ has Iitaka dimension at most 1. It follows that $\mathcal{D}'' \cdot \det(\mathcal{F}) \leq 0$ or $\mathcal{D}'' \cdot \mathcal{D}'' \leq 0$.[2] This completes the proof. □

Assuming $c_2(\mathcal{F})$ is positive for the time being, we can obtain an upper bound on c_2 provided the sheaf $\mathcal{F} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D})$ has no sections. This can then be used to derive a contradiction. To be more precise, one needs a modified version of Proposition 2.3, in which the condition on the sheaf $\mathcal{F}\otimes\mathcal{O}_{\mathcal{X}}(-\mathcal{D})$ having a section is replaced by the condition that some symmetric power $S^m\mathcal{F} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D})$ having a section.

Theorem 2.4. *Let $\mathcal{F} \subset \Omega^1_{\mathcal{X}}$ be a locally free sheaf of rank 2 and assume that $\det(\mathcal{F})^{\otimes n}$ is generated by global sections for some $n > 0$. If $S^m\mathcal{F} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{D})$ has a non-trivial section, then*

$$\mathcal{D} \cdot \det(\mathcal{F}) \leq \max\{mc_2(\mathcal{F}), 0\}.$$

The proof of Theorem 2.4 follows from Proposition 2.3 and the following easy lemma (which is analogous to [9, Lemma 11]).

Lemma 2.5. *Let $p\colon \mathcal{V} = \mathbb{P}(\mathcal{F}) \to \mathcal{X}$ be the projective bundle of a locally free sheaf of rank 2. Let $\mathcal{W} \in |m\mathcal{H} - p^*\mathcal{D}|$. Then there is a surjective morphism $\beta\colon \mathcal{X}' \to \mathcal{X}$ such that $\beta^*\mathcal{W}$ is decomposed to $\mathcal{W}_1 + \cdots + \mathcal{W}_m$ where $\mathcal{W}_i$ is an effective divisor linear equivalent to $\mathcal{H}' - p^*\mathcal{D}_i$.*

Proof of Theorem 2.4. The following argument is taken from [9, Theorem 3]. Let f be a global section of $S^m\mathcal{F}\otimes\mathcal{O}_{\mathcal{X}}(-\mathcal{D})$. Lemma 2.5 implies that after a suitable cover $\beta\colon \mathcal{Y} \to \mathcal{X}$, we can decompose β^*f can be written as $f_1 f_2 \cdots f_m \in H^0(\mathcal{Y}, S^m\beta^*\mathcal{F}\otimes\mathcal{O}_{\mathcal{Y}}(-\beta^*\mathcal{D}))$, where $f_i \in H^0(\mathcal{Y}, \beta^*\mathcal{F} \otimes \mathcal{O}_{\mathcal{Y}}(-\beta^*\mathcal{D}_i))$ and $(\det \beta^*\mathcal{F})^{\otimes m} \cong (\beta^* \det(\mathcal{F})^{\otimes m})$ is generated by global sections. From Proposition 2.3, it follows that $\beta^*\mathcal{D}_i \cdot (\det(\beta^*\mathcal{F})) \leq \max\{c_2(\beta^*\mathcal{F}), 0\}$. Summing over all i's, we have $\beta^*\mathcal{D}\cdot\det(\beta^*\mathcal{F}) \leq \max\{mc_2(\beta^*\mathcal{F}), 0\}$. Let d be the mapping degree of β. Clearly, $\beta^*\mathcal{D} \cdot \det(\beta^*\mathcal{F}) = dD \cdot \det(\mathcal{F})$ and $c_2(\beta^*\mathcal{F}) = d\beta^* c_2(\mathcal{F})$. □

[2] Arguing as in [9, Lemma 10].

We now come to (1.2).

Theorem 2.6. *Let $\mathcal{X}$ be a non-singular Deligne-Mumfors stack with the projective coarse space X of general type and $c_1(\mathcal{X})$ nef. Then $c_1^2(\mathcal{X}) \le 3c_2(\mathcal{X})$ holds.*

Proof. As in [9], we consider two cases: (1) $c_1^2(\mathcal{X}) \le 2c_2(\mathcal{X})$ and (2) $c_1^2(\mathcal{X}) > 2c_2(\mathcal{X})$. The first case is obvious. For the second case, set $\alpha := c_2(\mathcal{X})/c_1^2(\mathcal{X})$. Note that $\alpha < 1/2$. Pick $\delta > 0$ sufficiently small and rational. By Theorem 2.4 applied to $\mathcal{D} = m(\alpha+\delta)K_{\mathcal{X}}$, $\mathcal{F} = \Omega^1_{\mathcal{X}}$, we can find a positive integer m such that $m(\alpha+\delta) \in \mathbb{Z}$, and

$$h^0(\mathcal{X}, S^m\Omega^1_{\mathcal{X}} \otimes \mathcal{O}_{\mathcal{X}}(-m(\alpha+\delta)K_{\mathcal{X}})) = 0.$$

By Serre duality for smooth projective Deligne-Mumford stacks [11, Theorem 2.22], we have

$$h^2(\mathcal{X}, S^m\Omega^1_{\mathcal{X}} \otimes \mathcal{O}_{\mathcal{X}}(-m(\alpha+\delta)K_{\mathcal{X}})) = h^0(\mathcal{X}, S^m\Omega^1_{\mathcal{X}} \otimes \mathcal{O}_{\mathcal{X}}(-m(1-\alpha-\delta)K_{\mathcal{X}}) \otimes K_{\mathcal{X}}).$$

As $\alpha < 1/2$ and δ is small, we have $1-\alpha-\delta > \alpha$. We apply Theorem 2.4 to $\mathcal{D} = m(2-\alpha-\delta)K_{\mathcal{X}}$, $\mathcal{F} = \Omega^1_{\mathcal{X}}$, to get

$$h^2(\mathcal{X}, S^m\Omega^1_{\mathcal{X}} \otimes \mathcal{O}_{\mathcal{X}}(-m(\alpha+\delta)K_{\mathcal{X}})) = 0.$$

Hence

$$\chi(\mathcal{X}, S^m\Omega_{\mathcal{X}} \otimes \mathcal{O}(-m(\alpha+\delta)K_{\mathcal{X}})) = -h^1(\mathcal{X}, S^m\Omega^1_{\mathcal{X}} \otimes \mathcal{O}_{\mathcal{X}}(-m(\alpha+\delta)K_{\mathcal{X}})) \le 0.$$

Note that to compute the cohomology groups of a (subsheaf of) symmetric power of a vector bundle, one can work on the the projectivized vector bundle and computing the cohomology groups of relevant line bundles. Thus

$$0 \ge \chi(\mathcal{X}, S^m\Omega_{\mathcal{X}} \otimes \mathcal{O}(-m(\alpha+\delta)K_{\mathcal{X}})) = \chi(\mathcal{V}, \mathcal{O}_{\mathcal{V}}(-m(\mathcal{H}-(\alpha+\delta)\pi^*K_{\mathcal{X}}))).$$

By Riemann-Roch for stacks [12], we have $\chi(\mathcal{V}, \mathcal{O}_{\mathcal{V}}(-m(\mathcal{H}-(\alpha+\delta)\pi^*K_{\mathcal{X}})))$ grows like $\frac{1}{6}(\mathcal{H}-(\alpha+\delta)\pi^*K_{\mathcal{X}})^3m^3$ as $m \to \infty$. It implies that $(\mathcal{H}-(\alpha+\delta)\pi^*K_{\mathcal{X}})^3 \le 0$. Taking δ to 0, we obtain

$$\begin{aligned} 0 \ge (\mathcal{H}-\alpha\pi^*K_{\mathcal{X}})^3 &= c_1^2(\mathcal{X}) - c_2(\mathcal{X}) - 3\alpha c_1^2(\mathcal{X}) + 3\alpha^2 c_1^2(\mathcal{X}) \\ &= (1-\alpha-3\alpha+3\alpha^2)c_1^2(\mathcal{X}) \\ &= (1-\alpha)(1-3\alpha)c_1^2(\mathcal{X}). \end{aligned}$$

Since $\alpha < 1/2$ and $c_1^2(\mathcal{X})$ is non-negative, we get $1-3\alpha \le 0$ as desired. ☐

3. Examples of (1.2)

3.1. General discussion

According to the main result of [4], the map $\mathcal{X} \to X$ from the stack $\mathcal{X}$ to its coarse moduli space X (which we assume to be a variety with quotient singularities) can be factored as

$$\mathcal{X} \to \mathcal{X}_1 \to \mathcal{X}_2 \to X,$$

where

(1) $\mathcal{X}_1$ has trivial generic stabilizers;

(2) $\mathcal{X}_2$ is the canonical stack associated to the variety X (see e.g., [3, Definition 4.4]) and has stack structures in codimension at least 2;

(3) $\mathcal{X} \to \mathcal{X}_1$ is a *gerbe*;

(4) $\mathcal{X}_1 \to \mathcal{X}_2$ is a composition of *root constructions* along divisors, thus introducing codimension-1 stack structures to $\mathcal{X}_2$.

Since $\mathcal{X} \to \mathcal{X}_1$ is a gerbe, the tangent bundle of $\mathcal{X}_1$ pulls back to the tangent bundle of $\mathcal{X}$. So the inequality (1.2) for $\mathcal{X}$ is equivalent to (1.2) for $\mathcal{X}_1$. Therefore when considering examples, we may restrict our attention to $\mathcal{X}$ whose stack structures are in codimension at least 1. In the rest of this section we present two examples of (1.2): the example in Section 3.2 is obtained by root constructions, and the examples in Section 3.3 are canonical stacks associated to quotient varieties. In these examples we show that (1.2) coincides with previous results.

3.2. Codimension 1 stack structure

We consider (1.2) for an example of stack $\mathcal{X}$ with stack structures in codimension 1.

Let X be a smooth complex projective surface and D a simple normal crossing $\mathbb{Q}$-divisor of the form $D = \sum_i (1 - 1/r_i) D_i$ with $r_i \geq 2$ integers. Let $\mathcal{X}$ be the natural stack cover of the pair (X, D). By construction the coarse moduli space of $\mathcal{X}$ is X. The natural map $\pi\colon \mathcal{X} \to X$ is an isomorphism outside $\pi^{-1}(\operatorname{Supp} D)$, which is where $\mathcal{X}$ has non-trivial stack structures. The stack $\mathcal{X}$ can be constructed from X by applying root constructions along components of D. Furthermore we have the following formula for the canonical bundle:

$$K_{\mathcal{X}} = \pi^*(K_X + D). \tag{3.1}$$

We now examine (1.2) for this $\mathcal{X}$. By (3.1),

$$c_1(T_{\mathcal{X}})^2 = c_1(K_{\mathcal{X}})^2 = (K_X + D)^2.$$

By Gauss-Bonnet theorem for Deligne-Mumford stacks [13, Corollaire 3.44] we have

$$c_2(T_{\mathcal{X}}) = \chi(\mathcal{X}),$$

the Euler characteristic of $\mathcal{X}$ as defined in [13, Definition 3.43] (note that the notation χ^{orb} is used in [13]). Put

$$\mathcal{D}_i := \pi^{-1}(D_i), \quad \mathcal{D}_i^\circ := \mathcal{D}_i \setminus \left(\bigcup_{j \neq i} (\mathcal{D}_i \cap \mathcal{D}_j) \right).$$

Then we have

$$\chi(\mathcal{X} \setminus \pi^{-1}(\operatorname{Supp} D)) = \chi(\mathcal{X}) - \sum_i \chi(\mathcal{D}_i^\circ) - \sum_{p \in \mathcal{D}_i \cap \mathcal{D}_j} \chi(p).$$

Similarly, put $D_i^\circ = D_i \setminus \left(\bigcup_{j \neq i} (D_i \cap D_j) \right)$, we have

$$\chi(X \setminus \operatorname{Supp} D) = \chi(X) - \sum_i \chi(D_i^\circ) - \sum_{\overline{p} \in D_i \cap D_j} \chi(\overline{p}).$$

Since $\mathcal{X} \setminus \pi^{-1}(\operatorname{Supp} D) \simeq X \setminus \operatorname{Supp} D$, we have $\chi(\mathcal{X} \setminus \pi^{-1}(\operatorname{Supp} D)) = \chi(X \setminus \operatorname{Supp} D)$. Equivalently,

$$\chi(\mathcal{X}) = \chi(X) - \sum_i \chi(D_i^\circ) - \sum_{\overline{p} \in D_i \cap D_j} \chi(\overline{p}) + \sum_i \chi(\mathcal{D}_i^\circ) + \sum_{p \in \mathcal{D}_i \cap \mathcal{D}_j} \chi(p).$$

Since the map $\mathcal{D}_i^\circ \to D_i^\circ$ is of degree $1/r_i$ and the map $\mathcal{D}_i \cap \mathcal{D}_j \to D_i \cap D_j$ is of degree $1/(r_i r_j)$, we have

$$\chi(\mathcal{D}_i) = \frac{1}{r_i} \chi(D_i), \quad \chi(\mathcal{D}_i \cap \mathcal{D}_j) = \frac{1}{r_i r_j} \chi(D_i \cap D_j).$$

This implies that

$$\chi(\mathcal{X}) = \chi(X) - \sum_i \left(1 - \frac{1}{r_i}\right) \chi(D_i^\circ) + \sum_{\overline{p} \in D_i \cap D_j} \left(\frac{1}{r_i r_j} - 1 \right). \tag{3.2}$$

By [7, Theorem 8.7], for $\overline{p} \in D_i \cap D_j$ the local orbifold Euler number of the pair (X, D) at $\overline{p}$ is given by $e_{\mathrm{orb}}(\overline{p}; X, D) = 1/(r_i r_j)$. Together with (3.2) this implies that $\chi(\mathcal{X})$ coincides with the orbifold Euler number $e_{\mathrm{orb}}(X, D)$ of the pair (X, D), as defined in [7]. Thus if $K_{\mathcal{X}}$ is numerically effective, then (1.2) is equivalent to [7, Theorem 0.1] applied to the pair (X, D).

3.3. Codimension 2 stack structure

Let $\mathcal{X}$ be a smooth proper Deligne-Mumford $\mathbb{C}$-stack of dimension 2 with isolated stack structures. Let $\pi\colon \mathcal{X} \to X$ be the natural map to the coarse moduli space X. Let $p_1, p_2, \dots, p_k \in \mathcal{X}$ be the stacky points. Suppose that $\mathcal{X}$ is Gorenstein, i.e., each stacky point p_i has a neighborhood $p_i \in U_i \subset \mathcal{X}$ of the form $U_i \simeq [\mathbb{C}^2/G_i]$ with $G_i \subset \mathrm{SU}(2)$ a finite subgroup, identifying p_i with $[0/G_i] \in [\mathbb{C}^2/G_i]$. It is a standard fact that the coarse moduli space X is a projective surface with canonical singularities.

Suppose further that $K_{\mathcal{X}}$ is numerically effective. We consider (1.2) for such $\mathcal{X}$.

By assumption we have $K_{\mathcal{X}} = \pi^* K_X$. Thus

$$c_1(T_{\mathcal{X}})^2 = c_1(K_{\mathcal{X}})^2 = c_1(K_X)^2.$$

We now consider the term $c_2(T_{\mathcal{X}})$. The first step is to consider $\chi(\mathcal{O}_{\mathcal{X}})$ by using Riemann-Roch theorem for stacks [12, 13]. We follow [14, Appendix A] for the presentation of the Riemann-Roch theorem. We have

$$\chi(\mathcal{O}_{\mathcal{X}}) = \int_{I\mathcal{X}} \widetilde{\mathrm{ch}}(\mathcal{O}_{\mathcal{X}}) \widetilde{\mathrm{Td}}(T_{\mathcal{X}}).$$

Here $I\mathcal{X}$ is the inertia stack of $\mathcal{X}$. By our assumption on $\mathcal{X}$, we have the following description of $I\mathcal{X}$:

$$I\mathcal{X} = \mathcal{X} \cup \bigcup_{i=1}^{k} (Ip_i \setminus p_i).$$

Here the term $Ip_i \setminus p_i$ is the inertia stack of $p_i \simeq BG_i$ with the main component removed, namely

$$Ip_i \setminus p_i \simeq \bigcup_{(g)\neq(1)\text{: conjugacy class of } G_i} BC_{G_i}(g).$$

By the definition of the Chern character $\widetilde{\mathrm{ch}}$, we have $\widetilde{\mathrm{ch}}(\mathcal{O}_{\mathcal{X}}) = 1$ on every component of $I\mathcal{X}$. Hence

$$\chi(\mathcal{O}_{\mathcal{X}}) = \int_{I\mathcal{X}} \widetilde{\mathrm{Td}}(T_{\mathcal{X}}) = \int_{\mathcal{X}} \widetilde{\mathrm{Td}}(T_{\mathcal{X}})\big|_{\mathcal{X}} + \sum_{i=1}^{k} \int_{Ip_i\setminus p_i} \widetilde{\mathrm{Td}}(T_{\mathcal{X}})\big|_{Ip_i\setminus p_i}. \tag{3.3}$$

Note that $\widetilde{\mathrm{Td}}(T_{\mathcal{X}})\big|_{\mathcal{X}} = \mathrm{Td}(T_{\mathcal{X}})$, and we only need its degree 2 component. Hence

$$\int_{\mathcal{X}} \widetilde{\mathrm{Td}}(T_{\mathcal{X}})\big|_{\mathcal{X}} = \frac{1}{12} \int_{\mathcal{X}} (c_2(T_{\mathcal{X}}) + c_1(T_{\mathcal{X}})^2). \tag{3.4}$$

The contribution coming from $Ip_i \setminus p_i$ can be also evaluated.

Lemma 3.1. *Let E_i be the exceptional divisor of the minimal resolution of $\mathbb{C}^2/G_i$. Then*

$$\int_{Ip_i\setminus p_i} \widetilde{\mathrm{Td}}(T_{\mathcal{X}})\big|_{Ip_i\setminus p_i} = \frac{1}{12}\left(\chi(E_i) - \frac{1}{|G_i|}\right).$$

An elementary proof of this lemma is given in the next section. Next, we reinterpret the term $\chi(\mathcal{O}_{\mathcal{X}})$. By definition, $\chi(\mathcal{O}_{\mathcal{X}}) := \sum_{l\geq 0}(-1)^l \dim H^l(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Since $\pi_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$ (see e.g., [1, Theorem 2.2.1]), we have $H^l(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = H^l(X, \mathcal{O}_X)$ and

$$\chi(\mathcal{O}_{\mathcal{X}}) = \chi(\mathcal{O}_X). \tag{3.5}$$

Combining (3.3), (3.4), (3.5), and Lemma 3.1, we obtain the following expression of $c_2(T_{\mathcal{X}})$:

$$\int_{\mathcal{X}} c_2(T_{\mathcal{X}}) = 12\chi(\mathcal{O}_X) - \int_{\mathcal{X}} c_1(T_{\mathcal{X}})^2 - \sum_{i=1}^{k}\left(\chi(E_i) - \frac{1}{|G_i|}\right).$$

Using this, we see that in the present situation, (1.2) is equivalent to

$$12\chi(\mathcal{O}_X) \geq \frac{4}{3}c_1(K_X)^2 + \sum_{i=1}^{k}\left(\chi(E_i) - \frac{1}{|G_i|}\right). \tag{3.6}$$

On the other hand, it is clear that (3.6) is a special case of [10, Corollary 1.3].

4. Proof of Lemma 3.1

In this section we prove Lemma 3.1. By our assumption on $\mathcal{X}$, for $g \in G_i$, the g-action on the tangent space $T_{p_i}\mathcal{X}$ has two eigenvalues ξ_g and ξ_g^{-1}, where ξ_g is a certain root of unity. By the definition of $\widetilde{\mathrm{Td}}(T_{\mathcal{X}})$ we have

$$\int_{Ip_i\setminus p_i} \widetilde{\mathrm{Td}}(T_{\mathcal{X}})\big|_{Ip_i\setminus p_i} = \sum_{(g)\neq(1):\text{ conjugacy class of } G_i} \frac{1}{|C_{G_i}(g)|}\frac{1}{2-\xi_g-\xi_g^{-1}}. \tag{4.1}$$

We now evaluate (4.1) using the ADE classification of $\mathbb{C}^2/G_i$.

4.1. Type A

If $\mathbb{C}^2/G_i$ is of type A_{n-1}, then $G_i \simeq \mathbb{Z}_n$ and the action on $\mathbb{C}^2$ is given as follows. If we identify $\mathbb{Z}_n$ with the group of n-th roots of 1, then an element $\xi \in \mathbb{Z}_n$ acts on $\mathbb{C}^2$ via the matrix

$$\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}.$$

It follows that (4.1) is given by

$$\frac{1}{n}\sum_{l=1}^{n-1}\frac{1}{2-\exp(2\pi\sqrt{-1}l/n)-\exp(2\pi\sqrt{-1}l/n)^{-1}}. \tag{4.2}$$

By [8, Lemma 3.3.2.1], (4.2) is equal to

$$\frac{n^2-1}{12n} = \frac{1}{12}\left(n - \frac{1}{n}\right).$$

Since the exceptional divisor of the minimal resolution of $\mathbb{C}^2/\mathbb{Z}_n$ is a chain of $(n-1)$ copies of $\mathbb{CP}^1$, its Euler characteristic is n. This proves Lemma 3.1 in type A case.

4.2. Type D

If $\mathbb{C}^2/G_i$ is of type D_{n+2} (here $n \geq 2$), then G_i is isomorphic to the binary dihedral group Dic_n. The group Dic_n is of order $4n$ and may be presented as follows:

$$\mathrm{Dic}_n = \langle a, x \mid a^{2n} = 1, x^2 = a^n, x^{-1}ax = a^{-1} \rangle.$$

The action of Dic_n on $\mathbb{C}^2$ is given as follows:

$$a \mapsto \begin{pmatrix} \exp(\pi\sqrt{-1}/n) & 0 \\ 0 & \exp(-\pi\sqrt{-1}/n) \end{pmatrix}, \quad x \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.3}$$

An elementary calculation shows that the conjugacy classes of Dic_n and the orders of their centralizer subgroups are given as follows:

(4.4)

$\{1\}, \quad \{a^n\},$ (order of centralizer group $= 4n$)

$\{a^l, a^{-l}\}, \quad 1 \leq l \leq n-1,$ (order of centralizer group $= 2n$)

$\{xa, xa^3, xa^5, \ldots, xa^{2n-1}\}, \ \{x, xa^2, xa^4, \ldots, xa^{2n-2}\},$ (order of centralizer group $= 4$).

Using (4.3) and (4.4) it is easy to identify the contribution from each conjugacy class. It follows that (4.1) is given by

$$\frac{1}{2n}\sum_{k=1}^{n-1} \frac{1}{2 - \exp(\pi\sqrt{-1}k/n) - \exp(\pi\sqrt{-1}k/n)^{-1}} + \frac{1}{16n} + \frac{1}{8} + \frac{1}{8}.$$

We need to evaluate the first sum above. Again by [8, Lemma 3.3.2.1], we have

$$\begin{aligned}
\frac{(2n)^2-1}{12} &= \sum_{k=1}^{2n-1} \frac{1}{2 - \exp(2\pi\sqrt{-1}k/(2n)) - \exp(2\pi\sqrt{-1}k/(2n))^{-1}} \\
&= \sum_{k=1}^{n-1} \frac{1}{2 - \exp(\pi\sqrt{-1}k/n) - \exp(\pi\sqrt{-1}k/n)^{-1}} + \frac{1}{4} \\
&\quad + \sum_{k=1}^{n-1} \frac{1}{2 - \exp(2\pi\sqrt{-1}(n+k)/(2n)) - \exp(2\pi\sqrt{-1}(n+k)/(2n))^{-1}}.
\end{aligned}$$

Note that

$$\begin{aligned}
&2 - \exp(2\pi\sqrt{-1}(n+k)/(2n)) - \exp(2\pi\sqrt{-1}(n+k)/(2n))^{-1} \\
&= 2 + \exp(\pi\sqrt{-1}k/n) + \exp(\pi\sqrt{-1}k/n)^{-1} \\
&= 2 + 2\cos(\pi k/n) = 4\cos^2(\pi k/(2n)) = 4\sin^2(\pi(k+n)/(2n)),
\end{aligned}$$

$$2 - \exp(\pi\sqrt{-1}k/n) - \exp(\pi\sqrt{-1}k/n)^{-1} = 2 - 2\cos(\pi k/n) = 4\sin^2(\pi k/(2n)).$$

Since $\sin(\pi(k+n)/(2n)) = -\sin(\pi(k-n)/(2n))$, we see that

$$\sum_{k=1}^{n-1} \frac{1}{2-\exp(\pi\sqrt{-1}k/n)-\exp(\pi\sqrt{-1}k/n)^{-1}}$$
$$=\sum_{k=1}^{n-1} \frac{1}{2-\exp(2\pi\sqrt{-1}(n+k)/(2n))-\exp(2\pi\sqrt{-1}(n+k)/(2n))^{-1}},$$

from which it follows that

$$2\sum_{k=1}^{n-1} \frac{1}{2-\exp(\pi\sqrt{-1}k/n)-\exp(\pi\sqrt{-1}k/n)^{-1}}+\frac{1}{4}=\frac{(2n)^2-1}{12}.$$

This shows that

$$\sum_{k=1}^{n-1} \frac{1}{2-\exp(\pi\sqrt{-1}k/n)-\exp(\pi\sqrt{-1}k/n)^{-1}}=\frac{n^2-1}{6}$$

and (4.1) is given by

$$\frac{n^2-1}{12n}+\frac{1}{16n}+\frac{1}{8}+\frac{1}{8}=\frac{1}{12}\left(n+3-\frac{1}{4n}\right).$$

Since the exceptional divisor of the minimal resolution of $\mathbb{C}^2/\operatorname{Dic}_n$ is a tree of $\mathbb{CP}^1$ whose dual graph is the Dynkin diagram D_{n+2}, its Euler characteristic is $n+3$ and Lemma 3.1 is proved in this case.

4.3. Type E

If $\mathbb{C}^2/G_i$ is of type E, then there are three possibilities: E_6, E_7, E_8. The group G_i is isomorphic to the binary tetrahedral group (for E_6), the binary octahedral group (for E_7), or the binary icosahedral group (for E_8). In each case the group and its action on $\mathbb{C}^2$ can be explicitly described, and Lemma 3.1 can be proved by computing (4.1) using this information. We work out the details for E_6 and leave the other two cases to the reader.

In the E_6 case, the group G_i is isomorphic to the binary tetrahedral group $2T$. This group is of order 24 and its elements can be identified with the following quaternion numbers:

$$\frac{1}{2}(\pm 1 \pm i \pm j \pm k), \quad \pm i, \quad \pm j, \quad \pm k, \quad \pm 1.$$

The group $2T$ has 7 conjugacy classes:

Conjugacy class	(1)	(-1)	(i)	$(\frac{1}{2}(1+i+j+k))$
Size	1	1	6	4
Conjugacy class	$(\frac{1}{2}(1+i+j-k))$	$(\frac{1}{2}(-1+i+j+k))$	$(\frac{1}{2}(-1+i+j-k))$	
Size	4	4	4	

The action of $2T$ on $\mathbb{C}^2$ can be described using the following identification

$$x+yi+zj+wk \mapsto \begin{pmatrix} x+yi & z+wi \\ -z+wi & x-yi \end{pmatrix}.$$

Now it is straightforward to see that (4.1) is given by

$$\begin{aligned}&\frac{1}{24}\frac{1}{2-(-2)}+\frac{1}{4}\frac{1}{2-0}+\frac{1}{6}\frac{1}{2-1}+\frac{1}{6}\frac{1}{2-1}+\frac{1}{6}\frac{1}{2-(-1)}+\frac{1}{6}\frac{1}{2-(-1)}\\ &=\frac{167}{288}=\frac{1}{12}\left(7-\frac{1}{24}\right).\end{aligned}$$

Since 7 is the Euler characteristic of the exceptional divisor of the minimal resolution of $\mathbb{C}^2/2T$, the result follows.

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