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A Note on Special Fibers of Shimura Curves and Special Representations

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Abstract. We study the geometry of the special fibers of certain Shimura curves and give a direct proof of global-to-local Jacquet-Langlands compatibility by Čerednik-Drinfel'd uniformizations theorem.

1. Introduction

Let F be a number field with absolute Galois group G_F, and let F_v be the completion at some place v with Weil group W_{F_v}. The Langlands program for GL_2 predicts the existence of a diagram of the following form:

$$\begin{array}{ccc}
\text{“}\mathrm{Rep}^2(G_F)\text{”} & \xleftrightarrow{\text{Global Langlands}} & \text{“}\{\text{Automorphic repres of } \mathrm{GL}_2(\mathbb{A}_F)\}\text{”} \\
\Big\downarrow{\scriptstyle \mathrm{Res}^{G_F}_{W_{F_v}}} & & \Big\downarrow{\scriptstyle \otimes_w \pi_w \mapsto \pi_v} \\
\mathrm{Rep}^2(W_{F_v})^{F\text{-ss}} & \xleftrightarrow{\text{Local Langlands}} & \{\text{Irreducible admissible repres of } \mathrm{GL}_2(F_v)\}
\end{array}$$

where $\mathrm{Rep}^2(W_{F_v})^{F\text{-ss}}$ is the collection of two dimensional Frobenius semi-simple continuous representations of W_{F_v} over $\overline{\mathbb{Q}}_\ell$ ($v \nmid \ell$), and $\mathrm{Rep}^2(G_F)$ is the collection of two dimensional continuous G_F-representations over $\overline{\mathbb{Q}}_\ell$. The local Langlands correspondence is often stated in terms of Weil-Deligne representations; however, Grothendieck's ℓ-adic monodromy theorem says that these are the same as Frobenius semi-simple continuous W_{F_v}-representations.

The first row of the above diagram, that is, the global Langlands correspondence, is still conjectural even for 2-dimensional representations with $F = \mathbb{Q}$. For example, one knows that most of the time there is no corresponding Galois representation for Maass forms. There are certain cases in which the upper arrow is defined, e.g., for automorphic or Galois representations corresponding to cusp forms. In this case, whether or not the local and global Langlands correspondences could be normalized so that the above diagram commutes? The existence and description of such a normalization is the *local-global compatibility*.

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One of the first monumental examples of Langlands program is the Jacquet-Langlands correspondence which is a correspondence between automorphic forms on GL_2 and its twisted forms [19, §16]. Roughly speaking, two groups sharing many conjugacy classes should share a large part of their representation theory. For the moment, let F be a local field, and B a quaternion algebra over F; let $\mathrm{Tr}_{B/F}$ and $\mathrm{Nm}_{B/F}$ be the reduced trace and norm on B respectively. Let $\omega\colon F^\times \to \mathbb{C}^\times$ be a smooth character.

Theorem 1.1 (Local Jacquet-Langlands Correspondence)**.** *There is a unique bijection*

$$\begin{aligned} &\{\textit{irreducible smooth repres } \pi' \textit{ of } B^\times \textit{ with central character } \omega\} \\ \longleftrightarrow\quad &\{\textit{irreducible discrete series repres } \pi \textit{ of } \mathrm{GL}_2(F) \textit{ with central character } \omega\} \end{aligned}$$

such that for $\pi' \leftrightarrow \pi$ the characters $\Theta_{\pi'}$ and Θ_π of π' and π satisfy the relation

$$\Theta_{\pi'}(t') = -\Theta_\pi(t),$$

each time t' and t are regular semi-simple elements of $B^\times$ and $\mathrm{GL}_2(F)$ related by the identities

$$\mathrm{Tr}_{B/F}(t') = \mathrm{Tr}(t), \quad \mathrm{Nm}_{B/F}(t') = \det(t).$$

Since F is a non-Archimedean local field, the discrete series representations of $\mathrm{GL}_2(F)$ are special (or twisted Steinberg) representations and supercuspidal representations. The trivial representation of $B^\times$ corresponds to the Steinberg representation of $\mathrm{GL}_2(F)$. More generally, each one dimensional character of $B^\times$ corresponds to special (or twisted Steinberg) representation of $\mathrm{GL}_2(F)$.

The correspondence between division algebras of dimension n^2 and GL_n was proved by Jacquet and Langlands in both the local and global settings in [19], hence the name. Rogawski [25] extended the local Jacquet-Langlands correspondence to division algebras of higher dimension in characteristic 0. Deligne, Kazhdan and Vignéras in [10] carried out the case of a general inner form of $\mathrm{GL}_n(F)$ in characteristic 0, and Badulescu [1] in characteristic p. Each of these cases was accomplished by embedding the local problem into a global one and then applying Selberg trace formula methods.

Now let F be a number field and B be a quaternion algebra over F ramified exactly at places S. Let $\omega\colon F^\times \setminus \mathbb{A}_F^\times \to \mathbb{C}$ be a smooth character.

Theorem 1.2 (Global Jacquet-Langlands Correspondence)**.** *There is a unique injection*

$$\begin{aligned} &\{\textit{irreducible automorphic repres } \pi' \textit{ of } \mathbb{A}_B^\times \textit{ of } \dim > 1 \textit{ with central character } \omega\} \\ \hookrightarrow\quad &\{\textit{irreducible cuspidal automorphic repres } \pi \textit{ of } \mathrm{GL}_2(\mathbb{A}_F) \textit{ with central character } \omega\} \end{aligned}$$

such that for $\pi' \leftrightarrow \pi$ if and only if $\pi'_v \simeq \pi_v$ for all $v \notin S$, and $\pi'_v \leftrightarrow \pi_v$ for all $v \in S$ in the sense of local Jacquet-Langlands correspondence. The image of this injection consists of those cuspidal automorphic π of $\mathrm{GL}_2(\mathbb{A}_F)$ with π_v in the discrete series for all $v \in S$.

In other words, global Jacquet-Langlands correspondence defines a bijection between the cuspidal automorphic representations of B and the set of cuspidal automorphic representations π of GL_2 such that π_v is in the discrete series for all $v \in S$. The statement about the local components of π at the end of the above theorem is exactly what is meant by "*local-global compatibility*". The proof of the global Jacquet-Langlands correspondence is now considered the simplest non-trivial application of the Selberg trace formula to Langlands functoriality.

Let $N = p_1 \cdots p_n$ be a product of distinct primes with n odd. Let $B_{\mathbb{Q}}$ be the definite quaternion algebra ramified at primes dividing N. Global Jacquet-Langlands correspondence gives a bijection between normalized cuspidal Hecke eigen newforms of weight 2 and of level $\Gamma_0(N)$, and automorphic representations π' of $\mathbb{A}_B^\times$ of dimension > 1, with the fixed central character (the norm map $|\cdot|\colon \mathbb{A}_B^\times \to \mathbb{R}_{>0}$) such that π'_∞ is trivial on $\mathrm{SU}(2)$ and π'_p is trivial when restricted to $\mathcal{O}_{B_p}^\times$ at $p \mid N$, and unramified elsewhere. Let $\mathcal{O}_{B_p}$ be a maximal order in B_p (unique if $p \mid N$), and let $h(B)$ be the class number of B (the number of maximal orders in B up to left multiplication by $B^\times$). Counting numbers of newforms for GL_2 and all forms for B, we obtain

$$\dim S_2^{\mathrm{new}}(\Gamma_0(N)) = g(X_0(N))^{\mathrm{new}} = \#\left(B^\times A_{\mathbb{Q}}^\times \setminus A_{B,f}^\times / \prod_p \mathcal{O}_{B_p}^\times\right) - 1 = h(D) - 1.$$

We subtract by one in the end of the right-hand side in order to exclude the unique 1-dimensional representation with given central character $|\cdot|$. For indefinite quaternion algebra $B_{\mathbb{Q}}$ ramified at primes dividing $N = p_1 \cdots p_n$ with n even, we can similarly get a relation between the new part of the genus of $X_0(N)$ and the genus of a certain Shimura curve.

The aim of this paper is to give a direct proof of global-to-local Jacquet-Langlands compatibility (cf. Theorem 7.2). To this end, we need to study the cohomology of certain Shimura curves using the spectral sequence of vanishing cycles. In §2 and §3, we will be recalling constructions of Shimura curves and of Čerednik-Drinfel'd uniformizations. The uniformization theorem of Čerednik-Drinfel'd will be used to study the generic fiber of Shimura curves in question (cf. Theorem 3.2). Moreover, the cohomology of Shimura curves in question can be interpreted as the cohomology of the dual graph associated with Shimura curves (cf. Corollary 3.3). This dual graph can be uniformized by a Schottky group which is associated with the open compact subgroup defining the Shimura curve. Therefore, we will establish a comparison isomorphism between the group cohomology of Schottky groups and the equivariant cohomology of dual graphs in §4 (cf. Proposition 4.2). After recollecting the theory of vanishing cycles in §6, the cohomology of Shimura curves in question can be determined by the exact sequence of specialization (cf. Proposition 6.3) and harmonic cocycles (cf. Lemma 7.1).

2. Shimura curves

In this and next sections, we will be recalling constructions of Shimura curves and of Čerednik-Drinfel'd uniformizations, and we then take up the study of the reduction modulo a prime of the Shimura curves. We rely on Boutot [2], Boutot-Carayol [3], Diamond-Taylor [11, Chapter 4], and Rapoport [22, Chapter 1] for the basic theory used in this section.

Fix a prime p. Let $D > 1$ be a product of an even number of primes, and let M be a square-free integer prime to D. Suppose that $N = MD$. We let B be an indefinite quaternion algebra over $\mathbb{Q}$ with discriminant D.

We fix a maximal order $\mathcal{O}_B$ of B, and an isomorphism $B \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathrm{M}_2(\mathbb{R})$. For any $\ell \nmid D$, we choose an isomorphism $\phi_\ell \colon B_\ell \xrightarrow{\sim} \mathrm{M}_2(\mathbb{Q}_\ell)$ such that $\phi_\ell(\mathcal{O}_{B_\ell}) = \mathrm{M}_2(\mathbb{Z}_\ell)$. Let $u_M \colon \widehat{\mathcal{O}}_B^\times \to \mathrm{GL}_2(\mathbb{Z}/M\mathbb{Z})$ induced by ϕ_ℓ. We let

- $\widehat{\Gamma}_0^D(M)$ denote the preimage of $\left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}/M\mathbb{Z}) \;\middle|\; c = 0 \right\}$ under u_M;
- $\widehat{\Gamma}_1^D(M)$ denote the preimage of $\left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}/M\mathbb{Z}) \;\middle|\; c = 0, d = 1 \right\}$ under u_M;
- $\widehat{\Gamma}^D(M)$ denote the kernel of u_M.

We associate with the quaternion algebra B an algebraic group $G = \underline{B}^\times$ defined over $\mathbb{Q}$ as follows:

$$\underline{B}^\times(R) := (B \otimes_{\mathbb{Q}} R)^\times \quad \text{for all } \mathbb{Q}\text{-algebras } R.$$

We may extend $\underline{B}^\times$ to a group scheme $G = \underline{\mathcal{O}}_B^\times$ defined over $\mathbb{Z}$ by

$$\underline{\mathcal{O}}_B^\times(R) := (\mathcal{O}_B \otimes_{\mathbb{Z}} R)^\times \quad \text{for all } \mathbb{Z}\text{-algebras } R.$$

Accordingly, G extends to a group scheme over $\mathbb{Z}$ by

$$G_{/\mathbb{Z}} = \underline{\mathcal{O}}_B^\times.$$

The center Z of G is a group scheme satisfying $Z(R) = (\mathbb{Z} \otimes_{\mathbb{Z}} R)^\times$.

Let $G(\mathbb{R})^+$ be the identity component of $G(\mathbb{R})$. The group $G(\mathbb{R})^+ = \mathrm{GL}_2(\mathbb{R})^+$ acts on the Poincaré upper half-plane $\mathcal{H}$ by the linear fractional transformation. Denote by C_{i} the stabilizer of $\mathrm{i} = \sqrt{-1}$ in $(B \otimes_{\mathbb{Q}} \mathbb{R})^\times$. Note that we have $\mathcal{H} \simeq G(\mathbb{R})^+/C_{\mathrm{i}}$. Let $R_{M,D}$ be the Eichler order of level M. We see that $\widehat{R}_{M,D}^\times = \widehat{\Gamma}_0^D(M)$. The *Shimura curve* $X^D(M)$ associated with these data is the following:

$$X^D(M) := X(\widehat{R}_{M,D})(\mathbb{C}) = G(\mathbb{Q}) \setminus G(\mathbb{A})/\mathbb{R}^\times \widehat{R}_{M,D}^\times \cdot C_{\mathrm{i}}.$$

By Shimura's theory (see Shimura [26, Chapter 8]), the curve $X^D(M)$ has a canonical model $X^D(M)$ defined over $\mathbb{Q}$: $X^D(M) \times_{\mathbb{Q}} \operatorname{Spec} \mathbb{C} = X^D(M)(\mathbb{C})$. Since $D > 1$, this model is already proper.

2.1. Moduli problems

Let S be a $\mathbb{Z}[1/D]$-scheme. A *false elliptic curve* over S is a pair (A, ι) where A is an abelian scheme of relative dimension two over S, and $\iota\colon \mathcal{O}_B \hookrightarrow \mathrm{End}_S(A)$ is an action of $\mathcal{O}_B$ on $A_{/S}$. Note that once we have fixed $u \in \mathcal{O}_B$ with $u^2 = -D$, any false elliptic curve admits a canonical principal polarization.

If (A_1, ι_1) and (A_2, ι_2) are false elliptic curves, then by an *isogeny* $\pi\colon A_1 \to A_2$ of false elliptic curves, we mean an isogeny $\pi\colon A_1 \to A_2$ over S in the usual sense, such that $\iota_2(x) \circ \pi = \pi \circ \iota_1(x)$ for all $x \in \mathcal{O}_B$. If $\pi\colon A_1 \to A_2$ is an isogeny of false elliptic curves, then π induces a dual isogeny $\pi^\vee\colon A_2^\vee \to A_1^\vee$ which, because we have defined principal polarisations on A_1 and A_2, induces a map $\pi^t\colon A_2 \to A_1$ which is also an isogeny of false elliptic curves, and we call it the *dual isogeny.* The composite $\pi^t \circ \pi\colon A_1 \to A_1$ is locally multiplication by an integer; if this integer is constant on S, we call it the *false degree* of π.

Let S be a $\mathbb{Z}[1/MD]$-scheme. A *(naive) full level M structure* on a false elliptic curve $(A/S, \iota)$ is an isomorphism

$$\alpha\colon (\mathcal{O}_B \otimes \mathbb{Z}/M\mathbb{Z})_S \xrightarrow{\simeq} A[M]$$

of schemes which preserves the left action of $\mathcal{O}_B$. Let $H \subseteq (\mathcal{O}_B \otimes \mathbb{Z}/M\mathbb{Z})^\times$ be a subgroup. Consider the contravariant functor $F_{A/S,M}$ from S-schemes to sets, sending T/S to the set of full level M-structures on A_T. Then $F_{A/S,M}$ is represented by a closed subscheme of $A[M]$, and this representing scheme is an étale $(\mathcal{O}_B \otimes \mathbb{Z}/M\mathbb{Z})^\times$-torsor. For each T, there is a left action of $(\mathcal{O}_B \otimes \mathbb{Z}/M\mathbb{Z})^\times$, and hence of H, on $F_{A/S,M}(T)$. If $H \setminus F_{A/S,M}(T)$ denotes the orbit space, then define $F_{A/S,H}$ to be the sheafification with respect to the étale topology of the functor $T \rightsquigarrow H \setminus F_{A/S,M}(T)$ from S-schemes to sets. Then $F_{A/S,H}$ is represented by a quotient of the S-scheme representing $F_{A/S,M}$ and is again a finite étale covering of S. If $\alpha \in F_{A/S,M}(S)$, we say that it is a *naive level H structure* on $(A/S, \iota)$.

2.2. Representability

Suppose that U is an open compact subgroup of $\widehat{\mathcal{O}}_B^\times$ satisfying the following three properties:

(i) $\det(U) = \widehat{\mathbb{Z}}^\times$.

(ii) U is maximal at primes dividing D; that is, $U = \prod_{p|D}(\mathcal{O}_B \otimes \mathbb{Z}_p)^\times \times U^D$, where U^D is the projection of U onto $\prod_{p\nmid D}(\mathcal{O}_B \otimes \mathbb{Z}_p)^\times$.

(iii) $U \subset \widehat{\Gamma}_1^D(l)$ for some $l \geq 4$ prime to D.

Remark 2.1. The assumption (i) implies that the complex curve attached to U is connected. The assumption (iii) is a smallness criterion which will be needed for representability results.

Let M_U denote the smallest positive prime-to-D integer such that $\widehat{\Gamma}^D(M_U) \subseteq U$, and let S be a $\mathbb{Z}[1/M_U D]$-scheme. We consider the following moduli problem $\mathcal{X}^D(U)$ which associates with $\mathbb{Z}[1/DM_U]$-scheme S the category of isomorphism classes of triples $(A, \iota, \nu)_{/S}$ such that

$$\begin{array}{rcl} \mathcal{X}^D(U)\colon \mathrm{Schemes}_{\mathbb{Z}[1/DM_U]} & \rightsquigarrow & \mathrm{Sets} \\ S & \rightsquigarrow & (A, \iota, \nu)_{/S}/\simeq \end{array}$$

(a) (A, ι) is a false elliptic curve over S.

(b) $\nu \in F_{A/S,U}(S)$ is a level U structure.

Theorem 2.2. *Let U be an open compact subgroup of $\widehat{\mathcal{O}}_B^\times$, and satisfy properties* (i)–(iii) *as stated earlier. Let M_U denote the smallest positive integer M prime to D such that $\widehat{\Gamma}^D(M) \subseteq U$. The moduli problem $\mathcal{X}^D(U)$ is representable by a scheme $X^D(U)$ which is flat of relative dimension one over* $\mathrm{Spec}\,\mathbb{Z}$. *The scheme $X^D(U)$ is smooth over* $\mathrm{Spec}\,\mathbb{Z}[1/DM_U]$. *If B is a division algebra (i.e., $D > 1$), then $X^D(U)$ is proper over* $\mathrm{Spec}\,\mathbb{Z}$.

Proof. For the representability of $X^D(U)$, the key point is to show the existence and the uniqueness of a polarization of (A, ι) of a certain type (cf. [3]). To construct $X^D(U)$, one imposes a level structure and use the relative representability over the principally polarized abelian varieties of dimension two with level M_U structure (also cf. [3]).

The smoothness of $X^D(U)$ over $\mathrm{Spec}\,\mathbb{Z}[1/DM_U]$ can be shown by using deformation theory and Serre-Tate theorem (cf. Katz [20]). If $D > 1$, using the semi-stable reduction theorem one can show the properness of $X^D(U)$ through the valuative criterion for properness. For the projectivity of $X^D(U)$ if $D > 1$, one can use the quasi-projectivity of the moduli space of principally polarized abelian varieties. □

Corollary 2.3. *Let U be an open compact subgroup of $\widehat{\mathcal{O}}_B^\times$. Suppose that U satisfies properties* (i)–(ii) *only. Then the moduli problem $\mathcal{X}^D(U)$ may not be representable but admits a coarse moduli scheme $X^D(U)$ which can be defined as the quotient of $X^D(\widetilde{U})$ by $U/\widetilde{U}$ for any normal subgroup $\widetilde{U} \subset U$ satisfying* (iii). *$X^D(U)$ is smooth over $\mathbb{Z}[1/DM_U]$.*

Suppose that r is a prime not dividing MD. We fix an isomorphism $\phi_r\colon \mathcal{O}_B \otimes \mathbb{Z}_r \simeq \mathrm{M}_2(\mathbb{Z}_r)$, and let e be the idempotent in $\mathcal{O}_B/r\mathcal{O}_B$ corresponding to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ by ϕ_r. Following Buzzard [7], we define a $\Gamma_0(r)$-structure (resp. $\Gamma_1(r)$-structure) on a false elliptic curve A

as a finite flat group scheme K_1 of rank r inside $(1-e)A[r]$ (resp. a Drinfel'd generator of this subgroup).

Let U be an open compact subgroup of $\widehat{\mathcal{O}}_B^\times$ of level M, and let $\Gamma_i = U \cap \widehat{\Gamma}_i^D(r)$ for $i = 0, 1$. The Shimura curve $X^D(\Gamma_0)$ is proper and semisteble over $\mathbb{Z}_r$. In fact, there are exactly two smooth irreducible components, X^m and X^e, in the special fiber; they can be described as the Zariski closure of the locus $X^{m,0}$ where K_1 is of multiplicative type, resp. of $X^{e,0}$ where K_1 is étale (cf. [29, §3.2]). The map $\pi\colon X^D(\Gamma_0) \to X^D(\Gamma)$ forgetting the Γ_0-structure is finite and flat (see [29, Proposition 3.3]). Moreover, one can use Tate-Oort theory to show the followings:

Proposition 2.4. (1) *The model $X^D(\Gamma_1)$ of $X^D(\Gamma_1)_{\mathbb{Q}_r}$ is regular and flat over $\mathbb{Z}_r$.*

(2) *The map $\pi_{10}\colon X^D(\Gamma_1) \to X^D(\Gamma_0)$ is finite flat; the special fiber of $X^D(\Gamma_1)$ is a divisor with normal crossings, with exactly two irreducible components $Y^e = \pi_{10}^{-1}(X^e)$ and $Y^m = \pi_{10}^{-1}(X^m)$ with multiplicity 1 and $r-1$ respectively, whose underlying reduced subschemes are smooth.*

(3) *The two components cross (transversally) at the supersingular points and nowhere else.*

Proof. Let us consider the finite group scheme $C = K_1^u$ of rank r over the $\mathbb{Z}_r$-scheme $X^D(\Gamma_0)$ and C^+ the complement of the zero section in C.

If $s \in X^{e,0}$, then C^+ is étale over $X^D(\Gamma_0)$ in a neighbourhood of s by [29, Proposition 3.3]. If $s \in X^{m,0}$, then C^+ is of multiplicative type, hence is isomorphic to μ_r on an étale neighbourhood of s. Hence, the statement about multiplicities is obvious, as the map π_{10} is an isomorphism. Let $s \in X^m \cap X^e$. By [29, Proposition 3.3] again, the completed local ring $\widehat{\mathcal{O}}_{X^D(\Gamma_0),s}$ at s is isomorphic to $R = \mathbb{Z}_r[\![u,v]\!]/(uv-r)$ in such a way that

(2.1) the completion at s of X^e (resp. X^m) has equation $v=0$ (resp. $u=0$).

Tate-Oort theory [28] classifies finite flat group schemes of rank r over any $\mathbb{Z}_r$-algebra; in particular, the pull-back C_R of C over $\operatorname{Spf} R$ is isomorphic to $G_R(x,y)$, the Tate-Oort group scheme of rank r over R for some parameters $x, y \in R$, where, in the notations of Tate-Oort, $a = \nu x$, $b = y$ and $w_r = \nu r = ab$, where $w_r \in \mathbb{Z}_r$ is an explicit Gauss sum. Note that $G_R(x,y) \simeq G_R(x',y')$ if and only if $x'x^{-1}$ is the $(r-1)^{\text{st}}$ power of a unit in R. As in [17, Corollary 3.3.5], we deduce from (2.1) that $x = \alpha u$ and $y = \alpha^{-1}v$ for some $\alpha \in R^\times$.

Therefore, $C_R \simeq G_R(u,v)$ if and only if the unit α is a $(r-1)^{\text{th}}$ power in R. Now the extension $R[(\alpha)^{1/(r-1)}]$ is finite étale over R. Since the problem is local in the étale topology, we may assume $C_R \simeq G_R(u,v)$.

Now we recall the Tate-Oort equations for $G_R(u,v)$ over R. Put $X_1 = \nu u$, $X_2 = v$, then $G_R(u,v) = \mathrm{Spf}(R[Y][\![X_1, X_2]\!]/(X_1X_2 - w_r, Y^r - X_1Y))$ (see [28, p. 13]).

Now the factorization $Y^r - X_1Y = Y(Y^{r-1} - X_1)$ provides an embedding of the algebra of formal functions on $G_R(u,v)$ into the ring $R_0 \times R^*$, where

$$R_0 = \mathbb{Z}_r[\![X_1, X_2]\!]/(X_1X_2 - w_r), \quad R^* = \mathbb{Z}_r[\![X_2, Y]\!]/(Y^{r-1}X_2 - w_r).$$

Dually, we have a surjective morphism

$$\mathrm{Spf}(R_0) \sqcup \mathrm{Spf}(R^*) \to G_R(u,v).$$

The image of $\mathrm{Spf}(R_0)$ corresponds to the zero section, while the image of $\mathrm{Spf}(R^*)$ is the scheme-theoretic closure of C^+. It follows that $\mathrm{Spf}(R^*)$ is a local model for C^+ over a neighbourhood of the singular point s. Obviously, $\mathrm{Spf}(R^*)$ has the properties required. Moreover $\mathrm{Spf}(R^*)$ is normal, and it follows that $\mathrm{Spf}(R^*)$ is a formal local model for $X^D(\Gamma_1)$. □

2.3. The moduli problem for primes dividing the discriminant

Fix a maximal order $\mathcal{O}_B$ of B which is stable under the main involution of B. Let p be a prime dividing D.

Definition 2.5. Let κ be an algebraic closed field of characteristic p. Let A be an abelian surface over κ, and $\iota\colon \mathcal{O}_B \to \mathrm{End}_\kappa(A)$ an $\mathcal{O}_B$-action on $A_{/\kappa}$. We say that the pair (A,ι) is *special* if the induced action of the Witt ring $W(\mathbb{F}_{p^2}) \subset \mathcal{O}_{B_p}$ on $\mathrm{Lie}(A)$ decomposes into two direct summands where $W(\mathbb{F}_{p^2})$ acts via the two embeddings

$$W(\mathbb{F}_{p^2}) \otimes_{\mathbb{Z}_p} \mathbb{F}_p \hookrightarrow \kappa.$$

In other words, the characteristic polynomial of $\iota(b)$ on $\mathrm{Lie}(A)$ can be written as follows:

$$\mathrm{char}(\iota(b)|\,\mathrm{Lie}(A))(T) = (T-b)\cdot(T-b').$$

Remark 2.6. (a) By the Honda-Tate's theorem, there exists only one isogeny class over $\overline{\mathbb{F}}_p$ of special pairs (A,ι). Let (A_0,ι_0) be a fixed member of this class. We have

$$\mathrm{End}_{\mathcal{O}_B}(A_0) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq B'$$

where B' is the quaternion $\mathbb{Q}$-algebra obtained by interchanging the local invariants p and ∞, i.e., $B_\ell \simeq B'_\ell$ for $\ell \notin \{p,\infty\}$ and $B_\ell \not\simeq B'_\ell$ for $\ell \in \{p,\infty\}$.

(b) For a scheme S flat over $\mathbb{Z}_p$, the action of $\mathcal{O}_B$ on an abelian surface A/S is always special over all geometric points of S over $\overline{\mathbb{F}}_p$.

Let $V = B$ and $W = \mathcal{O}_B$ the standard lattice of V stable by left multiplication by $\mathcal{O}_B$. We consider the following moduli problem over $\mathbb{Z}_p$. Let U be an open compact subgroup of $\widehat{\mathcal{O}}_B^\times$ satisfying the three conditions described in §2.2. Let M_U denote the smallest positive integer M prime to D such that $\widehat{\Gamma}(M) \subseteq U$. The moduli problem $\mathcal{S}_U$ associates with $\mathbb{Z}_p$-scheme S the set of isomorphism classes of triples $(A_{/S}, \iota, \nu)$ such that

$$\begin{array}{rcl} \mathcal{S}_U \colon \mathrm{Schemes}_{\mathbb{Z}_p} & \to & \mathrm{Sets} \\ S & \rightsquigarrow & (A_{/S}, \iota, \nu)/\sim \end{array}$$

(i) A is an abelian scheme of relative dimension two over S;

(ii) $\iota \colon \mathcal{O}_B \to \mathrm{End}_S(A)$ is an action of $\mathcal{O}_B$ which is *special* over each characteristic p geometric point $\overline{s}$ of S;

(iii) ν is a level U structure.

We then recall the following theorem in [3, Theorem 3.4, Chapter III]:

Theorem 2.7. *Let U be an open compact subgroup of $\widehat{\mathcal{O}}_B^\times$, and satisfying condition* (i)–(iii) *as in §1.2. Let M_U denote the smallest positive integer M prime to D such that $\widehat{\Gamma}^D(M) \subseteq U$. Then the moduli problem $\mathcal{S}_U$ is representable by a projective flat $\mathbb{Z}_p$-scheme S_U.*

As a consequence, the generic fiber of the $\mathbb{Z}_p$-scheme S_U is the base change to $\mathbb{Q}_p$ of the Shimura curve $X^D(U)$ of level U:

$$S_U \otimes \mathbb{Q}_p \simeq X^D(U) \otimes \mathbb{Q}_p.$$

3. Čerednik-Drinfel'd uniformization

Čerednik was the first who observed that the Shimura curves associated with the quaternion division algebra B admit a p-adic uniformization in the sense of Mumford's construction of degenerating curves over complete local rings. Drinfel'd clarified and improved the result of Čerednik by constructing a universal family of formal groups over the p-adic upper half-plane which corresponds to the formal group of the universal abelian scheme over the Shimura curves. Čerednik-Drinfel'd uniformization theorem will be used to study the generic fiber of Shimura curves. We refer to the original papers [8, 12, 13] of Čerednik and Drinfel'd, and to the paper of Boutot and Carayol [3] for details and proofs.

Let $M' = pM$ and $D' = D/p$. Let B' be the *definite* quaternion algebra ramified precisely at the primes dividing D', and the archimedean place. Let G' be the group scheme over $\mathbb{Z}$ associated with $B'^\times$. We fix an isomorphism $\phi_p \colon B' \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\simeq} \mathrm{M}_2(\mathbb{Q}_p)$.

There is a *unique* Eichler $\mathbb{Z}[1/p]$-order $R_{M,D'}$ of *level* M in B', up to conjugation by $B'^{\times}$. Let $r \geq 4$ be an integer prime to N, and put $\Gamma' = \widehat{R}^{\times}_{M,D'} \cap \widehat{\Gamma}^{D'}_1(r)$. The group Γ' is torsion-free. Write Γ'_+ to be the subgroup of Γ' consisting of elements whose p-adic valuation of its reduced norm is even. We see that Γ'_+ is a normal subgroup of Γ' of index two. Let $\Gamma'/\Gamma'_+ = \{1, W_p\}$. Hence W_p may be represented by an element of reduced norm p. We write Δ for the image by ϕ_p of $\Gamma'_+ \cap G'(\mathbb{Q})_+$. It is a Schottky group in the sense of [14,15].

Via the isomorphism ϕ_p, the group Δ acts properly discontinuously on p-adic upper half-plane $\Omega := \mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{P}^1(\mathbb{Q}_p)$ with compact quotient, and this quotient $\Delta \setminus \Omega$ is a rigid curve defined over $\mathbb{Q}_p$. We denote the curve $\Delta \setminus \Omega$ by X.

We write $\mathbb{A}^{p\infty}$ for the ring of finite adèles without p-component. From the definition of B', we have an anti-isomorphism

$$B^{\mathrm{opp}} \otimes_{\mathbb{Q}} \mathbb{A}^{p\infty} \simeq B' \otimes_{\mathbb{Q}} \mathbb{A}^{p\infty}. \tag{3.1}$$

We thus obtain a group isomorphism

$$B^{\times}(\mathbb{A}^{p\infty}) \simeq B'^{\times}(\mathbb{A}^{p\infty})$$

after composition of (3.1) by the inversion $g \mapsto g^{-1}$. We write $\widehat{\Gamma}(M) = U^0_p \cdot U^p_M$ ($= U$ for short) where U^0_p denotes the group of units in the maximal ideal $\mathcal{O}_{B_p}$ and U^p_M is an open compact subgroup of $B^{\times}(\mathbb{A}^p)$. We may consider U^p_M as a subgroup of $B'^{\times}(\mathbb{A}^p)$. Define

$$Z_U = U^p_M \setminus \widehat{B'}^{\times} / B'^{\times}.$$

The group $\mathrm{GL}_2(\mathbb{Q}_p)$ acts naturally on the left on the Bruhat-Tits tree $\mathcal{T}$, the rigid analytic space Ω^{rig}, the formal scheme $\widehat{\Omega}$, and on Z_U. Let $\mathbb{Q}^{\mathrm{nr}}_p$ be the maximal unramified extension of $\mathbb{Q}_p$ and $\widehat{\mathbb{Z}}^{\mathrm{nr}}_p$ the completion of the ring of integers in $\mathbb{Q}^{\mathrm{nr}}_p$. An element $g \in \mathrm{GL}_2(\mathbb{Q}_p)$ acts on $\mathbb{Q}^{\mathrm{nr}}_p$ and $\widehat{\mathbb{Z}}^{\mathrm{nr}}_p$ via $\widehat{\mathrm{Frob}}_p^{\,-v_p(\det g)}$ where $\widehat{\mathrm{Frob}}_p$ denotes the arithmetic Frobenius automorphism.

We have defined a scheme S_U representing a Kottwitz moduli problem $\mathcal{S}_U$ in §1.3. We denote by $\widehat{S}_U$ the completion of S_U along its special fiber and by S^{an}_U the rigid analytic space over $\mathbb{Q}_p$ associates with $\widehat{S}_U$.

Theorem 3.1 (Čerednik-Drinfel'd)**.** *There is a canonical isomorphism of formal schemes over* $\mathbb{Z}_p$

$$\mathrm{GL}_2(\mathbb{Q}_p) \setminus [(\widehat{\Omega} \widehat{\otimes}_{\mathbb{Z}_p} \widehat{\mathbb{Z}}^{\mathrm{nr}}_p) \times Z_U] \xrightarrow{\simeq} \widehat{S}_U \tag{3.2}$$

and a canonical isomorphism of rigid analytic spaces over $\mathbb{Q}_p$

$$\mathrm{GL}_2(\mathbb{Q}_p) \setminus [(\Omega \otimes_{\mathbb{Q}_p} \mathbb{Q}^{\mathrm{nr}}_p) \times Z_U] \xrightarrow{\simeq} \widehat{S}^{\mathrm{an}}_U.$$

Proof. The details of the proof could be found in [3, Chapter III, §6]. We only sketch it right now.

Let S be a $\mathbb{Z}_p$-scheme where the image of p is nilpotent, and Φ be a formal special $\mathcal{O}_{B_p}$-module over S. An *algebrization* of Φ is a pair (A, ϵ) where A is an abelian scheme over S with an $\mathcal{O}_B$-action, and $\epsilon\colon \widehat{A} \to \Phi$ is an $\mathcal{O}_B$-equivariant isomorphism from the formal group associated with A to Φ. If A is equipped with a level U structure, we call the pair (A, ϵ) an algebrization with level U-structure. Let $\mathrm{Alg}_U(\Phi)$ be the set of isomorphism classes of algebrizations with level U structure of Φ.

The group $\widehat{B}'^{\times} = B_p'^{\times} \times B'^{\times}(\mathbb{A}^{p\infty})$ acts on $\mathrm{Alg}_U(\Phi)$ from the left: $B_p'^{\times}$ acts by composition with ϵ and $B'(\mathbb{A}^{p\infty})^{\times}$ acts by composition with $\overline{\nu}$. Given a triple $(A_0, \epsilon_0, \overline{\nu}_0) \in \mathrm{Alg}_U(\Phi)$, the stabilizer of $(A_0, \epsilon_0, \overline{\nu}_0)$ is $B'^{\times}$. Hence, we can deduce a bijection

$$\mathrm{Alg}_U(\Phi) \xrightarrow{\sim} U_M^p \setminus \widehat{B'}^{\times} / B'^{\times} = Z_U.$$

Following Chapter II §8.4 and Chapter III §6.2 of [3], a section of $(\widehat{\Omega} \otimes \overline{\mathbb{F}}_p) \times Z_U$ over a connected scheme $S = \operatorname{Spec} k$ of characteristic p is given by:

(a) A homomorphism $\phi\colon \overline{\mathbb{F}}_p \to k$.

(b) An isomorphism class of pairs (X, ρ), where X is a formal special $\mathcal{O}_B$-module over S, and $\rho\colon \phi_*\Phi \to X$ is a quasi-isogeny of height zero.

(c) An algebrization $(A, \epsilon, \overline{\nu})$ of Φ with level U structure.

This datum gives a point in $S_U(k)$. Hence we obtain a morphism of $\mathbb{F}_p$-schemes

$$\Theta\colon (\widehat{\Omega} \otimes \overline{\mathbb{F}}_p) \times Z_U \to S_U \otimes \overline{\mathbb{F}}_p.$$

Note that Θ is invariant under the left action of $\mathrm{GL}_2(\mathbb{Q}_p)$; thus Θ factors through a morphism of $\mathbb{F}_p$-schemes:

$$\overline{\Theta}\colon\ \mathrm{GL}_2(\mathbb{Q}_p) \setminus [(\widehat{\Omega} \otimes \overline{\mathbb{F}}_p) \times Z_U] \to S_U \otimes \overline{\mathbb{F}}_p.$$

One can also show that $\overline{\Theta}$ is actual an isomorphism (cf. [3, Chapter III §6.4]). By deformation theory and Serre-Tate theorem (cf. Katz [20]), we may prolong the isomorphism $\overline{\Theta}$ to an isomorphism form the formal scheme $\mathrm{GL}_2(\mathbb{Q}_p) \setminus [(\widehat{\Omega} \widehat{\otimes}_{\mathbb{Z}_p} \widehat{\mathbb{Z}}_p^{\mathrm{nr}}) \times Z_U]$ to the formal scheme $\widehat{S}_U$ as well. $\square$

3.1. Torsors

Let S be a scheme. Suppose that $C \to S$ is a curve over S, and $T \to S$ is a Galois étale covering. Denote G by $\mathrm{Gal}(T/S)$. We briefly recall the definition and some properties of *torsor* for G over C.

Let $\chi \in \mathrm{H}^1(G, \mathrm{Aut}(C \times_S T_{/T}))$ be represented by a 1-cocycle $\widetilde{\chi}$. Define a *torsor* $\mathcal{C}^\chi\colon \mathrm{Schemes}_S \to \mathrm{Sets}$ by

$$\begin{array}{rcl} \mathcal{C}^\chi\colon \mathrm{Schemes}_{/S} & \to & \mathrm{Sets} \\ Y_{/S} & \rightsquigarrow & G \setminus (C \times_S T)(Y), \end{array}$$

where $\sigma \in G$ acts on the left on $C \times_S T$ by $\sigma \cdot (x,t) = \big((1 \times \sigma) \circ \widetilde{\chi}(\sigma)\big)(x,t)$. Then $\mathcal{C}^\chi$ is represented by a scheme $C^\chi_{/S}$. Such an C^χ is unique and is called the *twist* of $C_{/S}$ by χ. We thus have

$$C^\chi \times_S T \simeq_T C \times_S T$$

and

$$C^\chi(S) = \big\{x \in C(T) \mid \widetilde{\chi}(\sigma)x = \sigma \cdot x \text{ for all } \sigma \in G\big\}.$$

3.2. Variant of Čerednik-Drinfel'd uniformization

The book of Gerritzen-Van der Put [15] provides the theory of p-adic Schottky groups and Mumford's theory of p-adic uniformization (see also Fresnel-Van der Put [14]). We also refer to the origin papers by Mumford [21] and Tate [27].

The action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on Z_U decomposes the latter space into finitely many orbits. Each orbit contains an element x_i whose p-component is one; the stabilizer H_i of x_i is a discrete cocompact subgroup of $B'^\times(\mathbb{Q}_p) = \mathrm{GL}_2(\mathbb{Q}_p)$ containing the ${n_i}^{\mathrm{th}}$ power of $\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$ for some $n_i \gg 0$. Dividing by the action of the elements $\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)^{n_i}$, we get

$$\mathrm{GL}_2(\mathbb{Q}_p) \setminus [(\widehat{\Omega} \widehat{\otimes}_{\mathbb{Z}_p} \widehat{\mathbb{Z}}_p^{\mathrm{nr}}) \times Z_U] = \coprod_i H_i \setminus (\widehat{\Omega} \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p^{(2n_i)})$$

where $\widehat{\mathbb{Z}}_p^{(2n_i)}$ is the ring of integers in the unramified extension $\mathbb{Q}_p^{(2n_i)}$ of degree $2n_i$ of $\mathbb{Q}_p$. Let H_i' be the image in $\mathrm{PGL}_2(\mathbb{Q}_p)$ of the subgroup of all elements in H_i whose determinant is a unit. Then H_i' is a Schottky group, since we always assume U satisfies the condition (i)–(iii) as in §1.5 (cf. [21, 23, 24]). After a base extension to $\mathbb{Z}_p^{(2n_i)}$, the quotient

$$H_i \setminus (\widehat{\Omega} \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p^{(2n_i)})$$

becomes as a $\mathbb{Z}_p^{(2n_i)}$-scheme isomorphic to a finite union of Mumford quotients $H_i' \setminus \widehat{\Omega}$. Thus we may recover (3.2) from this finite union by Galois descent with respect to the extension $\mathbb{Z}_p^{(2n_i)}$ of $\mathbb{Z}_p$. This shows that the left hand side in (3.2) is a Galois twisted form of a finite disjoint union of Mumford curves.

Let K_p be the unique unramified quadratic extension of $\mathbb{Q}_p$. Let $\Gamma = \widehat{\Gamma}_0^D(M) \cap \widehat{\Gamma}_1^D(r)$ and let $X^D(\Gamma)$ be the Shimura curve associated with Γ. Let $X^D(\Gamma)^\chi$ be the curve over

$\mathbb{Q}_p$ obtained by twisting $X^D(\Gamma)$ by the 1-cocyle

$$\chi \in \mathrm{H}^1\big(\mathrm{Gal}(K_p/\mathbb{Q}_p), \mathrm{Aut}(X^D(\Gamma))\big)$$

which sends the generator of $\mathrm{Gal}(K_p/\mathbb{Q}_p)$ to the Atkin-Lehner operator W_p. Then we have

Theorem 3.2 (Čerednik-Drinfel'd). *Let $X = \Delta \setminus \Omega$ be the rigid analytic curve over $\mathbb{Q}_p$ associated with $\Delta = \phi_p(\Gamma'_+ \cap G'(\mathbb{Q})_+)$ (cf. second and third paragraphs in §3). The curve $X^D(\Gamma)^\chi$ is isomorphic, as an rigid curve over $\mathbb{Q}_p$, to X. In particular, the rigid curve X is isomorphic to $X^D(\Gamma)$ over K_p.*

For a dual graph $\mathcal{G}$ of a semi-stable curve C over a strict henselian trait, since the residue field is finite, the Frobenius morphism on the special fiber of C induces an automorphism of dual graphs $F\colon \mathcal{G} \to \mathcal{G}$. By Mumford's theory (cf. [14, §5.4]), the dual graph of X is canonical isomorphic to the graph $\Delta \setminus \mathcal{T}_\Delta$ where $\mathcal{T}_\Delta$ is a locally finite tree associated with Δ. Hence, the theorem of Čerednik-Drinfel'd shows that the dual graph can be described by the following:

Corollary 3.3. *There is an isomorphism of pairs*

$$(\mathcal{G}(X^D(\Gamma) \times \mathbb{Z}_{p/\mathbb{Z}_p}), F(X^D(\Gamma) \times \mathbb{Z}_{p/\mathbb{Z}_p})) \simeq (\Delta \setminus \mathcal{T}, W_p).$$

4. Equivariant cohomology theory

Let $\mathcal{T}$ be the Bruhat-Tits tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$, whose vertices corresponds to homothety classes of rank two $\mathbf{Z}_p$-latices in $\mathbf{Q}_p^2$. Let Δ be a discrete subgroup of $\mathrm{PGL}_2(\mathbb{Q}_p)$ acting on $\mathcal{T}$ properly discontinuously and freely. More precisely, we assume that Δ does not invert edges, and that stabilizers of vertices (and hence of edges) are finite. Let $\mathrm{pr}\colon \mathcal{T} \to \Delta \setminus \mathcal{T} = \mathcal{G}$ be the quotient map. Let $\mathcal{O}$ be a ring and let Λ be an $\mathcal{O}$-module on which Δ acts $\mathcal{O}$-linearly. The goal of this section is to establish a comparison isomorphism between the group cohomology $\mathrm{H}^\bullet(\Delta, \Lambda)$ and the equivariant cohomology $\mathrm{H}^\bullet_\Delta(\mathcal{T}, \Lambda)$ (see Proposition 4.2). The main references here are [4–6].

If $\mathcal{G}$ is any graph, we denote by $\Sigma_i(\mathcal{G})$ the set of i-simplices of $\mathcal{G}$ for $i = 0, 1$. Hence, $\Sigma_0(\mathcal{G})$ is the set of vertices $\mathrm{Ver}(\mathcal{G})$ of $\mathcal{G}$ and $\Sigma_1(\mathcal{G})$ is the set of edges $\mathrm{Ed}(\mathcal{G})$ of $\mathcal{G}$.

Let $\mathcal{G} = \Delta \setminus \mathcal{T}$. For any i-simplex $\sigma \in \Sigma_i(\mathcal{G})$, we denote by $\widetilde{\sigma} \in \Sigma_i(\mathcal{T})$ an i-simplex of $\mathcal{T}$ lying above σ. We denote by $\Delta(\widetilde{\sigma})$ the stabilizer of $\widetilde{\sigma}$ in Δ. We also denote by $\Delta\widetilde{\sigma}$ the Δ-orbit of any i-simplex $\widetilde{\sigma}$. Let $\Sigma_1^o(\mathcal{T}) = \mathrm{Ed}^o\,\mathcal{T}$ be the oriented edges of $\mathcal{T}$. For every oriented edge $e \in \Sigma_1^o(\mathcal{T})$, there is an *opposite edge* $\overline{e} \in \Sigma_1^o(\mathcal{T})$. An oriented edge $e \in \Sigma_1^o(\mathcal{T})$

has an *initial vertex* $o(e)$ and a *terminal vertex* $t(e)$. We set

$$
\begin{aligned}
C^0(\mathcal{T},\Lambda) &:= \mathrm{Funct}_f(\Sigma_0(\mathcal{T}),\Lambda),\\
C^1(\mathcal{T},\Lambda) &:= \big\{f \in \mathrm{Funct}_f(\Sigma_1^o(\mathcal{T}),\Lambda) \mid f(e) = -f(\overline{e})\big\},
\end{aligned}
$$

where Funct_f denotes the set of functions with finite support. A choice of an orientation for $\mathcal{T}$ gives an identification of $C^1(\mathcal{T},\Lambda)$ with $\mathrm{Funct}_f(\Sigma_1(\mathcal{T}),\Lambda)$. We thus have the cochain complex

$$
\begin{array}{rccc}
d\colon & C^0(\mathcal{T},\Lambda) & \longrightarrow & C^1(\mathcal{T},\Lambda)\\
& f & \longmapsto & e \mapsto f(t(e)) - f(o(e)).
\end{array} \tag{4.1}
$$

We denote the *equivariant cochain groups* $C^i_\Delta(\mathcal{T},\Lambda)$ by the Δ-invariant of $C^i(\mathcal{T},\Lambda)$ for $i = 1, 2$. That is,

$$C^i_\Delta(\mathcal{T},\Lambda) := C^i(\mathcal{T},\Lambda)^\Delta.$$

Taking Δ-invariant of (4.1) gives the complex $C^\bullet_\Delta(\mathcal{T},\Lambda)$

$$d\colon C^0_\Delta(\mathcal{T},\Lambda) \to C^1_\Delta(\mathcal{T},\Lambda),$$

whose cohomology yields the *equivariant cohomology* $\mathrm{H}^\bullet_\Delta(\mathcal{T},\Lambda)$, i.e.,

$$0 \longrightarrow \mathrm{H}^0_\Delta(\mathcal{T},\Lambda) \longrightarrow C^0_\Delta(\mathcal{T},\Lambda) \xrightarrow{d} C^1_\Delta(\mathcal{T},\Lambda) \longrightarrow \mathrm{H}^1_\Delta(\mathcal{T},\Lambda) \longrightarrow 0. \tag{4.2}$$

On the other hand, since $\mathcal{T}$ is contractible, the complex (4.1) gives an exact sequence of Δ-modules

$$0 \longrightarrow \Lambda \longrightarrow C^0(\mathcal{T},\Lambda) \xrightarrow{d} C^1(\mathcal{T},\Lambda) \longrightarrow 0. \tag{4.3}$$

Note that

$$C^i(\mathcal{T},\Lambda) \simeq \bigoplus_{\sigma_i \in \Sigma_i(\mathcal{G})} \mathrm{Hom}(\Delta\sigma_i,\Lambda).$$

We now recall *Shapiro's Lemma*:

Lemma 4.1 (Shapiro's Lemma)**.** *Let H be a subgroup of G, and let A be an H-module. Then for all $m \geq 0$,*

$$\mathrm{H}^m(G, \mathrm{Ind}_H^G(A)) \simeq \mathrm{H}^m(H,A),$$

where $\mathrm{Ind}_H^G(A)$ consists of all maps $x\colon G \to A$ such that $x(\tau\gamma) = \tau x(\gamma)$ for all $\tau \in H$.

Applying Shapiro's Lemma,

$$\mathrm{H}^m(\Delta, C^i(\mathcal{T},\Lambda)) \simeq \bigoplus_{\sigma_i \in \Sigma_i(\mathcal{G})} \mathrm{H}^m(\Delta(\widetilde{\sigma}),\Lambda)$$

for $m \geq 1$. Therefore, from (4.3) we get the following long exact sequence:

$$\cdots \to \mathrm{H}^m(\Delta, \Lambda) \to \bigoplus_{\sigma_0 \in \Sigma_0(\mathcal{G})} \mathrm{H}^m(\Delta(\widetilde{\sigma}), \Lambda) \to \bigoplus_{\sigma_1 \in \Sigma_1(\mathcal{G})} \mathrm{H}^m(\Delta(\widetilde{\sigma}), \Lambda) \to \mathrm{H}^{m+1}(\Delta, \Lambda) \to \cdots .$$

Since Δ acts without fixed points, $\mathrm{H}^m(\Delta, C^i(\mathcal{T}, \Lambda)) = 0$ for $m \geq 1$, and the long exact sequences becomes

$$0 \longrightarrow \mathrm{H}^0(\Delta, \Lambda) \longrightarrow C^0_\Delta(\mathcal{T}, \Lambda) \xrightarrow{d} C^1_\Delta(\mathcal{T}, \Lambda) \longrightarrow \mathrm{H}^1(\Delta, \Lambda) \longrightarrow 0. \tag{4.4}$$

Hence, by comparing (4.2) and (4.4), we obtain

Proposition 4.2. *The group cohomology* $\mathrm{H}^\bullet(\Delta, \Lambda)$ *is canonical isomorphic to equivariant cohomology* $\mathrm{H}^\bullet_\Delta(\mathcal{T}, \Lambda)$.

5. Harmonic cocycles

Let $\mathcal{O}$ be a subring of $\mathbb{C}$ preserved under complex conjugation. Let $\langle \cdot, \cdot \rangle$ be a positive definite hermitian form on Λ, and suppose that Δ acts on Λ unitarily. Define an $\mathcal{O} \otimes \mathbb{Q}$-valued inner product $\langle \cdot, \cdot \rangle^i_\Delta$ on $C^i_\Delta(\mathcal{T}, \Lambda)$ by

$$\langle f, g \rangle^i_\Delta = \sum_{\sigma \in \Sigma_i(\mathcal{G})} \frac{\langle f(\widetilde{\sigma}), g(\widetilde{\sigma}) \rangle}{|\Delta(\widetilde{\sigma})|}.$$

The inner product $\langle \cdot, \cdot \rangle^i_\Delta$, $i = 0, 1$, are positive definite.

Define $\delta\colon C^1_\Delta(\mathcal{T}, \Lambda) \to C^0_\Delta(\mathcal{T}, \Lambda)$ by

$$(\delta f)(\widetilde{v}) = \sum_{\widetilde{e} \in \mathrm{St}(\widetilde{v})} f(\widetilde{e})$$

for $\widetilde{v} \in \Sigma_0(\mathcal{T})$ and $f \in C^1_\Delta(\mathcal{T}, \Lambda)$, where $\mathrm{St}(\widetilde{v}) = \{\widetilde{e} \in \mathrm{Ed}^{\mathrm{o}}(\mathcal{T}) \mid t(\widetilde{e}) = \widetilde{v}\}$. We define the *laplacian* $\Box^i$ by $\Box^i = d\delta + \delta d$. So $\Box^0 = \delta d$ and $\Box^1 = d\delta$. An i-cochain $c \in C^i_\Delta(\mathcal{T}, \Lambda)$ is called *harmonic* if $\Box^i c = 0$. We denote by $\mathbf{H}^i = \mathbf{H}^i(\Lambda) \subseteq C^i_\Delta(\mathcal{T}, \Lambda)$ the $\mathcal{O}$-module of all harmonic i-cochains.

Lemma 5.1. *We have* $\mathbf{H}^0 = \ker d$ *and* $\mathbf{H}^1 = \ker \delta$*; hence* $\mathbf{H}^0 = \mathrm{H}^0(\Delta, \Lambda)$.

Proof. It is clear that $\ker d \subseteq \mathbf{H}^0$ and $\ker \delta \subseteq \mathbf{H}^1$. Suppose $c \in \mathbf{H}^0$, then

$$\langle dc, dc \rangle^1_\Delta = \langle \delta dc, c \rangle^0_\Delta = 0.$$

Hence $dc = 0$ and $\ker d \supseteq \mathbf{H}^0$. Similarly, if $c \in \mathbf{H}^1$, then

$$\langle \delta c, \delta c \rangle^0_\Delta = \langle d\delta c, c \rangle^1_\Delta = 0.$$

Thus, $\ker \delta \supseteq \mathbf{H}^1$. $\Box$

6. Vanishing cycles

Suppose that $M \geq 4$ and let $R_{M,D}$ be an Eichler order of level M. Denote by $X^D(M)$ the Shimura curve $G(\mathbb{Q}) \setminus G(\mathbb{A})/\mathbb{R}^\times \widehat{R}^\times_{M,D} C_{\mathrm{i}}$ associated with $R_{M,D}$. Let X be the proper semi-stable model of $X^D(M)$ over $\mathbb{Z}[1/DM]$. Let $\overline{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$. We denote by $\overline{\mathbb{Z}}_p$ the normalization of $\mathbb{Z}_p$ in $\overline{\mathbb{Q}}_p$. Let $I_p = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{\mathrm{nr}})$ be the inertia subgroup at p.

Let $S = \operatorname{Spec} \overline{\mathbb{Z}}_p$, and η resp. s the generic resp. closed point of S. We still denote X by the base change to S of X. We consider the commutative diagram

$$\begin{tikzcd} X_s \arrow[r, "i"] \arrow[d] & X \arrow[d] & X_\eta \arrow[l, "j"'] \arrow[d] \\ s \arrow[r] & S & \eta \arrow[l] \end{tikzcd}$$

We denote by $\mathcal{G}$ the dual graph of X_s, and by Σ_i the i-simplices of $\mathcal{G}$ for $i = 0, 1$. We assume that a neighbourhood of each point $x \in \Sigma_1$ in X is locally S-isomorphic to the subscheme of $\mathbb{A}^2_S = \mathbb{Z}_q[t_1, t_2]$ with $t_1 t_2 = p \neq 0$.

Let $Y = X_s$. For each $x \in \Sigma_1$, let $Y_{(x)}$ denote the henselization of Y at x, and let C_x denote the set of the two branches of Y at x. As in Illusie [18, §1.1], we define $\mathbb{Z}(x)$ and $\mathbb{Z}'(x)$ according to the following two dual exact sequences:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{(1)} \mathbb{Z}^{C_x} \longrightarrow \mathbb{Z}(x) \longrightarrow 0, \quad 0 \longrightarrow \mathbb{Z}'(x) \longrightarrow \mathbb{Z}^{C_x} \xrightarrow{(2)} \mathbb{Z} \longrightarrow 0$$

where (1) is the diagonal map, and (2) is the sum. Choosing an ordering for C_x for each $x \in \Sigma_1$ defines a base $\delta'_x = (1, -1)$ for $\mathbb{Z}'(x)$ and the dual base for $\mathbb{Z}(x)$ will be denoted by δ_x. Let

$$\Lambda = \mathbb{Z}_\ell, \quad \Lambda(x) = \mathbb{Z}(x) \otimes \Lambda, \quad \Lambda'(x) = \mathbb{Z}'(x) \otimes \Lambda.$$

Let $\mathcal{F}$ be a constructible $\mathbb{Z}_\ell$-sheaf on a proper semi-stable curve X/S. We have the following well-known results, for which we refer to SGA7 [9, 16], especially to the Exposés I, XIII, XIV, XV by Deligne.

(1) We have the *Leray spectral sequence*

$$\mathrm{H}^m(X, \mathrm{R}^n j_* \mathcal{F}) \Longrightarrow \mathrm{H}^{m+n}(X_\eta, \mathcal{F}),$$

which is I_p-equivariant.

(2) By the *Proper Base Change Theorem*, the morphism of functors i^* induces an isomorphism

$$\mathrm{H}^m(X, \mathrm{R}^n j_* \mathcal{F}) \simeq \mathrm{H}^m(X_s, i^* \mathrm{R}^n j_* \mathcal{F}).$$

Let

$$\mathrm{R}^n \Psi(\mathcal{F}) := i^* \mathrm{R}^n j_* \mathcal{F}$$

be the n^{th} *sheaf of nearby cycles.*

(3) The following facts are well-known:

(i) $\mathrm{R}^0\Psi(\mathcal{F}) = \mathcal{F}$.

(ii) $\tau_{\geq 1}\mathrm{R}^\bullet\Psi(\mathcal{F}) = \mathrm{R}^\bullet\Phi(\mathcal{F})$, where $\mathrm{R}\Phi(\mathcal{F}) = \mathrm{Cone}(\mathcal{F} \to \mathrm{R}\Psi(\mathcal{F}))$ is the *sheaf of vanishing cycles.*

(iii) $\mathrm{R}^1\Phi(\mathcal{F})$ is a sheaf concentrated on the set Σ_1 of singular points of X_s.

(iv) $\mathrm{R}^n\Psi(\mathcal{F}) = 0$ for $n \geq 2$.

(4) Let $\mathcal{F} = \Lambda$. For any singular point $x \in \Sigma_1$, $\mathrm{H}^n_x(X_s, \mathrm{R}\Psi(\Lambda)) = 0$ for $n \neq 1, 2$, and for $n = 2$ we have the trace isomorphism:

$$\mathrm{Tr}\colon \mathrm{H}^2_x(X_s, \mathrm{R}\Psi(\Lambda)) \xrightarrow{\simeq} \Lambda(-1)$$

whereas for $n = 1$ we have

$$\mathrm{H}^1_x(X_s, \mathrm{R}\Psi(\Lambda)) \xrightarrow{\simeq} \Lambda(x).$$

So for any singular points $x \in \Sigma_1$, we get a *vanishing cycle* $\delta_x \in \mathrm{H}^1_x(X_s, \mathrm{R}\Psi(\Lambda))$. Similarly, we have

$$\Lambda'(x) \xrightarrow{\simeq} \mathrm{R}^1\Phi(\Lambda_x)(1),$$

and $\delta'_x \in \mathrm{R}^1\Phi(\Lambda_x)(1)$. These cycles are perpendicular to each other with respect to the canonical pairing on $\mathrm{H}^1(X_{\eta_s}, \Lambda)$ to $\Lambda(-1)$. That is, this pairing

$$\begin{array}{ccc} \mathrm{R}^1\Phi(\Lambda)_x \times \mathrm{H}^1_x(X_{s,(x)}, \mathrm{R}\Psi(\Lambda)) & \longrightarrow & \Lambda(-1) \\ (a, b) & \longmapsto & \mathrm{Tr}(ab) \end{array}$$

is perfect between free rank one modules.

(5) *Picard-Lefschetz formula*: If $I_{\mathrm{t}} \subset I_p$ denotes the subgroup corresponding to the maximal tamely ramified field extension of $\mathbb{Q}_p$, one has the isomorphism

$$\begin{array}{ccc} \epsilon\colon I_p/I_{\mathrm{t}} & \longrightarrow & \mathbb{Z}_p(1) \text{ via the maps} \\ I_p/I_{\mathrm{t}} & \longrightarrow & \boldsymbol{\mu}_{p^n} \\ \sigma & \longmapsto & \frac{\sigma(\sqrt[p^n]{\pi})}{(\sqrt[p^n]{\pi})} \end{array}$$

for some uniformizing element π. Then I_p acts on $a \in \mathrm{H}^1(X_{\eta_s}, \Lambda)$ via

$$\sigma(a) = a - \sum_{x \in \Sigma_1} (\epsilon_x(\sigma)\langle a, \delta_x\rangle \delta_x),$$

where ϵ_x is the isomorphism $I_p/I_{\mathrm{t}} \simeq \mathbb{Z}_p(1)$ at the singular point x, and $\langle\,\cdot\,,\cdot\,\rangle$ is the pairing on $\mathrm{H}^1(X_\eta, \Lambda)$ to $\Lambda(-1)$.

Using these results, for any constructible $\mathbb{Z}_\ell$-sheaf $\mathcal{F}$ on X/S we obtain the following *exact sequence of specialization*:

$$0 \longrightarrow \mathrm{H}^1(X_s, \mathrm{R}^0\Psi(\mathcal{F}))(1) \xrightarrow{\mathrm{sp}} \mathrm{H}^1(X_\eta, \mathcal{F})(1) \xrightarrow{\beta} \mathrm{H}^0(X_s, \mathrm{R}^1\Phi(\mathcal{F}))(1)$$
$$\xrightarrow{d_2} \mathrm{H}^2(X_s, \mathrm{R}^0\Psi(\mathcal{F}))(1) \xrightarrow{\mathrm{sp}} \mathrm{H}^2(X_\eta, \mathcal{F})(1) \longrightarrow 0.$$

We thus have the following:

Proposition 6.1. *One has*

(i) $\mathrm{H}^1(X_s, \mathrm{R}^0\Psi(\Lambda)) \simeq \mathrm{H}^1(\Delta, \Lambda)$ *with a trivial action of inertia.*

(ii) $\mathrm{H}^0(X_s, \mathrm{R}^1\Psi(\Lambda)) = \bigoplus_{x\in\Sigma_1} \Lambda(-1)$.

(iii) $\mathrm{H}^2(X_s, \mathrm{R}^0\Psi(\Lambda)) = \bigoplus_{x\in\Sigma_0} \Lambda(-1)$.

Proof. (ii) and (iii) are clear from (1)–(5) above. (i) follows, because we have the following isomorphism

$$\mathrm{H}^1(X_s, \mathrm{R}^0\Psi(\Lambda)) \simeq \mathrm{H}^1(\mathcal{G}, \Lambda).$$

Recall that $\mathcal{G} = \Delta \setminus \mathcal{T}$. We hence have

$$\mathrm{H}^1(\mathcal{G}, \Lambda) = \mathrm{H}^1(\Delta \setminus \mathcal{T}, \Lambda).$$

Since $\mathcal{T}$ is contractible, the Hochschild-Serre spectral sequence degenerates at E_1 and

$$\mathrm{H}^1(\Delta \setminus \mathcal{T}, \Lambda) = \mathrm{H}^1(\Delta, \Lambda). \qquad \square$$

Let us consider ϕ: $\bigoplus_{x\in\Sigma_1} \Lambda(-1) \to \bigoplus_{y\in\Sigma_0} \Lambda(-1)$ for $x \in \Sigma_1$ with vertices y_1, y_2:

$$\phi(\underline{1}_x) = \epsilon_x(y_1)y_1 + \epsilon_x(y_2)y_2$$

with $\epsilon_x(y_i) = \pm 1$ for $i = 1, 2$ and $\epsilon_x(y_1) + \epsilon_x(y_2) = 0$ holds, where $\underline{1}_x$ denotes the constant function supported at x.

Lemma 6.2. *Via the identifications of Proposition* 6.1(ii) *and* (iii), *we have the following commutative diagram:*

$$\begin{tikzcd}
\mathrm{H}^0(X_{\overline{s}}, \mathrm{R}^1\Psi\Lambda) \arrow[r, "d_2"] \arrow[d, equal] & \mathrm{H}^2(X_{\overline{s}}, \mathrm{R}^0\Psi\Lambda) \arrow[d, equal] \\
\bigoplus_{x\in\Sigma_1} \Lambda(-1) \arrow[r, "\phi"'] & \bigoplus_{x\in\Sigma_0} \Lambda(-1)
\end{tikzcd}$$

Proof. We put an orientation on each edge $x \in \Sigma_1$ by putting $x = (y_1, y_2)$, if $\epsilon_x(y_1) = -1$, $\epsilon_x(y_2) = +1$. This lemma is nothing but the Picard-Lefschetz formula. $\square$

By the exact sequence of specialization, Proposition 6.1, and Lemma 6.2, it is easy to derive the following result:

Proposition 6.3. *We have the following exact sequence:*

$$0 \longrightarrow \mathrm{H}^1(\Delta, \Lambda) \longrightarrow \mathrm{H}^1(X_\eta, \Lambda) \longrightarrow \bigoplus_{x \in \Sigma_1} \Lambda(-1) \xrightarrow{d_2} \bigoplus_{x \in \Sigma_0} \Lambda(-1) \xrightarrow{\deg} \Lambda(-1) \longrightarrow 0.$$

7. Applications to the cohomology of special fibers

According to Corollary 3.3, Lemma 5.1 and Proposition 6.3, we have

Lemma 7.1. *One has these two isomorphisms*

$$\mathbf{H}^1 \simeq \ker(d_2) \quad \textit{and} \quad \mathrm{H}^1(\Delta, \mathbb{Q}_\ell(-1)) \simeq \mathbf{H}^1.$$

Proof. Consider the map

$$\begin{array}{ccc} \mathbf{H}^1(\Lambda(-1)) & \longrightarrow & \ker(d_2) \\ c & \longmapsto & \bigoplus_{x \in \Sigma_1} c(x) \underline{1}_x \end{array}$$

where we use the orientation imposed on $x \in \Sigma_1$. By Lemma 6.2 (in fact, the Picard-Lefschetz formula), this map is an isomorphism.

Any element of $\mathbf{H}^1$ induces a 1-cochain on $\mathcal{T}$ which is Δ-invariant. Hence it induces a cohomology class in $\mathrm{H}^1(\Delta, \mathbb{Q}_\ell(-1))$. □

Recall that we have the following exact sequence of Galois groups:

$$1 \to \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{t}}/\mathbb{Q}_p^{\mathrm{nr}}) \to \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{t}}/\mathbb{Q}_p) \to \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}/\mathbb{Q}_p) \to 1.$$

Let σ be a lifting of Frob_p. Consider the module $M = \mathbb{Z}_\ell \oplus \mathbb{Z}_\ell(-1)$, where σ acts on M by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \mathrm{Frob}_p \end{smallmatrix}\right)$. The action of $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{t}}/\mathbb{Q}_p^{\mathrm{nr}})$ is trivial on M. Consider the bilinear pairing

$$\begin{array}{ccc} I_p \times \mathbb{Z}_\ell(-1) & \longrightarrow & \mathbb{Z}_\ell \\ (\tau, \alpha) & \longmapsto & \tau(\alpha) - \alpha. \end{array}$$

It turns out that this pairing would be

$$\begin{array}{ccc} \mathbb{Z}_\ell(1) \times \mathbb{Z}_\ell(-1) & \xrightarrow{\simeq} & \mathbb{Z}_\ell \\ (\tau, \alpha) & \longmapsto & \tau(\alpha) - \alpha, \end{array}$$

where $\tau \in I_{\mathrm{t}}$ and $\alpha \in \mathbb{Z}_\ell(-1)$. This gives an action of $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{t}}/\mathbb{Q}_p)$ on $M \otimes \mathbb{Q}_\ell$ which is called the *special representation* V_{sp}.

Theorem 7.2. *We have the following isomorphism as* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$*-modules:*

$$\mathrm{H}^1(X_\eta, \mathbb{Q}_\ell) \simeq \mathrm{H}^1(\Delta, \mathbb{Q}_\ell) \otimes V_{\mathrm{sp}}.$$

Proof. By Lemma 7.1, we have the following exact sequence

$$0 \longrightarrow \mathrm{H}^1(\Delta, \mathbb{Q}_\ell) \longrightarrow \mathrm{H}^1(X_\eta, \mathbb{Q}_\ell) \longrightarrow \mathrm{H}^1(\Delta, \mathbb{Q}_\ell) \otimes \mathbb{Q}_\ell(-1) \longrightarrow 0.$$

We already know the Galois actions on the terms on the left and on the right, and we ought to get an I_p/I_{t}-action.

Suppose that $\sigma \in I_p/I_{\mathrm{t}}$ and $u \in \mathrm{H}^1(\Delta, \mathbb{Q}_\ell(-1))$. Let $\widetilde{u} \in \mathrm{H}^1(X_\eta, \mathbb{Q}_\ell)$ be a lifting of u. Consider

$$(\sigma, u) \mapsto \sigma(\widetilde{u}) - \widetilde{u},$$

which is a bilinear pairing from $\mathrm{H}^1(\Delta, \mathbb{Q}_\ell(-1)) \times \mathbb{Q}_\ell(1)$ into $\mathrm{H}^1(\Delta, \mathbb{Q}_\ell)$. The Picard-Lefschetz formula

$$\sigma(u) - u = \sum_{x \in \Sigma_1} \epsilon_x(\sigma) \langle u, \delta_x \rangle \delta_x$$

can be identified with the cohomology class in $\mathrm{H}^1(\Delta, \mathbb{Q}_\ell)$ given by the 1-cochains c_x:

$$c_x(y) = \begin{cases} 0 & \text{if } y \in \Sigma,\ y \neq x, \\ 1 & \text{if } y = x. \end{cases}$$

The trace map restricted to $\mathrm{H}^1(\Delta, \mathbb{Q}_\ell) \times \ker(d_2)$ gives the required isomorphism. □

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