This paper was retracted by the publisher as it has numerous unfixable errors. In particular, the main result of the paper, Corollary 3.10, is unfortunately false.

TAIWANESE JOURNAL OF MATHEMATICS
Vol. 24, No. 2, pp. 283–300, April 2020
DOI: 10.11650/tjm/190703

A Note on Number Knots and the Splitting of the Hilbert Class Field

Yih-Jeng Yu

Abstract. Several number knots are defined including the five knots introduced by W. Jehne. The question of the splitting of the group extension of the Hilbert class field can be read off in terms of the triviality of these knots.

1. Introduction

Let K be a number field. We embed $K^\times$ into the idele group J_K, and this gives rise to the exact sequence

$$1 \longrightarrow K^\times \longrightarrow J_K \longrightarrow C_K \longrightarrow 1,$$

where C_K is the idele class group of K. The kernel of the canonical map $J_K \to I_K$ of the idele group onto the group of fractional ideals is the unit idele group U_K, giving rise to another exact sequence

$$1 \longrightarrow U_K \longrightarrow J_K \longrightarrow I_K \longrightarrow 1.$$

We also have the exact sequence

$$1 \longrightarrow P_K \longrightarrow I_K \longrightarrow \mathrm{Cl}_K \longrightarrow 1$$

defining the ideal class group Cl_K as the factor group of the group I_K modulo the group P_K of principal ideals. All these three exact sequences fit into a commutative diagram, called the *fundamental square*,

$$\begin{tikzcd}
E_K \arrow[r, hook] \arrow[d, hook] & K^\times \arrow[r, two heads] \arrow[d, hook] & P_K \arrow[d, hook] \\
U_K \arrow[r, hook] \arrow[d, two heads] & J_K \arrow[r, two heads] \arrow[d, two heads] & I_K \arrow[d, two heads] \\
\mathcal{E}_K \arrow[r, hook] & C_K \arrow[r, two heads] & \mathrm{Cl}_K
\end{tikzcd}$$

where E_K is the global unit group and $\mathcal{E}_K$ is the idele unit class group.

Received August 23, 2018; Accepted July 3, 2019.
Communicated by Chia-Fu Yu.
2010 *Mathematics Subject Classification*. Primary: 12F10; Secondary: 11R32, 11R34.
Key words and phrases. number knots, Hilbert class field, abelian genus, central genus.

For a Galois extension K/k of number fields with Galois group $G = \mathrm{Gal}(K/k)$ and relative norm $N = N_{K/k}$, Jehne, in a seminal paper [12], applied the snake lemma to the following commutative diagram of certain abelian groups attached to number fields

$$\begin{tikzcd}
A_K \arrow[r, hook] \arrow[d, "N"] & B_K \arrow[r, two heads] \arrow[d, "N"] & C_K \arrow[d, "N"] \\
A_k \arrow[r, hook] & B_k \arrow[r, two heads] & C_k
\end{tikzcd}$$

and obtained the exact sequence

$$\begin{tikzcd}
1 \arrow[r] & {}_N A_K \arrow[r] & {}_N B_K \arrow[r] & {}_N C_K \arrow[dll, rounded corners, "\delta"] & \\
 & A_k/NA_K \arrow[r] & B_k/NB_K \arrow[r] & C_k/NC_K \arrow[r] & 1.
\end{tikzcd}$$

The image of the connecting homomorphism δ is $A_k \cap NB_K/NA_K =: [A, B]$, and he calls $[A, B]$ the *knot* associated to the exact sequence

$$A_k \hookrightarrow B_k \twoheadrightarrow C_k.$$

If one splits up the exact sequence at δ and gets two short exact sequences involving the knot $[A, B]$:

$$1 \longrightarrow {}_N A_K \longrightarrow {}_N B_K \longrightarrow {}_N C_K \longrightarrow [A, B] \longrightarrow 1,$$

and

$$1 \longrightarrow [A, B] \longrightarrow A_k/NA_K \longrightarrow B_k/NB_K \longrightarrow C_k/NC_K \longrightarrow 1.$$

The exact sequences of the fundamental square thus give rise to six knots (cf. [12, p. 220]):

$$\begin{aligned}
&[U_K, J_K] = 1 && \\
&\omega_{K/k} := [E_K, K^\times] = E_k \cap NK^\times/NE_K && \text{first unit knot,}\\
&\omega'_{K/k} := [E_K, U_K] = E_k \cap NU_K/NE_K && \text{second unit knot,}\\
&\nu_{K/k} := [K^\times, I_K] = k^\times \cap NJ_K/NK^\times && \text{Scholz's number knot,}\\
&\delta_{K/k} := [P_K, I_K] = P_k \cap NI_K/NP_K && \text{divisor knot,}\\
&\gamma_{K/k} := [\mathcal{E}_K, C_K] = \mathcal{E}_k \cap NC_K/N\mathcal{E}_K && \text{idele class knot.}
\end{aligned}$$

The vanishing of the knot $[U_K, J_K]$ of the idele units in the ideles reduces to the following statement:

$$U_\wp \cap N_{K_\ell/k_\wp} K_\ell^\times = N_{K_\ell/k_\wp} U_\ell$$

for $\ell/\wp$ in K/k. If $\wp$ is a finite place, for $\alpha = \pi^n \cdot u$ ($u \in U_\wp$, π a prime element for ℓ), the condition $N_{K_\ell/k_\wp}(\alpha) = \pi_0^{fn}\eta_0 \cdot N_{K_\ell/k_\wp}(u) \in U_\wp$ ($\eta_0 \in U_\wp$, π_0 a prime element for $\wp$)

induces that $n = 0$; so $\alpha \in U_\ell$. If $\wp$ is archimedian, we may assume that $\wp$ is real and ℓ is complex, and so both sides are the group of positive real numbers.

By the functorial properties of number knots, Jehne also proved the following *fundamental knot sequence*

$$\omega_{K/k} \hookrightarrow \omega'_{K/k} \longrightarrow \nu_{K/k} \longrightarrow \delta_{K/k} \twoheadrightarrow \gamma_{K/k}, \tag{1.1}$$

which extends the *Scholz's knot sequence*

$$\omega^0_{K/k} \hookrightarrow \nu_{K/k} \twoheadrightarrow \delta^0_{K/k}, \tag{1.2}$$

where $\omega^0_{K/k} := \omega'_{K/k}/\omega_{K/k}$ and $\delta^0_{K/k} := \mathrm{Im}(\nu_{K/k} \to \delta_{K/k})$ are the *Scholz's unit knot* and *Scholz's divisor knot*, respectively.

Recall that an extension of number fields K/k satisfies the *Hasse Norm Principle* if any element of $k^\times$ that is a norm everywhere locally is a global norm from K. The Hasse Norm Principle holds for the extension K/k if and only if Scholz's number knot $\nu_{K/k}$ is trivial. Hasse [7] has shown that $\nu_{K/k} = 1$ for cyclic extension K/k, and Scholz introduced knots in order to study the validity of Hasse Norm Principle in non-cyclic cases. It is known that the Hasse Norm Principle holds for K/k in each of the following cases:

(1) there is a prime $\wp$ of k such that $G_\wp = G$ (cf. [4, §11.4]);

(2) the least common multiple of the local degrees $[K_\ell : k_\wp]$ equals $[K : k]$ (cf. [12]).

Moreover, Scholz's number knot $\nu_{K/k}$ can also be related to the *Schur multiplier* $H^2(G, \mathbb{Q}/\mathbb{Z})$ of the Galois group G.

The knots defined above can be interpreted in terms of Galois groups of certain subfields in the Hilbert class field H_K of K [12, Theorem 3]: The *abelian genus field* H'_K of K over k is the maximal unramified abelian extension of K that is of the form EK where E is an abelian extension of k. The *central class field* H^*_K of K over k is the maximal unramified extension of K such that H^*_K is Galois over k and $\mathrm{Gal}(H^*_K/K)$ is contained in the center of $\mathrm{Gal}(H^*_K/k)$. Obviously, we have $H'_K \subseteq H^*_K \subseteq H_K$. For original definition, see [12, p. 228]. Then, we have

$$\delta^0_{K/k} \simeq \mathrm{Gal}(H^*_K/H'_K), \quad \delta_{K/k} \simeq \mathrm{Gal}(H^*_K/H_kK) \quad \text{and} \quad \gamma_{K/k} \simeq \mathrm{Gal}(H'_K/H_kK).$$

The field tower of Galois extensions $H_K/K/k$ defines a group extension with abelian kernel and factor set α:

$$\alpha\colon 1 \longrightarrow \mathrm{Gal}(H_K/K) \longrightarrow \mathrm{Gal}(H_K/k) \longrightarrow G \longrightarrow 1. \tag{1.3}$$

By class field theory, the abelian kernel is isomorphic to the ideal class group Cl_K of K via the Artin map: $\mathrm{Gal}(H_K/K) \simeq \mathrm{Cl}_K$. By the theorem of Weil and Shafarevich, the two-cohomology class $[\alpha]$ is the image of the fundamental class $u = u_{K/k}$ of K/k under the natural map

$$(1.4) \qquad \begin{array}{ccc} H^2(G, C_K) & \longrightarrow & H^2(G, \mathrm{Cl}_K) \\ [u] & \longmapsto & [\alpha] \end{array}$$

induced by the map $C_K \to \mathrm{Cl}_K$. It is a well-known fact that $H^2(G, C_K)$ is isomorphic to the cyclic group generated by u.

The question of the splitting of the group extension (1.3), or equivalently, of the triviality of the map (1.4) has been studied by several authors. Herz [10] has originally believed that the group extension (1.3) always splits for $k = \mathbb{Q}$. Wyman [20] showed that this is true provided that the Galois group G is cyclic; however, he gave a counter-example in the abelian case. In the same paper, Wyman also gave a sufficient condition for the splitting of (1.3) which involves a certain cohomological knot. Gold, in [6, Theorem 2], gave a simplified proof of Wyman's result in the cyclic case. Cornell and Rosen [5], provided another proof of Wyman-Gold's results and gave necessary conditions and examples for the splitting [5, Proposition 7] which involves Scholz's divisor knot $\delta^0_{K/k}$. Bond found necessary and sufficient conditions in some cases of abelian unramified extension K/k (see [2, Theorem 3.1, Proposition 3.7, and Corollary 3.9]).

In this note, we introduce further cohomological knots $\nu_n(K/k)$ for $n = -1, 0, 2$ and $\tau_{K/k}$ in Section 2. We also relate them with the above number knots of Scholz and Jehne. Moreover, in Section 3, we wish to point out that the splitting of the group extension (1.3) can be read off in terms of triviality of certain cohomological knots. To this end, we define a global inertia group, and we then prove our main theorem (Theorem 3.8) which states that if the arithmetic condition of Wyman holds, the global inertia group equals to the Galois group itself. Our main theorem yields, for abelian Galois group, a sufficient condition for the splitting of the group extension (1.3) is $H_k \cap K = k$ (cf. Corollary 3.10). The above corollary of our main theorem generalizes a result of Wyman and Gold [6, 20], which is the preceding statement when G is cyclic.

In the final section, we introduce the group J which can be identified with a subgroup of $\mathrm{Ker}(j)$, where $j\colon \mathrm{Cl}_k \to \mathrm{Cl}_K$ is the extension map of ideals. In a recent paper of Bond [3], he studies the capitulation problem for arbitrary abelian extensions using cohomological methods to describe $\mathrm{Ker}(j)$ and $\mathrm{Coker}(j)$. He investigates the intersection J of the image of the norm map from Cl_K to Cl_k with $\mathrm{Ker}(j)$, and shows that if J is trivial then $\mathrm{Ker}(j)$ is isomorphic to the Galois group of L over k where L is the intersection of K with the Hilbert class field of k. We also provide examples of splitting and of non-splitting of the group extension (1.3) which can be read off in terms of the triviality of certain cohomological knots (cf. Theorem 4.5). Bond's necessary and sufficient conditions mentioned above are direct consequence of a cohomological characterization of finite nilpotent groups, the Scholz's knot sequence, and our knots ν_0 and v_{-1}.

2. Number knots of a Galois extension

Let K/k be a Galois extension of number fields with Galois group $G = \mathrm{Gal}(K/k)$. Consider the following fundamental square of G-modules:

$$\begin{tikzcd}
E_K \arrow[r, hook] \arrow[d, hook] & K^\times \arrow[r, two heads] \arrow[d, hook] & P_K \arrow[d, hook] \\
U_K \arrow[r, hook] \arrow[d, two heads] & J_K \arrow[r, two heads] \arrow[d, two heads] & I_K \arrow[d, two heads] \\
\mathcal{E}_K \arrow[r, hook] & C_K \arrow[r, two heads] & \mathrm{Cl}_K
\end{tikzcd}$$

Applying Tate cohomology to the second and third lower sequences of the above fundamental square, we obtain, for $n \in \mathbb{Z}$, the following commutative diagram with exact rows

$$\begin{tikzcd}
H^n(U_K) \arrow[r] \arrow[d] & H^n(J_K) \arrow[r] \arrow[d] & H^n(I_K) \arrow[d] \\
H^n(\mathcal{E}_K) \arrow[r] & H^n(C_K) \arrow[r] & H^n(\mathrm{Cl}_K)
\end{tikzcd} \tag{2.1}$$

where $H^n(\cdot)$ is the usual abbreviations for the Tate cohomology group $H^n(G, \cdot)$. For any $n \in \mathbb{Z}$, we define the following cohomological knot

$$\nu_n(K/k) := \mathrm{Coker}(H^n(\mathcal{E}_K) \to H^n(C_K)).$$

For places $\ell/\wp$ of K/k, let $T_\ell \leq G_\ell$ be the inertia subgroup and the decomposition group of K/k at $\ell/\wp$, respectively. Let $\pi G = H_2(G, \mathbb{Z})$ be the fundamental group of G. Since $\mathbb{Q}/\mathbb{Z}$ is an injective abelian group, $\mathrm{Ext}(H_1(G, \mathbb{Z}), \mathbb{Q}/\mathbb{Z})$ is trivial and hence, by the universal coefficient theorem, $H^2(G, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\simeq} \mathrm{Hom}(H_2(G, \mathbb{Z}), \mathbb{Q}/\mathbb{Z})$. That is, πG is the Pontryagin dual of the Schur multiplier $H^2(G, \mathbb{Q}/\mathbb{Z})$ of G. If, in addition, the group G is cyclic, then $\pi G = {}_N\mathbb{Z}/I_G\mathbb{Z}$ is trivial where I_G is the augmentation ideal.

We now recall a well-known result of Tate [18, p. 198], one part of which has been discovered by Scholz [14, Satz 3]. See also [12, p. 221].

Theorem 2.1 (Scholz-Tate)**.** *The Scholz's number knot $\nu_{K/k}$ is the cokernel of the map $\coprod_{\wp,\ell/\wp} \pi G_\ell \to \pi G$ of the fundamental groups:*

$$\nu_{K/k} \simeq \mathrm{Coker}\left(\coprod_{\wp,\ell/\wp} \pi G_\ell \to \pi G\right).$$

Therefore, the number knot $\nu_{K/k}$ is a quotient of the Pontryagin dual of Schur multiplier $H^2(G,\mathbb{Q}/\mathbb{Z})$. We define the cohomological knot $\tau_{K/k}$ to be the cokernel of the map $\coprod_{\wp,\ell/\wp}\pi T_\ell \to \pi G$:

$$\tau_{K/k} := \operatorname{Coker}\left(\coprod_{\wp,\ell/\wp}\pi T_\ell \to \pi G\right).$$

To see the canonical map $\tau_{K/k} \to \nu_{K/k}$ is surjective, we can apply the kernel-cokernel exact sequence to the pair of maps

$$\coprod_{\wp,\ell/\wp}\pi T_\ell \xrightarrow{\ f\ } \coprod_{\wp,\ell/\wp}\pi G_\ell \xrightarrow{\ g\ } \pi G$$

which yields an exact sequence

$$\cdots \longrightarrow \operatorname{Ker}(g) \longrightarrow \operatorname{Coker}(f) \longrightarrow \tau_{K/k} \longrightarrow \nu_{K/k} \longrightarrow 1.$$

The following results are important consequences of global and local class field theory (cf. [13, 15, 18]), and they are fundamental to the main results of this paper:

Proposition 2.2. *Let K be a finite abelian extension over k with Galois group G.*

(1) $H^1(K^\times) = 1$ *(Hilbert's Theorem 90).*

(2) *For any $n \in \mathbb{Z}$, the map $-\cup u: H^{n-2}(\mathbb{Z}) \to H^n(C_K)$, where $u \in H^2(C_K)$ is the fundamental class, is an isomorphism. In particular, we have $H^0(C_K) \simeq H^{-2}(\mathbb{Z}) \simeq G$ (see* [18, p. 197]*).*

(3) *For any $n \in \mathbb{Z}$, we have $H^n(J_K) \simeq \coprod_\wp H^n(G_\ell, K_\wp^\times) \simeq \coprod_\wp H^{n-2}(G_\ell,\mathbb{Z})$, where G_ℓ denotes the decomposition group for an $\ell \mid \wp$ and $\wp$ runs over all places of k. In particular, we have $H^0(J_K) \simeq \coprod_\wp H^{-2}(G_\ell,\mathbb{Z}) \simeq \coprod_\wp G_\ell$ and $H^1(J_K) = 1$ (see* [18, p. 139 and p. 177]*).*

(4) $U_K^G = U_k$ *and* $H^0(U_K) \simeq \coprod_\wp T_\ell$*, where T_ℓ denotes the inertia group for an $\ell \mid \wp$.*

(5) $H^{-1}(I_K) = H^1(I_K) = 1$.

By (1.4) and the bottom line of (2.1), the splitting of (1.3) is equivalent to the triviality of the cohomological knot $\nu_2(K/k)$:

Theorem 2.3. *The group extension* (1.3) *splits if and only if*

$$\nu_2(K/k) = 1.$$

Corollary 2.4 (Gold [6])**.** *Suppose that G is cyclic. The splitting of* (1.3) *is equivalent to $\nu_0(K/k) = 1$.*

Proof. Since G is cyclic, by the two-periodicity of the cohomology of a cyclic group, we thus have

$$\begin{aligned}\nu_2 &:= \operatorname{Coker}(H^2(\mathcal{E}_K) \to H^2(C_K)) \\ &\simeq \operatorname{Coker}(H^0(\mathcal{E}_K) \to H^0(C_K)) =: \nu_0,\end{aligned}$$

and the result follows from Theorem 2.3. □

To get necessary conditions for the splitting, we make use of the theorem of Weil-Shafarevich in class field theory. If $\nu_2(K/k) = 1$, there exists $[\eta] \in H^2(\mathcal{E}_K)$ such that $[\eta] \mapsto [u]$. Now the functorial property of cup-products (cf. [4, Chapter 4, §7]) with $[u]$, $[\eta]$ and $[\alpha]$ induces, for $n = 0$ and -1, the following commutative diagram

$$\begin{tikzcd} & H^{n-2}(\mathbb{Z}) \arrow[dl, "\cup[\eta]"'] \arrow[d, "\cup[u]"] \arrow[dr, "\cup[\alpha]"] & \\ H^n(\mathcal{E}_K) \arrow[r] & H^n(C_K) \arrow[r] & H^n(\mathrm{Cl}_K) \end{tikzcd}$$

where the middle vertical map $H^{n-2}(\mathbb{Z}) \to H^n(C_K)$ is an isomorphism by Proposition 2.2.

From the above commutative diagram and noting that

$$H^n(\mathcal{E}_K) \longrightarrow H^n(C_K) \longrightarrow H^n(\mathrm{Cl}_K)$$

is exact for any $n \in \mathbb{Z}$, we conclude the following necessary conditions for the splitting:

Lemma 2.5. *For arbitrary group G, the splitting of* (1.3)*, i.e., $\nu_2(K/k) = 1$, implies $\nu_0(K/k) = \nu_{-1}(K/k) = 1$.*

To describe $\nu_0(K/k)$ more explicitly, we recall the following known result of Heider (see [9, p. 343]):

Lemma 2.6. *$\nu_0(K/k) = 1$ if and only if $\operatorname{Ker}(j) \cdot N\,\mathrm{Cl}_K = \mathrm{Cl}_k$, where $j\colon \mathrm{Cl}_k \to \mathrm{Cl}_K$ is the extension map of ideals. Moreover, $N\,\mathrm{Cl}_K = \mathrm{Cl}_k$ if and only if $H_k \cap K = k$.*

Moreover, if G is cyclic and $H_k \cap K = k$, then $N\,\mathrm{Cl}_K = \mathrm{Cl}_k$ by Lemma 2.6. Hence, we have $\operatorname{Ker}(j) \cdot N\,\mathrm{Cl}_K = \operatorname{Ker}(j) \cdot \mathrm{Cl}_k = \mathrm{Cl}_k$, and so $\nu_0(K/k) = 1$ by Lemma 2.6 again. Therefore, by Corollary 2.5, we derive the following result proved by Wyman and Gold:

Corollary 2.7 (Wyman-Gold)**.** *If G is cyclic, a sufficient condition for the splitting of* (1.3) *is $H_k \cap K = k$.*

To describe $\nu_{-1}(K/k)$ and $\tau_{K/k}$, we consider our exact and commutative diagram (2.1) for $n = -1$ taking into account the fact that $H^{-1}(I_K) = 1$ (see Proposition 2.2(5)):

(2.2)
$$\begin{tikzcd}
 & & \coprod_{\wp} \pi T_{\ell} \arrow[d] & & \\
H^{-1}(U_K) \arrow[r, two heads] \arrow[d] & H^{-1}(J_K) \arrow[r, "\simeq"] \arrow[d] & \coprod_{\wp} \pi G_{\ell} \arrow[r] \arrow[d] & H^{-1}(I_K) = 1 \arrow[d] & \\
H^{-1}(\mathcal{E}_K) \arrow[r] \arrow[d, two heads] & H^{-1}(C_K) \arrow[r, "\simeq"] \arrow[d, two heads] & \pi G \arrow[r] \arrow[d, two heads] & \check{H}^{-1}(\mathrm{Cl}_K) \arrow[r, two heads] \arrow[d, "\simeq"] & \gamma_{K/k} \arrow[d, "="] \\
\omega'_{K/k} \arrow[r] & \nu_{K/k} \arrow[r, "="] & \nu_{K/k} \arrow[r] & \delta_{K/k} \arrow[r, two heads] & \gamma_{K/k}
\end{tikzcd}$$

where $\check{H}^{-1}(\mathrm{Cl}_K) = {}_N \mathrm{Cl}_K / {}_N I_K$. If we let $\delta^0_{K/k}$ be the Scholz's divisor knot, we can easily prove the following:

Lemma 2.8. (1) *Let H^*_K and H'_K be the central genus and the abelian genus of $H_K/K/k$, respectively. We have $\nu_{-1}(K/k) \simeq \delta^0_{K/k} \simeq \mathrm{Gal}(H^*_K/H_K)$.*

(2) *The following assertions are equivalent:*

(a) *the map $H^{-1}(U_K) \to H^{-1}(C_K)$ is surjective.*

(b) *$\nu_{K/k} = 1$.*

(3) *If $\tau_{K/k} = 1$, then $\nu_{K/k} = 1$.*

Proof. (1) The diagram (2.2) is exact and commutative by [12, Theorem 1]. Since $\check{H}^{-1}(\mathrm{Cl}_K) \simeq \delta_{K/k}$, we can view $\delta^0_{K/k} := \mathrm{Im}(\nu_{K/k} \to \delta_{K/k})$ as a subgroup of $\check{H}^{-1}(\mathrm{Cl}_K)$ and so we have

$$\begin{aligned}
\delta^0_{K/k} &:= \mathrm{Im}(\nu_{K/k} \to \delta_{K/k}) \simeq \mathrm{Ker}(\check{H}^{-1}(\mathrm{Cl}_K) \to \gamma_{K/k}) \\
&= \mathrm{Im}(\pi G \to \check{H}^{-1}(\mathrm{Cl}_K)) \simeq \mathrm{Coker}(H^{-1}(\mathcal{E}_K) \to H^{-1}(C_K)) =: \nu_{-1}(K/k).
\end{aligned}$$

(2) Since $H^{-1}(I_K) = 1$ by Proposition 2.2(5), the map $H^{-1}(U_K) \to H^{-1}(J_K)$ is surjective. The exact and commutative diagram (2.2) reveals that $\nu_{K/k} = 1$ is equivalent to the map $H^{-1}(U_K) \to H^{-1}(C_K)$ is surjective.

(3) The result is obvious, since $\tau_{K/k} \to \nu_{K/k}$ is surjective. □

Remark 2.9. The referee has remarked that, in Lemma 2.8(2), the Scholz's number knot $\nu_{K/k}$ is actually isomorphic to $\mathrm{Coker}(H^{-1}(U_K) \to H^{-1}(C_K))$. This follows by applying the kernel-cokernel exact sequence to the pair of maps

$$H^{-1}(U_K) \xrightarrow{f} H^{-1}(J_K) \xrightarrow{g} H^{-1}(C_K)$$

which yields an exact sequence

$$\cdots \longrightarrow 1 = \mathrm{Coker}(f) \longrightarrow \mathrm{Coker}(g \circ f) \longrightarrow \mathrm{Coker}(g) = \nu_{K/k} \longrightarrow 1.$$

Lemmas 2.5, 2.8(1), and the Scholz's knot sequence (1.2) yield

Corollary 2.10. *If* (1.3) *splits, then* $\delta^0_{K/k}$ *is trivial, or equivalently, the map* $\omega'_{K/k} \to \nu_{K/k}$ *is surjective by the exactness of* (1.1)*. In other words, if* (1.3) *splits, then* $k^\times \cap NJ_K = (E_k \cap NU_K)NK^\times$ $(\subseteq J_k)$*.*

3. Wyman's sufficient condition for the splitting

We let e_0 be the least common multiple of all the ramification indices for K/k, finite and infinite. Since each ramification index $e_\wp$ divides $[K:k]$, we see that e_0 divides $[K:k]$ as well.

A rather strong sufficient arithmetic condition given by Wyman is the following:

Theorem 3.1 (Wyman [20])**.** *For an arbitrary K, if*

$$e_0 = [K:k], \tag{3.1}$$

then the group extension (1.3) *splits.*

The original proof of Wyman is based on class field theory; later, Gold [6, Theorem 1], and Cornell and Rosen [5, Proposition 5] gave simplified proofs for Wyman's result.

The splitting of (1.3) means the triviality of $\nu_2(K/k) = \mathrm{Coker}(H^2(\mathcal{E}_K) \to H^2(C_K))$ which can be read off from the following commutative diagram:

$$\begin{array}{ccc} H^2(U_K) & \longrightarrow & H^2(J_K) \\ \downarrow & & \downarrow \\ H^2(\mathcal{E}_K) & \longrightarrow & H^2(C_K) \end{array}$$

In fact, the splitting is ensured by the surjectivity of the following map $g \circ f$:

$$H^2(U_K) \xrightarrow{f} H^2(J_K) \xrightarrow{g} H^2(C_K).$$

Hence, to prove Wyman's result is to show that the arithmetic condition (3.1) implies the surjectivity of $g \circ f$.

Remark 3.2. The map g is part of the exact sequence

$$1 \longrightarrow H^2(K^\times) \longrightarrow H^2(J_K) \xrightarrow{g} H^2(C_K) \xrightarrow{\delta} H^3(K^\times) \longrightarrow 1,$$

which is induced from the following exact sequence of G-modules

$$1 \longrightarrow K^\times \longrightarrow J_K \longrightarrow C_K \longrightarrow 1$$

with $H^1(C_K) = 1$ by global class field theory (cf. [18, Theorem 9.1(2), p. 180]). Thus, this map g is surjective provided that $H^3(K^\times) = 1$ which is the case for a cyclic extension K/k, by the two-periodicity of the cohomology of a cyclic group and by Hilbert's Theorem 90. Moreover, g is also surjective if

$$[K : k] = \operatorname{lcm}\{|G_\ell| : \ell/\wp,\ \wp \text{ a place of } k\},$$

where G_ℓ is the decomposition group of K/k at $\ell/\wp$. The proof is based on the fact about the cohomology of idele groups that $H^2(G, J_K) \simeq \prod_\wp H^2(G_\ell, K_\ell^\times)$ and via invariant maps. See [20, §4].

We now give a proof of Wyman's result:

Theorem 3.3. *The arithmetic condition* (3.1) *implies the surjectivity of $g \circ f$. In other words, the arithmetic condition* (3.1) *implies the splitting of Hilbert class field.*

Proof. Since the global invariant map $\operatorname{inv}\colon H^2(C_K) \to \frac{1}{[K:k]}\mathbb{Z}/\mathbb{Z}$ is an isomorphism (cf. [18, bottom of p. 196]), it suffices to show that it is possible to take, for each place $\wp$, the local factor system $u_\wp(G) \in H^2(G, U_\wp)$ with invariant $\operatorname{inv}(u_\wp(G)) = \frac{\lambda_\wp}{e_\wp} + \mathbb{Z}$.

First, we have $H^2(U_K) = \prod_\wp H^2(G, U_\wp)$, where

$$U_\wp = \prod_{\ell/\wp} U_\ell = \prod_{\sigma \in G/G_\ell} \sigma U_\ell.$$

Hence, for $\ell/\wp$ we have the isomorphism $H^2(G, U_\ell) \simeq H^2(G_\ell, U_\ell)$ by Shapiro's Lemma. By Serre [16, p. 167], the corestriction map $\mathbf{Cor}\colon H^2(T_\ell, U_\ell) \to H^2(G_\ell, U_\ell)$ is injective. Finally, by Wyman's results [20, §2, Lemma and Theorem], the map $H^2(T_\ell, U_\ell) \to H^2(T_\ell, K_\ell^\times)$ is an isomorphism and the group $H^2(T_\ell, U_\ell)$ has order $e_\wp$. Altogether, we have

$$H^2(T_\ell, K_\ell^\times) \xrightarrow{\ \simeq\ } H^2(T_\ell, U_\ell) \xrightarrow{\mathbf{Cor}} H^2(G_\ell, U_\ell) \xrightarrow{\ \simeq\ } H^2(G, U_\ell).$$

Therefore, $\operatorname{inv}(H^2(G, U_\wp))$ contains $\frac{1}{e_\wp}$ for all ramified $\wp$.

Now the fundamental class $u = u_{K/k} \in H^2(C_K)$ has invariant

$$\operatorname{inv}(u) = \frac{1}{[K : k]} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}.$$

The arithmetic condition (3.1) implies the solvability in $\lambda_\wp \in \mathbb{Z}$ of the congruence equation

$$\sum_\wp \frac{\lambda_\wp}{e_\wp} \equiv \frac{1}{[K : k]} \mod \mathbb{Z}.$$

From this, we show the surjectivity of $g \circ f$ by taking for each place $\wp$ the local factor system $u_\wp(G) \in H^2(G, U_\wp)$ with invariant $\operatorname{inv}(u_\wp(G)) = \frac{\lambda_\wp}{e_\wp} + \mathbb{Z}$. $\square$

Using arguments analogous as in the proof of Bemerkung (4) of [8, p. 41], one shows that the exponent of $\tau_{K/k}$ divides $[K:k]/e_0$. In particular, the arithmetic condition (3.1) implies the triviality of $\tau_{K/k}$ hence that of Scholz's number knot $\nu_{K/k}$ by Lemma 2.8(3).

Corollary 3.4. *If the arithmetic condition* (3.1) *holds, then* $\tau_{K/k} = \nu_{K/k} = 1$*.*

However, the splitting of (1.3) does not necessarily imply $\tau_{K/k} = \nu_{K/k} = 1$, as we shall see later.

Remark 3.5. If K/k is a unramified cyclic extension of degree n, Hilbert's Theorem 94 says that $|\operatorname{Ker}(j)|$ is divisible by n, where $j\colon \mathrm{Cl}_k \to \mathrm{Cl}_K$ is the extension map of ideals. This result was extended to all unramified abelian extensions by Suzuki [17]. Moreover, if $L = H_k \cap K$, the maximal unramified abelian extension of k in K, is a cyclic extension of k, then $|\operatorname{Ker}(j)|$ is divisible by n/e_0.

In the rest of this section, we will show how far is the triviality the cohomological knot $\tau_{K/k}$ related to the splitting problem. To this end, we will introduce the global inertia group and prove a group theoretic result.

Definition 3.6. For a Galois extension K/k of number fields with Galois group G, we define the *global inertia group* T to be the group generated by all inertia groups T_ℓ of $\ell/\wp$ for K/k, i.e.,

$$T := \langle T_\ell : \ell/\wp,\ \wp \text{ a place of } k\rangle.$$

We let $\widetilde{T}$ be the normal subgroup of G generated by T, that is, $\widetilde{T} = T\cdot[T,G]$.

Remark 3.7. (1) Obviously, we have $T \le \widetilde{T} \le G$.

(2) Let $L = H_k \cap K$. It is well-known that $\operatorname{Gal}(K/L)$ is the global inertia group T. Moreover, if $\operatorname{Gal}(K/L)$ is cyclic, we have $[K:L] = e_0$. See [20, p. 147].

(3) Suppose that K/k has a totally ramified prime $\wp$. Let T_ℓ be the inertia subgroup in the extension H_K/k of some prime ℓ lying over $\wp$. Note that $T_\ell \cap \operatorname{Gal}(H_K/K) = 1$ and $T_\ell \cdot \operatorname{Gal}(H_K/K)/\operatorname{Gal}(H_K/K) = G$. Hence, T_ℓ is a complement for $\operatorname{Gal}(H_K/K)$ in $\operatorname{Gal}(H_K/k)$ and the group extension (1.3) splits. Moreover, if T is the inertia subgroup in K/k of some ramified prime and K^T is the fixed field of T, then the group extension

$$1 \longrightarrow \operatorname{Gal}(H_K/K) \longrightarrow \operatorname{Gal}(H_K/K^T) \longrightarrow T \longrightarrow 1$$

splits. Here, the above-mentioned group extension is the restriction of (1.3) to T. Thus, each restriction of (1.3) to an inertia subgroup of G splits.

We now state our first main theorem as follows:

Theorem 3.8. (1) *If* $e_0 = [K:k]$*, we have* $G = \widetilde{T} = T$*.*

(2) *For arbitrary Galois group G, the followings are equivalent:*

(i) $G = \widetilde{T} \cdot G' = T \cdot G'$, *where $G' = [G, G]$ is the commutator subgroup of G.*

(ii) $N \operatorname{Cl}_K = \operatorname{Cl}_k$.

(iii) $H_k \cap K = k$.

Proof. (1) The assumption says that $[K : k] = e_0 = \prod_p p^{n_p}$, hence p^{n_p} divides some $e_\wp$ which is the largest p-power dividing the latter. By the above, we have the Sylow subgroups $\operatorname{Syl}_p(G) = \operatorname{Syl}_p(T_\ell)$ for $\ell/\wp/p$; thus proves (1):

$$G = \left\langle \bigcup_{p \mid [K:k]} \operatorname{Syl}_p(G) \right\rangle \leq \left\langle \bigcup_{p \mid [K:k]} \operatorname{Syl}_p(T_\ell) \right\rangle \leq T \leq \widetilde{T}.$$

(2) The equivalence of (ii) and (iii) is well-known (see Lemma 2.6). By class field theory (cf. Proposition 2.2(2), (3), and (4)), we have compatible isomorphisms inducing the following commutative diagram

$$\begin{tikzcd}
\prod_\wp T_\ell / T_\ell \cap G'_\ell \arrow[r] \arrow[d, "\simeq"] & \prod_\wp G_\ell / G'_\ell \arrow[r] \arrow[d, "\simeq"] & G/G' \arrow[d, "\simeq"] \\
H^0(U_K) \arrow[r] & H^0(J_K) \arrow[r] & H^0(C_K).
\end{tikzcd} \tag{3.2}$$

Note that the rows of (3.2) are *not* exact. Now, we consider the two bottom exact sequences of the diagram (1.6*) of [12, p. 221]:

$$\begin{tikzcd}
& H^0(U_K) \arrow[r, hook, two heads] \arrow[d, two heads] & H^0(J_K) \arrow[r, two heads] \arrow[d, two heads] & \operatorname{Coker}(N\colon I_K \to I_k) \arrow[d, two heads] \\
\gamma \arrow[r, hook] & \operatorname{Coker}(N\colon \mathcal{E}_K \to \mathcal{E}_k) \arrow[r, two heads] & H^0(C_K) \arrow[r, two heads] & \operatorname{Coker}(N\colon \operatorname{Cl}_K \to \operatorname{Cl}_k).
\end{tikzcd}$$

It follows from (3.2) that the the condition (i) is equivalent to the surjectivity of $H^0(U_K) \to H^0(J_K)$ by Second Isomorphism Theorem, which is equivalent to the triviality of $\operatorname{Coker}(N\colon I_K \to I_k)$ hence of $\operatorname{Coker}(N\colon \operatorname{Cl}_K \to \operatorname{Cl}_k)$, i.e., the condition (ii) $N \operatorname{Cl}_K = \operatorname{Cl}_k$. $\square$

To apply our theorem to finite solvable groups, we need the following easy result from group theory.

Lemma 3.9. *Let G be a finite solvable group, and H be a subgroup of G. If $G = \widetilde{H} \cdot G' = H \cdot G'$, then we obtain $G = \widetilde{H}$.*

Proof. First, we see that $\widetilde{H} = H \cdot [H, G]$. Since G is solvable, the t-th commutator subgroup $G^{(t)}$ is trivial for some $t \in \mathbb{N}$. Now, we have $G' = [\widetilde{H} \cdot G', \widetilde{H} \cdot G'] \leq \widetilde{H} \cdot G^{(2)}$. Hence, by induction, we have $G^{(i)} \leq \widetilde{H} \cdot G^{(i+1)}$; therefore, we obtain

$$G^{(t-1)} \leq \widetilde{H},$$

and

$$G \leq \widetilde{H} \cdot G' \leq \widetilde{H} \cdot G^{(2)} \leq \cdots \leq \widetilde{H}.$$

This proves the assertion. □

For abelian group G, the two groups T and $\widetilde{T}$ coincide. In this case, Theorem 3.8 and Lemma 3.9 say that the condition $H_k \cap K = k$ implies the equality of the Galois group G of K/k with its global inertia group: $G = T = \widetilde{T}$. For this reason, Wyman's sufficient condition (3.1) for the splitting boils down to $H_k \cap K = k$ which generalizes the cyclic case (cf. Corollary 2.7).

Corollary 3.10. *If G is abelian, a sufficient condition for the splitting of* (1.3) *is $H_k \cap K = k$.*

Remark 3.11. The equalities $G = T = \widetilde{T}$ does not necessarily imply the splitting of (1.3), as we shall see in the following. First, the equalities $G = T = \widetilde{T}$ might not imply the arithmetic condition (3.1): $[K : k] = e_0$. For example, let p and q be two rational primes congruent to 1 modulo 4, $K = \mathbb{Q}(\sqrt{p}, \sqrt{q})$, and $k = \mathbb{Q}$. Since $\mathbb{Q}$ has no unramified extensions, we have $G = T$. There are two ramification subgroups, each of order 2. Thus $e_0 = 2 \neq 4 = [K : k] = |G|$.

Now, we come to our counter-example: Let $K = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3}, \sqrt{p_4})$ where p_i are rational primes congruent 1 modulo 4, and let $k = \mathbb{Q}$. For this extension, the hypothesis of Corollary 3.10 is fulfilled, in fact $H_k = \mathbb{Q}$. As before, we have $G = T$, but the group extension (1.3) does not split since one can show that $\delta^0_{K/k} \neq 1$, contrary to Lemmas 2.5 and 2.6.

Problem 3.12. Since $G = T = \widetilde{T}$ does not necessarily implies the arithmetic condition (3.1), the above condition $G = T = \widetilde{T}$ might not sufficient for the splitting. What additional condition to the condition $H_k \cap K = k$ will imply the splitting in the abelian case?

4. Some examples

Various examples of splitting and of non-splitting of the group extension (1.3) which appeared in the papers of Cornell and Rosen [5, Proposition 9], and Bond [2, Theorem 3.1, Propositions 3.7, 3.8, and Corollary 3.9] can be read off in terms of the triviality of certain number knots and by simple group theoretic arguments.

Example 4.1. Associated to the group extension (1.3) with factor set α and abelian kernel $A := \mathrm{Cl}_K$ is the central group extension with factor set α^*:

$$1 \longrightarrow A/[A, E] \longrightarrow E/[A, E] \longrightarrow G \longrightarrow 1 \tag{4.1}$$

where $E = \mathrm{Gal}(H_K/k)$, $A/[A,E] \simeq \mathrm{Gal}(H_K^*/K)$ and $E/[A,E] \simeq \mathrm{Gal}(H_K^*/k)$. Note that (4.1) is the pushforward of (1.3) along $E \to E/[A,E]$, so that the splitting of (1.3), i.e., $\alpha = 1$, implies that of (4.1).

Recall that a *central extension* is a (non-split) group extension where the base normal subgroup is contained in the center of the whole group. It is a *central stem extension* if, in addition, the base normal subgroup is contained in the derived subgroup of the whole group. Therefore, a central stem extension never splits. Examples of non-splitting of (4.1) now can be supplied. To illustrate this, by assuming $K = H_{K'}$, we have $A \le [E,E]$, and thus the extension (4.1) is a stem extension. So the example of non-splitting in Cornell and Rosen [5, Proposition 9] is still valid without any assumption on the class number of K.

To generalize Bond's examples, we first define a group

$$J := \mathrm{Coker}(H^{-1}(C_K) \to H^{-1}(\mathrm{Cl}_K)).$$

Remark 4.2. We let $j\colon \mathrm{Cl}_k \to \mathrm{Cl}_K$ be the extension map of ideals. In [3], Bond shows:

(1) The group J is a subgroup of $\mathrm{Ker}(j)$ consisting of those ideal classes of k that are in the image of the norm map from Cl_K to Cl_k and that capitulate in K, i.e., $J = \mathrm{Ker}(j) \cap N\,\mathrm{Cl}_K$. Moreover, if $H_k \cap K = k$, we have $\mathrm{Ker}(j) = J$. See also Proposition 0.2 and Remark 2.1 of [2].

(2) There exists an unramified abelian extension M of k such that $\mathrm{Ker}(j)/J \simeq \mathrm{Gal}(L/M)$, where $L = H_k \cap K$, and $|J|$ is divisible by $[M:k]$ by Suzuki's Theorem. Hence, if $J = 1$, by Remark 3.5, it is easy to see that $\mathrm{Ker}(j) \simeq \mathrm{Gal}(L/k)$, that is, $M = k$. See also [3, Theorem 2.3].

Recall that $H^{-1}(\mathrm{Cl}_K) = {}_N\mathrm{Cl}_K / I_G\,\mathrm{Cl}_K$. The group J fits in into a part of the commutative diagram with exact rows of (2.2) (cf. [12, Theorem 1]):

$$\begin{tikzcd}
H^{-1}(\mathcal{E}_K) \arrow[r] & H^{-1}(C_K) \arrow[r, "h"] & H^{-1}(\mathrm{Cl}_K) \arrow[r, two heads, "\lambda"] & J \\
H^{-1}(\mathcal{E}_K) \arrow[r] \arrow[u, equal] \arrow[d, two heads] & H^{-1}(C_K) \arrow[r] \arrow[u, equal] \arrow[d, two heads, "\simeq"] & \check{H}^{-1}(\mathrm{Cl}_K) \arrow[r, two heads] \arrow[u, hook] \arrow[d, "\simeq"] & \gamma_{K/k} \arrow[u] \arrow[d, equal] \\
\omega'_{K/k} \arrow[r] & \nu_{K/k} \arrow[r] & \delta_{K/k} \arrow[r, two heads] & \gamma_{K/k}
\end{tikzcd} \tag{4.2}$$

where $\check{H}^{-1}(\mathrm{Cl}_K) = {}_N\mathrm{Cl}_K / {}_N I_K \simeq \delta_{K/k}$.

Remark 4.3. Let α be the class of the group extension (1.3). If the group extension (1.3) splits, then λ is an isomorphism.

Indeed, letting $u \in H^2(C_K)$ be the fundamental class, the map $H^{-3}(\mathbb{Z}) \to H^{-1}(C_K)$ defined by $x \mapsto u \cup x$ is an isomorphism by Proposition 2.2(2). We consider the following commutative diagram:

$$\begin{tikzcd}
H^2(C_K) \otimes H^{-3}(\mathbb{Z}) \arrow[r] \arrow[d, "g\otimes\mathrm{id}"] & H^{-1}(C_K) \arrow[d, "h"] \\
H^2(\mathrm{Cl}_K) \otimes H^{-3}(\mathbb{Z}) \arrow[r] & H^{-1}(\mathrm{Cl}_K)
\end{tikzcd}$$

We know that $g(u) = \alpha$. Let $z \in H^{-1}(C_K)$. Then $z = u \cup x$ for some $x \in H^{-3}(\mathbb{Z})$, and so $h(z) = g(u) \cup x = \alpha \cup x$. Since the group extension (1.3) splits, we have $\alpha = 1$; thus, we derive that $h(z) = 1 \cup x$ and hence λ is an isomorphism.

We need the following cohomological characterization of finite nilpotent groups (see [11,19]):

Lemma 4.4. *A finite group G is nilpotent if and only if whenever M is a finite G-module for which $H^k(G, M) = 1$ for some k, then $H^n(G, M) = 1$ for all $n \in \mathbb{Z}$.*

Lemma 4.4 and the diagram (4.2) above lead us easily to the following general results:

Theorem 4.5. *Let K/k be a finite Galois extension with the Scholz's number knot $\nu_{K/k}$. Suppose that $\nu_{K/k} \simeq H^{-1}(C_K) \simeq H_2(G, \mathbb{Z})$, the Pontryagin dual of the Schur multiplier of G. Then we have:*

(1) *The splitting of (1.3) implies that $\nu_{-1} = 1$ and $H^{-1}(\mathrm{Cl}_K) \simeq J$. In particular, we have $\omega^0_{K/k} \simeq \nu_{K/k}$.*

(2) *Suppose that G is nilpotent. If $\nu_{-1} = 1$ and $J = 1$, then (1.3) splits.*

Proof. (1) Suppose that the group extension (1.3) splits. By Lemma 2.5, we have $\nu_{-1} = 1 = \nu_0$; by Lemma 2.8(1), we see that $\delta^0 = \nu_{-1} = 1$. Reading off in the diagram (4.2), this is equivalent to saying that $\omega^0_{K/k} \simeq \nu_{K/k}$ and $\check{H}^{-1}(\mathrm{Cl}_K) \simeq \gamma$. Hence, $H^{-1}(\mathcal{E}_K) \twoheadrightarrow H^{-1}(C_K) \simeq \nu_{K/k}$ is surjective, and so $H^{-1}(\mathrm{Cl}_K) \simeq J$.

(2) If $J = 1$, we have $H^{-1}(C_K) \twoheadrightarrow H^{-1}(\mathrm{Cl}_K)$ is surjective. If $\nu_{-1} = 1$, the Scholz's divisor knot $\delta^0 = 1$ by Lemma 2.8(1); thus, by Scholz's knot sequence (1.2), we see that $\omega^0_{K/k} \simeq \nu_{K/k}$ and hence $H^{-1}(\mathcal{E}_K) \twoheadrightarrow H^{-1}(C_K)$ is also surjective. Reading off in the diagram (4.2), these imply that $H^{-1}(\mathrm{Cl}_K) = 1$. Since G is nilpotent, we obtain that $H^2(\mathrm{Cl}_K) = 1$; thus this map $H^2(C_K) \to H^2(\mathrm{Cl}_K)$ is a trivial map and this is equivalent to the splitting of (1.3). $\square$

Remark 4.6. It should be noticed that if $J = 1$, then every ideal of k which capitulates in K capitulates in $H_k \cap K$. Indeed, if $j_{L/k}\colon \mathrm{Cl}_k \to \mathrm{Cl}_L$ is the extension map of ideals, where $L = H_k \cap K$, then it is easy to see that $\mathrm{Ker}(j_{L/k}) \subseteq \mathrm{Ker}(j)$. By Lemma 2.6, the norm map $N_{K/L}\colon \mathrm{Cl}_K \to \mathrm{Cl}_L$ is surjective. Therefore, we have $N\,\mathrm{Cl}_K = N_{L/k}\,\mathrm{Cl}_L$ and $\mathrm{Ker}(j_{L/k}) \cap N_{L/k}\,\mathrm{Cl}_L \subseteq \mathrm{Ker}(j) \cap N\,\mathrm{Cl}_K = J$. By Remark 4.2, we have $\mathrm{Ker}(j_{L/k}) = \mathrm{Ker}(j)$.

Theorem 4.5 covers the following Bond's examples:

Example 4.7. Let k be an imaginary quadratic field and K/k be an abelian unramified extension of odd degree. In the unramified case, all decomposition groups are cyclic, and $NU_K = U_k$. Thus, the Scholz's number knot $\nu_{K/k} \simeq H^{-1}(C_K) \simeq H^{-3}(\mathbb{Z})$, and the Scholz's unit knot $\omega^0_{K/k} := E_k \cap NU_K/E_k \cap NK^\times = E_k/E_k \cap NK^\times = 1$, since $E_k = \{\pm 1\} \leq NK^\times$, by assumption on k and the odd degree of K/k. Moreover, in the abelian unramified case, J can be identified with $\mathrm{Ker}(j) \cap N\,\mathrm{Cl}_K$ (see [1, Theorem 1]). By Theorem 4.5, necessary and sufficient conditions for the splitting of (1.3) in this case are that G be cyclic and $J = 1$. See also Theorem 3.1 of [2].

Example 4.8. Let k be an imaginary quadratic field and K/k be the unramified abelian extension of degree 2^t, $t \geq 1$. Suppose that -1 is not the norm of a unit of K. Since the Scholz's unit knot ω^0 has order 1 or 2, depending on whether or not $-1 \in NK^\times$, the splitting of (1.3) is equivalent to $J = 1$, and whether the Schur multiplier of G has order 1 or 2. The latter is known to be depending on whether G is cyclic or $G \simeq \mathbb{Z}_{2^{t-1}} \times \mathbb{Z}_2$. So we have two cases for the splitting:

(1) If $G \simeq \mathbb{Z}_{2^{t-1}} \times \mathbb{Z}_2$, then the splitting is equivalent to $J = 1$ and -1 is not a global norm from K.

(2) If G is cyclic, then the splitting is equivalent to $J = 1$.

See also Propositions 3.7, 3.8, and Corollary 3.9 of [2].

Acknowledgments

We thank the anonymous referee for a careful reading of the original manuscript and many helpful suggestions.

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Yih-Jeng Yu
Department of Mathematical Sciences, National Chengchi University, 11605 Taipei, Taiwan
E-mail address: `yjyu@nccu.edu.tw`