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Singularity Formation of the Non-barotropic Compressible Magnetohydrodynamic Equations Without Heat Conductivity

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Abstract. We study the singularity formation of strong solutions to the three-dimensional full compressible magnetohydrodynamic equations with zero heat conduction in a bounded domain. We show that for the initial density allowing vacuum, the strong solution exists globally if the density ρ, the magnetic field $\mathbf{b}$, and the pressure P satisfy $\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\mathbf{b}\|_{L^\infty(0,T;L^6)} + \|P\|_{L^\infty(0,T;L^\infty)} < \infty$ and the coefficients of viscosity verity $3\mu > \lambda$. This extends the corresponding results in Duan (2017), Fan et al. (2018) [1,2] where a blow-up criterion in terms of the upper bounds of the density, the magnetic field and the temperature was obtained under the condition $2\mu > \lambda$. Our proof relies on some delicate energy estimates.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a domain, the motion of a viscous, compressible, and heat conducting magnetohydrodynamic (MHD) flow in Ω can be described by the full compressible MHD equations

$$
\begin{cases}
\rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \\
(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) - \mu\Delta\mathbf{u} - (\lambda+\mu)\nabla\operatorname{div}\mathbf{u} + \nabla P = \mathbf{b}\cdot\nabla\mathbf{b} - \frac{1}{2}\nabla|\mathbf{b}|^2, \\
c_\nu[(\rho\theta)_t + \operatorname{div}(\rho\mathbf{u}\theta)] + P\operatorname{div}\mathbf{u} - \kappa\Delta\theta = 2\mu|\mathfrak{D}(\mathbf{u})|^2 + \lambda(\operatorname{div}\mathbf{u})^2 + \nu|\operatorname{curl}\mathbf{b}|^2, \\
\mathbf{b}_t - \mathbf{b}\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{b} + \mathbf{b}\operatorname{div}\mathbf{u} = \nu\Delta\mathbf{b}, \\
\operatorname{div}\mathbf{b} = 0.
\end{cases}
\tag{1.1}
$$

Here, $t \geq 0$ is the time, $x \in \Omega$ is the spatial coordinate, and ρ, $\mathbf{u}$, $P = R\rho\theta$ $(R > 0)$, θ, $\mathbf{b}$ are the fluid density, velocity, pressure, absolute temperature, and the magnetic field respectively; $\mathfrak{D}(\mathbf{u})$ denotes the deformation tensor given by

$$\mathfrak{D}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{tr}}).$$

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The constant viscosity coefficients μ and λ satisfy the physical restrictions

$$\mu > 0, \quad 2\mu + 3\lambda \geq 0. \tag{1.2}$$

Positive constants c_ν, κ, and ν are respectively the heat capacity, the ratio of the heat conductivity coefficient over the heat capacity, and the magnetic diffusive coefficient.

There is huge literature on the studies about the theory of well-posedness of solutions to the Cauchy problem and the initial boundary value problem (IBVP) for the compressible MHD system due to the physical importance, complexity, rich phenomena, and mathematical challenges, refer to [3, 6–8, 15, 18, 19, 24] and references therein. However, many physical important and mathematical fundamental problems are still open due to the lack of smoothing mechanism and the strong nonlinearity. Kawashima [14] first obtained the global existence and uniqueness of classical solutions to the multi-dimensional compressible MHD equations when the initial data are close to a non-vacuum equilibrium in H^3-norm. When the initial density allows vacuum, the local well-posedness of strong solutions to the initial boundary value problem of 3D non-isentropic MHD equations has been obtained by Fan-Yu [3]. For general large initial data, Hu-Wang [7, 8] proved the global existence of weak solutions with finite energy in Lions' framework for compressible Navier-Stokes equations [4, 17] provided the adiabatic exponent is suitably large. Recently, Li-Xu-Zhang [15] established the global existence and uniqueness of classical solutions to the Cauchy problem for the isentropic compressible MHD system in 3D with smooth initial data which are of small energy but possibly large oscillations and vacuum, which generalized the result for compressible Navier-Stokes equations obtained by Huang-Li-Xin [11]. Very recently, Hong-Hou-Peng-Zhu [6] improved the result in [15] to allow the initial energy large as long as the adiabatic exponent is close to 1 and ν is suitably large. Furthermore, Lv-Shi-Xu [19] established the global existence and uniqueness of strong solutions to the 2D MHD equations provided that the smooth initial data are of small total energy. Nevertheless, it is an outstanding challenging open problem to investigate the global well-posedness for general large strong solutions with vacuum.

Therefore, it is important to study the mechanism of blow-up and structure of possible singularities of strong (or classical) solutions to the compressible MHD equations. The pioneering work can be traced to [5], where He and Xin proved Serrin's criterion for strong solutions to the incompressible MHD system, that is,

$$\lim_{T\to T^*} \|\mathbf{u}\|_{L^s(0,T;L^r)} = \infty \quad \text{for } \frac{2}{s} + \frac{3}{r} = 1,\ 3 < r \leq \infty, \tag{1.3}$$

here T^* is the finite blow up time. For the three-dimensional compressible isentropic MHD system, Xu-Zhang [21] obtained the following Serrin type criterion

$$\lim_{T\to T^*} \left(\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\mathbf{u}\|_{L^s(0,T;L^r)}\right) = \infty, \tag{1.4}$$

where r and s are as in (1.3). Surprisingly, Huang-Li [9] showed (1.4) also holds true for the Cauchy problem and the IBVP of 3D full compressible MHD system. Recently, under the assumption

$$2\mu > \lambda, \tag{1.5}$$

Huang-Wang [12] showed that

$$\lim_{T\to T^*}\left(\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\theta\|_{L^\infty(0,T;L^\infty)} + \|\mathbf{b}\|_{L^\infty(0,T;L^\infty)}\right) = \infty \tag{1.6}$$

for the system (1.1) with $\kappa = \nu = 0$. Then the authors [1, 2] established (1.6) to the system (1.1) with $\kappa = 0$ provided that (1.5) holds true. Very recently, for the Cauchy problem of the system (1.1) with $\kappa = 0$, Zhong [23] proved that

$$\lim_{T\to T^*}\left(\|\mathfrak{D}(\mathbf{u})\|_{L^1(0,T;L^\infty)} + \|P\|_{L^\infty(0,T;L^\infty)}\right) = \infty$$

provided that $3\mu > \lambda$. For more information on the blow-up criteria of compressible MHD equations, we refer to [1, 2, 22, 25] and the references therein.

When $\kappa = 0$, and without loss of generality, take $c_\nu = R = 1$, the system (1.1) can be written as

$$\begin{cases} \rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \\ (\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) - \mu\Delta\mathbf{u} - (\lambda+\mu)\nabla\operatorname{div}\mathbf{u} + \nabla P = \mathbf{b}\cdot\nabla\mathbf{b} - \frac{1}{2}\nabla|\mathbf{b}|^2, \\ P_t + \operatorname{div}(P\mathbf{u}) + P\operatorname{div}\mathbf{u} = 2\mu|\mathfrak{D}(\mathbf{u})|^2 + \lambda(\operatorname{div}\mathbf{u})^2 + \nu|\operatorname{curl}\mathbf{b}|^2, \\ \mathbf{b}_t - \mathbf{b}\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{b} + \mathbf{b}\operatorname{div}\mathbf{u} = \nu\Delta\mathbf{b}, \\ \operatorname{div}\mathbf{b} = 0. \end{cases} \tag{1.7}$$

The present paper aims at giving a blow-up criterion of strong solutions to the system (1.7) in a bounded simply connected smooth domain $\Omega \subset \mathbb{R}^3$ with the initial condition

$$(\rho, \rho\mathbf{u}, P, \mathbf{b})(x, 0) = (\rho_0, \rho_0\mathbf{u}_0, P_0, \mathbf{b}_0)(x), \quad x \in \Omega, \tag{1.8}$$

and the boundary condition

$$\mathbf{u} = \mathbf{0}, \quad \mathbf{b} = \mathbf{0} \quad \text{on } \partial\Omega. \tag{1.9}$$

Before stating our main result, we first explain the notations and conventions used throughout this paper. We denote by

$$\int \cdot\, \mathrm{d}x = \int_\Omega \cdot\, \mathrm{d}x.$$

For $1 \le p \le \infty$ and integer $k \ge 0$, the standard Sobolev spaces are denoted by

$$L^p = L^p(\Omega), \quad W^{k,p} = W^{k,p}(\Omega), \quad H^k = H^{k,2}(\Omega),$$
$$H_0^1 = \{u \in H^1 \mid u = 0 \text{ on } \partial\Omega\}, \quad D^{k,p} = \{u \in L^1_{\text{loc}} \mid \nabla^k u \in L^p\}.$$

Now we define precisely what we mean by strong solutions to the problem (1.7)–(1.9).

Definition 1.1 (Strong solutions)**.** $(\rho, \mathbf{u}, P, \mathbf{b})$ is called a strong solution to (1.7)–(1.9) in $\Omega \times (0,T)$, if for some $q_0 > 3$,

$$\begin{cases} \rho \ge 0, \quad \rho \in C([0,T]; W^{1,q_0}), \quad \rho_t \in C([0,T]; L^{q_0}), \\ (\mathbf{u}, \mathbf{b}) \in C([0,T]; H_0^1 \cap H^2) \cap L^2(0,T; D^{2,q_0}), \quad \mathbf{b} \in C([0,T]; H^2), \\ (\mathbf{u}_t, \mathbf{b}_t) \in L^2(0,T; D^{1,2}), \quad (\sqrt{\rho}\mathbf{u}_t, \mathbf{b}_t) \in L^\infty(0,T; L^2), \\ P \ge 0, \quad P \in C([0,T]; W^{1,q_0}), \quad P_t \in C([0,T]; L^{q_0}), \end{cases}$$

and $(\rho, \mathbf{u}, P, \mathbf{b})$ satisfies both (1.7) almost everywhere in $\Omega \times (0,T)$ and (1.8) almost everywhere in Ω.

Our main result reads as follows:

Theorem 1.2. *For constant $q \in (3,6]$, assume that the initial data $(\rho_0 \ge 0, \mathbf{u}_0, P_0 \ge 0, \mathbf{b}_0)$ satisfies*

$$(\rho_0, P_0) \in W^{1,q}(\Omega), \quad (\mathbf{u}_0, \mathbf{b}_0) \in H_0^1(\Omega) \cap H^2(\Omega), \quad \operatorname{div} \mathbf{b}_0 = 0, \tag{1.10}$$

and the compatibility conditions

$$-\mu\Delta\mathbf{u}_0 - (\lambda+\mu)\nabla \operatorname{div} \mathbf{u}_0 + \nabla P_0 - (\operatorname{curl} \mathbf{b}_0) \times \mathbf{b}_0 = \sqrt{\rho_0}\mathbf{g} \tag{1.11}$$

for some $\mathbf{g} \in L^2(\Omega)$. Let $(\rho, \mathbf{u}, P, \mathbf{b})$ be the strong solution to the problem (1.7)–(1.9). If $T^* < \infty$ is the maximal time of existence for that solution, then we have

$$\lim_{T \to T^*} \left(\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\mathbf{b}\|_{L^\infty(0,T;L^6)} + \|P\|_{L^\infty(0,T;L^\infty)} \right) = \infty \tag{1.12}$$

provided that

$$3\mu > \lambda. \tag{1.13}$$

Several remarks are in order.

Remark 1.3. The local existence of a unique strong solution with initial data as in Theorem 1.2 has been established in [3]. Hence, the maximal time T^* is well-defined.

Remark 1.4. Compared with [1,2], according to (1.12), the L^∞ bound of the temperature θ is not the key point to make sure that the solution $(\rho, \mathbf{u}, P, \mathbf{b})$ is a global one, and it may go to infinity in the vacuum region within the life span of our strong solution.

Remark 1.5. In [1, 2], to obtain higher order derivatives of the solutions, the restriction $2\mu > \lambda$ plays a crucial role in the analysis. In fact, the condition $2\mu > \lambda$ is only used to get the upper bound of $\int \rho|\mathbf{u}|^4 \,\mathrm{d}x$. Here, we derive the upper bound of $\int \rho|\mathbf{u}|^4 \,\mathrm{d}x$ under the assumption $3\mu > \lambda$ (see Lemma 3.2), which is weaker than $2\mu > \lambda$ due to $\mu > 0$. Moreover, since Ω is bounded, we see that the bound of $\|\mathbf{b}\|_{L^\infty(0,T;L^\infty)}$ implies that $\|\mathbf{b}\|_{L^\infty(0,T;L^6)}$ is bounded. Thus, the blow-up criterion (1.12) is an extension towards (1.6) in [1, 2].

If $\mathbf{b} \equiv \mathbf{b}_0 \equiv \mathbf{0}$, Theorem 1.2 directly yields the following blow-up criterion of the non-isentropic Navier-Stokes equations without heat-conductivity.

Theorem 1.6. *For constant $q \in (3, 6]$, assume that the initial data $(\rho_0 \geq 0, \mathbf{u}_0, P_0 \geq 0)$ satisfies* (1.10) *and the compatibility conditions* (1.11)*. Let $(\rho, \mathbf{u}, P)$ be the strong solution to the problem* (1.7)–(1.9) *with $\mathbf{b} = \mathbf{0}$. If $T^* < \infty$ is the maximal time of existence for that solution, then we have*

$$\lim_{T\to T^*} \left(\|\rho\|_{L^\infty(0,T;L^\infty)} + \|P\|_{L^\infty(0,T;L^\infty)}\right) = \infty$$

provided that $3\mu > \lambda$.

Remark 1.7. It should be noted that Theorem 1.6 generalizes the result obtained by Huang and Xin [13]. Compared with their result, the coefficients of viscosity is relaxed to $3\mu > \lambda$ in our work, while $\mu > 4\lambda$ is needed in [13].

We now make some comments on the analysis of this paper. We mainly make use of continuation argument to prove Theorem 1.2. That is, suppose that (1.12) were false, i.e.,

$$\lim_{T\to T^*} \left(\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\mathbf{b}\|_{L^\infty(0,T;L^6)} + \|P\|_{L^\infty(0,T;L^\infty)}\right) \leq M_0 < \infty.$$

We want to show that

$$\sup_{0\leq t\leq T^*} (\|(\rho, P)\|_{W^{1,q}} + \|\nabla \mathbf{u}\|_{H^1} + \|\mathbf{b}\|_{H^2}) \leq C < +\infty.$$

It should be pointed out that the crucial techniques of proofs in [1,2] cannot be adapted directly to the situation treated here, since their arguments depend crucially on the boundedness of the magnetic field and $2\mu > \lambda$. Moreover, technically, since the magnetic field is strongly coupled with the velocity field of the fluid in the compressible MHD system, some new difficulties arise in comparison with the problem for the compressible Navier-Stokes equations studied in [13].

To overcome these difficulties mentioned above, some new ideas are needed. First, motivated by [20], we derive the upper bound of $\int \rho|\mathbf{u}|^4 \,\mathrm{d}x$ under the condition (1.13), As a byproduct, we also get the upper bound of $\int_0^T \int |\mathbf{u}|^2|\nabla \mathbf{u}|^2 \,\mathrm{d}x\mathrm{d}t$, which plays a crucial role in the proof of $L^\infty(0, T; L^2)$-norm of $\nabla \mathbf{u}$ and $\nabla \mathbf{b}$ (see Lemma 3.3). Secondly, the key a priori estimates on the $L_t^\infty L_x^q$-norm of $(\nabla\rho, \nabla P)$ and the $L_t^1 L_x^\infty$-norm of the velocity gradient can be obtained (see Lemma 3.5) simultaneously by solving a logarithm Gronwall inequality based on a logarithm estimate for the Lamé system (see Lemma 2.2) and the a priori estimates we have derived.

The rest of this paper is organized as follows. In Section 2, we collect some elementary facts and inequalities that will be used later. Section 3 is devoted to the proof of Theorem 1.2.

2. Preliminaries

In this section, we will recall some known facts and elementary inequalities that will be used frequently later.

First, the following logarithm estimate will be used to estimate $\|\nabla \mathbf{u}\|_{L^\infty}$.

Lemma 2.1. *For $q \in (3,\infty)$, there is a constant $C(q) > 0$ such that for all $\nabla \mathbf{v} \in L^2 \cap D^{1,q}$, it holds that*

$$\|\nabla \mathbf{v}\|_{L^\infty} \le C\left(\|\operatorname{div} \mathbf{v}\|_{L^\infty} + \|\operatorname{curl} \mathbf{v}\|_{L^\infty}\right)\log(e + \|\nabla^2 \mathbf{v}\|_{L^q}) + C\|\nabla \mathbf{v}\|_{L^2} + C.$$

Proof. See [10, Lemma 2.3]. □

Finally, we consider the following Lamé system

$$\begin{cases} -\mu\Delta \mathbf{U} - (\lambda+\mu)\nabla \operatorname{div} \mathbf{U} = \mathbf{F} & \text{if } x \in \Omega, \\ \mathbf{U} = \mathbf{0} & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\mathbf{U} = (U^1, U^2, U^3)$, $\mathbf{F} = (F^1, F^2, F^3)$, and μ, λ satisfy (1.2).

Next, the following logarithm estimate for the Lamé system (2.1) will be used to estimate $\|\nabla \mathbf{u}\|_{L^\infty}$ and $\|\nabla\rho\|_{L^2\cap L^q}$.

Lemma 2.2. *Let μ, λ satisfy (1.2). Assume that $\mathbf{F} = \operatorname{div} \mathbf{g}$ where $\mathbf{g} = (g_{kj})_{3\times 3}$ with $g_{kj} \in L^2 \cap L^r \cap D^{1,q}$ for $k, j = 1, \dots, 3$, $r \in (1,\infty)$, and $q \in (3,\infty)$. Then the Lamé system (2.1) has a unique solution $\mathbf{U} \in H_0^1 \cap D^{1,r} \cap D^{2,q}$, and there exists a generic positive constant C depending only on μ, λ, q, and r such that*

$$\|\nabla \mathbf{U}\|_{L^r} \le C\|\mathbf{g}\|_{L^r}$$

and

$$\|\nabla \mathbf{U}\|_{L^\infty} \le C\left(1 + \log(e + \|\nabla \mathbf{g}\|_{L^q})\|\mathbf{g}\|_{L^\infty} + \|\mathbf{g}\|_{L^r}\right).$$

Proof. See [9, Lemma 2.3]. □

3. Proof of Theorem 1.2

Let $(\rho, \mathbf{u}, P, \mathbf{b})$ be a strong solution described in Theorem 1.2. Suppose that (1.12) were false, that is, there exists a constant $M_0 > 0$ such that

$$\lim_{T\to T^*} \left(\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\mathbf{b}\|_{L^\infty(0,T;L^6)} + \|P\|_{L^\infty(0,T;L^\infty)}\right) \le M_0 < \infty. \tag{3.1}$$

First of all, we have the following standard estimate.

Lemma 3.1. *Under the condition* (3.1), *it holds that for any* $T \in [0, T^*)$,

$$\sup_{0\le t\le T} \left(\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 + \|\mathbf{b}\|_{L^2}^2\right) + \int_0^T \left(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{b}\|_{L^2}^2\right) \mathrm{d}t \le C, \tag{3.2}$$

where and in what follows, C, C_1, C_2 stand for generic positive constants depending only on M_0, λ, μ, ν, T^*, and the initial data.

Proof. Multiplying $(1.7)_2$ by $\mathbf{u}$ and integrating (by parts) over Ω, we derive that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|\mathbf{u}|^2\,\mathrm{d}x + \int \left[\mu|\nabla\mathbf{u}|^2 + (\lambda+\mu)(\operatorname{div}\mathbf{u})^2\right]\,\mathrm{d}x\\ &= \int \left(P + \frac{1}{2}|\mathbf{b}|^2\right)\operatorname{div}\mathbf{u}\,\mathrm{d}x + \int \mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{u}\,\mathrm{d}x.\end{aligned} \tag{3.3}$$

Multiplying $(1.7)_4$ by $\mathbf{b}$ and integrating (by parts) over Ω, we get that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\mathbf{b}|^2\,\mathrm{d}x + \nu\int|\nabla\mathbf{b}|^2\,\mathrm{d}x + \int|\mathbf{b}|^2\operatorname{div}\mathbf{u}\,\mathrm{d}x = \int \mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}\,\mathrm{d}x - \int \mathbf{u}\cdot\nabla\mathbf{b}\cdot\mathbf{b}\,\mathrm{d}x. \tag{3.4}$$

Due to $\operatorname{div}\mathbf{b} = 0$ and $\mathbf{u}|_{\partial\Omega} = \mathbf{0}$, we have

$$\int \mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{u}\,\mathrm{d}x = \int b^i\partial_i b^j u^j\,\mathrm{d}x = -\int \mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}\,\mathrm{d}x. \tag{3.5}$$

Similarly, one obtains

$$-\int \mathbf{u}\cdot\nabla\mathbf{b}\cdot\mathbf{b}\,\mathrm{d}x = -\int u^i\partial_i b^j b^j\,\mathrm{d}x = \int|\mathbf{b}|^2\operatorname{div}\mathbf{u}\,\mathrm{d}x + \int \mathbf{u}\cdot\nabla\mathbf{b}\cdot\mathbf{b}\,\mathrm{d}x,$$

and thus

$$-\int \mathbf{u}\cdot\nabla\mathbf{b}\cdot\mathbf{b}\,\mathrm{d}x = \frac{1}{2}\int|\mathbf{b}|^2\operatorname{div}\mathbf{u}\,\mathrm{d}x. \tag{3.6}$$

Combining (3.3)–(3.6), we deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\rho|\mathbf{u}|^2 + |\mathbf{b}|^2\right)\,\mathrm{d}x + \int\left[\mu|\nabla\mathbf{u}|^2 + (\lambda+\mu)(\operatorname{div}\mathbf{u})^2 + \nu|\nabla\mathbf{b}|^2\right]\,\mathrm{d}x = \int P\operatorname{div}\mathbf{u}\,\mathrm{d}x. \tag{3.7}$$

By Cauchy-Schwarz inequality, the right-hand side of (3.7) can be bounded by

$$\frac{\mu}{2}\int(\operatorname{div}\mathbf{u})^2\,\mathrm{d}x+\frac{1}{2\mu}\int P^2\,\mathrm{d}x.$$

This together with (3.7) and (3.1) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+\|\mathbf{b}\|_{L^2}^2\right)+\mu\|\nabla\mathbf{u}\|_{L^2}^2+\nu\|\nabla\mathbf{b}\|_{L^2}^2\le C. \tag{3.8}$$

So the desired (3.2) follows from (3.8) integrated with respect to t. This completes the proof of Lemma 3.1. □

We next improve the regularity of the density ρ and the velocity $\mathbf{u}$. We start with a high energy estimate for the velocity under the conditions (1.13) and (3.1).

Lemma 3.2. *Under the condition* (3.1), *it holds that for any* $T\in[0,T^*)$,

$$\int\rho|\mathbf{u}|^4\,\mathrm{d}x+\int_0^T\int|\mathbf{u}||\nabla\mathbf{u}|^2\,\mathrm{d}x\mathrm{d}t\le C \tag{3.9}$$

provided that $3\mu>\lambda$.

Proof. We first show that the pressure P is always nonnegative before the blow-up time T^*. Indeed, it follows from $(1.7)_3$ that

$$P_t+\mathbf{u}\cdot\nabla P+2P\operatorname{div}\mathbf{u}=F\triangleq 2\mu|\mathfrak{D}(\mathbf{u})|^2+\lambda(\operatorname{div}\mathbf{u})^2+\nu|\operatorname{curl}\mathbf{b}|^2\ge 0. \tag{3.10}$$

Define particle path before blowup time

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}(x,t)=\mathbf{u}(\mathbf{X}(x,t),t),\quad \mathbf{X}(x,0)=x.$$

Thus, along particle path, we obtain from (3.10) that

$$\frac{\mathrm{d}}{\mathrm{d}t}P(\mathbf{X}(x,t),t)=-2P\operatorname{div}\mathbf{u}+F,$$

which implies

$$P(\mathbf{X}(x,t),t)=\exp\left(-2\int_0^t\operatorname{div}\mathbf{u}\,\mathrm{d}s\right)\left[P_0+\int_0^t\exp\left(2\int_0^s\operatorname{div}\mathbf{u}\,\mathrm{d}\tau\right)F\,\mathrm{d}s\right]\ge 0.$$

Next, inspired by [20], multiplying $(1.7)_2$ by $4|\mathbf{u}|^2\mathbf{u}$ and integrating the resulting equation over Ω yield that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|\mathbf{u}|^4\,\mathrm{d}x+4\int\left[\mu|\mathbf{u}|^2|\nabla\mathbf{u}|^2+(\lambda+\mu)|\mathbf{u}|^2(\operatorname{div}\mathbf{u})^2+\frac{\mu|\nabla|\mathbf{u}|^2|^2}{2}\right]\mathrm{d}x\\&=4\int\operatorname{div}(|\mathbf{u}|^2\mathbf{u})P\,\mathrm{d}x-8(\lambda+\mu)\int\operatorname{div}\mathbf{u}|\mathbf{u}|\mathbf{u}\cdot\nabla|\mathbf{u}|\,\mathrm{d}x\\&\quad+4\int|\mathbf{u}|^2\mathbf{u}\cdot\left(\operatorname{div}(\mathbf{b}\otimes\mathbf{b})-\nabla\left(\frac{|\mathbf{b}|^2}{2}\right)\right)\mathrm{d}x\\&\le 4\int\operatorname{div}(|\mathbf{u}|^2\mathbf{u})P\,\mathrm{d}x-8(\lambda+\mu)\int\operatorname{div}\mathbf{u}|\mathbf{u}|\mathbf{u}\cdot\nabla|\mathbf{u}|\,\mathrm{d}x+12\int|\mathbf{u}|^2|\nabla\mathbf{u}||\mathbf{b}|^2\,\mathrm{d}x.\end{aligned} \tag{3.11}$$

For the last term of the right-hand side of (3.11), one obtains from Hölder's inequality, Sobolev's inequality, and (3.1) that, for any $\varepsilon_1 \in (0,1)$,

$$\begin{aligned}12\int |\mathbf{u}|^2|\nabla \mathbf{u}||\mathbf{b}|^2\,\mathrm{d}x &\le 4\mu\varepsilon_1\int |\mathbf{u}|^2|\nabla\mathbf{u}|^2\,\mathrm{d}x + C(\mu,\varepsilon_1)\int |\mathbf{u}|^2|\mathbf{b}|^4\,\mathrm{d}x\\ &\le 4\mu\varepsilon_1\int |\mathbf{u}|^2|\nabla\mathbf{u}|^2\,\mathrm{d}x + C(\varepsilon_1)\|\mathbf{u}\|_{L^6}^2\|\mathbf{b}\|_{L^6}^4\\ &\le 4\mu\varepsilon_1\int |\mathbf{u}|^2|\nabla\mathbf{u}|^2\,\mathrm{d}x + C(\varepsilon_1)\|\nabla\mathbf{u}\|_{L^2}^2,\end{aligned}$$

which together with (3.11) leads to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|\mathbf{u}|^4\,\mathrm{d}x + 4\int\left[\mu(1-\varepsilon_1)|\mathbf{u}|^2|\nabla\mathbf{u}|^2 + (\lambda+\mu)|\mathbf{u}|^2(\operatorname{div}\mathbf{u})^2 + \frac{\mu|\nabla|\mathbf{u}|^2|^2}{2}\right]\mathrm{d}x\\ &\le 4\int \operatorname{div}(|\mathbf{u}|^2\mathbf{u})P\,\mathrm{d}x - 8(\lambda+\mu)\int \operatorname{div}\mathbf{u}|\mathbf{u}|\mathbf{u}\cdot\nabla|\mathbf{u}|\,\mathrm{d}x + C(\varepsilon_1)\|\nabla\mathbf{u}\|_{L^2}^2.\end{aligned}$$

Consequently, we have

(3.12)
$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|\mathbf{u}|^4\,\mathrm{d}x + 4\int_{\Omega\cap\{|\mathbf{u}|>0\}}\left[\mu(1-\varepsilon_1)|\mathbf{u}|^2|\nabla\mathbf{u}|^2 + (\lambda+\mu)|\mathbf{u}|^2(\operatorname{div}\mathbf{u})^2 + \frac{\mu|\nabla|\mathbf{u}|^2|^2}{2}\right]\mathrm{d}x\\ &\le 4\int_{\Omega\cap\{|\mathbf{u}|>0\}} \operatorname{div}(|\mathbf{u}|^2\mathbf{u})P\,\mathrm{d}x - 8(\lambda+\mu)\int_{\Omega\cap\{|\mathbf{u}|>0\}} \operatorname{div}\mathbf{u}|\mathbf{u}|\mathbf{u}\cdot\nabla|\mathbf{u}|\,\mathrm{d}x + C(\varepsilon_1)\|\nabla\mathbf{u}\|_{L^2}^2.\end{aligned}$$

Direct calculations give that for $x \in \Omega\cap\{|\mathbf{u}|>0\}$,

$$|\mathbf{u}|^2|\nabla\mathbf{u}|^2 = |\mathbf{u}|^4\left|\nabla\left(\frac{\mathbf{u}}{|\mathbf{u}|}\right)\right|^2 + |\mathbf{u}|^2|\nabla|\mathbf{u}||^2, \tag{3.13}$$

$$|\mathbf{u}|\operatorname{div}\mathbf{u} = |\mathbf{u}|^2\operatorname{div}\left(\frac{\mathbf{u}}{|\mathbf{u}|}\right) + \mathbf{u}\cdot\nabla|\mathbf{u}|. \tag{3.14}$$

For $\varepsilon_1, \varepsilon_2 \in (0,1)$, we now define a nonnegative function as follows

$$k(\varepsilon_1,\varepsilon_2) = \begin{cases}\frac{\mu\varepsilon_2(3-\varepsilon_1)}{\lambda+\varepsilon_1\mu} & \text{if } \lambda+\varepsilon_1\mu>0,\\ 0 & \text{if } \lambda+\varepsilon_1\mu\le 0.\end{cases} \tag{3.15}$$

We prove (3.9) in two cases.

Case 1: we assume that

$$\int_{\Omega\cap\{|\mathbf{u}|>0\}}|\mathbf{u}|^4\left|\nabla\left(\frac{\mathbf{u}}{|\mathbf{u}|}\right)\right|^2\mathrm{d}x \le k(\varepsilon_1,\varepsilon_2)\int_{\Omega\cap\{|\mathbf{u}|>0\}}|\mathbf{u}|^2|\nabla|\mathbf{u}||^2\,\mathrm{d}x. \tag{3.16}$$

It follows from (3.12) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|\mathbf{u}|^4\,\mathrm{d}x + 4\int_{\Omega\cap\{|\mathbf{u}|>0\}}\Psi\,\mathrm{d}x \le 4\int_{\Omega\cap\{|\mathbf{u}|>0\}}\operatorname{div}(|\mathbf{u}|^2\mathbf{u})P\,\mathrm{d}x + C(\varepsilon_1)\|\nabla\mathbf{u}\|_{L^2}^2, \tag{3.17}$$

where

$$\Psi \triangleq \mu(1-\varepsilon_1)|\mathbf{u}|^2|\nabla \mathbf{u}|^2 + (\lambda+\mu)|\mathbf{u}|^2(\operatorname{div}\mathbf{u})^2 + 2\mu|\mathbf{u}|^2|\nabla|\mathbf{u}||^2 + 2(\lambda+\mu)\operatorname{div}\mathbf{u}|\mathbf{u}|\mathbf{u}\cdot\nabla|\mathbf{u}|.$$

Employing (3.13) and (3.14), we find that

$$\begin{aligned}
\Psi &= \mu(1-\varepsilon_1)|\mathbf{u}|^2|\nabla \mathbf{u}|^2 + (\lambda+\mu)|\mathbf{u}|^2(\operatorname{div}\mathbf{u})^2 + 2\mu|\mathbf{u}|^2|\nabla|\mathbf{u}||^2 \\
&\quad + 2(\lambda+\mu)|\mathbf{u}|^2\operatorname{div}\left(\frac{\mathbf{u}}{|\mathbf{u}|}\right)\mathbf{u}\cdot\nabla|\mathbf{u}| + 2(\lambda+\mu)|\mathbf{u}\cdot\nabla|\mathbf{u}||^2 \\
&= \mu(1-\varepsilon_1)\left(|\mathbf{u}|^4\left|\nabla\left(\frac{\mathbf{u}}{|\mathbf{u}|}\right)\right|^2 + |\mathbf{u}|^2|\nabla|\mathbf{u}||^2\right) + (\lambda+\mu)\left(|\mathbf{u}|^2\operatorname{div}\left(\frac{\mathbf{u}}{|\mathbf{u}|}\right) + \mathbf{u}\cdot\nabla|\mathbf{u}|\right)^2 \\
&\quad + 2\mu|\mathbf{u}|^2|\nabla|\mathbf{u}||^2 + 2(\lambda+\mu)|\mathbf{u}|^2\operatorname{div}\left(\frac{\mathbf{u}}{|\mathbf{u}|}\right)\mathbf{u}\cdot\nabla|\mathbf{u}| + 2(\lambda+\mu)|\mathbf{u}\cdot\nabla|\mathbf{u}||^2 \\
&= \mu(1-\varepsilon_1)|\mathbf{u}|^4\left|\nabla\left(\frac{\mathbf{u}}{|\mathbf{u}|}\right)\right|^2 + \mu(3-\varepsilon_1)|\mathbf{u}|^2|\nabla|\mathbf{u}||^2 - \frac{\lambda+\mu}{3}|\mathbf{u}|^4\left|\operatorname{div}\left(\frac{\mathbf{u}}{|\mathbf{u}|}\right)\right|^2 \\
&\quad + 3(\lambda+\mu)\left(\frac{2}{3}|\mathbf{u}|^2\operatorname{div}\left(\frac{\mathbf{u}}{|\mathbf{u}|}\right) + \mathbf{u}\cdot\nabla|\mathbf{u}|\right)^2 \\
&\geq -(\lambda+\varepsilon_1\mu)|\mathbf{u}|^4\left|\nabla\left(\frac{\mathbf{u}}{|\mathbf{u}|}\right)\right|^2 + \mu(3-\varepsilon_1)|\mathbf{u}|^2|\nabla|\mathbf{u}||^2.
\end{aligned}$$

Here we have used the facts that $\lambda+\mu>0$[1] and

$$\left|\operatorname{div}\left(\frac{\mathbf{u}}{|\mathbf{u}|}\right)\right|^2 \leq 3\left|\nabla\left(\frac{\mathbf{u}}{|\mathbf{u}|}\right)\right|^2.$$

Then we derive from (3.16) and (3.15) that

$$\begin{aligned}
4\int_{\Omega\cap\{|\mathbf{u}|>0\}}\Psi\,\mathrm{d}x &\geq [-4(\lambda+\varepsilon_1\mu)k(\varepsilon_1,\varepsilon_2) + 4\mu(3-\varepsilon_1)]\int_{\Omega\cap\{|\mathbf{u}|>0\}}|\mathbf{u}|^2|\nabla|\mathbf{u}||^2\,\mathrm{d}x \\
&\geq 4\mu(3-\varepsilon_1)(1-\varepsilon_2)\int_{\Omega\cap\{|\mathbf{u}|>0\}}|\mathbf{u}|^2|\nabla|\mathbf{u}||^2\,\mathrm{d}x.
\end{aligned} \tag{3.18}$$

Thus, substituting (3.18) into (3.17) and using (3.1), (3.13), and (3.16) yield

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|\mathbf{u}|^4\,\mathrm{d}x + 4\mu(3-\varepsilon_1)(1-\varepsilon_2)\int_{\Omega\cap\{|\mathbf{u}|>0\}}|\mathbf{u}|^2|\nabla|\mathbf{u}||^2\,\mathrm{d}x \\
&\leq 4\int_{\Omega\cap\{|\mathbf{u}|>0\}}\operatorname{div}(|\mathbf{u}|^2\mathbf{u})P\,\mathrm{d}x + C(\varepsilon_1)\|\nabla\mathbf{u}\|_{L^2}^2 \\
&\leq C\int_{\Omega\cap\{|\mathbf{u}|>0\}}|\mathbf{u}|^2|\nabla\mathbf{u}|P\,\mathrm{d}x + C(\varepsilon_1)\|\nabla\mathbf{u}\|_{L^2}^2
\end{aligned}$$

[1] From (1.2) and $3\mu-\lambda>0$, we have $5\mu+2\lambda>0$. Then by (1.2) again one gets $7\mu+5\lambda>0$, which combined with (1.2) again implies $9\mu+8\lambda>0$. This together with (1.2) once more gives $11\mu+11\lambda>0$. Thus the result follows.

$$\le \varepsilon \int_{\Omega\cap\{|\mathbf{u}|>0\}} |\mathbf{u}|^2|\nabla\mathbf{u}|^2 \,\mathrm{d}x + C(\varepsilon)\|\mathbf{u}\|_{L^2}^2\|P\|_{L^\infty}^2 + C(\varepsilon_1)\|\nabla\mathbf{u}\|_{L^2}^2$$

$$\le \varepsilon(1+k(\varepsilon_1,\varepsilon_2)) \int_{\Omega\cap\{|\mathbf{u}|>0\}} |\mathbf{u}|^2|\nabla|\mathbf{u}||^2 \,\mathrm{d}x + C(\varepsilon,\varepsilon_1)\|\nabla\mathbf{u}\|_{L^2}^2.$$

Taking $\varepsilon = \frac{2\mu(3-\varepsilon_1)(1-\varepsilon_2)}{1+k(\varepsilon_1,\varepsilon_2)}$, we then arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|\mathbf{u}|^4 \,\mathrm{d}x + 2\mu(3-\varepsilon_1)(1-\varepsilon_2)\int |\mathbf{u}|^2|\nabla|\mathbf{u}||^2 \,\mathrm{d}x \le C(\varepsilon_1,\varepsilon_2)\|\nabla\mathbf{u}\|_{L^2}^2,$$

which combined with (3.13) and (3.16) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|\mathbf{u}|^4 \,\mathrm{d}x + \varepsilon\int |\mathbf{u}|^2|\nabla\mathbf{u}|^2 \,\mathrm{d}x \le C(\varepsilon_1,\varepsilon_2)\|\nabla\mathbf{u}\|_{L^2}^2. \tag{3.19}$$

Case 2: we assume that

$$\int_{\Omega\cap\{|\mathbf{u}|>0\}} |\mathbf{u}|^4 \left|\nabla\left(\frac{\mathbf{u}}{|\mathbf{u}|}\right)\right|^2 \,\mathrm{d}x > k(\varepsilon_1,\varepsilon_2)\int_{\Omega\cap\{|\mathbf{u}|>0\}} |\mathbf{u}|^2|\nabla|\mathbf{u}||^2 \,\mathrm{d}x. \tag{3.20}$$

It follows from (3.12) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|\mathbf{u}|^4 \,\mathrm{d}x + 4\int_{\Omega\cap\{|\mathbf{u}|>0\}} \left[\mu(1-\varepsilon_1)|\mathbf{u}|^2|\nabla\mathbf{u}|^2 + (\lambda+\mu)|\mathbf{u}|^2(\operatorname{div}\mathbf{u})^2 + \frac{\mu|\nabla|\mathbf{u}|^2|^2}{2}\right] \mathrm{d}x$$
$$\le 4\int_{\Omega\cap\{|\mathbf{u}|>0\}} \operatorname{div}(|\mathbf{u}|^2\mathbf{u})P \,\mathrm{d}x - 8(\lambda+\mu)\int_{\Omega\cap\{|\mathbf{u}|>0\}} \operatorname{div}\mathbf{u}|\mathbf{u}|\mathbf{u}\cdot\nabla|\mathbf{u}| \,\mathrm{d}x + C(\varepsilon_1)\|\nabla\mathbf{u}\|_{L^2}^2$$
$$\le C\int_{\Omega\cap\{|\mathbf{u}|>0\}} P|\mathbf{u}|^2|\nabla\mathbf{u}| \,\mathrm{d}x + 4(\lambda+\mu)\int_{\Omega\cap\{|\mathbf{u}|>0\}} |\mathbf{u}|^2|\nabla|\mathbf{u}||^2 \,\mathrm{d}x$$
$$+ 4(\lambda+\mu)\int_{\Omega\cap\{|\mathbf{u}|>0\}} |\mathbf{u}|^2|\operatorname{div}\mathbf{u}|^2 \,\mathrm{d}x + C(\varepsilon_1)\|\nabla\mathbf{u}\|_{L^2}^2,$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|\mathbf{u}|^4 \,\mathrm{d}x + 4\mu(1-\varepsilon_1)\int_{\Omega\cap\{|\mathbf{u}|>0\}} |\mathbf{u}|^2|\nabla\mathbf{u}|^2 \,\mathrm{d}x + 4(\mu-\lambda)\int_{\Omega\cap\{|\mathbf{u}|>0\}} |\mathbf{u}|^2|\nabla|\mathbf{u}||^2 \,\mathrm{d}x$$
$$\le C\int_{\Omega\cap\{|\mathbf{u}|>0\}} P|\mathbf{u}|^2|\nabla\mathbf{u}| \,\mathrm{d}x + C(\varepsilon_1)\|\nabla\mathbf{u}\|_{L^2}^2. \tag{3.21}$$

Inserting (3.13) into (3.21) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|\mathbf{u}|^4 \,\mathrm{d}x + [8\mu - 4(\varepsilon_1\mu+\lambda)]\int_{\Omega\cap\{|\mathbf{u}|>0\}} |\mathbf{u}|^2|\nabla|\mathbf{u}||^2 \,\mathrm{d}x$$
$$+ 4\mu(1-\varepsilon_1)\int_{\Omega\cap\{|\mathbf{u}|>0\}} |\mathbf{u}|^4\left|\nabla\left(\frac{\mathbf{u}}{|\mathbf{u}|}\right)\right|^2 \,\mathrm{d}x$$
$$\le C\int_{\Omega\cap\{|\mathbf{u}|>0\}} P|\mathbf{u}|^2|\nabla|\mathbf{u}|| \,\mathrm{d}x + C\int_{\Omega\cap\{|\mathbf{u}|>0\}} P|\mathbf{u}|^3\left|\nabla\left(\frac{\mathbf{u}}{|\mathbf{u}|}\right)\right| \,\mathrm{d}x + C(\varepsilon_1)\|\nabla\mathbf{u}\|_{L^2}^2$$

$$
\begin{aligned}
&\le C\int_{\Omega\cap\{|\mathbf{u}|>0\}} P|\mathbf{u}|^2|\nabla|\mathbf{u}||\,\mathrm{d}x + 4\mu(1-\varepsilon_1)\varepsilon_3\int_{\Omega\cap\{|\mathbf{u}|>0\}}|\mathbf{u}|^4\left|\nabla\left(\frac{\mathbf{u}}{|\mathbf{u}|}\right)\right|^2\mathrm{d}x\\
&\quad + C(\varepsilon_1,\varepsilon_3)\|\mathbf{u}\|_{L^2}^2\|P\|_{L^\infty}^2 + C(\varepsilon_1)\|\nabla\mathbf{u}\|_{L^2}^2\\
&\le C\int_{\Omega\cap\{|\mathbf{u}|>0\}} P|\mathbf{u}|^2|\nabla|\mathbf{u}||\,\mathrm{d}x + 4\mu(1-\varepsilon_1)\varepsilon_3\int_{\Omega\cap\{|\mathbf{u}|>0\}}|\mathbf{u}|^4\left|\nabla\left(\frac{\mathbf{u}}{|\mathbf{u}|}\right)\right|^2\mathrm{d}x\\
&\quad + C(\varepsilon_1,\varepsilon_3)\|\nabla\mathbf{u}\|_{L^2}^2
\end{aligned}
$$

with $\varepsilon_3\in(0,1)$. Hence we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|\mathbf{u}|^4\,\mathrm{d}x + [8\mu-4(\lambda+\varepsilon_1\mu)]\int_{\Omega\cap\{|\mathbf{u}|>0\}}|\mathbf{u}|^2|\nabla|\mathbf{u}||^2\,\mathrm{d}x\\
&\quad + 4\mu(1-\varepsilon_1)(1-\varepsilon_3)\int_{\Omega\cap\{|\mathbf{u}|>0\}}|\mathbf{u}|^4\left|\nabla\left(\frac{\mathbf{u}}{|\mathbf{u}|}\right)\right|^2\mathrm{d}x\\
&\le C\int_{\Omega\cap\{|\mathbf{u}|>0\}} P|\mathbf{u}|^2|\nabla|\mathbf{u}||\,\mathrm{d}x + C(\varepsilon_1,\varepsilon_3)\|\nabla\mathbf{u}\|_{L^2}^2.
\end{aligned}
$$

This together with (3.20) leads to

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|\mathbf{u}|^4\,\mathrm{d}x + k_1(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)\int_{\Omega\cap\{|\mathbf{u}|>0\}}|\mathbf{u}|^2|\nabla|\mathbf{u}||^2\,\mathrm{d}x\\
&\quad + k_2(\varepsilon_1,\varepsilon_3,\varepsilon_4)\int_{\Omega\cap\{|\mathbf{u}|>0\}}|\mathbf{u}|^4\left|\nabla\left(\frac{\mathbf{u}}{|\mathbf{u}|}\right)\right|^2\mathrm{d}x\\
&\le C\int_{\Omega\cap\{|\mathbf{u}|>0\}} P|\mathbf{u}|^2|\nabla|\mathbf{u}||\,\mathrm{d}x + C(\varepsilon_1,\varepsilon_3)\|\nabla\mathbf{u}\|_{L^2}^2,
\end{aligned}
\tag{3.22}
$$

where

$$
\begin{aligned}
k_1(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) &\triangleq 4\mu(1-\varepsilon_1)(1-\varepsilon_3)(1-\varepsilon_4)k(\varepsilon_1,\varepsilon_2)+8\mu-4(\lambda+\varepsilon_1\mu),\\
k_2(\varepsilon_1,\varepsilon_3,\varepsilon_4) &\triangleq 4\mu(1-\varepsilon_1)(1-\varepsilon_3)\varepsilon_4
\end{aligned}
$$

with $\varepsilon_i\in(0,1)$, $i=1,2,3,4$. For all $(\varepsilon_1,\varepsilon_3,\varepsilon_4)\in(0,1)\times(0,1)\times(0,1)$, due to $\mu>0$, one has $k_2(\varepsilon_1,\varepsilon_3,\varepsilon_4)>0$. We next show that there exists $(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)\in(0,1)\times(0,1)\times(0,1)\times(0,1)$ such that

$$k_1(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)>0.$$

In fact, if $\lambda<0$, take $\varepsilon_1=-\frac{\lambda}{m\mu}\in(0,1)$ with the positive integer m large enough, then we have

$$\lambda+\varepsilon_1\mu=\frac{m-1}{m}\lambda<0,$$

which combined with (3.15) implies that $k(\varepsilon_1,\varepsilon_2)=0$, and hence

$$k_1(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)=8\mu-4(\lambda+\varepsilon_1\mu)>8\mu>0.$$

If $\lambda = 0$, then $\lambda + \varepsilon_1\mu > 0$, thus we infer from (3.15) that

$$k_1(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = \frac{4\mu(1-\varepsilon_1)(1-\varepsilon_3)(1-\varepsilon_4)(3-\varepsilon_1)\varepsilon_2}{\varepsilon_1} + 8\mu - 4\varepsilon_1\mu > 4\mu > 0.$$

If $0 < \lambda < 3\mu$, then we have $\lambda + \varepsilon_1\mu > 0$, so we deduce from (3.15) that

$$k_1(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = \frac{4\mu^2(1-\varepsilon_1)(1-\varepsilon_3)(1-\varepsilon_4)(3-\varepsilon_1)\varepsilon_2}{\lambda + \varepsilon_1\mu} + 8\mu - 4(\lambda + \varepsilon_1\mu).$$

Notice that $k_1(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ is continuous over $[0,1]\times[0,1]\times[0,1]\times[0,1]$, and

$$k_1(0,1,0,0) = \frac{12\mu^2}{\lambda} + 8\mu - 4\lambda = 4\lambda^{-1}(\lambda+\mu)(3\mu-\lambda) > 0,$$

so there exists $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \in (0,1)\times(0,1)\times(0,1)\times(0,1)$ such that

$$k_1(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) > 0.$$

Consequently, one obtains from (3.22) and (3.1) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|\mathbf{u}|^4\,\mathrm{d}x + k_1(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)\int_{\Omega\cap\{|\mathbf{u}|>0\}} |\mathbf{u}|^2|\nabla|\mathbf{u}||^2\,\mathrm{d}x \\
&+ k_2(\varepsilon_1, \varepsilon_3, \varepsilon_4)\int_{\Omega\cap\{|\mathbf{u}|>0\}} |\mathbf{u}|^4\left|\nabla\left(\frac{\mathbf{u}}{|\mathbf{u}|}\right)\right|^2\,\mathrm{d}x \\
\le\ & \frac{k_1(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)}{2}\int_{\Omega\cap\{|\mathbf{u}|>0\}} |\mathbf{u}|^2|\nabla|\mathbf{u}||^2\,\mathrm{d}x \\
&+ C(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)\|\mathbf{u}\|_{L^2}^2\|P\|_{L^\infty}^2 + C(\varepsilon_1, \varepsilon_3)\|\nabla\mathbf{u}\|_{L^2}^2 \\
\le\ & \frac{k_1(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)}{2}\int_{\Omega\cap\{|\mathbf{u}|>0\}} |\mathbf{u}|^2|\nabla|\mathbf{u}||^2\,\mathrm{d}x + C(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)\|\nabla\mathbf{u}\|_{L^2}^2.
\end{aligned}$$

Therefore, one gets

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|\mathbf{u}|^4\,\mathrm{d}x + \frac{k_1(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)}{2}\int |\mathbf{u}|^2|\nabla|\mathbf{u}||^2\,\mathrm{d}x \\
&+ k_2(\varepsilon_1, \varepsilon_3, \varepsilon_4)\int |\mathbf{u}|^4\left|\nabla\left(\frac{\mathbf{u}}{|\mathbf{u}|}\right)\right|^2\,\mathrm{d}x \\
\le\ & C(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)\|\nabla\mathbf{u}\|_{L^2}^2.
\end{aligned} \tag{3.23}$$

From (3.19), (3.23), and (3.13), we conclude that if $3\mu > \lambda$, there exists a constant $\overline{C} > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|\mathbf{u}|^4\,\mathrm{d}x + \overline{C}\int |\mathbf{u}|^2|\nabla\mathbf{u}|^2\,\mathrm{d}x \le C\|\nabla\mathbf{u}\|_{L^2}^2,$$

which together with (3.2) and Gronwall's inequality gives the desired (3.9). □

Let E be the specific energy defined by

$$E = \theta + \frac{|\mathbf{u}|^2}{2}. \tag{3.24}$$

Let G be the effective viscous flux, $\boldsymbol{\omega}$ be vorticity given by

$$G = (\lambda + 2\mu)\operatorname{div}\mathbf{u} - \left(P + \frac{|\mathbf{b}|^2}{2}\right), \quad \boldsymbol{\omega} = \operatorname{curl}\mathbf{u}. \tag{3.25}$$

Then the momentum equations $(1.7)_2$ can be rewritten as

$$\rho\dot{\mathbf{u}} - \mathbf{b}\cdot\nabla\mathbf{b} = \nabla G - \operatorname{curl}\boldsymbol{\omega}, \tag{3.26}$$

where $\dot{\mathbf{u}} \triangleq \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}$.

Then, we derive the following crucial estimate on the $L^\infty(0,T;L^2)$-norm of both $\nabla\mathbf{u}$ and $\nabla\mathbf{b}$.

Lemma 3.3. *Under the condition* (3.1), *it holds that for any* $T \in [0, T^*)$,

$$\sup_{0\le t\le T}\left(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{b}\|_{L^2}^2\right) + \int_0^T \left(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \|\mathbf{b}_t\|_{L^2}^2 + \|\nabla^2\mathbf{b}\|_{L^2}^2\right) \mathrm{d}t \le C. \tag{3.27}$$

Proof. Multiplying $(1.7)_2$ by $\mathbf{u}_t$ and integrating the resulting equation over Ω give rise to

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \left(\mu|\nabla\mathbf{u}|^2 + (\lambda+\mu)(\operatorname{div}\mathbf{u})^2\right) \mathrm{d}x + \int \rho|\dot{\mathbf{u}}|^2 \,\mathrm{d}x \\
&= \int \rho\dot{\mathbf{u}}\cdot(\mathbf{u}\cdot\nabla)\mathbf{u}\,\mathrm{d}x + \int \left(P + \frac{|\mathbf{b}|^2}{2}\right)\operatorname{div}\mathbf{u}_t\,\mathrm{d}x - \int (\mathbf{b}\otimes\mathbf{b}) : \nabla\mathbf{u}_t\,\mathrm{d}x \\
&\le \eta_1 \int \rho|\dot{\mathbf{u}}|^2\,\mathrm{d}x + C(\eta_1)\int |\mathbf{u}|^2|\nabla\mathbf{u}|^2\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int \left[\left(P + \frac{|\mathbf{b}|^2}{2}\right)\operatorname{div}\mathbf{u} - (\mathbf{b}\otimes\mathbf{b}) : \nabla\mathbf{u}\right] \mathrm{d}x \\
&\quad - \int \left(P + \frac{|\mathbf{b}|^2}{2}\right)_t \operatorname{div}\mathbf{u}\,\mathrm{d}x + \int (\mathbf{b}\otimes\mathbf{b})_t : \nabla\mathbf{u}\,\mathrm{d}x \\
&\le \frac{\mathrm{d}}{\mathrm{d}t}\int \left[\left(P + \frac{|\mathbf{b}|^2}{2}\right)\operatorname{div}\mathbf{u} - (\mathbf{b}\otimes\mathbf{b}) : \nabla\mathbf{u} - \frac{\left(P + |\mathbf{b}|^2/2\right)^2}{2(\lambda+2\mu)}\right] \mathrm{d}x + \eta_1\int \rho|\dot{\mathbf{u}}|^2\,\mathrm{d}x \\
&\quad + C(\eta_1)\int |\mathbf{u}|^2|\nabla\mathbf{u}|^2\,\mathrm{d}x - \frac{1}{\lambda+2\mu}\int \left(P + \frac{|\mathbf{b}|^2}{2}\right)_t G\,\mathrm{d}x + \int (\mathbf{b}\otimes\mathbf{b})_t : \nabla\mathbf{u}\,\mathrm{d}x.
\end{aligned} \tag{3.28}$$

Here we have used $\lambda + 2\mu > 0$. Indeed, we obtain this result from $\lambda + \mu > 0$ (see footnote on page 612) and $\mu > 0$.

It follows from (1.7) that E satisfies

$$\left(\rho E + \frac{|\mathbf{b}|^2}{2}\right)_t + \operatorname{div}(\rho\mathbf{u}E) = \operatorname{div}\mathbf{H} \tag{3.29}$$

with

$$\mathbf{H} \triangleq (\mathbf{u}\times\mathbf{b})\times\mathbf{b} + \nu(\operatorname{curl}\mathbf{b})\times\mathbf{b} + (2\mu\mathfrak{D}(\mathbf{u}) + \lambda\operatorname{div}\mathbf{u}\mathbb{I}_3)\mathbf{u} - P\mathbf{u}.$$

Then we infer from (3.24), (3.29), and $(1.7)_1$ that

$$\begin{aligned}
&-\int\left(P+\frac{|\mathbf{b}|^2}{2}\right)_t G\,\mathrm{d}x\\
&= -\int\left(\rho E+\frac{|\mathbf{b}|^2}{2}\right)_t G\,\mathrm{d}x + \frac{1}{2}\int(\rho|\mathbf{u}|^2)_t G\,\mathrm{d}x\\
&= \int \operatorname{div}(\rho\mathbf{u}E-\mathbf{H})G\,\mathrm{d}x + \frac{1}{2}\int\rho_t|\mathbf{u}|^2G\,\mathrm{d}x + \int\rho\mathbf{u}\cdot\mathbf{u}_tG\,\mathrm{d}x\\
&= -\int(\rho\mathbf{u}E-\mathbf{H})\cdot\nabla G\,\mathrm{d}x - \frac{1}{2}\int\operatorname{div}(\rho\mathbf{u})|\mathbf{u}|^2G\,\mathrm{d}x + \int\rho\mathbf{u}\cdot\mathbf{u}_tG\,\mathrm{d}x\\
&= -\int(P\mathbf{u}-\mathbf{H})\cdot\nabla G\,\mathrm{d}x + \frac{1}{2}\int\rho\mathbf{u}\cdot\nabla(|\mathbf{u}|^2)G\,\mathrm{d}x + \int\rho\mathbf{u}\cdot\mathbf{u}_tG\,\mathrm{d}x \triangleq \sum_{i=1}^{3} I_i.
\end{aligned}\tag{3.30}$$

From Hölder's inequality, Sobolev's inequality, and (3.1), we have

$$\begin{aligned}
I_1 &\le \int(P|\mathbf{u}|+|\mathbf{H}|)|\nabla G|\,\mathrm{d}x\\
&\le C\int(P|\mathbf{u}|+|\mathbf{u}||\mathbf{b}|^2+|\mathbf{b}||\nabla\mathbf{b}|+|\mathbf{u}||\nabla\mathbf{u}|)|\nabla G|\,\mathrm{d}x\\
&\le \eta_1\|\nabla G\|_{L^2}^2\\
&\quad + C(\eta_1)\left(\|P\|_{L^\infty}^2\|\mathbf{u}\|_{L^2}^2 + \|\mathbf{u}\|_{L^6}^2\|\mathbf{b}\|_{L^6}^4 + \|\mathbf{b}\|_{L^3}^2\|\nabla\mathbf{b}\|_{L^6}^2 + \||\mathbf{u}||\nabla\mathbf{u}|\|_{L^2}^2\right)\\
&\le \eta_1\|\nabla G\|_{L^2}^2 + C(\eta_1)\left(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{b}\|_{H^1}^2 + \||\mathbf{u}||\nabla\mathbf{u}|\|_{L^2}^2\right).
\end{aligned}\tag{3.31}$$

By (3.25), (3.1), Hölder's inequality, and Sobolev's inequality, one gets

$$\begin{aligned}
I_2 &\le \int\rho|\mathbf{u}|^2|\nabla\mathbf{u}||G|\,\mathrm{d}x \le C\int\rho|\mathbf{u}|^2|\nabla\mathbf{u}|(|\nabla\mathbf{u}|+|\mathbf{b}|^2+P)\,\mathrm{d}x\\
&\le C\int(|\mathbf{u}|^2|\nabla\mathbf{u}|^2+|\mathbf{u}|^2|\nabla\mathbf{u}||\mathbf{b}|^2+\rho|\mathbf{u}|^2|\nabla\mathbf{u}|)\,\mathrm{d}x\\
&\le C(\||\mathbf{u}||\nabla\mathbf{u}|\|_{L^2}^2 + \|\mathbf{u}\|_{L^6}\||\mathbf{u}||\nabla\mathbf{u}|\|_{L^2}\|\mathbf{b}\|_{L^6}^2 + \|\rho\|_{L^\infty}\|\mathbf{u}\|_{L^2}\||\mathbf{u}||\nabla\mathbf{u}|\|_{L^2})\\
&\le C\||\mathbf{u}||\nabla\mathbf{u}|\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2}^2.
\end{aligned}\tag{3.32}$$

Similar to I_2, we find that

$$\begin{aligned}
I_3 &= \int\left[\rho\mathbf{u}\cdot\dot{\mathbf{u}} - \rho\mathbf{u}\cdot(\mathbf{u}\cdot\nabla)\mathbf{u}\right]G\,\mathrm{d}x\\
&\le C\int\rho(|\mathbf{u}||\dot{\mathbf{u}}|+|\mathbf{u}|^2|\nabla\mathbf{u}|)(|\nabla\mathbf{u}|+|\mathbf{b}|^2+P)\,\mathrm{d}x\\
&\le C\left(\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}\||\mathbf{u}||\nabla\mathbf{u}|\|_{L^2} + \|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}\|\mathbf{u}\|_{L^6}\|\mathbf{b}\|_{L^6}^2\right)\\
&\quad + C\left(\|\rho\|_{L^\infty}\||\mathbf{u}||\nabla\mathbf{u}|\|_{L^2}^2 + \|\sqrt{\rho}\|_{L^\infty}\|P\|_{L^\infty}\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}\|\mathbf{u}\|_{L^2}\right)\\
&\quad + C\left(\|\rho\|_{L^\infty}\||\mathbf{u}||\nabla\mathbf{u}|\|_{L^2}\|\mathbf{u}\|_{L^6}\|\mathbf{b}\|_{L^6}^2 + \|\rho\|_{L^\infty}\|P\|_{L^\infty}\||\mathbf{u}||\nabla\mathbf{u}|\|_{L^2}\|\mathbf{u}\|_{L^2}\right)\\
&\le \eta_1\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + C(\eta_1)(\||\mathbf{u}||\nabla\mathbf{u}|\|_{L^2}^2 + \|\nabla\mathbf{u}\|_{L^2}^2).
\end{aligned}\tag{3.33}$$

Inserting (3.31)–(3.33) into (3.30), we arrive at

$$
\begin{aligned}
-\int \left(P+\frac{|\mathbf{b}|^2}{2}\right)_t G\,\mathrm{d}x &\le \eta_1\|\nabla G\|_{L^2}^2+\eta_1\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 \\
&\quad + C(\eta_1)\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{b}\|_{H^1}^2+\||\mathbf{u}||\nabla\mathbf{u}|\|_{L^2}^2\right).
\end{aligned}\tag{3.34}
$$

In view of (3.25), (3.26), (1.7), and (1.9), we see that G satisfies

$$
\begin{cases}
\Delta G=\operatorname{div}(\rho\dot{\mathbf{u}}-\mathbf{b}\cdot\nabla\mathbf{b}) & \text{in } \Omega,\\
\nabla G\cdot\mathbf{n}=0 & \text{on } \partial\Omega,
\end{cases}
$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$. Applying the standard L^p-estimate for Neumann problem to the above elliptic equation (see e.g., [16]), we have for any $p\ge 2$,

$$
\|\nabla G\|_{L^p}\le C\left(\|\rho\dot{\mathbf{u}}\|_{L^p}+\|\mathbf{b}\cdot\nabla\mathbf{b}\|_{L^p}\right).\tag{3.35}
$$

In particular, taking $p=2$ in (3.35), we deduce from (3.1) that

$$
\begin{aligned}
\|\nabla G\|_{L^2}^2 &\le C\left(\|\rho\dot{\mathbf{u}}\|_{L^2}^2+\|\mathbf{b}\cdot\nabla\mathbf{b}\|_{L^2}^2\right)\le C\left(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2+\|\mathbf{b}\|_{L^3}^2\|\nabla\mathbf{b}\|_{L^6}^2\right)\\
&\le C\left(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2+\|\nabla\mathbf{b}\|_{H^1}^2\right),
\end{aligned}
$$

which combined with (3.34) implies that

$$
\begin{aligned}
-\int \left(P+\frac{|\mathbf{b}|^2}{2}\right)_t G\,\mathrm{d}x &\le C\eta_1\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 \\
&\quad + C(\eta_1)\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{b}\|_{H^1}^2+\||\mathbf{u}||\nabla\mathbf{u}|\|_{L^2}^2\right).
\end{aligned}\tag{3.36}
$$

For the last term on the right-hand side of (3.28), we obtain from Hölder's inequality and (3.1) that

$$
\begin{aligned}
\int (\mathbf{b}\otimes\mathbf{b})_t:\nabla\mathbf{u}\,\mathrm{d}x &\le C\int |\mathbf{b}_t||\mathbf{b}||\nabla\mathbf{u}|\,\mathrm{d}x\\
&\le \widetilde{\eta}\|\mathbf{b}_t\|_{L^2}^2+C(\widetilde{\eta})\|\mathbf{b}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^4}^2\\
&\le \widetilde{\eta}\|\mathbf{b}_t\|_{L^2}^2+C(\widetilde{\eta})\|\nabla\mathbf{u}\|_{L^2}^{1/2}\|\nabla\mathbf{u}\|_{L^6}^{3/2}\\
&\le \widetilde{\eta}\|\mathbf{b}_t\|_{L^2}^2+\eta_1\|\nabla\mathbf{u}\|_{L^6}^2+C(\widetilde{\eta},\eta_1)\|\nabla\mathbf{u}\|_{L^2}^2.
\end{aligned}\tag{3.37}
$$

We need to estimate $\|\nabla\mathbf{u}\|_{L^6}$. To this end, inspired by [9], let $\mathbf{u}=\mathbf{v}+\mathbf{w}$ such that

$$
\begin{cases}
\mu\Delta\mathbf{v}+(\lambda+\mu)\nabla\operatorname{div}\mathbf{v}=\nabla\left(P+\frac{|\mathbf{b}|^2}{2}\right),\\
\mathbf{v}(x,t)=\mathbf{0}\quad \text{on } \partial\Omega;
\end{cases}
$$

and

$$
\begin{cases}
\mu\Delta\mathbf{w}+(\lambda+\mu)\nabla\operatorname{div}\mathbf{w}=\rho\dot{\mathbf{u}}-\mathbf{b}\cdot\nabla\mathbf{b},\\
\mathbf{w}(x,t)=\mathbf{0}\quad \text{on } \partial\Omega,
\end{cases}
$$

which implies that

$$\|\nabla \mathbf{v}\|_{L^6} \le C \left\| P + \frac{|\mathbf{b}|^2}{2} \right\|_{L^6} \le C,$$

and

$$\|\nabla \mathbf{w}\|_{L^6} + \|\nabla^2 \mathbf{w}\|_{L^2} \le C(\|\rho \dot{\mathbf{u}}\|_{L^2} + \|\mathbf{b}\cdot\nabla \mathbf{b}\|_{L^2}) \le C(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2} + \|\nabla \mathbf{b}\|_{H^1}).$$

Then we have

$$\|\nabla \mathbf{u}\|_{L^6}^2 \le \|\nabla \mathbf{v}\|_{L^6}^2 + \|\nabla \mathbf{w}\|_{L^6}^2 \le C(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla \mathbf{b}\|_{H^1}^2 + 1). \tag{3.38}$$

Substituting (3.38) into (3.37) leads to

$$\int (\mathbf{b}\otimes\mathbf{b})_t : \nabla \mathbf{u}\,\mathrm{d}x \le \widetilde{\eta}\|\mathbf{b}_t\|_{L^2}^2 + C\eta_1(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla \mathbf{b}\|_{H^1}^2 + 1) + C\|\nabla \mathbf{u}\|_{L^2}^2. \tag{3.39}$$

Inserting (3.36) and (3.39) into (3.28) and choosing η_1 suitably small, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int \Phi\,\mathrm{d}x + \|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 &\le \widetilde{\eta}\|\mathbf{b}_t\|_{L^2}^2 + C_2\|\nabla^2 \mathbf{b}\|_{L^2}^2 \\ &\quad + C\left(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{b}\|_{L^2}^2 + \||\mathbf{u}||\nabla \mathbf{u}|\|_{L^2}^2 + 1\right),\end{aligned} \tag{3.40}$$

where

$$\Phi \triangleq \mu|\nabla \mathbf{u}|^2 + (\mu+\lambda)(\operatorname{div}\mathbf{u})^2 - (2P + |\mathbf{b}|^2)\operatorname{div}\mathbf{u} + 2(\mathbf{b}\otimes\mathbf{b}) : \nabla \mathbf{u} + \frac{\left(P + |\mathbf{b}|^2/2\right)^2}{\lambda + 2\mu}$$

satisfies

$$\frac{\mu}{2}\|\nabla \mathbf{u}\|_{L^2}^2 - C \le \int \Phi\,\mathrm{d}x \le \mu\|\nabla \mathbf{u}\|_{L^2}^2 + C \tag{3.41}$$

due to (3.1).

It follows from $(1.7)_4$, Hölder's inequality, (3.1), and (3.38) that

$$\begin{aligned}&\nu\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla \mathbf{b}|^2\,\mathrm{d}x + \int |\mathbf{b}_t|^2\,\mathrm{d}x + \nu^2\int |\Delta \mathbf{b}|^2\,\mathrm{d}x \\ &= \int |\mathbf{b}_t - \nu\Delta \mathbf{b}|^2\,\mathrm{d}x = \int |\mathbf{b}\cdot\nabla \mathbf{u} - \mathbf{u}\cdot\nabla \mathbf{b} - \mathbf{b}\operatorname{div}\mathbf{u}|^2\,\mathrm{d}x \\ &\le C\int |\mathbf{b}|^2|\nabla \mathbf{u}|^2\,\mathrm{d}x + C\int |\mathbf{u}|^2|\nabla \mathbf{b}|^2\,\mathrm{d}x \le C\|\mathbf{b}\|_{L^4}^2\|\nabla \mathbf{u}\|_{L^4}^2 + C\|\mathbf{u}\|_{L^6}^2\|\nabla \mathbf{b}\|_{L^3}^2 \\ &\le C\|\nabla \mathbf{u}\|_{L^2}^{1/2}\|\nabla \mathbf{u}\|_{L^6}^{3/2} + C\|\nabla \mathbf{u}\|_{L^2}^2\|\nabla \mathbf{b}\|_{L^2}\|\nabla \mathbf{b}\|_{H^1} \\ &\le C\|\nabla \mathbf{u}\|_{L^2}^2 + \frac{1}{8}\left(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla \mathbf{b}\|_{H^1}^2 + 1\right) + C\|\nabla \mathbf{u}\|_{L^2}^2\|\nabla \mathbf{b}\|_{L^2}(\|\nabla \mathbf{b}\|_{L^2} + \|\nabla^2 \mathbf{b}\|_{L^2}) \\ &\le \eta_2\|\nabla^2 \mathbf{b}\|_{L^2}^2 + C(\eta_2)\left(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{b}\|_{L^2}^2\right)\left(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{b}\|_{L^2}^2 + 1\right) + \frac{1}{8}\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + C.\end{aligned}$$

Noting that the standard L^2 estimate of elliptic system gives

$$\|\nabla^2\mathbf{b}\|_{L^2} \le C_3\|\Delta\mathbf{b}\|_{L^2} + C_3\|\nabla\mathbf{b}\|_{L^2},$$

hence we deduce after choosing η_2 suitably small that

$$\begin{aligned}&2\nu\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla\mathbf{b}|^2\,\mathrm{d}x + 2\|\mathbf{b}_t\|_{L^2}^2 + C_3^{-1}\nu^2\|\nabla^2\mathbf{b}\|_{L^2}^2\\ &\le C(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{b}\|_{L^2}^2)(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{b}\|_{L^2}^2 + 1) + \frac{1}{8}\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + C.\end{aligned} \tag{3.42}$$

Then adding (3.42) to (3.40) and choosing $\widetilde{\eta}$ small enough, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int(\Phi + 2C_4\nu|\nabla\mathbf{b}|^2)\,\mathrm{d}x + \frac{1}{2}(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \|\mathbf{b}_t\|_{L^2}^2 + \|\nabla^2\mathbf{b}\|_{L^2}^2)\\ &\le C(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{b}\|_{L^2}^2)(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{b}\|_{L^2}^2 + 1)\\ &\quad + C(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{b}\|_{L^2}^2 + \||\mathbf{u}||\nabla\mathbf{u}|\|_{L^2}^2 + 1).\end{aligned}$$

Thus we obtain the desired (3.27) after using Gronwall's inequality, (3.2), (3.9), and (3.41). This completes the proof of Lemma 3.3. □

Next, we have the following estimates on the material derivatives of the velocity which are important for the higher order estimates of strong solutions.

Lemma 3.4. *Under the condition* (3.1)*, it holds that for any* $T \in [0, T^*)$*,*

$$\sup_{0\le t\le T}\left(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \|\mathbf{b}_t\|_{L^2}^2 + \|\nabla^2\mathbf{b}\|_{L^2}^2\right) + \int_0^T\left(\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla\mathbf{b}_t\|_{L^2}^2\right)\mathrm{d}t \le C. \tag{3.43}$$

Proof. By the definition of $\dot{\mathbf{u}}$, we can rewrite $(1.7)_2$ as follows:

$$\rho\dot{\mathbf{u}} + \nabla P = \mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\operatorname{div}\mathbf{u} + \operatorname{curl}\mathbf{b}\times\mathbf{b}. \tag{3.44}$$

Differentiating (3.44) with respect to t and using $(1.7)_1$, we have

$$\begin{aligned}\rho\dot{\mathbf{u}}_t + \rho\mathbf{u}\cdot\nabla\dot{\mathbf{u}} + \nabla P_t &= \mu\Delta\dot{\mathbf{u}} + (\lambda+\mu)\operatorname{div}\dot{\mathbf{u}} - \mu\Delta(\mathbf{u}\cdot\nabla\mathbf{u}) - (\lambda+\mu)\operatorname{div}(\mathbf{u}\cdot\nabla\mathbf{u})\\ &\quad + (\operatorname{curl}\mathbf{b}\times\mathbf{b})_t + \operatorname{div}(\rho\dot{\mathbf{u}}\otimes\mathbf{u}).\end{aligned} \tag{3.45}$$

Multiplying (3.45) by $\dot{\mathbf{u}}$ and integrating by parts over Ω, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|\dot{\mathbf{u}}|^2\,\mathrm{d}x + \mu\int |\nabla\dot{\mathbf{u}}|^2\,\mathrm{d}x + (\lambda+\mu)\int |\operatorname{div}\dot{\mathbf{u}}|^2\,\mathrm{d}x \\
&= \int [P_t \operatorname{div}\dot{\mathbf{u}} + (\nabla P\otimes\mathbf{u}) : \nabla\dot{\mathbf{u}}]\,\mathrm{d}x \\
&\quad - \int [\operatorname{div}(\operatorname{curl}\mathbf{b}\times\mathbf{b})\otimes\mathbf{u} - (\operatorname{curl}\mathbf{b}\times\mathbf{b})_t]\cdot\dot{\mathbf{u}}\,\mathrm{d}x \\
&\quad + \mu\int [\operatorname{div}(\Delta\mathbf{u}\otimes\mathbf{u}) - \Delta(\mathbf{u}\cdot\nabla\mathbf{u})]\cdot\dot{\mathbf{u}}\,\mathrm{d}x \\
&\quad + (\lambda+\mu)\int [(\nabla\operatorname{div}\mathbf{u})\otimes\mathbf{u} - \nabla\operatorname{div}(\mathbf{u}\cdot\nabla\mathbf{u})]\cdot\dot{\mathbf{u}}\,\mathrm{d}x \\
&\triangleq \sum_{i=1}^{4} J_i,
\end{aligned}
\tag{3.46}
$$

where J_i can be bounded as follows.

It follows from $(1.7)_3$ that

$$
\begin{aligned}
J_1 &= \int \big(-\operatorname{div}(P\mathbf{u})\operatorname{div}\dot{\mathbf{u}} - P\operatorname{div}\mathbf{u}\operatorname{div}\dot{\mathbf{u}} + \mathcal{T}(\mathbf{u}) : \nabla\mathbf{u}\operatorname{div}\dot{\mathbf{u}} \\
&\qquad + \nu|\operatorname{curl}\mathbf{b}|^2\operatorname{div}\dot{\mathbf{u}} + (\nabla P\otimes\mathbf{u}) : \nabla\dot{\mathbf{u}}\big)\,\mathrm{d}x \\
&= \int \big(P\mathbf{u}\cdot\nabla\operatorname{div}\dot{\mathbf{u}} - P\operatorname{div}\mathbf{u}\operatorname{div}\dot{\mathbf{u}} + \mathcal{T}(\mathbf{u}) : \nabla\mathbf{u}\operatorname{div}\dot{\mathbf{u}} + \nu|\operatorname{curl}\mathbf{b}|^2\operatorname{div}\dot{\mathbf{u}}\big)\,\mathrm{d}x \\
&\quad - \int \Big(P\nabla\mathbf{u}^\top : \nabla\dot{\mathbf{u}} + P\mathbf{u}\cdot\nabla\operatorname{div}\dot{\mathbf{u}}\Big)\,\mathrm{d}x \\
&= \int \Big(-P\operatorname{div}\mathbf{u}\operatorname{div}\dot{\mathbf{u}} + \mathcal{T}(\mathbf{u}) : \nabla\mathbf{u}\operatorname{div}\dot{\mathbf{u}} + \nu|\operatorname{curl}\mathbf{b}|^2\operatorname{div}\dot{\mathbf{u}} - P\nabla\mathbf{u}^\top : \nabla\dot{\mathbf{u}}\Big)\,\mathrm{d}x \\
&\le C\int \big(|\nabla\mathbf{u}||\nabla\dot{\mathbf{u}}| + |\nabla\mathbf{u}|^2|\nabla\dot{\mathbf{u}}| + |\nabla\mathbf{b}|^2|\nabla\dot{\mathbf{u}}|\big)\,\mathrm{d}x \\
&\le C\left(\|\nabla\mathbf{u}\|_{L^2} + \|\nabla\mathbf{u}\|_{L^4}^2 + \|\nabla\mathbf{b}\|_{L^4}^2\right)\|\nabla\dot{\mathbf{u}}\|_{L^2},
\end{aligned}
\tag{3.47}
$$

where $\mathcal{T}(\mathbf{u}) = 2\mu\mathfrak{D}(\mathbf{u}) + \lambda\operatorname{div}\mathbf{u}\mathbb{I}_3$. Integrating by parts leads to

$$
\begin{aligned}
J_2 &= \int \left[\operatorname{div}(\mathbf{b}\otimes\mathbf{b})_t - \nabla\left(\frac{|\mathbf{b}|^2}{2}\right)_t - \operatorname{div}(\operatorname{curl}\mathbf{b}\times\mathbf{b})\otimes\mathbf{u}\right]\cdot\dot{\mathbf{u}}\,\mathrm{d}x \\
&\le C\int (|\mathbf{b}||\mathbf{b}_t| + |\mathbf{b}||\nabla\mathbf{b}||\mathbf{u}|)|\nabla\dot{\mathbf{u}}|\,\mathrm{d}x \\
&\le C(\|\mathbf{b}\|_{L^6}\|\mathbf{b}_t\|_{L^3} + \|\mathbf{b}\|_{L^6}\|\nabla\mathbf{b}\|_{L^6}\|\mathbf{u}\|_{L^6})\|\nabla\dot{\mathbf{u}}\|_{L^2} \\
&\le C\left(\|\mathbf{b}_t\|_{L^2}^{1/2}\|\nabla\mathbf{b}_t\|_{L^2}^{1/2} + \|\nabla\mathbf{b}\|_{H^1}\right)\|\nabla\dot{\mathbf{u}}\|_{L^2}.
\end{aligned}
\tag{3.48}
$$

For J_3 and J_4, notice that for all $1 \le i, j, k \le 3$, one has

$$
\begin{aligned}
&\partial_j(\partial_{kk}u_i u_j) - \partial_{kk}(u_j\partial_j u_i) = \partial_k(\partial_j u_j\partial_k u_i) - \partial_k(\partial_k u_j\partial_j u_i) - \partial_j(\partial_k u_j\partial_k u_i), \\
&\partial_j(\partial_{ik}u_k u_j) - \partial_{ij}(u_k\partial_k u_j) = \partial_i(\partial_j u_j\partial_k u_k) - \partial_i(\partial_j u_k\partial_k u_j) - \partial_k(\partial_i u_k\partial_j u_j).
\end{aligned}
$$

So integrating by parts gives

$$
\begin{aligned}
J_3 &= \mu \int [\partial_k(\partial_j u_j \partial_k u_i) - \partial_k(\partial_k u_j \partial_j u_i) - \partial_j(\partial_k u_j \partial_k u_i)]\dot{u}_i \, \mathrm{d}x \\
&\le C\|\nabla \mathbf{u}\|_{L^4}^2 \|\nabla \dot{\mathbf{u}}\|_{L^2},
\end{aligned}
\tag{3.49}
$$

$$
\begin{aligned}
J_4 &= (\lambda+\mu) \int [\partial_i(\partial_j u_j \partial_k u_k) - \partial_i(\partial_j u_k \partial_k u_j) - \partial_k(\partial_i u_k \partial_j u_j)]\dot{u}_i \, \mathrm{d}x \\
&\le C\|\nabla \mathbf{u}\|_{L^4}^2 \|\nabla \dot{\mathbf{u}}\|_{L^2}.
\end{aligned}
\tag{3.50}
$$

Inserting (3.47)–(3.50) into (3.46) and applying (3.27) lead to

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \mu\|\nabla \dot{\mathbf{u}}\|_{L^2}^2 + (\lambda+\mu)\|\operatorname{div} \dot{\mathbf{u}}\|_{L^2}^2 \\
&\le C\left(\|\nabla \mathbf{u}\|_{L^2} + \|\nabla \mathbf{u}\|_{L^4}^2 + \|\nabla \mathbf{b}\|_{L^4}^2 + \|\mathbf{b}_t\|_{L^2}^{1/2}\|\nabla \mathbf{b}_t\|_{L^2}^{1/2} + \|\nabla \mathbf{b}\|_{H^1}\right)\|\nabla \dot{\mathbf{u}}\|_{L^2} \\
&\le \delta_1\|\nabla \dot{\mathbf{u}}\|_{L^2}^2 + \delta_2\|\nabla \mathbf{b}_t\|_{L^2}^2 \\
&\quad + C(\delta_1,\delta_2)\left(\|\nabla \mathbf{u}\|_{L^4}^4 + \|\nabla \mathbf{b}\|_{L^4}^4 + \|\mathbf{b}_t\|_{L^2}^2 + \|\nabla^2 \mathbf{b}\|_{L^2}^2 + 1\right).
\end{aligned}
\tag{3.51}
$$

Using the standard H^2 estimate of elliptic equations to $(1.7)_4$, then we get from (3.1), (3.27), and (3.38) that

$$
\begin{aligned}
\|\nabla^2 \mathbf{b}\|_{L^2}^2 &\le C(\|\mathbf{b}_t\|_{L^2}^2 + \||\mathbf{u}||\nabla \mathbf{b}|\|_{L^2}^2 + \||\mathbf{b}||\nabla \mathbf{u}|\|_{L^2}^2) \\
&\le C(\|\mathbf{b}_t\|_{L^2}^2 + \|\mathbf{u}\|_{L^6}^2\|\nabla \mathbf{b}\|_{L^3}^2 + \|\mathbf{b}\|_{L^6}^2\|\nabla \mathbf{u}\|_{L^2}\|\nabla \mathbf{u}\|_{L^6}) \\
&\le C(\|\mathbf{b}_t\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^2}^2\|\nabla \mathbf{b}\|_{L^2}\|\nabla \mathbf{b}\|_{H^1} + \|\nabla \mathbf{u}\|_{L^6}) \\
&\le \frac{1}{2}\|\nabla^2 \mathbf{b}\|_{L^2}^2 + C\|\mathbf{b}_t\|_{L^2}^2 + C\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + C,
\end{aligned}
$$

which implies that

$$
\|\nabla^2 \mathbf{b}\|_{L^2}^2 \le C\|\mathbf{b}_t\|_{L^2}^2 + C\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + C.
\tag{3.52}
$$

Differentiating $(1.7)_4$ with respect to t, we have

$$
\mathbf{b}_{tt} - \nu\Delta \mathbf{b}_t = \mathbf{b}_t\cdot\nabla \mathbf{u} - \mathbf{u}\cdot\nabla \mathbf{b}_t - \mathbf{b}_t \operatorname{div} \mathbf{u} + \mathbf{b}\cdot\nabla \mathbf{u}_t - \mathbf{u}_t\cdot\nabla \mathbf{b} - \mathbf{b}\operatorname{div} \mathbf{u}_t.
\tag{3.53}
$$

Multiplying (3.53) by $\mathbf{b}_t$ and integrating by parts lead to

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\mathbf{b}_t|^2 \, \mathrm{d}x + \nu\int |\nabla \mathbf{b}_t|^2 \, \mathrm{d}x &= \int (\mathbf{b}_t\cdot\nabla \mathbf{u} - \mathbf{u}\cdot\nabla \mathbf{b}_t - \mathbf{b}_t \operatorname{div} \mathbf{u})\cdot \mathbf{b}_t \, \mathrm{d}x \\
&\quad + \int (\mathbf{b}\cdot\nabla \mathbf{u}_t - \mathbf{u}_t\cdot\nabla \mathbf{b} - \mathbf{b}\operatorname{div} \mathbf{u}_t)\cdot \mathbf{b}_t \, \mathrm{d}x \\
&\triangleq L_1 + L_2.
\end{aligned}
\tag{3.54}
$$

Integrating by parts implies that

$$
\begin{aligned}
L_1 &= \int \left(\mathbf{b}_t\cdot\nabla \mathbf{u}\cdot \mathbf{b}_t - \frac{1}{2}|\mathbf{b}_t|^2 \operatorname{div} \mathbf{u}\right) \mathrm{d}x \le C\|\mathbf{b}_t\|_{L^4}^2\|\nabla \mathbf{u}\|_{L^2} \\
&\le C\|\mathbf{b}_t\|_{L^2}^{1/2}\|\nabla \mathbf{b}_t\|_{L^2}^{3/2} \le \delta_1\|\nabla \mathbf{b}_t\|_{L^2}^2 + C(\delta_1)\|\mathbf{b}_t\|_{L^2}^2
\end{aligned}
\tag{3.55}
$$

and

(3.56)

$$
\begin{aligned}
L_2 &= \int (\mathbf{b}\cdot\nabla\dot{\mathbf{u}} - \dot{\mathbf{u}}\cdot\nabla\mathbf{b} - \mathbf{b}\,\mathrm{div}\,\dot{\mathbf{u}})\cdot\mathbf{b}_t\,\mathrm{d}x \\
&\quad - \int [\mathbf{b}\cdot\nabla(\mathbf{u}\cdot\nabla\mathbf{u}) - (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\nabla\mathbf{b} - \mathbf{b}\,\mathrm{div}(\mathbf{u}\cdot\nabla\mathbf{u})]\cdot\mathbf{b}_t\,\mathrm{d}x \\
&= \int (\mathbf{b}\cdot\nabla\dot{\mathbf{u}} - \dot{\mathbf{u}}\cdot\nabla\mathbf{b} - \mathbf{b}\,\mathrm{div}\,\dot{\mathbf{u}})\cdot\mathbf{b}_t\,\mathrm{d}x \\
&\quad + \int [(\mathbf{u}\cdot\nabla\mathbf{u})\cdot(\mathbf{b}\cdot\nabla\mathbf{b}_t) + (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\nabla\mathbf{b}_t\cdot\mathbf{b}]\,\mathrm{d}x \\
&\le C\int (|\mathbf{b}||\mathbf{b}_t||\nabla\dot{\mathbf{u}}| + |\dot{\mathbf{u}}||\nabla\mathbf{b}||\mathbf{b}_t| + |\mathbf{u}||\nabla\mathbf{u}||\mathbf{b}||\nabla\mathbf{b}_t|)\,\mathrm{d}x \\
&\le C(\|\mathbf{b}\|_{L^6}\|\mathbf{b}_t\|_{L^3}\|\nabla\dot{\mathbf{u}}\|_{L^2} + \|\dot{\mathbf{u}}\|_{L^6}\|\nabla\mathbf{b}\|_{L^2}\|\mathbf{b}_t\|_{L^3} + \|\mathbf{u}\|_{L^6}\|\nabla\mathbf{u}\|_{L^6}\|\mathbf{b}\|_{L^6}\|\nabla\mathbf{b}_t\|_{L^2}) \\
&\le C(\|\mathbf{b}_t\|_{L^3}\|\nabla\dot{\mathbf{u}}\|_{L^2} + \|\nabla\mathbf{u}\|_{L^6}\|\nabla\mathbf{b}_t\|_{L^2}) \\
&\le C(\|\mathbf{b}_t\|_{L^2}^{1/2}\|\nabla\mathbf{b}_t\|_{L^2}^{1/2}\|\nabla\dot{\mathbf{u}}\|_{L^2} + \|\nabla\mathbf{u}\|_{L^6}\|\nabla\mathbf{b}_t\|_{L^2}) \\
&\le \delta_1\|\nabla\mathbf{b}_t\|_{L^2}^2 + \delta_2\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + C(\delta_1,\delta_2)\|\mathbf{b}_t\|_{L^2} + C(\delta_1)\|\nabla\mathbf{u}\|_{L^6}^2.
\end{aligned}
$$

Inserting (3.55) and (3.56) into (3.54), we have

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{b}_t\|_{L^2}^2 + \nu\|\nabla\mathbf{b}_t\|_{L^2}^2 &\le 2\delta_1\|\nabla\mathbf{b}_t\|_{L^2}^2 + \delta_2\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 \\
&\quad + C(\delta_1,\delta_2)\|\mathbf{b}_t\|_{L^2} + C(\delta_1)\|\nabla\mathbf{u}\|_{L^6}^2.
\end{aligned}
\tag{3.57}
$$

Adding (3.57) to (3.51) and applying (3.52), we obtain after choosing δ_1, δ_2 suitably small that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \|\mathbf{b}_t\|_{L^2}^2\right) + \widetilde{C}\left(\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla\mathbf{b}_t\|_{L^2}^2\right) \\
&\le C(\|\mathbf{b}_t\|_{L^2}^2 + \|\nabla\mathbf{u}\|_{L^4}^4 + \|\nabla\mathbf{b}\|_{L^4}^4 + \|\nabla\mathbf{u}\|_{L^6}^2 + 1).
\end{aligned}
\tag{3.58}
$$

By (3.38), (3.27), and (3.52), one has

$$
\|\nabla\mathbf{u}\|_{L^6}^2 \le C(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla\mathbf{b}\|_{H^1}^2 + 1) \le C(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \|\mathbf{b}_t\|_{L^2}^2 + 1).
\tag{3.59}
$$

It follows from Hölder's inequality, (3.27), and (3.59) that

$$
\begin{aligned}
\|\nabla\mathbf{u}\|_{L^4}^4 &\le C\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^6}^3 \le C(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^3 + \|\mathbf{b}_t\|_{L^2}^3 + 1) \\
&\le C\left(1 + \|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \|\mathbf{b}_t\|_{L^2}^2\right)\left(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \|\mathbf{b}_t\|_{L^2}^2\right) + C.
\end{aligned}
\tag{3.60}
$$

Similarly, we get

$$
\begin{aligned}
\|\nabla\mathbf{b}\|_{L^4}^4 &\le C\|\nabla\mathbf{b}\|_{L^2}\|\nabla\mathbf{b}\|_{H^1}^3 \le C(\|\nabla^2\mathbf{b}\|_{L^2}^3 + 1) \le C(\|\mathbf{b}_t\|_{L^2}^3 + \|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^3 + 1) \\
&\le C(1 + \|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \|\mathbf{b}_t\|_{L^2}^2)(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \|\mathbf{b}_t\|_{L^2}^2) + C.
\end{aligned}
\tag{3.61}
$$

Substituting (3.59)–(3.61) into (3.58) and then applying Gronwall's inequality and (3.27) give the desired (3.43). Hence we complete the proof of Lemma 3.4. □

Finally, the following lemma will treat the higher order derivatives of the solutions which are needed to guarantee the extension of local strong solution to be a global one under the conditions (1.10), (1.11) and (3.1).

Lemma 3.5. *Under the condition* (3.1)*, and let* $q \in (3,6]$ *be as in Theorem* 1.2*, then it holds that for any* $T \in [0, T^*)$*,*

$$\sup_{0\le t\le T}\left(\|(\rho,P)\|_{W^{1,q}}+\|\nabla\mathbf{u}\|_{H^1}+\|\mathbf{b}\|_{H^2}\right)\le C. \tag{3.62}$$

Proof. First, in view of (3.2), (3.27), and (3.43), one has

$$\|\mathbf{b}\|_{H^2}\le C. \tag{3.63}$$

It follows from (3.38), (3.43), and (3.63) that

$$\|\nabla\mathbf{u}\|_{L^6}\le C. \tag{3.64}$$

By virtue of Gagliardo-Nirenberg inequality, Sobolev's inequality, (3.27), and (3.64), we arrive at

$$\|\mathbf{u}\|_{L^\infty}\le C\|\mathbf{u}\|_{L^6}^{1/2}\|\nabla\mathbf{u}\|_{L^6}^{1/2}\le C\|\nabla\mathbf{u}\|_{L^2}^{1/2}\|\nabla\mathbf{u}\|_{L^6}^{1/2}\le C.$$

Direct calculations show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\rho\|_{L^q}\le C(1+\|\nabla\mathbf{u}\|_{L^\infty})\|\nabla\rho\|_{L^q}+C\|\nabla^2\mathbf{u}\|_{L^q}. \tag{3.65}$$

Similarly,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla P\|_{L^q}\le C(1+\|\nabla\mathbf{u}\|_{L^\infty})(\|\nabla P\|_{L^q}+\|\nabla^2\mathbf{u}\|_{L^q})+C\|\nabla\mathbf{b}\|_{L^\infty}\|\nabla^2\mathbf{b}\|_{L^q}. \tag{3.66}$$

Applying the standard L^p-estimate of elliptic system to (3.26), (3.1), and (3.63) yield

$$\|\nabla G\|_{L^6}+\|\nabla\boldsymbol{\omega}\|_{L^6}\le C(\|\rho\dot{\mathbf{u}}\|_{L^6}+\|\mathbf{b}\cdot\nabla\mathbf{b}\|_{L^6})\le C+C\|\nabla\dot{\mathbf{u}}\|_{L^2},$$

which combined with Gagliardo-Nirenberg inequality implies

$$\begin{aligned}\|G\|_{L^\infty}&\le\|G\|_{L^2}^{\beta}\|\nabla G\|_{L^6}^{1-\beta}\le C+C\|\nabla\dot{\mathbf{u}}\|_{L^2}^{1-\beta},\\ \|\boldsymbol{\omega}\|_{L^\infty}&\le\|\boldsymbol{\omega}\|_{L^2}^{\beta}\|\nabla\boldsymbol{\omega}\|_{L^6}^{1-\beta}\le C+C\|\nabla\dot{\mathbf{u}}\|_{L^2}^{1-\beta}\end{aligned}$$

for some $\beta\in(0,1)$.

For $2\le p\le q$, employing the standard L^p-estimate of elliptic system to $(1.7)_2$ leads to

$$\begin{aligned}\|\nabla^2\mathbf{u}\|_{L^p}&\le C\left(\|\rho\dot{\mathbf{u}}\|_{L^p}+\|\nabla P\|_{L^p}+\|\mathbf{b}\cdot\nabla\mathbf{b}\|_{L^p}\right)\\ &\le C\left(1+\|\rho\dot{\mathbf{u}}\|_2^{\alpha}\|\rho\dot{\mathbf{u}}\|_{L^6}^{1-\alpha}+\|\nabla P\|_{L^p}\right)\\ &\le C\left(1+\|\rho\dot{\mathbf{u}}\|_2^{\alpha}\|\nabla\dot{\mathbf{u}}\|_{L^2}^{1-\alpha}+\|\nabla P\|_{L^p}\right)\\ &\le C\left(1+\|\nabla\dot{\mathbf{u}}\|_{L^2}^{1-\alpha}+\|\nabla P\|_{L^p}\right)\end{aligned} \tag{3.67}$$

for some $\alpha \in (0,1)$. This, together with Lemma 2.1, gives

$$\|\nabla \mathbf{u}\|_{L^\infty} \le C\left(1+\|\nabla \dot{\mathbf{u}}\|_{L^2}^{1-\beta}\right)\log\left(e+\|\nabla \dot{\mathbf{u}}\|_{L^2}+\|\nabla P\|_{L^q}\right)+C\|\nabla \dot{\mathbf{u}}\|_{L^2}.$$

Applying the standard L^p-estimate to $(1.7)_4$ yields

$$\begin{aligned}\|\nabla^2\mathbf{b}\|_{L^q} &\le C\left(\|\mathbf{b}_t\|_{L^q}+\||\mathbf{u}||\nabla\mathbf{b}|\|_{L^q}+\||\mathbf{b}||\nabla\mathbf{u}|\|_{L^q}\right)\\ &\le C\left(\|\mathbf{b}_t\|_{L^2}^{(6-q)/(2q)}\|\nabla\mathbf{b}_t\|_{L^2}^{(3q-6)/(2q)}+\|\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{b}\|_{L^q}+\|\mathbf{b}\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^q}\right)\\ &\le C\Big(\|\mathbf{b}_t\|_{L^2}^{(6-q)/(2q)}\|\nabla\mathbf{b}_t\|_{L^2}^{(3q-6)/(2q)}+\|\nabla\mathbf{b}\|_{L^2}^{(6-q)/(2q)}\|\nabla^2\mathbf{b}\|_{L^2}^{(3q-6)/(2q)}\\ &\qquad+\|\nabla\mathbf{u}\|_{L^2}^{(6-q)/(2q)}\|\nabla\mathbf{u}\|_{L^6}^{(3q-6)/(2q)}\Big)\\ &\le C\left(1+\|\nabla\mathbf{b}_t\|_{L^2}^{(3q-6)/(2q)}\right).\end{aligned} \tag{3.68}$$

It follows from Gagliardo-Nirenberg inequality that

$$\|\nabla\mathbf{b}\|_{L^\infty} \le C(\|\nabla^2\mathbf{b}\|_{L^q}+1). \tag{3.69}$$

Substituting (3.68) and (3.69) into (3.65)–(3.66) yields that

$$f'(t) \le Cg(t)f(t)\log f(t)+Cg(t)f(t)+Cg(t),$$

where

$$\begin{aligned}f(t) &\triangleq e+\|\nabla\rho\|_{L^q}+\|\nabla P\|_{L^q},\\ g(t) &\triangleq (1+\|\nabla\dot{\mathbf{u}}\|_{L^2})\log(e+\|\nabla\dot{\mathbf{u}}\|_{L^2})+\|\nabla\mathbf{b}_t\|_{L^2}^2.\end{aligned}$$

This yields

$$(\log f(t))' \le Cg(t)+Cg(t)\log f(t) \tag{3.70}$$

due to $f(t)>1$. Thus it follows from (3.70), (3.43), and Gronwall's inequality that

$$\sup_{0\le t\le T}\|(\nabla\rho,\nabla P)\|_{L^q} \le C, \tag{3.71}$$

which together with (3.67) yields that

$$\sup_{0\le t\le T}\|\nabla^2\mathbf{u}\|_{L^2} \le C.$$

This combined with (3.71), (3.27), and (3.63) finishes the proof of Lemma 3.5. □

With Lemmas 3.1–3.5 at hand, we are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. We argue by contradiction. Suppose that (1.12) were false, that is, (3.1) holds. Note that the general constant C in Lemmas 3.1–3.5 is independent of $t < T^*$, that is, all the a priori estimates obtained in Lemmas 3.1–3.5 are uniformly bounded for any $t < T^*$. Hence, the function

$$(\rho, \mathbf{u}, P, \mathbf{b})(x, T^*) \triangleq \lim_{t \to T^*} (\rho, \mathbf{u}, P, \mathbf{b})(x, t)$$

satisfy the initial condition (1.10) at $t = T^*$.

Furthermore, standard arguments yield that $\rho\dot{\mathbf{u}} \in C([0, T]; L^2)$, which implies

$$\rho\dot{\mathbf{u}}(x, T^*) = \lim_{t \to T^*} \rho\dot{\mathbf{u}} \in L^2.$$

Hence,

$$-\mu\Delta\mathbf{u} - (\lambda + \mu)\nabla \operatorname{div} \mathbf{u} + \nabla P - \operatorname{curl} \mathbf{b} \times \mathbf{b}\big|_{t=T^*} = \sqrt{\rho}(x, T^*)g(x)$$

with

$$g(x) \triangleq \begin{cases} \rho^{-1/2}(x, T^*)(\rho\dot{\mathbf{u}})(x, T^*) & \text{for } x \in \{x \mid \rho(x, T^*) > 0\}, \\ 0 & \text{for } x \in \{x \mid \rho(x, T^*) = 0\} \end{cases}$$

satisfying $g \in L^2$ due to (3.62). Therefore, one can take $(\rho, \mathbf{u}, P, \mathbf{b})(x, T^*)$ as the initial data and extend the local strong solution beyond T^*. This contradicts the assumption on T^*. Thus we finish the proof of Theorem 1.2. □

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