

Some Properties of the Signless Laplacian and Normalized Laplacian Tensors of General Hypergraphs

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Abstract. In this paper, we obtain some properties of signless Laplacian eigenvalues of general hypergraphs. We give the upper and the lower bound of edge connectivity of general hypergraphs in terms of average degree, minimum degree, the rank and the number of vertices, or analytic connectivity $\alpha(G)$, respectively. We also give the upper bound of analytic connectivity $\alpha(G)$ of general hypergraphs in terms of the degrees of vertices. Finally, we obtain the bounds of the smallest H^+ -eigenvalue of the normalized Laplacian sub-tensors of general hypergraphs.

1. Introduction

A hypergraph H is a pair (V, E) , where $E \subseteq P(V)$ and $P(V)$ stands for the power set of V . The elements of $V = V(H)$ are called vertices, and the elements of $E = E(H)$ are called edges. If each edge $e \in E$ contains precisely m vertices, then the hypergraph H is called m -uniform. If $H = (V, E)$ and $H' = (V', E')$ are hypergraphs such that $V' \subseteq V$ and $E' \subseteq E$, then H' is called a subhypergraph of H . For a vertex $v \in V$, we denote by E_v the set of edges containing v , that is, $E_v = \{e \in E \mid v \in e\}$. The cardinality $|E_v|$ of the set E_v is defined as the degree $d_v = d_H(v)$ of the vertex v . Similarly, for a subset $S \subset [n] = \{1, 2, \dots, n\}$, we denoted by E_S the set of edges $\{e \in E \mid S \cap e \neq \emptyset\}$. The rank of H , denoted by $\text{rank}(H)$, is the maximum cardinality of the edges in the hypergraph H . A nearly m -uniform supertree is both connected and acyclic, in which each edge contains either $m - 1$ or m vertices.

Spectral graph theory has a long history behind its development [2, 3, 5]. It is natural to generalize spectral graph theory from graphs to hypergraphs. In 2005, Qi [18] and Lim [12] independently introduced the concept of the eigenvalues of tensors and the spectra of tensors. In 2008, Lim [13] proposed the study of the spectra of hypergraphs by using the spectra of tensors. Recently, there are lots of papers concerning spectral hypergraph theory [4, 6, 8–10, 17, 19, 21, 24]. Li et al. [9] gave some operations such as

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moving edges, edge-releasing and total grafting on hypergraphs, and determined the first two spectral radius of m -uniform supertrees with n vertices. Lu and Man [16] generalized Smith's theorem to m -uniform hypergraphs, and classified all connected m -uniform hypergraphs with spectral radius at most $\rho_m = \sqrt[m]{4}$ which is the smallest limit point of the spectral radii of connected m -uniform hypergraphs. Xiao et al. [23] investigated the spectral radius of the k -uniform hypergraph when the uniform hypergraph is perturbed by 2-switch operation, and determined the maximum spectral radius of k -uniform supertrees with given a degree sequence. Xiao et al. [22] determined the first two largest spectral radii of uniform supertrees with given diameter. Liu et al. [15] investigated the bounds of the principal ratio $\gamma(H) = x_{\max}/x_{\min}$ and gave an estimate of the gap of spectral radii between H and its proper sub-hypergraph H' . Many scholars also started to investigate the signless Laplacian spectral radius of m -uniform hypergraphs [7, 14, 25].

In 2003, Qi [20] investigated the H^+ -eigenvalues of Laplacian tensors and signless Laplacian tensors. Banerjee et al. [1] gave some spectral properties of those hypermatrices of a general hypergraph, and found that these properties are similar for graphs and uniform hypergraphs. Zhang et al. [26] studied Perron-Frobenius theorem, the techniques of weighted incidence matrix and moving edges of general hypergraphs. And they determined the maximum spectral radius in two classes of nearly uniform supertrees. Inspired by the idea of Qi [20] and Banerjee et al. [1], we continue to study properties of the signless Laplacian and normalized Laplacian tensors of general hypergraphs.

In Section 2, some necessary notations and definitions are given. In Section 3, we mainly give some properties of signless Laplacian spectral of general hypergraphs, such as the bounds of H^+ -eigenvalues of the signless Laplacian tensors and the relationship of the signless Laplacian spectral radius of cartesian product of two hypergraphs and so on. In Section 4, we mainly give the bounds of edge connectivity of general hypergraphs, the bound of analytic connectivity $\alpha(G)$ of general hypergraphs and the bounds of the smallest H^+ -eigenvalue of the normalized Laplacian sub-tensors of general hypergraphs, respectively.

2. Preliminaries

A tensor \mathcal{A} with order m and dimension n over the real field \mathbb{R} is a multidimensional array

$$\mathcal{A} = (a_{i_1 i_2 \dots i_m}), \quad 1 \leq i_1, i_2, \dots, i_m \leq n.$$

The tensor \mathcal{A} is called symmetric if its entries are invariant under any permutation of their indices.

For a vector $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$, $\mathcal{A}x^{m-1}$ is a vector in \mathbb{R}^n with its i -th compo-

ment defined as

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2, \dots, i_m=1}^n a_{ii_2 \dots i_m} x_{i_2} \cdots x_{i_m}, \quad \forall i \in [n].$$

Also, a tensor \mathcal{A} of order m and dimension n uniquely determines an m th degree homogeneous polynomial function $\mathcal{A}x^m$, which is

$$x^T(\mathcal{A}x^{m-1}) = \sum_{i_1, i_2, \dots, i_m=1}^n a_{i_1 i_2 \dots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}.$$

Definition 2.1. [20] Let \mathcal{A} be a nonzero tensor with order m and dimension n . A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called eigenvalue and eigenvector (or simply an eigenpair) if they satisfy the following equation

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}.$$

For $x^{[m-1]}$ is a vector with i -th entry x_i^{m-1} . We call (λ, x) an H -eigenpair (i.e., λ and x are called H -eigenvalue and H -eigenvector, respectively) if they are both real. An H -eigenvalue λ is called an H^+ (H^{++})-eigenvalue if the corresponding eigenvector $x \in \mathbb{R}_+^n$ (\mathbb{R}_{++}^n).

Definition 2.2. [1] Let $H = (V, E)$ be a hypergraph with $V = \{v_1, v_2, \dots, v_n\}$ and $E = \{e_1, e_2, \dots, e_k\}$. Suppose that $\text{rank}(H) = m$. The adjacency tensor \mathcal{A}_H of H is as follows:

$$\mathcal{A}_H = (a_{i_1, i_2, \dots, i_m}), \quad 1 \leq i_1, i_2, \dots, i_m \leq n.$$

For all edges $e = \{v_{l_1}, v_{l_2}, \dots, v_{l_s}\} \in E$ of cardinality $s \leq m$,

$$a_{i_1, i_2, \dots, i_m} = \frac{s}{\alpha(s)}, \quad \text{where } \alpha(s) = \sum_{\substack{k_1, \dots, k_s \geq 1 \\ k_1 + \dots + k_s = m}} \frac{m!}{k_1! k_2! \cdots k_s!},$$

and i_1, i_2, \dots, i_m are chosen in all possible ways from $L = \{l_1, \dots, l_s\}$ with at least once for each element of the set L , while each k_i represents the times that l_i appears in $\{i_1, i_2, \dots, i_m\}$. The other positions of the tensor are zero.

The adjacency tensor is symmetric, while the nonzero elements of adjacency tensor may have different values. Even if the subscript of the elements contains two same indices, the value may be nonzero. Specially, when H is an m -uniform hypergraph, we know that the nonzero elements of adjacency tensor are $1/(m-1)!$.

Let $H = (V, E)$ be a hypergraph, where $V = \{v_1, v_2, \dots, v_n\}$ and $E = \{e_1, e_2, \dots, e_k\}$. The degree of a vertex v_i is given by

$$d_i = d_H(v_i) = \sum_{i_2, i_3, \dots, i_m=1}^n a_{ii_2 i_3 \dots i_m}.$$

Let E_i be the set of all the edges containing v_i and $e = \{v_i, v_{i_2}, \dots, v_{i_s}\} \in E_i$. We denote an i -th component of a vector $\mathcal{A}(e)x$ by

$$(\mathcal{A}(e)x)_i = \frac{s}{\alpha(s)} \sum_{\substack{k_1 \geq 0, k_2, \dots, k_s \geq 1 \\ k_1 + \dots + k_s = m-1}} \frac{(m-1)!}{k_1!k_2! \dots k_s!} x_i^{k_1} x_{i_2}^{k_2} \dots x_{i_s}^{k_s},$$

then we have

$$(\mathcal{A}x)_i = \sum_{e \in E_i} (\mathcal{A}(e)x)_i.$$

And we also have

$$x^T \mathcal{A}(e)x = \frac{s}{\alpha(s)} \sum_{\substack{k_1, k_2, \dots, k_s \geq 1 \\ k_1 + \dots + k_s = m}} \frac{m!}{k_1!k_2! \dots k_s!} x_{i_1}^{k_1} x_{i_2}^{k_2} \dots x_{i_s}^{k_s},$$

then we have

$$x^T \mathcal{A}x = \sum_{e \in E} x^T \mathcal{A}(e)x.$$

Definition 2.3. [1] Let $H = (V, E)$ be a (general) hypergraph without isolated vertex where $V = \{v_1, v_2, \dots, v_n\}$ and $E = \{e_1, e_2, \dots, e_k\}$. Let $\text{rank}(H) = m$. We define the Laplacian hypermatrix, \mathcal{L}_H , of $H = (V, E)$ as $\mathcal{L}_H = \mathcal{D}_H - \mathcal{A}_H = (l_{i_1 i_2 \dots i_m})$, $1 \leq i_1, i_2, \dots, i_m \leq n$, where $\mathcal{D}_H = (d_{i_1 i_2 \dots i_m})$ is the m order n dimensional hypermatrix with $d_{i_1 \dots i_1} = d_i$ and others are zero. The signless Laplacian hypermatrix of G is defined as $\mathcal{Q}_H = \mathcal{D}_H + \mathcal{A}_H$.

Let $H = (V, E)$ be a general hypergraph with $\text{rank}(H) = m$. For any edge $e = \{v_{i_1}, v_{i_2}, \dots, v_{i_s}\}$, we have

$$\mathcal{L}(e)x^m = \sum_{j=1}^s x_{i_j}^m - \frac{s}{\alpha(s)} x_m^e,$$

where $x_m^e = \sum_{\substack{k_1, \dots, k_s \geq 1 \\ k_1 + \dots + k_s = m}} \frac{m!}{k_1!k_2! \dots k_s!} x_{i_1}^{k_1} \dots x_{i_s}^{k_s}$.

And we also have

$$\mathcal{L}x^m = \sum_{e \in E} \mathcal{L}(e)x^m.$$

Similarly, we have

$$\mathcal{Q}(e)x^m = \sum_{j=1}^s x_{i_j}^m + \frac{s}{\alpha(s)} x_m^e.$$

And we also have

$$\mathcal{Q}x^m = \sum_{e \in E} \mathcal{Q}(e)x^m.$$

3. Some properties of signless Laplacian eigenvalues of general hypergraphs

In this section, we mainly give some basic properties of signless Laplacian tensors of general hypergraphs.

Theorem 3.1. *Let G be a general hypergraph with n vertices and maximum degree Δ such that $\text{rank}(G) = m$, and let q be an H -eigenvalue of \mathcal{Q}_G . Then we have $|q| \leq 2\Delta$.*

Proof. Let $x = (x_1, x_2, \dots, x_n)^T$ be an eigenvector of a hypergraph G corresponding to q and $x_t = \max\{|x_1|, |x_2|, \dots, |x_n|\}$. Without loss of generality, we can assume that $x_t = 1$.

$$\begin{aligned} |q| &= |qx_t^{m-1}| = \left| d_t x_t^{m-1} + \sum_{i_2, i_3, \dots, i_m=1}^n a_{ti_2i_3\dots i_m} x_{i_2} x_{i_3} \cdots x_{i_m} \right| \\ &\leq |d_t x_t^{m-1}| + \left| \sum_{i_2, i_3, \dots, i_m=1}^n a_{ti_2i_3\dots i_m} x_t^{m-1} \right| \\ &\leq |d_t| |x_t^{m-1}| + \sum_{i_2, i_3, \dots, i_m=1}^n |a_{ti_2i_3\dots i_m}| |x_t|^{m-1} \\ &\leq 2d_t \leq 2\Delta. \end{aligned} \quad \square$$

A hypergraph is called k -regular if every vertex has the same degree k . By Theorem 3.1, we have $|q| \leq 2k$ for a k -regular hypergraph.

Corollary 3.2. *Let G be a k -regular general hypergraph with $\text{rank}(G) = m$ and n vertices. Then $2k$ is an H^{++} -eigenvalue of \mathcal{Q}_G .*

Proof. For a vector $x = (1, 1, \dots, 1)^T \in \mathbb{R}^n$, and $1 \leq i \leq n$, we have

$$(\mathcal{Q}_G x^{m-1})_i = d_i + \sum_{i_2, i_3, \dots, i_m=1}^n a_{ii_2i_3\dots i_m} = 2k. \quad \square$$

Definition 3.3. [1] Let \mathcal{A} be a nonzero hypermatrix. A pair $(\lambda, x) \in (\mathbb{C} \times \mathbb{C}^n \setminus \{0\})$ is called an E -eigenpair (where λ and x are called E -eigenvalue and E -eigenvector, respectively) if they satisfy the following equations

$$\mathcal{A}x^{m-1} = \lambda x, \quad \sum_{i=1}^n x_i^2 = 1.$$

We call (λ, x) a Z -eigenpair if both of them are real.

Theorem 3.4. *Let G be a k -regular general hypergraph with $\text{rank}(G) = m$ and n vertices. Then $2k(1/\sqrt{n})^{m-2}$ is a Z -eigenvalue of \mathcal{Q}_G .*

Proof. For a vector $x = (1/\sqrt{n}, 1/\sqrt{n}, \dots, 1/\sqrt{n})^T \in \mathbb{R}^n$, we have $\sum_{i=1}^n x_i^2 = 1$. And for $1 \leq i \leq n$, we have

$$\begin{aligned} (\mathcal{Q}_G x^{m-1})_i &= d_i x^{m-1} + \sum_{i_2, i_3, \dots, i_m=1}^n a_{ii_2 i_3 \dots i_m} x_{i_2} x_{i_3} \cdots x_{i_m} \\ &= k \left(\frac{1}{\sqrt{n}}\right)^{m-1} + \left(\frac{1}{\sqrt{n}}\right)^{m-1} \sum_{i_2, i_3, \dots, i_m=1}^n a_{ii_2 i_3 \dots i_m} \\ &= 2k \left(\frac{1}{\sqrt{n}}\right)^{m-2} \cdot \left(\frac{1}{\sqrt{n}}\right). \quad \square \end{aligned}$$

By a similar argument to the proof of Theorem 3.4 in [1], we have the following result.

Theorem 3.5. *Let G be a general hypergraph with n vertices and maximum degree Δ such that $\text{rank}(G) = m$. Let $x = (x_1, x_2, \dots, x_n)^T$ be a Z -eigenvector of \mathcal{Q}_G corresponding to an eigenvalue q . If $x_t = \max\{|x_1|, |x_2|, \dots, |x_n|\}$, then we have $|q| \leq 2\Delta/x_t$.*

Let $x = (x_1, x_2, \dots, x_n)^T$ be a vector in \mathbb{R}^n and $m \geq s - 1$ be an integer. For an edge $e = \{v_{i_1}, v_{i_2}, \dots, v_{i_s}\}$ and a vertex v_{i_i} , we define

$$x_m^{e/v_{i_i}} := \sum x_{i_1} x_{i_2} \cdots x_{i_m},$$

where the sum is over i_1, i_2, \dots, i_m chosen in all possible way from $\{l_1, l_2, \dots, l_s\}$, such that, all l_j ($j \neq i$) occur at least once.

Definition 3.6. [1] Let G and H be two hypergraphs. The Cartesian product, $G \times H$, of G and H is defined by the vertex set $V(G \times H) = V(G) \times V(H)$ and the edge set $E(G \times H) = \{\{v\} \times e : v \in V(G), e \in E(H)\} \cup \{e \times \{v\} : e \in E(G), v \in V(H)\}$.

The following theorem is similar to the proof of uniform hypergraphs in [4].

Theorem 3.7. *Let G and H be two general hypergraphs with $\text{rank}(G) = \text{rank}(H)$. If q_G and q_H are H -eigenvalues for G and H , respectively, then $q_G + q_H$ is an H -eigenvalue for $G \times H$.*

Proof. Let n_1 and n_2 be the number of vertices in G and H , respectively. Let (q_G, \mathbf{u}) and (q_H, \mathbf{v}) be H -eigenpairs of \mathcal{Q}_G and \mathcal{Q}_H , respectively. Let $\mathbf{w} \in \mathbb{C}^{n_1 n_2}$ be a vector with the entries indexed by the pairs $(a, b) \in [n_1] \times [n_2]$ such that $w(a, b) = u(a)v(b)$. We have

$$\begin{aligned} & d_{(a,b)}^{G \times H} w^{m-1}(a, b) + \sum_{e \in E(a,b)} a_e^{G \times H} w_{m-1}^{e/(a,b)} \\ &= (d_a^G + d_b^H) w^{m-1}(a, b) + \sum_{e \times \{b\} \in E(a,b), e \in E_a} a_e^{G \times H} w_{m-1}^{e \times \{b\}/(a,b)} \end{aligned}$$

$$\begin{aligned}
 & + \sum_{\{a\} \times e \in E_{(a,b)}, e \in E_b} a_e^{G \times H} w_{m-1}^{\{a\} \times e / (a,b)} \\
 = & d_a^G w^{m-1}(a, b) + \sum_{e \in E_a(G)} a_e^{G \times H} u_{m-1}^{e/a} v^{m-1}(b) \\
 & + d_b^H w^{m-1}(a, b) + \sum_{e \in E_b(H)} a_e^{G \times H} u^{m-1}(a) v_{m-1}^{e/b} \\
 = & d_a^G u^{m-1}(a) v^{m-1}(b) + v^{m-1}(b) \sum_{e \in E_a(G)} a_e^G u_{m-1}^{e/a} \\
 & + d_b^H u^{m-1}(a) v^{m-1}(b) + u^{m-1}(a) \sum_{e \in E_b(H)} a_e^H v_{m-1}^{e/b} \\
 = & v^{m-1}(b) q_G u^{m-1}(a) + u^{m-1}(a) q_H v^{m-1}(b) \\
 = & (q_G + q_H) w^{m-1}(a, b),
 \end{aligned}$$

where $d_{(a,b)}^{G \times H}$, d_a^G and d_b^H , respectively, are the degree of (a, b) in $G \times H$, the degree of a in G and the degree of b in H . □

Denote by $\mathbf{e}^{(j)}$ the j th unit vector in \mathbb{R}^n , i.e., $\mathbf{e}_i^{(j)} = 1$ if $i = j$ and $\mathbf{e}_i^{(j)} = 0$ if $i \neq j$, for $i, j = 1, 2, \dots, n$.

Theorem 3.8. *Let G be a general hypergraph and $s = \min_{e \in E(G)} |e| \geq 2$. For $j = 1, 2, \dots, n$, d_j is an H^+ -eigenvalue of \mathcal{Q}_G with eigenvector $\mathbf{e}^{(j)}$.*

Proof. For a vector $\mathbf{e}^{(j)}$, we can easily verify that for $j = 1, 2, \dots, n$, d_j is an H^+ -eigenvalue of \mathcal{Q}_G with eigenvector $\mathbf{e}^{(j)}$ by the characteristic equation of the signless Laplacian tensor. □

Theorem 3.9. *Let G be a general hypergraph with $\text{rank}(G) = m$ and n vertices. Then we have*

$$\max\{\Delta, 2\bar{d}\} \leq q_1 \leq 2\Delta,$$

where $\bar{d} = \frac{1}{n} \sum_{i=1}^n d_i$.

Proof. By the characteristic equation of the signless Laplacian tensor, it is obvious that $\Delta \leq q_1 \leq 2\Delta$. Let $x = \left(\frac{1}{n^{1/m}}, \frac{1}{n^{1/m}}, \dots, \frac{1}{n^{1/m}}\right)^T$, we have

$$q_1 \geq \mathcal{Q}_G x^m = \sum_{i=1}^n d_i x_i^m + \sum_{i=1}^n \sum_{i_2, i_3, \dots, i_m=1}^n a_{ii_2 i_3 \dots i_m} x_i x_{i_2} x_{i_3} \cdots x_{i_m} = 2 \frac{1}{n} \sum_{i=1}^n d_i.$$

Thus,

$$\max\{\Delta, 2\bar{d}\} \leq q_1 \leq 2\Delta. \quad \square$$

By a similar argument to the proof of Theorem 15 of [20], we have the following result.

Theorem 3.10. *The smallest signless Laplacian H^+ -eigenvalue of G is δ . We always have*

$$\delta = \min \left\{ \mathcal{Q}x^k : x \in \mathbb{R}_+^n, \sum_{i=1}^n x_i^k = 1 \right\}.$$

4. Edge connectivity and H^+ -eigenvalues of sub-tensors of the normalized Laplacian tensors

Let S be a proper nonempty subset of V , and $\bar{S} = V \setminus S$ be the complement of S in G . The edge set E is partitioned into three parts $E(S)$, $E(\bar{S})$ and $E(S, \bar{S})$. The edge set $E(S)$ (resp. $E(\bar{S})$) consists of edges whose vertices are all in S (resp. \bar{S}). The edge set $E(S, \bar{S})$ consists of edges whose vertices are in both S and \bar{S} . We call $E(S, \bar{S})$ an edge cut of G . If we delete all the edges of $E(S, \bar{S})$ from G , then G is separated into two hypergraphs $G[S] = (S, E(S))$ and $G[\bar{S}] = (\bar{S}, E(\bar{S}))$ (note that the ranks of $G[S]$ and $G[\bar{S}]$ may be different). For an edge $e_p \in E(S, \bar{S})$, $t(e_p)$ is the number of vertices contained in S of e_p . For all edges $e_p \in E(S, \bar{S})$, the average value of such $t(e_p)$ is denoted $t(S)$. We call the minimum (resp. maximum) cardinality of such an edge cut the edge connectivity (resp. maximum cut) of G , and denoted by $e(G)$ (resp. $c(G)$).

For each edge $e = \{v_{i_1}, v_{i_2}, \dots, v_{i_t}\}$, denote by S_e the collection of all the ordered multisets extended from e , that is $|S_e| = \alpha(t)$, and $S_{E(G)} = \bigcup_{e \in E(G)} S_e$ represent all the possible ordered multisets extended from the edges in $E(G)$.

In the following, let G be a hypergraph with n vertices and k edges. When the hypergraph G is uniform, Theorem 4.1 is exactly Proposition 16 of Qi [20].

Theorem 4.1. *Let G be a general hypergraph with n vertices, average degree \bar{d} , minimum degree δ and $\text{rank}(G) = m$. Let $m \geq s = \min_{e \in E(G)} |e| \geq 2$.*

(i) *If $m - 2s < 0$, then, for any nonempty proper subset S of $V(G)$, we have*

$$e(G) \leq |E(S, \bar{S})| \leq \frac{(2\bar{d} - \delta)n}{2s - m}.$$

(ii) *If $n \leq 2s - 1$, then $e(G) = \delta$.*

Proof. (i) Let S be a nonempty proper subset of V . Let $x = \frac{1}{|S|^{1/m}} \sum_{i \in S} \mathbf{e}^{(i)}$, where $\mathbf{e}^{(i)}$ is a vector whose i -th element is 1, and other elements are 0. For $e_p \in E(S)$, by Definition 2.3, we have

$$\begin{aligned} \mathcal{Q}_{G(e_p)}x^m &= |e_p| \cdot \frac{1}{|S|} + \frac{|e_p|}{\alpha(|e_p|)} \sum_{e'_p \in S_{e_p}} x^{e'_p} = |e_p| \cdot \frac{1}{|S|} + \frac{|e_p|}{\alpha(|e_p|)} \sum_{e'_p \in S_{e_p}} \frac{1}{|S|} \\ &= |e_p| \cdot \frac{1}{|S|} + \frac{|e_p|}{\alpha(|e_p|)} \cdot \alpha(|e_p|) \cdot \frac{1}{|S|} = \frac{2|e_p|}{|S|}. \end{aligned}$$

For $e_p \in E(\bar{S})$, it is obvious that

$$\mathcal{Q}_G(e_p)x^m = 0.$$

For $e_p \in E(S, \bar{S})$, it is obvious that

$$\mathcal{Q}_G(e_p)x^m = \frac{t(e_p)}{|S|}.$$

Thus,

$$\begin{aligned} \mathcal{Q}_G x^m &= \left(\sum_{e_p \in E(S)} + \sum_{e_p \in E(\bar{S})} + \sum_{e_p \in E(S, \bar{S})} \right) \mathcal{Q}_G(e_p)x^m \\ &= \sum_{e_p \in E(S)} \frac{2|e_p|}{|S|} + \sum_{e_p \in E(S, \bar{S})} \frac{t(e_p)}{|S|} = \sum_{e_p \in E(S)} \frac{2|e_p|}{|S|} + \frac{t(S)}{|S|} |E(S, \bar{S})|. \end{aligned}$$

Moreover, by Theorem 3.10, we have

$$(4.1) \quad \delta|S| \leq \sum_{e_p \in E(S)} 2|e_p| + t(S)|E(S, \bar{S})|.$$

Similarly, letting $y = \frac{1}{|\bar{S}|^{1/m}} \sum_{i \in \bar{S}} \mathbf{e}^{(i)}$, we have

$$\mathcal{Q}_G x^m = \left(\sum_{e_p \in E(S)} + \sum_{e_p \in E(\bar{S})} + \sum_{e_p \in E(S, \bar{S})} \right) \mathcal{Q}_G(e_p)x^m = \sum_{e_p \in E(\bar{S})} \frac{2|e_p|}{|\bar{S}|} + \frac{t(\bar{S})}{|\bar{S}|} |E(S, \bar{S})|.$$

By Theorem 3.10, we have

$$(4.2) \quad \delta|\bar{S}| \leq \sum_{e_p \in E(\bar{S})} 2|e_p| + t(\bar{S})|E(S, \bar{S})|.$$

By Inequalities (4.1) and (4.2), we have

$$\begin{aligned} \delta n &\leq \sum_{e_p \in E(S)} 2|e_p| + t(S)|E(S, \bar{S})| + \sum_{e_p \in E(\bar{S})} 2|e_p| + t(\bar{S})|E(S, \bar{S})| \\ &= \sum_{e_p \in E(G)} 2|e_p| - \sum_{e_p \in E(S, \bar{S})} 2|e_p| + (t(S) + t(\bar{S}))|E(S, \bar{S})| \\ &\leq 2\bar{d}n - 2s|E(S, \bar{S})| + m|E(S, \bar{S})| \\ &= 2\bar{d}n - (2s - m)|E(S, \bar{S})|. \end{aligned}$$

If $m - 2s < 0$, then we have

$$e(G) \leq |E(S, \bar{S})| \leq \frac{(2\bar{d} - \delta)n}{2s - m}.$$

(ii) If $n \leq 2s - 1$, then either $|S| < s$ or $|\overline{S}| < s$. Without loss of generality, we may assume that $|S| < s$. Then $E(S) = \emptyset$, that is, $|E(S)| = 0$. Thus, by (4.1), we have

$$|S|\delta \leq t(S)|E(S, \overline{S})|.$$

Since $t(S) \leq |S|$, we have

$$\delta \leq |E(S, \overline{S})|.$$

Thus, $e(G) \geq \delta$.

Assume that $d_j = \delta$. Let $S = \{j\}$, then

$$|E(S, \overline{S})| = d_j = \delta.$$

Thus, $e(G) = \delta$. □

Let $G = (V, E)$ be a general hypergraph with $\text{rank}(G) = m$. Similar to Qi [20] the analytic connectivity $\alpha(G)$ of G is defined as

$$\alpha(G) = \min_{j=1,2,\dots,n} \left\{ \min \left\{ \mathcal{L}x^m \mid x \in \mathbb{R}_+^n, \sum_{i=1}^n x_i^m = 1, x_j = 0 \right\} \right\}.$$

When the hypergraph G is uniform, Theorem 4.2 is exactly Proposition 18 of Qi [20].

Theorem 4.2. *Let G be a general hypergraph with n vertices and $\text{rank}(G) = m$. Then we have*

$$e(G) \geq \frac{n}{m} \alpha(G).$$

Proof. Let $s = \min_{e \in E(G)} |e|$, and let S be a nonempty proper subset of V . Then there is a $j \notin S$ such that

$$(4.3) \quad \min \left\{ \mathcal{L}_G x^m : x \in \mathbb{R}_+^n, \sum_{i=1}^n x_i^m = 1, x_j = 0 \right\} \geq \alpha(G).$$

Let $x = \frac{1}{|S|^{1/m}} \sum_{i \in S} e^{(i)}$. It is clear that x is a feasible point of the minimization problem in (4.3). For $e_p \in E(S)$, by Definition 2.3, we have

$$\mathcal{L}_G(e_p)x^m = |e_p| \cdot \frac{1}{|S|} - \frac{|e_p|}{\alpha(|e_p|)} \cdot \alpha(|e_p|) \cdot \frac{1}{|S|} = 0.$$

For $e_p \in E(\overline{S})$, it is obvious that

$$\mathcal{L}_G(e_p)x^m = 0.$$

For $e_p \in E(S, \overline{S})$, it is obvious that

$$\mathcal{L}_G(e_p)x^m = \frac{t(e_p)}{|S|}.$$

Thus,

$$\begin{aligned} \mathcal{L}_G x^m &= \left(\sum_{e_p \in E(S)} + \sum_{e_p \in E(\bar{S})} + \sum_{e_p \in E(S, \bar{S})} \right) \mathcal{L}_G(e_p) x^m \\ &= \sum_{e_p \in E(S, \bar{S})} \frac{t(e_p)}{|S|} = \frac{t(S)}{|S|} |E(S, \bar{S})|. \end{aligned}$$

By Inequality (4.3), we have

$$(4.4) \quad \alpha(G)|S| \leq t(S)|E(S, \bar{S})|.$$

Similarly, letting $y = \frac{1}{|\bar{S}|^{1/m}} \sum_{i \in \bar{S}} \mathbf{e}^{(i)}$, we have

$$\mathcal{L}_G x^m = \left(\sum_{e_p \in E(S)} + \sum_{e_p \in E(\bar{S})} + \sum_{e_p \in E(S, \bar{S})} \right) \mathcal{L}_G(e_p) x^m = \frac{t(\bar{S})}{|\bar{S}|} |E(S, \bar{S})|$$

and

$$(4.5) \quad \alpha(G)|\bar{S}| \leq t(\bar{S})|E(S, \bar{S})|.$$

Thus, by Inequalities (4.4) and (4.5), we have

$$\alpha(G)n \leq t(S)|E(S, \bar{S})| + t(\bar{S})|E(S, \bar{S})| \leq m|E(S, \bar{S})|.$$

Then, we have

$$|E(S, \bar{S})| \geq \frac{n}{m} \alpha(G).$$

Hence,

$$e(G) \geq \frac{n}{m} \alpha(G). \quad \square$$

By Theorem 4.2 and $e(G) \leq \delta$, we note $\alpha(G) \leq \frac{m}{n} \delta$ for a hypergraph with rank = m and minimum degree δ .

Motivated by the idea of Li et al. [11], we have the following theorem. When G is an m -uniform hypergraph, the upper bound in Theorem 4.3 is $\min \left\{ \frac{d_{i_1} + d_{i_2} + \dots + d_{i_m} - m}{m} \mid \{v_{i_1}, v_{i_2}, \dots, v_{i_m}\} \in E(G) \right\}$, which is exactly the upper bound of Theorem 3.9 in Li et al. [11].

Theorem 4.3. *Let G be a general hypergraph with $\text{rank}(G) = m$, n vertices and more than one edge. Then we have*

$$\alpha(G) \leq \min \left\{ \frac{d_{i_1} + d_{i_2} + \dots + d_{i_l} - l}{l} \mid \{v_{i_1}, v_{i_2}, \dots, v_{i_l}\} \in E(G) \right\}.$$

Proof. Let $e = \{v_{i_1}, v_{i_2}, \dots, v_{i_l}\} \in E(G)$ be an edge of G and let $x = \frac{1}{l^{1/m}} \sum_{i \in e} \mathbf{e}^{(i)}$ be a vector. It is obvious $\sum_{i=1}^n x_i^m = 1$. Thus, we have

$$\begin{aligned} \alpha(G) &\leq \mathcal{L}_G x^m = \sum_{e_p \in E(G)} \mathcal{L}_G(e_p) x^m \\ &= d_{i_1} x_{i_1}^m + d_{i_2} x_{i_2}^m + \dots + d_{i_l} x_{i_l}^m - \sum_{e' \in S_e} \frac{l}{\alpha(l)} \frac{1}{l} \\ &= d_{i_1} \frac{1}{l} + d_{i_2} \frac{1}{l} + \dots + d_{i_l} \frac{1}{l} - 1 = \frac{d_{i_1} + d_{i_2} + \dots + d_{i_l} - l}{l}. \quad \square \end{aligned}$$

Let G be a general hypergraph with n vertices and k edges, and $\text{rank}(G) = m$. The normalized adjacency tensor \mathfrak{A} , which is an m -th order n -dimensional symmetric nonnegative tensor, is defined as: for any edge $e = \{v_{l_1}, v_{l_2}, \dots, v_{l_s}\} \in E$ of cardinality $s \leq m$,

$$a_{i_1, i_2, \dots, i_m} = \frac{s}{\alpha(s)} \prod_{j=1}^m \frac{1}{\sqrt[m]{d(v_{i_j})}}, \quad \text{where } \alpha(s) = \sum_{\substack{k_1, \dots, k_s \geq 1 \\ k_1 + \dots + k_s = m}} \frac{m!}{k_1! k_2! \dots k_s!},$$

and i_1, i_2, \dots, i_m are chosen in all possible ways from $\{l_1, l_2, \dots, l_s\}$ with at least once for each element of the set. The rest of the positions are zero.

The normalized Laplacian tensor \mathfrak{L} , which is an m -th order n -dimensional symmetric tensor, is defined as

$$\mathfrak{L} = \mathfrak{J} - \mathfrak{A},$$

where \mathfrak{J} is an m -th order n -dimensional diagonal tensor with the i -th diagonal element $j_{i \dots i} = 1$, and zero otherwise.

In the following, we gave some results of sub-tensors of the normalized Laplacian tensors.

Definition 4.4. [8] Let \mathcal{T} be a real tensor with order k dimensional n and $s \in [n]$. The k -th order s -dimensional tensor \mathcal{U} with entries $u_{i_1, i_2, \dots, i_k} = t_{i_1, i_2, \dots, i_k}$ for all $i_1, i_2, \dots, i_k \in [s]$ is called the sub-tensor of \mathcal{T} associated to the subset $S := \{j_1, j_2, \dots, j_s\}$. We usually denoted \mathcal{U} as \mathcal{T}_S .

Let $S \subseteq [n]$ be nonempty and let $\kappa(S)$ be the smallest H^+ -eigenvalue of normalized Laplacian sub-tensor \mathfrak{L}_S . The proof of the following theorem is similar to that of uniform hypergraphs in [8]. Hence, we omit its proof.

Theorem 4.5. *Let G be a general hypergraph on n vertices without isolated vertices and $\text{rank}(G) = m$. Let $S \subseteq [n]$ be nonempty. Then we have $\kappa(S) = 1 - \rho(\mathfrak{A}_S)$, and*

$$\kappa(S) = \min \left\{ \mathfrak{L} y^m \mid y \in \mathbb{R}_+^n, \sum_{i \in [n]} y_i^m = 1, y_i = 0, \forall i \in \bar{S} \right\},$$

where $\rho(\mathfrak{A}_S)$ is the spectral radius of normalized adjacency tensor \mathfrak{A}_S .

Corollary 4.6. *Let G be a general hypergraph on n vertices without isolated vertices and $\text{rank}(G) = m$. Let $S, T \subseteq [n]$ be nonempty such that $S \subset T$. Then we have $\kappa(T) \leq \kappa(S)$.*

Proof. By Theorem 4.5, it is obvious that Corollary 4.6 holds. \square

Let $\delta = \min_{i \in [n]} d_i$ and $\Delta = \max_{i \in [n]} d_i$. For a nonempty subset $S \subset V$, define $\text{vol}(S) = \sum_{i \in S} d_i$ as the volume of S . Let $\text{vol}(\emptyset) = 0$. The volume $\text{vol}([n])$ of a hypergraph is simply denoted as d_{vol} . For a subset $S \subset [n] = \{1, 2, \dots, n\}$, we use E_S to denote the set of edges $\{e \in E \mid S \cap e \neq \emptyset\}$.

Theorem 4.7. *Let G be a general hypergraph on n vertices without isolated vertices and $\text{rank}(G) = m$, and $s = \min_{e \in E(G)} |e| \geq 2$. For any nonempty $S \subseteq [n]$, we have*

$$\kappa(S) \leq \frac{m(s-1)\text{vol}(\bar{S}) + (m-s)d_{\text{vol}} - s(m-s)}{m\text{vol}(S)}.$$

In particular, for any $i \in [n]$, we have

$$\kappa(\overline{\{i\}}) \leq \frac{m(s-1)\Delta + (m-s)d_{\text{vol}} - s(m-s)}{m(d_{\text{vol}} - \Delta)}.$$

Proof. If $S = V$, then it is obvious that $\kappa(S) = \kappa(V) = 0$ and the result holds. Thus we may assume that $S \neq V$. Let $\mathbf{y} = (y_1, y_2, \dots, y_n)^T$ be the vector with its j -th element being $\sqrt[m]{d_j} / \sqrt[m]{\text{vol}(S)}$ for all $j \in S$ and $y_j = 0$ for $j \in \bar{S}$. Then, by Theorem 4.5, we have that

$$\begin{aligned} \kappa(S) &\leq \mathfrak{L}\mathbf{y}^m = 1 - \mathfrak{A}_S \mathbf{y}^m = 1 - \sum_{e \in E \setminus E_{\bar{S}}} \mathfrak{A}_S(e) \mathbf{y}^m \\ &= 1 - \sum_{e \in E \setminus E_{\bar{S}}, |e|=l} \frac{l}{\alpha(l)} \sum_{\substack{k_1, k_2, \dots, k_l \geq 1 \\ k_1 + k_2 + \dots + k_l = m}} \frac{m!}{k_1! \cdots k_l!} \frac{y_{i_1}^{k_1} y_{i_2}^{k_2} \cdots y_{i_l}^{k_l}}{d_{i_1}^{k_1/m} d_{i_2}^{k_2/m} \cdots d_{i_l}^{k_l/m}} \\ &\leq 1 - s \sum_{e \in E \setminus E_{\bar{S}}} \frac{1}{\text{vol}(S)} = 1 - s \frac{|E| - |E_{\bar{S}}|}{\text{vol}(S)} \\ &\leq \frac{\text{vol}(S) + s|E_{\bar{S}}| - \frac{sd_{\text{vol}} + s(m-s)}{m}}{\text{vol}(S)} \leq \frac{\text{vol}(S) + s\text{vol}(\bar{S}) - \frac{sd_{\text{vol}} + s(m-s)}{m}}{\text{vol}(S)} \\ &= \frac{m(s-1)\text{vol}(\bar{S}) + (m-s)d_{\text{vol}} - s(m-s)}{m\text{vol}(S)}. \end{aligned}$$

Since $d_{\text{vol}} \leq s + m(|E| - 1)$, that is, $|E| \geq (d_{\text{vol}} + m - s)/m$, we know that the inequality above is true.

Moreover, for any $i \in [n]$, it is obvious that

$$\kappa(\overline{\{i\}}) \leq \frac{m(s-1)\Delta + (m-s)d_{\text{vol}} - s(m-s)}{m(d_{\text{vol}} - \Delta)}. \quad \square$$

Example 4.8. Let $G = (V, E)$ be a general hypergraph, where $V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 2, 3, 4\}, \{2, 3, 4\}, \{3, 4, 5\}, \{1, 5\}\}$. We know that $m (= \text{rank}(G)) = 4$ and $s = 2$. Let $S = \{1, 2, 3, 4\}$, then we have $\text{vol}(S) = d_1 + d_2 + d_3 + d_4 = 2 + 2 + 3 + 3 = 10$, $\text{vol}(\bar{S}) = d_5 = 2$, and $d_{\text{vol}} = \text{vol}(S) + \text{vol}(\bar{S}) = 12$. By Theorem 4.7, we have that

$$\kappa(S) \leq \frac{4(2-1)\text{vol}(\bar{S}) + (4-2)d_{\text{vol}} - 2(4-2)}{4\text{vol}(S)} = \frac{8 + 24 - 4}{40} = \frac{7}{10}.$$

Theorem 4.9. Let G be a general hypergraph on n vertices without isolated vertices and $\text{rank}(G) = m$, and $s = \min_{e \in E(G)} |e| \geq 2$. Then we have

$$\min \kappa(\overline{\{i\}}) \leq \frac{d_{\text{vol}} + s(e(G) - |E|)}{d_{\text{vol}}}.$$

Proof. Let $S \subseteq [n]$ be nonempty. Let $\mathbf{y} = (y_1, y_2, \dots, y_n)^T$ be the vector with its j -th element being $\sqrt[m]{d_j} / \sqrt[m]{\text{vol}(S)}$ for all $i \in S$ and $y_j = 0$ for $j \in \bar{S}$. Then, by Theorem 4.5, we have that

$$\begin{aligned} \kappa(S) &\leq \mathfrak{L}\mathbf{y}^m = 1 - \sum_{e \in E(S), |e|=l} \frac{l}{\alpha(l)} \sum_{\substack{k_1, k_2, \dots, k_l \geq 1 \\ k_1 + k_2 + \dots + k_l = m}} \frac{m!}{k_1! \cdots k_l!} \frac{y_{i_1}^{k_1} y_{i_2}^{k_2} \cdots y_{i_l}^{k_l}}{d_{i_1}^{k_1/m} d_{i_2}^{k_2/m} \cdots d_{i_l}^{k_l/m}} \\ &\leq 1 - s \sum_{e \in E(S)} \frac{1}{\text{vol}(S)} = 1 - s \frac{|E(S)|}{\text{vol}(S)}. \end{aligned}$$

Similarly, we have that

$$\kappa(\bar{S}) \leq 1 - s \frac{|E(\bar{S})|}{\text{vol}(\bar{S})}.$$

Hence,

$$\begin{aligned} \kappa(S) \text{vol}(S) + \kappa(\bar{S}) \text{vol}(\bar{S}) &\leq \text{vol}(S) + \text{vol}(\bar{S}) - s(|E(S)| + |E(\bar{S})|) \\ &= d_{\text{vol}} - s(|E| - |E(S, \bar{S})|). \end{aligned}$$

It is obvious that $S \subseteq \overline{\{r\}}$ and $\bar{S} \subseteq \overline{\{t\}}$ for some r and t respectively. Without loss of generality, we assume that $\kappa(S) \leq \kappa(\bar{S})$. By Corollary 4.6, we have that

$$\begin{aligned} d_{\text{vol}} \min_{i \in [n]} \kappa(\overline{\{i\}}) &\leq d_{\text{vol}} \min \{ \kappa(\overline{\{r\}}), \kappa(\overline{\{t\}}) \} \leq d_{\text{vol}} \kappa(S) = (\text{vol}(S) + \text{vol}(\bar{S})) \kappa(S) \\ &\leq \text{vol}(S) \kappa(S) + \text{vol}(\bar{S}) \kappa(\bar{S}) \leq d_{\text{vol}} - s(|E| - |E(S, \bar{S})|). \end{aligned}$$

Thus,

$$\min \kappa(\overline{\{i\}}) \leq \frac{d_{\text{vol}} + s(e(G) - |E|)}{d_{\text{vol}}}.$$

□

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