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Extension Operators Preserving Janowski Classes of Univalent Functions

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Abstract. In this paper, our main interest is devoted to study the extension operator $\Phi_{n,\alpha,\beta}\colon \mathcal{L}S \to \mathcal{L}S_n$ given by $\Phi_{n,\alpha,\beta}(f)(z) = \big(f(z_1), \widetilde{z}(f(z_1)/z_1)^\alpha (f'(z_1))^\beta\big)$, $z = (z_1, \widetilde{z}) \in \mathbf{B}^n$, where $\alpha, \beta \geq 0$. We shall prove that if $f \in S$ can be embedded as the first element of a g-Loewner chain with $g\colon U \to \mathbb{C}$ given by $g(\zeta) = (1 + A\zeta)/(1 + B\zeta)$, $|\zeta| < 1$, and $-1 \leq B < A \leq 1$, then $F = \Phi_{n,\alpha,\beta}(f)$ can be embedded as the first element of a g-Loewner chain on the unit ball $\mathbf{B}^n$ for $\alpha \in [0, 1]$, $\beta \in [0, 1/2]$ and $\alpha + \beta \leq 1$. As a consequence, the operator $\Phi_{n,\alpha,\beta}$ preserves the notions of Janowski starlikeness on $\mathbf{B}^n$ and Janowski almost starlikeness on $\mathbf{B}^n$. Particular cases will be also mentioned.

On the other hand, we are also concerned about some radius problems related to the operator $\Phi_{n,\alpha,\beta}$ and the Janowski class $S^*(a, b)$. We compute the radius $S^*(a, b)$ of the class S (respectively S^*).

1. Introduction and preliminaries

Let $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, z_2, \ldots, z_n)$ where $z_j \in \mathbb{C}$, $1 \leq j \leq n$ with the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w}_j$ and the Euclidean norm $\|z\| = \sqrt{\langle z, z\rangle}$. The open unit ball $\{z \in \mathbb{C}^n : \|z\| = 1\}$ is denoted by $\mathbf{B}^n$ and, in the case of one complex variable, $\mathbf{B}^1$ is denoted by U. We denote by $H(\mathbf{B}^n)$ the set of holomorphic mappings from $\mathbf{B}^n$ into $\mathbb{C}^n$. We say that $f \in H(\mathbf{B}^n)$ is normalized if $f(0) = 0$ and $Df(0) = I_n$, where I_n is the $n \times n$-unitary matrix. We denote by $\mathcal{L}S_n$ the set of normalized locally biholomorphic mappings on $\mathbf{B}^n$ and, in the case of one complex variable, $\mathcal{L}S_1$ is denoted by $\mathcal{L}S$. We consider the following notations: $S(\mathbf{B}^n)$ the family of normalized biholomorphic mappings on $\mathbf{B}^n$, $S^*(\mathbf{B}^n)$ the family of normalized biholomorphic mappings that are starlike with respect to zero, respectively $K(\mathbf{B}^n)$ the family of normalized biholomorphic mappings on $\mathbf{B}^n$ that are convex. In the case of one complex variable, the above families will be denoted by S, S^*, respectively K.

Further we will introduce some subclasses of $H(\mathbf{B}^n)$ that will be useful in the next sections.

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The Carathéodory class of holomorphic functions with positive real part on U is defined by (see e.g., [24])

$$\mathcal{P} = \{p \in H(U) : p(0) = 1, \operatorname{Re} p(z) > 0, |z| < 1\}.$$

The above class was generalized to the unit ball $\mathbf{B}^n$ ($n \geq 2$) as follows (see [23]):

$$\mathcal{M} = \{h \in H(\mathbf{B}^n) : h(0) = 0, Dh(0) = I_n, \operatorname{Re}\langle h(z), z\rangle > 0, z \in \mathbf{B}^n \setminus \{0\}\}.$$

This class is related to some subclasses of biholomorphic mappings on $\mathbf{B}^n$, for example the class of biholomorphic mappings which have parametric representation, the class of starlike mappings and the class of spirallike mappings of type δ, where $\delta \in (-\pi/2, \pi/2)$ (see e.g., [10, 28] and the references therein).

Definition 1.1. Let $g\colon U \to \mathbb{C}$ be an univalent function on the unit disk U such that $g(0) = 1$, $\operatorname{Re} g(\zeta) > 0$, $\zeta \in U$ and the coefficients in its power series expansion are real (i.e., $g(\overline{\zeta}) = \overline{g(\zeta)}$ on U). Also, assume g satisfies the following conditions for all $r \in (0, 1)$:

$$\min_{|\zeta|=r} \operatorname{Re} g(\zeta) = \min\{g(r), g(-r)\},$$
$$\max_{|\zeta|=r} \operatorname{Re} g(\zeta) = \max\{g(r), g(-r)\}.$$

In this paper, our main concern is the case when the function g has the following particular form:

$$g(\zeta) = \frac{1 + A\zeta}{1 + B\zeta}, \quad |\zeta| < 1, \quad \text{where } -1 \leq B < A \leq 1. \tag{1.1}$$

Let $\mathcal{M}_g$ be the subclass of $\mathcal{M}$ given by (see [7])

$$\mathcal{M}_g = \{h \in H(\mathbf{B}^n) : h(0) = 0, Dh(0) = I_n, \langle h(z), z/\|z\|^2\rangle \in g(U), z \in \mathbf{B}^n \setminus \{0\}\},$$

where g is given as in Definition 1.1. Note that if $h \in \mathcal{M}_g$, then $\langle h(z), z/\|z\|^2\rangle\big|_{z=0} = 1$, since $h(0) = 0$ and $Dh(0) = I_n$.

Let $g\colon U \to \mathbb{C}$ be a function given by (1.1) then we obtain the following particular forms of $\mathcal{M}_g$ by choosing suitable values of parameters A and B.

Case I: $B = -1$. In this situation, we have that

$$\mathcal{M}_g = \left\{h \in H(\mathbf{B}^n) : h(0) = 0, Dh(0) = I_n, \operatorname{Re}\langle h(z), z\rangle > \frac{1 - A}{2}\|z\|^2, z \in \mathbf{B}^n \setminus \{0\}\right\}.$$

Moreover, if $A = 1$, then $\mathcal{M}_g = \mathcal{M}$.

Case II: $B \neq -1$. In this case, we have that

$$\mathcal{M}_g = \left\{h \in H(\mathbf{B}^n) : h(0) = 0, Dh(0) = I_n, \left|\frac{1}{\|z\|^2}\langle h(z), z\rangle - \frac{1 - AB}{1 - B^2}\right| < \frac{A - B}{1 - B^2}, z \in \mathbf{B}^n \setminus \{0\}\right\}.$$

Now, we assume that $A = (a-1)/b$ and $B = (a^2 - b^2 - a)/b$, where $|1-a| < b \leq a$. In the case that $b < a$, we obtain that

$$\mathcal{M}_g = \left\{ h \in H(\mathbf{B}^n) : h(0) = 0, Dh(0) = I_n, \left| \frac{1}{\|z\|^2} \langle h(z), z \rangle - \frac{a}{a^2 - b^2} \right| < \frac{b}{a^2 - b^2}, z \in \mathbf{B}^n \setminus \{0\} \right\}.$$

This special case is related to Janowski starlikeness on $\mathbf{B}^n$ (see [4]).

If $A = (a - a^2 + b^2)/b$ and $B = (1-a)/b$ with $|1-a| < b \leq a$, then

$$\mathcal{M}_g = \left\{ h \in H(\mathbf{B}^n) : h(0) = 0, Dh(0) = I_n, \left| \frac{1}{\|z\|^2} \langle h(z), z \rangle - a \right| < b, z \in \mathbf{B}^n \setminus \{0\} \right\}.$$

This case is related to Janowski almost starlikeness on $\mathbf{B}^n$ (see [4]).

The following subclasses of biholomorphic mappings on $\mathbf{B}^n$ were introduced by Curt (see [4]).

Definition 1.2. (see [4]) Assume $a, b \in \mathbb{R}$ such that $|1-a| < b \leq a$. Let

$$S^*(a, b, \mathbf{B}^n) = \left\{ f \in \mathcal{L}S_n : \left| \frac{\|z\|^2}{\langle [Df(z)]^{-1} f(z), z \rangle} - a \right| < b, z \in \mathbf{B}^n \setminus \{0\} \right\}$$

be the class of Janowski starlike mappings on $\mathbf{B}^n$ and let

$$\mathcal{A}S^*(a, b, \mathbf{B}^n) = \left\{ f \in \mathcal{L}S_n : \left| \frac{\langle [Df(z)]^{-1} f(z), z \rangle}{\|z\|^2} - a \right| < b, z \in \mathbf{B}^n \setminus \{0\} \right\}$$

be the class of Janowski almost starlike mappings on $\mathbf{B}^n$.

We remark that both classes $S^*(a, b, \mathbf{B}^n)$ and $\mathcal{A}S^*(a, b, \mathbf{B}^n)$ are subsets of $S^*(\mathbf{B}^n)$, since $|a-1| < b \leq a$.

The class $S^*(a, b, \mathbf{B}^1)$ reduces to the following subclass of S^*:

$$S^*(a, b) = \left\{ f \in H(U) : f(0) = 0, f'(0) = 1, \left| \frac{zf'(z)}{f(z)} - a \right| < b, z \in U \right\}.$$

Note that the class $S^*(a, b)$ was introduced by Silverman in [26] (see also [27]). This class is closely related to the following class of holomorphic functions on U, which was introduced by Janowski [16]

$$S^*[A, B] = \left\{ f \in H(U) : f(0) = 0, f'(0) = 1, \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}, z \in U \right\},$$

where $-1 \leq B < A \leq 1$ and "$\prec$" is the usual symbol of subordination.

Also, the class $\mathcal{A}S^*(a, b, \mathbf{B}^1)$ reduces to the following subclass of S^*:

$$\mathcal{A}S^*(a, b) = \left\{ f \in H(U) : f(0) = 0, f'(0) = 1, \left| \frac{f(z)}{zf'(z)} - a \right| < b, z \in U \right\}.$$

Next, we recall the definition of starlikeness of order γ on $\mathbf{B}^n$, where $\gamma \in [0, 1)$. This notion was introduced by Curt [3] and Kohr [17].

Definition 1.3. Let $f \in \mathcal{L}S_n$ and $\gamma \in [0,1)$. The mapping f is said to be starlike of order γ if

$$\operatorname{Re}\left\{\frac{\|z\|^2}{\langle [Df(z)]^{-1}f(z), z\rangle}\right\} > \gamma, \quad z \in \mathbf{B}^n \setminus \{0\}.$$

Let $S^*_\gamma(\mathbf{B}^n)$ be the set of starlike mappings of order γ on $\mathbf{B}^n$.

Now, we recall the notion of almost starlikeness of order γ on $\mathbf{B}^n$, where $\gamma \in [0,1)$. This notion was introduced by Kohr [18] for $\gamma = 1/2$, Feng [6], and by Xu and Liu [31].

Definition 1.4. Let $f \in \mathcal{L}S_n$ and $\gamma \in [0,1)$. The mapping f is said to be almost starlike of order γ on $\mathbf{B}^n$ if

$$\operatorname{Re}\left\{\frac{\langle [Df(z)]^{-1}f(z), z\rangle}{\|z\|^2}\right\} > \gamma, \quad z \in \mathbf{B}^n \setminus \{0\}.$$

Let $\mathcal{A}S^*_\gamma(\mathbf{B}^n)$ be the set of almost starlike mappings of order γ on $\mathbf{B}^n$.

Remark 1.5. It is easy to see that if $a = b = 1/(2\gamma)$, where $\gamma \in (0,1)$, then

$$\mathcal{A}S^*\left(\frac{1}{2\gamma}, \frac{1}{2\gamma}, \mathbf{B}^n\right) = S^*_\gamma(\mathbf{B}^n) \quad \text{and} \quad S^*\left(\frac{1}{2\gamma}, \frac{1}{2\gamma}, \mathbf{B}^n\right) = \mathcal{A}S^*_\gamma(\mathbf{B}^n).$$

Remark 1.6. Let $f \in \mathcal{L}S_n$ and $h(z) = [Df(z)]^{-1}f(z)$, $z \in \mathbf{B}^n$. Also, let $a, b \in \mathbb{R}$ be such that $|a-1| < b \le a$ and $\gamma \in [0,1)$. In view of [4, Remark 3.3], we deduce the following relations:

(i) $f \in S^*(a,b,\mathbf{B}^n) \iff h \in \mathcal{M}_g$, where $g(\zeta) = \frac{1+(a-1)/b\zeta}{1+(a^2-b^2-a)/b\zeta}$, $|\zeta| < 1$.

(ii) $f \in \mathcal{A}S^*(a,b,\mathbf{B}^n) \iff h \in \mathcal{M}_g$, where $g(\zeta) = \frac{1+(a-a^2+b^2)/b\zeta}{1+(1-a)/b\zeta}$, $|\zeta| < 1$.

(iii) $f \in S^*_\gamma(\mathbf{B}^n) \iff h \in \mathcal{M}_g$, where $g(\zeta) = \frac{1+\zeta}{1+(2\gamma-1)\zeta}$, $|\zeta| < 1$.

(iv) $f \in \mathcal{A}S^*_\gamma(\mathbf{B}^n) \iff h \in \mathcal{M}_g$, where $g(\zeta) = \frac{1+(1-2\gamma)\zeta}{1-\zeta}$, $|\zeta| < 1$.

Further, we present the notion of g-starlikeness on $\mathbf{B}^n$, introduced by Graham, Hamada and Kohr in [7] (see also [13]).

Definition 1.7. Let $g\colon U \to \mathbb{C}$ be a function given by Definition 1.1. A mapping $f \in \mathcal{L}S_n$ is said to be g-starlike on $\mathbf{B}^n$ if $h \in \mathcal{M}_g$ where $h(z) = [Df(z)]^{-1}f(z)$ for all $z \in \mathbf{B}^n$. We denote by $S^*_g(\mathbf{B}^n)$ the class of g-starlike mappings on $\mathbf{B}^n$ and $S^*_g(\mathbf{B}^1)$ by S^*_g.

Taking into account the analytical characterization of starlikeness on $\mathbf{B}^n$ due to Suffridge [28], it is easy to see that $S^*_g(\mathbf{B}^n)$ is a subset of $S^*(\mathbf{B}^n)$.

Next, we will present some observations regarding the case of g-starlikeness on the complex plane. The purpose of this remark is to point out how the class S^*_g can be related to the Janowski class $S^*[A,B]$, respectively to the class $S^*(a,b)$ introduced by Silverman, in the case that the function g is given by the relation (1.1).

Remark 1.8. Let $A, B \in \mathbb{R}$ be such that $-1 \le B < A \le 1$. Also, let $g(\zeta) = (1 + A\zeta)/(1 + B\zeta)$, $\zeta \in U$. The class S_g^* can be rewritten as follows:

(i) $S_g^* = \{f \in H(U) : f(0) = 0, f'(0) = 1, f(z)/(zf'(z)) \prec (1 + Az)/(1 + Bz), z \in U\}$.

(ii) $S_g^* = S^*[-B, -A]$.

(iii) If $A \neq 1$ then $S_g^* = S^*(a, b)$, where $a = (1 - AB)/(1 - A^2)$, $b = (A - B)/(1 - A^2)$. If $A = 1$ then $S_g^* = S^*_{(1+B)/2}$.

Proof. Indeed, we know from the definition of S_g^* that

$$S_g^* = \left\{ f \in H(U) : f(0) = 0, f'(0) = 1, \frac{f(z)}{zf'(z)} \in g(U), |z| < 1 \right\}.$$

The condition $f(z)/(zf'(z)) \in g(U)$ is equivalent to $f(z)/(zf'(z)) \prec (1 + Az)/(1 + Bz)$, so this justifies (i).

On the other hand, if $f \in H(U)$, $f(0) = 0$ and $f'(0) = 1$ then $f \in S_g^*$ if and only if $zf'(z)/f(z) \prec (1 - Bz)/(1 - Az)$. Also, it is easy to see that $-1 \le -A < -B \le 1$. This proves (ii). If $A \neq 1$ then q maps the unit disk U onto the open disk $U\big((1 - AB)/(1 - A^2), (A - B)/(1 - A^2)\big)$. If $A = 1$, the unit disk U is mapped onto $\{\zeta \in \mathbb{C} : \operatorname{Re} \zeta > (1 + B)/2\}$. Thus, the statement of Remark 1.8(iii) is now justified. □

Next, we take into consideration the case $n \ge 2$. In [4, Remark 3.3], Curt obtained the appropriate values of parameters A and B such that the classes $S^*(a, b, \mathbf{B}^n)$ and $\mathcal{A}S^*(a, b, \mathbf{B}^n)$ can be rewritten as $S_g^*(\mathbf{B}^n)$.

Remark 1.9. (see [4]) Let $a, b \in \mathbb{R}$ be such that $|1 - a| < b \le a$ and let the function g be given by $g(\zeta) = (1 + A\zeta)/(1 + B\zeta)$, $\zeta \in U$, where $-1 \le B < A \le 1$.

(i) If $A = (a - 1)/b$, $B = (a^2 - b^2 - a)/b$ then $S^*(a, b, \mathbf{B}^n) = S_g^*(\mathbf{B}^n)$.

(ii) If $A = (a - a^2 + b^2)/b$, $B = (1 - a)/b$ then $\mathcal{A}S^*(a, b, \mathbf{B}^n) = S_g^*(\mathbf{B}^n)$.

Chirilă [2] introduced the notion of g-spirallikeness of type δ, where $\delta \in (-\pi/2, \pi/2)$. The particular case when $g(\zeta) = (1 + \zeta)/(1 - \zeta)$, $\zeta \in U$ was studied in [14].

Definition 1.10. Let $f \in \mathcal{L}S_n$ and let $g \colon U \to \mathbb{C}$ be a function given by Definition 1.1. We say that f is g-spirallike of type δ, where $\delta \in (-\pi/2, \pi/2)$, if $h(\,\cdot\,, t) \in \mathcal{M}_g$, $t \ge 0$, where

$$h(z, t) = iaz + (1 - ia)e^{-iat}[Df(e^{iat}z)]^{-1} f(e^{iat}z), \quad z \in \mathbf{B}^n,\ t \ge 0,$$

where $a = \tan \delta$.

Remark 1.11. Let $g\colon U \to \mathbb{C}$ be the function given by $g(\zeta) = (1+A\zeta)/(1+B\zeta)$, $\zeta \in U$. The class of g-spirallike mappings of type δ reduces to some well known classes of biholomorphic mappings on $\mathbf{B}^n$ by choosing suitable values for the parameters A and B. In particular, if $A = 1$ and $B = -1$ then the class of g-spirallike mappings of type δ reduces to the class of spirallike mappings of type δ on $\mathbf{B}^n$, which is denoted by $\widehat{S}_\delta(\mathbf{B}^n)$. This class was introduced in [14]. For $A = 1$ and $B = 2\gamma - 1$ with $\gamma \in (0, 1)$, we obtain the class of spirallike mappings of type δ and order γ on $\mathbf{B}^n$ (see [20]). The case of $\delta = 0$ in Definition 1.10 leads us to the class of g-starlike mappings on $\mathbf{B}^n$.

We need to recall the definition of a Loewner chain prior introducing the notion of g-parametric representation. Many results related to Loewner chains in $\mathbb{C}^n$ may be found in [5, 7, 10, 12, 23, 29].

Definition 1.12. (see [23]) Let $f, g \in H(\mathbf{B}^n)$. We say that f is subordinate to g (write $f \prec g$) if there is a Schwarz mapping v (i.e., $v \in H(\mathbf{B}^n)$, $\|v(z)\| \le \|z\|$, $z \in \mathbf{B}^n$) such that $f(z) = g(v(z))$, $z \in \mathbf{B}^n$.

Definition 1.13. (see [23]) Let $f\colon \mathbf{B}^n \times [0, \infty) \to \mathbb{C}^n$. We say that f is a Loewner chain if $f(\,\cdot\,, t)$ is biholomorphic on $\mathbf{B}^n$, $f(0, t) = 0$, $Df(0, t) = e^t I_n$ for $t \ge 0$ and $f(\,\cdot\,, s) \prec f(\,\cdot\,, t)$ whenever $0 \le s \le t < \infty$.

Further, we will present a characterization of Loewner chain obtained by Pfaltzgraff [23].

Lemma 1.14. *Let* $f = f(z, t)\colon \mathbf{B}^n \times [0, \infty) \to \mathbb{C}^n$ *be a mapping such that* $f(\,\cdot\,, t) \in H(\mathbf{B}^n)$, $f(0, t) = 0$, $Df(0, t) = e^t I_n$ *for* $t \ge 0$ *and* $f(z, \cdot)$ *is locally absolutely continuous on* $[0, \infty)$ *locally uniformly with respect to* $z \in \mathbf{B}^n$. *Assume that there exists a mapping* $h = h(z, t)\colon \mathbf{B}^n \times [0, \infty) \to \mathbb{C}^n$ *which satisfies the following conditions:*

(i) $h(\,\cdot\,, t) \in \mathcal{M}$ *for* $t \ge 0$,

(ii) $h(z, \cdot)$ *is measurable on* $[0, \infty)$ *for* $z \in \mathbf{B}^n$,

and such that the following differential equation is fulfilled

$$\frac{\partial f}{\partial t}(z, t) = Df(z, t)h(z, t) \quad a.e.\ t \ge 0,\ \forall\, z \in \mathbf{B}^n.$$

Further, assume that $\{e^{-t} f(\,\cdot\,, t)\}_{t \ge 0}$ *is a locally uniformly bounded family on* $\mathbf{B}^n$. *Then* $f(z, t)$ *is a Loewner chain.*

Definition 1.15. (see [5]) A mapping $h(z, t)\colon \mathbf{B}^n \times [0, \infty) \to \mathbb{C}^n$ which satisfies the conditions (i) and (ii) in Lemma 1.14 is called a Herglotz vector field.

The notions of g-Loewner chain and g-parametric representation were introduced by Graham, Hamada and Kohr in [7], where the function g satisfies the conditions of Definition 1.1.

Definition 1.16. Given a mapping $f = f(z,t)\colon \mathbf{B}^n \times [0,\infty) \to \mathbb{C}^n$, one says that f is a g-Loewner chain if $f(z,t)$ is a Loewner chain such that $\{e^{-t}f(\,\cdot\,,t)\}_{t\geq 0}$ is a normal family on unit ball $\mathbf{B}^n$ and the mapping $h = h(z,t)$, which occurs in the following Loewner differential equation

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t) \quad \text{a.e. } t \geq 0,\ \forall\, z \in \mathbf{B}^n,$$

satisfies the condition $h(\,\cdot\,,t) \in \mathcal{M}_g$ for a.e. $t \geq 0$.

Definition 1.17. Given a normalized holomorphic mapping $f\colon \mathbf{B}^n \to \mathbb{C}^n$, we say that f has g-parametric representation if there exists a g-Loewner chain $f(z,t)$ such that f can be embedded as the first element of the g-Loewner chain $f(z,t)$ (i.e., $f = f(\,\cdot\,,0)$). We will denote by $S_g^0(\mathbf{B}^n)$ the set of mappings which have g-parametric representation on $\mathbf{B}^n$.

We remark that if $g(\zeta) = (1+\zeta)/(1-\zeta)$, $\zeta \in U$, then $S_g^0(\mathbf{B}^n)$ reduces to the set $S^0(\mathbf{B}^n)$ of mappings which have parametric representation, hence any Loewner chain $f(z,t)$ on $\mathbf{B}^n$, such that $\{e^{-t}f(\,\cdot\,,t)\}_{t\geq 0}$ is a normal family on $\mathbf{B}^n$, is a g-Loewner chain on $\mathbf{B}^n$ (see [7]). Also, the family $S_g^0(\mathbf{B}^n)$ where the function g is given by $g(\zeta) = \frac{1+\zeta}{1+(2\gamma-1)\zeta}$, $\zeta \in U$ and $\gamma \in (0,1)$ was studied by Chirilă in [1].

Remark 1.18. (i) Let $f \in \mathcal{L}S_n$. We have that $f \in S_g^*(\mathbf{B}^n)$ if and only if $f(z,t) = e^t f(z)$ is a g-Loewner chain for all $z \in \mathbf{B}^n$ and $t \geq 0$ (see [7]).

(ii) Chirilă in [2, Teorem 3.1] proved that if $f \in \mathcal{L}S_n$ and $\delta \in (-\pi/2, \pi/2)$, then f is g-spirallike of type δ if and only if $f(z,t) = e^{(1-ia)t}f(e^{iat}z)$ is a g-Loewner chain, where $a = \tan\delta$.

Let $\Phi_{n,\alpha,\beta}$ be the extension operator defined by the following relation (see [8]):

$$\Phi_{n,\alpha,\beta}(f)(z) = \left(f(z_1), \tilde{z}\left(\frac{f(z_1)}{z_1}\right)^{\alpha}(f'(z_1))^{\beta}\right), \quad z = (z_1,\tilde{z}) \in \mathbf{B}^n,$$

where $\alpha \geq 0$, $\beta \geq 0$ and $f \in \mathcal{L}S$ such that $f(z_1) \neq 0$, $z \in U \setminus \{0\}$.

The branches of the power functions are chosen such that

$$\left(\frac{f(z_1)}{z_1}\right)^{\alpha}\bigg|_{z_1=0} = 1, \quad (f'(z_1))^{\beta}\Big|_{z_1=0} = 1.$$

We observe that for $\alpha = 0$, $\beta = 1/2$, the operator $\Phi_{n,\alpha,\beta}$ reduces to the Roper-Suffridge operator $\Phi_n\colon \mathcal{L}S \to \mathcal{L}S_n$, given by (see [25])

$$\Phi_n(f)(z) = \big(f(z_1), \tilde{z}\sqrt{f'(z_1)}\big), \quad z = (z_1,\tilde{z}) \in \mathbf{B}^n.$$

It is known that the operator $\Phi_{n,\alpha,\beta}$ preserves the notions of starlikeness and parametric representation from unit disk U into the unit ball $\mathbf{B}^n$, for $n \geq 2$ and $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha+\beta \leq 1$ (see [8]). But $\Phi_{n,\alpha,\beta}$ preserves the notion of convexity from unit disk U into the unit ball $\mathbf{B}^n$ if and only if $(\alpha,\beta)=(0,1/2)$ (see [8,25]). Various properties of the operator $\Phi_{n,\alpha,\beta}$ were investigated in [11,21], in the case $\alpha=0$ and $\beta \in [0,1/2]$ (see also [9], in the case $\beta=0$). Also, the operator $\Phi_{n,\alpha,\beta}$ was studied in [1,8,10].

In this paper, we continue the work in [1,2,7,8] concerning extension operators and g-Loewner chains in $\mathbb{C}^n$. We consider the operator $\Phi_{n,\alpha,\beta}$ and the set $S_g^0(U)$ of normalized holomorphic functions on unit disk U that have g-parametric representation, where $g(\zeta)=(1+A\zeta)/(1+B\zeta)$, $|\zeta|<1$ and $-1 \leq B < A \leq 1$.

We shall prove that if $f \in S$ can be embedded as first element of a g-Loewner chain, where $g\colon U \to \mathbb{C}$ is given by the relation (1.1), then $F=\Phi_{n,\alpha,\beta}(f)$ can be embedded as first element of a g-Loewner chain on the unit ball $\mathbf{B}^n$ for $\alpha \in [0,1]$, $\beta \in [0,1/2]$ and $\alpha+\beta \leq 1$. As a consequence, the operator $\Phi_{n,\alpha,\beta}$ preserves the notions of Janowski starlikeness and Janowski almost starlikeness from the unit disk into the unit ball $\mathbf{B}^n$. Particular cases from [1,2] will be also mentioned.

In the last part of the paper, we obtain some radius of Janowski starlikeness associated to some classes of biholomorphic mappings on $\mathbf{B}^n$ generated by the extension operator $\Phi_{n,\alpha,\beta}$.

2. Main results

In this section we prove the following theorem, which is the main result of this paper. The following result was obtained in [8, Theorem 2.1] (see also [11] for $\alpha=0$), in the case that $g(\zeta)=(1+\zeta)/(1-\zeta)$, $\zeta \in U$. Also, Theorem 2.1 was recently obtained by Chirilă (see [1]) when the function g is given by $g(\zeta)=\frac{1+\zeta}{1+(2\gamma-1)\zeta}$, $\zeta \in U$, where $\gamma \in (0,1)$.

Theorem 2.1. *Let $g\colon U \to \mathbb{C}$ be the function given by $g(\zeta)=(1+A\zeta)/(1+B\zeta)$, $\zeta \in U$, where $-1 \leq B < A \leq 1$. If $f \in S$ has g-parametric representation, then $F=\Phi_{n,\alpha,\beta}(f)$ also has g-parametric representation on unit ball $\mathbf{B}^n$ for $\alpha \in [0,1]$, $\beta \in [0,1/2]$ and $\alpha+\beta \leq 1$.*

Proof. In order to prove the result, we shall use arguments similar to those used in [8, Theorem 2.1] and [1, Theorem 2.1]. It is obvious that it is enough to consider only the case $n=2$.

We know that f can be embedded as first element of a g-Loewner chain, therefore there exists a g-Loewner chain $f(z_1,t)$ such that $f(z_1,0)=f(z_1)$, $z_1 \in U$.

Let us consider the following mapping $F_{\alpha,\beta}\colon \mathbf{B}^2 \times [0,\infty) \to \mathbb{C}^2$, defined by

$$F_{\alpha,\beta}(z,t)=\left(f(z_1,t), e^{(1-\alpha-\beta)t} z_2 \left(\frac{f(z_1,t)}{z_1}\right)^{\alpha} (f'(z_1,t))^{\beta}\right) \tag{2.1}$$

for $z = (z_1, z_2) \in \mathbf{B}^2$, $t \geq 0$. As it follows from [8], $F_{\alpha,\beta}(z,t)$ is a Loewner chain, since $\alpha \in [0,1]$, $\beta \in [0,1/2]$ and $\alpha + \beta \leq 1$.

We know that given a Loewner chain $f(z_1,t)$ on U, there exists a function $p(\,\cdot\,,t)$ that belongs to $H(U)$ for $t \geq 0$, is measurable in $t \geq 0$, with $p(0,t) = 1$, $\operatorname{Re} p(z_1,t) > 0$, $z_1 \in U$, $0 \leq t < \infty$, and such that

$$\frac{\partial f}{\partial t}(z_1,t) = z_1 f'(z_1,t) p(z_1,t) \quad \text{a.e. } t \geq 0,\ \forall\, z_1 \in U.$$

Also, the fact that $f(z_1,t)$ is a g-Loewner chain implies that $p(z_1,t) \in g(U)$ for a.e. $t \geq 0$, $\forall\, z_1 \in U$. The vector field $h(z,t)$ associated with the Loewner chain $F_{\alpha,\beta}(z,t)$ has the following form (see [8]):

$$h(z,t) = \big(z_1 p(z_1,t), z_2(1 - \alpha - \beta + (\alpha+\beta)p(z_1,t) + \beta z_1 p'(z_1,t))\big)$$

for $z = (z_1,z_2) \in \mathbf{B}^2$ and $t \geq 0$. This expression was obtained from the Loewner differential equation

$$\frac{\partial F_{\alpha,\beta}}{\partial t}(z,t) = DF_{\alpha,\beta}(z,t)h(z,t) \quad \text{a.e. } t \geq 0,\ \forall\, z \in \mathbf{B}^2.$$

We shall prove that $h(\,\cdot\,,t) \in \mathcal{M}_g$ for a.e. $t \geq 0$. Therefore, it suffices to show that the following condition holds:

$$\frac{1}{\|z\|^2}\langle h(z,t), z\rangle \in g(U) \quad \text{a.e. } t \geq 0,\ z \in \mathbf{B}^2 \setminus \{0\}. \tag{2.2}$$

Next, we will consider the following cases:

Case 1. If $B = -1$ then the function g becomes $g(\zeta) = (1 + A\zeta)/(1-\zeta)$, $\zeta \in U$. In this case, the function g maps the unit disk onto the half-plane $\{\zeta \in \mathbb{C} : \operatorname{Re}\zeta > (1-A)/2\}$.

The relation (2.2) is equivalent to

$$\frac{1}{\|z\|^2}\operatorname{Re}\langle h(z,t), z\rangle > \frac{1-A}{2} \quad \text{a.e. } t \geq 0,\ z \in \mathbf{B}^2 \setminus \{0\}.$$

Also, in this case, the relation $p(z_1,t) \in g(U)$ is equivalent to $\operatorname{Re} p(z_1,t) > (1-A)/2$ for a.e. $t \geq 0$, $z_1 \in U$.

Without loss of generality, we may assume that $h(\,\cdot\,,t)$ is holomorphic on $\overline{\mathbf{B}}^2$. Otherwise, let $\rho \in (0,1)$. Also, let $h_\rho(z,t) = \frac{1}{\rho}h(\rho z,t)$ for all $z \in \overline{\mathbf{B}}^2$, $t \geq 0$. Then the mapping $h_\rho(\,\cdot\,,t)$ is well defined and holomorphic on $\overline{\mathbf{B}}^2$, $t \geq 0$. Next, if $w \in \partial\mathbf{B}^2$ is fixed and $t \geq 0$, then the function $q_\rho(\,\cdot\,,t)\colon \overline{U} \to \mathbb{C}$ given by $q_\rho(\zeta,t) = \frac{1}{\zeta}\langle h_\rho(\zeta w,t), w\rangle$ for $\zeta \in U \setminus \{0\}$, and $q_\rho(0) = 1$, is holomorphic on U and continuous on $\overline{U}$. Thus $\operatorname{Re} q_\rho(\,\cdot\,,t)$ is harmonic on U and continuous on $\overline{U}$.

In view of the minimum principle for harmonic functions, it suffices to prove that $\operatorname{Re} q_\rho(\zeta,t) \geq (1-A)/2$ for $|\zeta| = 1$. Then $\operatorname{Re} q_\rho(\zeta,t) > (1-A)/2$ for $|\zeta| < 1$, by the fact that $\operatorname{Re} q_\rho(0,t) = 1 > (1-A)/2$ and since $\operatorname{Re} q_\rho(\,\cdot\,,t)$ is harmonic on U.

Next, if $z \in \mathbf{B}^2 \setminus \{0\}$ and $w = z/\|z\|$, then $w \in \partial\mathbf{B}^2$, and if $\zeta = \|z\|$, then $\zeta \in U$, and $(1-A)/2 < \operatorname{Re} q_\rho(\|z\|, t) = \frac{1}{\|z\|^2} \operatorname{Re}\langle h_\rho(z,t), z\rangle$. Then letting $\rho \nearrow 1$, we deduce that $\operatorname{Re}\langle h(z,t), z/\|z\|^2\rangle > (1-A)/2$, by the same argument as above, based on the minimum principle for harmonic functions.

Consequently, in view of the above arguments, we have to prove that:

$$\operatorname{Re}\langle h(z,t), z\rangle \geq \frac{1-A}{2} \quad \text{a.e. } t \geq 0, \forall\, z = (z_1, z_2) \in \partial\mathbf{B}^2. \tag{2.3}$$

Indeed, if we fix $z = (z_1, z_2) \in \partial\mathbf{B}^2$ and using elementary computation, we obtain the following relation:

$$\begin{aligned}\operatorname{Re}\langle h(z,t), z\rangle - \frac{1-A}{2} &= \big[|z_1|^2 + (1-|z_1|^2)(\alpha+\beta)\big] \operatorname{Re} p(z_1, t) \\ &\quad + (1-|z_1|^2)\beta \operatorname{Re}(z_1 p'(z_1,t)) + (1-|z_1|^2)(1-\alpha-\beta) - \frac{1-A}{2}.\end{aligned}$$

It can be seen that for $|z_1| = 1$ (which implies $z_2 = 0$), the above expression is non-negative. Therefore, further we consider only the case $z_2 \neq 0$, thus $|z_1| < 1$.

Since $\operatorname{Re}(z_1 p'(z_1,t)) \geq -|z_1||p'(z_1,t)|$, this implies that

$$\begin{aligned}\operatorname{Re}\langle h(z,t), z\rangle - \frac{1-A}{2} &\geq \big[|z_1|^2 + (1-|z_1|^2)(\alpha+\beta)\big] \operatorname{Re} p(z_1, t) \\ &\quad - (1-|z_1|^2)\beta|z_1||p'(z_1,t)| \\ &\quad + (1-|z_1|^2)(1-\alpha-\beta) - \frac{1-A}{2}.\end{aligned} \tag{2.4}$$

Next, we wish to give an estimate of the right-hand side member of the above inequality. First, we give an upper bound for $|p'(z_1,t)|$.

It can be easily seen that

$$\frac{p(\,\cdot\,,t) - (1-A)/2}{1-(1-A)/2} \in \mathcal{P} \quad \text{for } t \geq 0. \tag{2.5}$$

It is known that for a function $q \in \mathcal{P}$, we have (see [24])

$$|q'(z)| \leq \frac{2\operatorname{Re} q(z)}{1-|z|^2}, \quad |z| < 1.$$

Therefore, from (2.5) and from the above inequality, we obtain

$$|p'(z_1,t)| \leq \frac{2\big(\operatorname{Re} p(z_1,t) - (1-A)/2\big)}{1-|z_1|^2}, \quad |z_1| < 1,\ t \geq 0. \tag{2.6}$$

Using the relation (2.6) and elementary computations on the right-hand side of (2.4), we obtain that

$$\begin{aligned}\operatorname{Re}\langle h(z,t),z\rangle - \frac{1-A}{2} &\geq \big[|z_1|^2 + (1-|z_1|^2)(\alpha+\beta)\big]\left(\operatorname{Re} p(z_1,t) - \frac{1-A}{2}\right)\\ &\quad - 2\beta|z_1|\left(\operatorname{Re} p(z_1,t) - \frac{1-A}{2}\right) + (1-|z_1|^2)(1-\alpha-\beta)\\ &\quad + \big[|z_1|^2 + (1-|z_1|^2)(\alpha+\beta)\big]\frac{1-A}{2} - \frac{1-A}{2}\\ &= \big[|z_1|^2 + (1-|z_1|^2)(\alpha+\beta) - 2\beta|z_1|\big]\left(\operatorname{Re} p(z_1,t) - \frac{1-A}{2}\right)\\ &\quad + (1-|z_1|^2)(1-\alpha-\beta)\left(1 - \frac{1-A}{2}\right).\end{aligned}$$

Next, we shall prove that the following inequality holds:

$$\begin{aligned}&\big[|z_1|^2 + (1-|z_1|^2)(\alpha+\beta) - 2\beta|z_1|\big]\left(\operatorname{Re} p(z_1,t) - \frac{1-A}{2}\right)\\ &+ (1-|z_1|^2)(1-\alpha-\beta)\left(1 - \frac{1-A}{2}\right) \geq 0.\end{aligned} \tag{2.7}$$

Indeed, in view of the following relations:

$$(1-|z_1|^2)(1-\alpha-\beta)\left(1 - \frac{1-A}{2}\right) = (1-|z_1|^2)(1-\alpha-\beta)\frac{1+A}{2} \geq 0$$

and

$$|z_1|^2 + (1-|z_1|^2)(\alpha+\beta) - 2\beta|z_1| = (1-|z_1|^2)\alpha + \beta(1-|z_1|)^2 + |z_1|^2(1-2\beta) \geq 0,$$

we obtain (2.7), as desired. Taking into account the above arguments, the inequality (2.3) holds.

Case 2. If $B \neq -1$ then the function $g\colon U \to \mathbb{C}$ given by $g(\zeta) = (1+A\zeta)/(1+B\zeta)$, $\zeta \in U$, maps the unit disk onto the disk $U\big((1-AB)/(1-B^2), (A-B)/(1-B^2)\big)$.

To simplify the calculations we make the following notations: $a \overset{\text{not}}{=} (1-AB)/(1-B^2)$ and $b \overset{\text{not}}{=} (A-B)/(1-B^2)$.

We remark that, for a.e. $t \geq 0$ and $\forall\, z_1 \in U$, the condition $p(z_1,t) \in g(U)$ is equivalent to $|p(z_1,t) - a| < b$.

In this case, the relation (2.2) is equivalent to

$$\left|\frac{1}{\|z\|^2}\langle h(z,t),z\rangle - a\right| < b \quad \text{a.e. } t \geq 0,\ \forall\, z \in \mathbf{B}^2 \setminus \{0\}. \tag{2.8}$$

Using an argument similar to that from the beginning of the first step, we may assume that $h(\,\cdot\,,t)$ is holomorphic on $\overline{\mathbf{B}}^2$, and show that

$$|\langle h(z,t),z\rangle - a| \leq b \quad \text{a.e. } t \geq 0,\ \forall\, z \in \partial\mathbf{B}^2.$$

Otherwise, we replace the mapping $h(\cdot,t)$ by the mapping $h_\rho(z,t)=\frac{1}{\rho}h(\rho z,t)$, for $z\in\overline{\mathbf{B}}^2$, a.e. $t\geq 0$, where $\rho\in(0,1)$. Using a similar argument as in the Case I, we have to prove that $|\langle h_\rho(z,t),z\rangle-a|\leq b$ for a.e. $t\geq 0$ and $\forall\, z\in\partial\mathbf{B}^2$. Then letting $\rho\to 1$, we obtain the conclusion.

Therefore, it suffices to prove that

$$|\langle h(z,t),z\rangle-a|\leq b\quad \text{a.e. } t\geq 0,\ \forall\, z=(z_1,z_2)\in\partial\mathbf{B}^2. \tag{2.9}$$

In the case $z_2=0$, it is easily seen that the relation (2.9) holds, since $|p(z_1,t)-a|\leq b$, $|z_1|=1$ and a.e. $t\geq 0$.

Next, we consider $z_2\neq 0$ which leads to $|z_1|<1$. Taking into account the following relations:

$$\begin{aligned}|\langle h(z,t),z\rangle-a|&=\Big|\big[|z_1|^2+(1-|z_1|^2)(\alpha+\beta)\big]p(z_1,t)+(1-|z_1|^2)\beta z_1p'(z_1,t)\\&\quad+(1-|z_1|^2)(1-\alpha-\beta)-a\Big|\\&=\Big|\big[|z_1|^2+(1-|z_1|^2)(\alpha+\beta)\big](p(z_1,t)-a)+(1-|z_1|^2)\beta z_1p'(z_1,t)\\&\quad+(1-|z_1|^2)(1-\alpha-\beta)+\big[|z_1|^2+(1-|z_1|^2)(\alpha+\beta)\big]\cdot a-a\Big|\\&=\Big|\big[|z_1|^2+(1-|z_1|^2)(\alpha+\beta)\big](p(z_1,t)-a)+(1-|z_1|^2)\beta z_1p'(z_1,t)\\&\quad+(1-|z_1|^2)(1-\alpha-\beta)(1-a)\Big|,\end{aligned}$$

we have the following estimate:

$$\begin{aligned}|\langle h(z,t),z\rangle-a|&\leq\big[|z_1|^2+(1-|z_1|^2)(\alpha+\beta)\big]|p(z_1,t)-a|\\&\quad+(1-|z_1|^2)\beta|z_1||p'(z_1,t)|+(1-|z_1|^2)(1-\alpha-\beta)|1-a|.\end{aligned}$$

Fix $t\geq 0$ and let the function $w(\cdot,t)\colon U\to\mathbb{C}$ be given by $w(z_1,t)=(p(z_1,t)-a)/b$, $z_1\in U$. Then $w(\cdot,t)\in H(U)$, $w(0,t)=0$ and $|w(z_1,t)|<1$, $|z_1|<1$. Hence the function $w(\cdot,t)$ satisfies the condition of Schwarz-Pick lemma and therefore

$$|p'(z_1,t)|\leq b\cdot\frac{1-|p(z_1,t)-a|^2/b^2}{1-|z_1|^2},\quad t\geq 0.$$

Using the above estimate, we obtain

$$\begin{aligned}|\langle h(z,t),z\rangle-a|&\leq\big[|z_1|^2+(1-|z_1|^2)(\alpha+\beta)\big]|p(z_1,t)-a|\\&\quad+b\beta|z_1|\left(1-\frac{|p(z_1,t)-a|^2}{b^2}\right)+(1-|z_1|^2)(1-\alpha-\beta)|1-a|.\end{aligned}$$

Next we will show that

$$\begin{aligned}&\big[|z_1|^2+(1-|z_1|^2)(\alpha+\beta)\big]|p(z_1,t)-a|+b\beta|z_1|\left(1-\frac{|p(z_1,t)-a|^2}{b^2}\right)\\&+(1-|z_1|^2)(1-\alpha-\beta)|1-a|-b\leq 0.\end{aligned}$$

It is clear that the above inequality is equivalent to the following:

$$\big[|z_1|^2 + (1-|z_1|^2)(\alpha+\beta)\big]\frac{|p(z_1,t)-a|}{b} + \beta|z_1|\left(1 - \frac{|p(z_1,t)-a|^2}{b^2}\right)$$
$$+ (1-|z_1|^2)(1-\alpha-\beta)\frac{|1-a|}{b} - 1 \le 0.$$

We make the following notation $x := |p(z_1,t)-a|/b$. Then $x \in [0,1]$. Also, let $E(x)$ be the following quantity

$$E(x) = -\beta|z_1|x^2 + \big[|z_1|^2 + (1-|z_1|^2)(\alpha+\beta)\big]x$$
$$+ (1-|z_1|^2)(1-\alpha-\beta)\frac{|1-a|}{b} + \beta|z_1| - 1.$$

We aim to show that $E(x) \le 0$ for $x \in [0,1]$.

Indeed, it can be easily seen that $E(x)$ is an increasing function on the variable x. Therefore $E(x) \le E(1)$, $x \in [0,1]$. Further, we need to evaluate the sign of $E(1)$.

$$E(1) = -\beta|z_1| + |z_1|^2 + (1-|z_1|^2)(\alpha+\beta) + (1-|z_1|^2)(1-\alpha-\beta)\frac{|1-a|}{b} + \beta|z_1| - 1$$
$$= (1-|z_1|^2)(1-\alpha-\beta)\left(\frac{|1-a|}{b} - 1\right).$$

Replacing the constant a by $(1-AB)/(1-B^2)$, and the constant b by $(A-B)/(1-B^2)$, we deduce that

$$E(1) = (1-|z_1|^2)(1-\alpha-\beta)(|B|-1) \le 0,$$

where we have used the fact that $|z_1| < 1$, $\alpha+\beta \le 1$ and $B \in (-1,1)$. Therefore, combining the above relations, we obtain that $E(x) \le 0$ for $x \in [0,1]$. Thus, we conclude that the condition (2.8) is fulfilled.

Finally, since $\{e^{-t}f(\,\cdot\,,t)\}_{t\ge 0}$ is a normal family on U, it suffices to use arguments similar to those used in the proof of [8, Theorem 2.1], to deduce that $\{e^{-t}F_{\alpha,\beta}(\,\cdot\,,t)\}_{t\ge 0}$ is a normal family on unit ball $\mathbf{B}^n$.

In view of the above arguments, we have proved that $F_{\alpha,\beta}(z,t)$ is a g-Loewner chain. Hence $\Phi_{n,\alpha,\beta}(f) = F_{\alpha,\beta}(\,\cdot\,,0)$ has g-parametric representation on $\mathbf{B}^n$. This completes the proof. □

As a consequence of Theorem 2.1, we shall prove that the operator $\Phi_{n,\alpha,\beta}$ preserves the notion of g-starlikeness on $\mathbf{B}^n$, where $g(\zeta) = (1+A\zeta)/(1+B\zeta)$, $\zeta \in U$, and $-1 \le B < A \le 1$. Particular cases of this result were obtained in [8] (for $A=1$ and $B=-1$) and [1] (for $A=1$ and $B=2\gamma-1$, where $\gamma \in (0,1)$).

Corollary 2.2. *Let $g\colon U \to \mathbb{C}$ be given by $g(\zeta) = (1+A\zeta)/(1+B\zeta)$, $\zeta \in U$, where $-1 \le B < A \le 1$. Also, let $f \in S_g^*$. Then $F = \Phi_{n,\alpha,\beta}(f) \in S_g^*(\mathbf{B}^n)$ for $\alpha \in [0,1]$, $\beta \in [0,1/2]$ and $\alpha+\beta \le 1$.*

Proof. Since $f \in S_g^*$, it follows that $f(z_1,t) = e^t f(z_1)$ is a g-Loewner chain (see [7]). The mapping $F_{\alpha,\beta}(z,t)$ given by (2.1) is a g-Loewner chain, accordingly to Theorem 2.1. But it is easy to deduce that $F_{\alpha,\beta}(z,t) = e^t F(z)$ for $z \in \mathbf{B}^n$ and $t \geq 0$. Hence $F \in S_g^*(\mathbf{B}^n)$. This completes the proof. □

By choosing suitable values for A, B, we obtain the following particular cases of Corollary 2.2. These particular cases have been approached in [30].

Corollary 2.3. (cf. [30]) *Let $a, b \in \mathbb{R}$ be such that $|1-a| < b \leq a$ and let $f \in S^*(a,b)$. Then $F = \Phi_{n,\alpha,\beta}(f) \in S^*(a,b,\mathbf{B}^n)$ for $\alpha \in [0,1]$, $\beta \in [0,1/2]$ and $\alpha+\beta \leq 1$.*

Proof. Indeed, if $A = (a-1)/b$, $B = (a^2-b^2-a)/b$ then $S_g^*(\mathbf{B}^n) = S^*(a,b,\mathbf{B}^n)$, where g is given by (1.1). From Corollary 2.2, we deduce that $\Phi_{n,\alpha,\beta}(f) \in S^*(a,b,\mathbf{B}^n)$ whenever $f \in S^*(a,b)$. □

Corollary 2.4. (cf. [30]) *Let $a, b \in \mathbb{R}$ be such that $|1-a| < b \leq a$ and let $f \in \mathcal{A}S^*(a,b)$. Then $F = \Phi_{n,\alpha,\beta}(f) \in \mathcal{A}S^*(a,b,\mathbf{B}^n)$ for $\alpha \in [0,1]$, $\beta \in [0,1/2]$ and $\alpha+\beta \leq 1$.*

Proof. Indeed, if the function g is given by (1.1) and A, B are $(a-a^2+b^2)/b$, respectively $(1-a)/b$, then $S_g^*(\mathbf{B}^n) = \mathcal{A}S^*(a,b,\mathbf{B}^n)$. In view of Corollary 2.2, we have that $\Phi_{n,\alpha,\beta}(f) \in \mathcal{A}S^*(a,b,\mathbf{B}^n)$ whenever $f \in \mathcal{A}S^*(a,b)$. □

On the other hand, we mention the following well known results, which can be obtained from Corollary 2.2 for suitable values of A and B.

Remark 2.5. (i) In the case that $A = 1$ and $B = 2\gamma - 1$, where $\gamma \in (0,1)$, it can be seen that $S_g^*(\mathbf{B}^n)$ reduces to the set $S_\gamma^*(\mathbf{B}^n)$ of starlike mappings of order γ on $\mathbf{B}^n$. Hence we deduce that $\Phi_{n,\alpha,\beta}(S_\gamma^*) \subset S_\gamma^*(\mathbf{B}^n)$. This result was obtained by Hamada, Kohr and Kohr [15], in the case of $\alpha = 0$, $\beta = \gamma = 1/2$, and by Liu [19], in the case $\gamma \in (0,1)$ and $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha+\beta \leq 1$. T. Chirilă proved this result by using g-Loewner chains (see [1]).

(ii) Also, if $A = 1$ and $B = -1$ then $S_g^*(\mathbf{B}^n)$ reduces to the set $S^*(\mathbf{B}^n)$ of normalized starlike mappings on $\mathbf{B}^n$. In view of Corollary 2.2, it can be seen that the operator $\Phi_{n,\alpha,\beta}$ has the property that $\Phi_{n,\alpha,\beta}(S^*) \subset S^*(\mathbf{B}^n)$. This result was obtained in [8].

The following result can be proved by arguments similar to those used in the proof of Corollary 2.2. We omit the proof of Corollary 2.6.

Corollary 2.6. *Let $g\colon U \to \mathbb{C}$ be given by $g(\zeta) = (1+A\zeta)/(1+B\zeta)$, $\zeta \in U$, where $-1 \leq B < A \leq 1$. Also, let f be a g-spirallike function of type δ on the unit disk U, where $\delta \in (-\pi/2,\pi/2)$. Then $F = \Phi_{n,\alpha,\beta}(f)$ is a g-spirallike mapping of type δ on $\mathbf{B}^n$, with $\alpha \in [0,1]$, $\beta \in [0,1/2]$ and $\alpha+\beta \leq 1$.*

Remark 2.7. If $A = 1$ and $B = 2\gamma - 1$, where $\gamma \in (0,1)$, then from Corollary 2.6 we deduce that the operator $\Phi_{n,\alpha,\beta}$ preserves the notion of spirallikenes of type δ and order γ with $\delta \in (-\pi/2, \pi/2)$. This result was obtained in [20] (see also [1, 19, 32]).

3. Radius problems

In this section we are concerned with certain radius problems which involve the operator $\Phi_{n,\alpha,\beta}$ and the notion of Janowski starlikeness. Other radius problems related to the subclasses of $S(\mathbf{B}^n)$ generated by the Roper-Suffridge extension operator and other extension operators were obtained in [1, 11].

The proof for the following result is immediate and we omit it. For $r \in (0,1]$, let us consider the following set of biholomorphic mappings on $\mathbf{B}^n_r = \{z \in \mathbb{C}^n : \|z\| < r\}$:

$$S^*(a,b,\mathbf{B}^n_r) = \Big\{ f \text{ a normalized locally biholomorphic mapping on } \mathbf{B}^n_r : \left| \frac{\|z\|^2}{\langle [Df(z)]^{-1} f(z), z\rangle} - a \right| < b, z \in \mathbf{B}^n_r \setminus \{0\} \Big\},$$

where $|1-a| < b \le a$. In the case $n = 1$, the set $S^*(a,b,\mathbf{B}^n_r)$ is denoted $S^*(a,b,U_r)$.

In the following remark (see also [8, Remark 5.1]), we assume that $\alpha, \beta \in [0,1]$ with $\beta \le 1/2$ and $\alpha + \beta \le 1$. Also, let $a, b \in \mathbb{R}$ be such that $|1-a| < b \le a$.

Remark 3.1. (i) If $\Phi_{n,\alpha,\beta}(f) \in S^*(a,b,B^n_r)$ then $f \in S^*(a,b,U_r)$ for all $r \in (0,1)$.

(ii) If $f \in S^*(a,b,U_r)$ then $\Phi_{n,\alpha,\beta}(f) \in S^*(a,b,\mathbf{B}^n_r)$ for all $r \in (0,1)$. This result is due to the following equality (see [8])

$$\Phi_{n,\alpha,\beta}(f_r)(z) = \frac{1}{r}\Phi_{n,\alpha,\beta}(f)(rz), \quad z \in \mathbf{B}^n,$$

where $f_r(\zeta) = \frac{1}{r} f(r\zeta)$, $\zeta \in U$.

We will consider $a, b \in \mathbb{R}$ such that $|1-a| < b \le a$. Further, we obtain the $S^*(a,b)$ radius of the class S (respectively S^*). For suitable values of the parameters a and b depending on A and B (see [27]), where $-1 \le B < A \le 1$, we can obtain the $S^*[A,B]$ radius of the class S (respectively S^*). The $S^*[A,B]$ radius was obtained in [22] on a wider class, namely the class of normalized analytical functions on the unit disk U with fixed second coefficient.

First, we recall the definition of the $S^*(a,b)$ radius of the class S (respectively S^*).

Definition 3.2. (cf. [27]) The $S^*(a,b)$ radius of S (respectively S^*), denoted by $\rho^*(a,b)$ (respectively $\rho_*(a,b)$), is the radius of the largest disk $|z| < \rho^*(a,b)$ (respectively $\rho_*(a,b)$) in which the condition

$$\left| \frac{zf'(z)}{f(z)} - a \right| < b \tag{3.1}$$

holds for all $f \in S$ (respectively S^*).

In order to prove the following result, we use the radius of starlikeness of the class S, i.e., the radius of the largest disk centered at the origin in which every function from S is starlike. We denote the radius of starlikeness of the class S by $r^*(S)$. It is well known that $r^*(S) = \tanh(\pi/4) \approx 0.65579\dots$ (see e.g., [24, Corollary 6.3]).

Theorem 3.3. *Let $a, b \in \mathbb{R}$ be such that $|1-a| < b \leq a$. Then the $S^*(a,b)$ radius of S is given by*

$$\rho^*(a,b) = \min\left\{\frac{1-a+b}{1+a-b}, \frac{-1+a+b}{1+a+b}, \tanh\left(\frac{\pi}{4}\right)\right\}. \tag{3.2}$$

Proof. We will use arguments similar to those in [27, Theorem 4]. We know that $S^*(a,b) \subset S^*$, thus $\rho^*(a,b) \leq r^*(S)$. Further, let $f \in S$. From the definition of $r^*(S)$, we have that $f \in S^*(U_{\tanh(\pi/4)})$, where $U_{\tanh(\pi/4)} = \{z \in \mathbb{C} : |z| < \tanh(\pi/4)\}$. This is equivalent to

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \quad |z| < \tanh(\pi/4).$$

Let $\rho \in (0,1)$. We say that the function $p\colon U \to \mathbb{C}$ belongs to $\mathcal{P}(U_\rho)$ if and only if $p_\rho \in \mathcal{P}$, where $p_\rho(z) = \frac{1}{\rho}p(\rho z)$, $z \in U$.

It is easy to see that $zf'(z)/f(z) \in \mathcal{P}(U_{\tanh(\pi/4)})$. It is known that a function p from $\mathcal{P}$ satisfies the property $p(\overline{U}_\rho) \subseteq \overline{U}\big((1+\rho^2)/(1-\rho^2), 2\rho/(1-\rho^2)\big)$ for all $\rho \in (0,1)$ (see e.g., [10, 24]).

Hence, $zf'(z)/f(z)$ fulfills the next condition

$$\left|\frac{zf'(z)}{f(z)} - \frac{1+\rho^2}{1-\rho^2}\right| \leq \frac{2\rho}{1-\rho^2}, \quad |z| \leq \rho,\ \rho \in (0, \tanh(\pi/4)).$$

Let $|z| < \rho$ with $\rho \in (0, \tanh(\pi/4)]$. The relation (3.1) holds if $\overline{U}\big((1+\rho^2)/(1-\rho^2), 2\rho/(1-\rho^2)\big) \subseteq \overline{U}(a,b)$. This implies that the following two conditions are simultaneously fulfilled

$$a - b \leq \frac{1+\rho^2}{1-\rho^2} - \frac{2\rho}{1-\rho^2},$$
$$a + b \geq \frac{1+\rho^2}{1-\rho^2} + \frac{2\rho}{1-\rho^2}.$$

The above inequalities are true if $\rho \leq \min\big\{(1-a+b)/(1+a-b), (-1+a+b)/(1+a+b)\big\}$. Since $\rho \leq \tanh(\pi/4)$, we obtain in view of the above arguments that $\rho \leq r$ where

$$r = \min\left\{\frac{1-a+b}{1+a-b}, \frac{-1+a+b}{1+a+b}, \tanh(\pi/4)\right\}.$$

This leads us to the fact that every $f \in S$ is also in $S^*(a,b,U_r)$. Moreover, there exists at least one function $f \in S$ such that $f \in S^*(a,b,U_r) \setminus S^*(a,b,U_R)$ for all $R > r$. This can be easily seen when the minimum in the expression of r is attained at $\tanh(\pi/4)$. There exists $f_0 \in S$ such that $f_0 \notin S^*(U_R)$ for all $R > \tanh(\pi/4)$. Hence, in this case, if $f_0 \in S^*(a, b, U_R)$ then this implies $f_0 \in S^*(U_R)$. Therefore we obtain a contradiction.

If $r \neq \tanh(\pi/4)$ then it suffices to prove the above statement by choosing the Koebe function:

$$k(\zeta) = \frac{\zeta}{(1-\zeta)^2}, \quad |\zeta| < 1.$$

Suppose that $k \in S^*(a, b, U_R)$ for some $R \in (r, 1)$ and derive a contradiction. In this case, the relation (3.1) is equivalent to the following:

$$\left|\frac{1+z}{1-z} - a\right| < b \tag{3.3}$$

for $|z| < R$. We will show that for some z_0 with $|z_0| = r < R$ the condition (3.3) is no longer true.

If the minimum in the expression of r is attained at $\rho = (1-a+b)/(1+a-b)$ then, for $z_0 = -\rho$, we have that

$$\begin{aligned}\left|\frac{1+z_0}{1-z_0} - a\right| &= \left|\frac{1-(1-a+b)/(1+a-b)}{1+(1-a+b)/(1+a-b)} - a\right| \\ &= |a-b-a| = |-b| = b.\end{aligned}$$

Otherwise, if the minimum is attained at $\rho = (-1+a+b)/(1+a+b)$ then for $z_0 = \rho$, it results that

$$\begin{aligned}\left|\frac{1+z_0}{1-z_0} - a\right| &= \left|\frac{1+(-1+a+b)/(1+a+b)}{1-(-1+a+b)/(1+a+b)} - a\right| \\ &= |a+b-a| = |b| = b.\end{aligned}$$

Hence, in both cases, we get a contradiction to (3.3). Therefore, there exists at least one function $f \in S^*(a, b, U_r)$ such that $f \notin S^*(a, b, U_R)$ for every $R > r$. Hence, we have proved that r is the radius of the largest disk in which the condition (3.1) holds. This completes the proof. □

In the case that $a = b$ in Theorem 3.3, we obtain the following particular case:

Corollary 3.4. *Let* $r = \frac{1}{2} \cdot e^{\pi/2} \approx 2.4052\ldots$. *Then we have that* $\rho^*(a, a) = (2a-1)/(2a+1)$ *for* $1/2 < a < r$ *and* $\rho^*(a, a) = \tanh(\pi/4)$ *for* $a \geq r$.

Proof. If we make the substitution $a = b$ in Theorem 3.3 then

$$\rho^*(a, a) = \min\left\{1, \frac{2a-1}{2a+1}, \tanh\left(\frac{\pi}{4}\right)\right\}.$$

It can easily be seen that $(2a-1)/(2a+1) < 1$. Also, the function $(2a-1)/(2a+1)$ is increasing and is equal to $\tanh(\pi/4)$ when $a = r$, where $r = \frac{1}{2} \cdot e^{\pi/2}$. □

This corollary shows that if $f \in S$ then f is not only starlike on $U_{\tanh(\pi/4)}$, but it is also in $S^*(a, a, U_{\tanh(\pi/4)})$, when $a \geq \frac{1}{2} \cdot e^{\pi/2}$.

In view of the proof of Theorem 3.3, we obtain the $S^*(a, b)$ radius of the class S^*.

Theorem 3.5. *Let $a, b \in \mathbb{R}$ be such that $|1 - a| < b \leq a$. Then the $S^*(a, b)$ radius of S^* is given by*

$$\rho_*(a, b) = \min \left\{ \frac{1 - a + b}{1 + a - b}, \frac{-1 + a + b}{1 + a + b} \right\}. \tag{3.4}$$

Moreover, if $a = b$ then $\rho_*(a, a) = (2a - 1)/(2a + 1)$.

In view of Remark 1.5 and Corollary 3.4, we obtain the following particular case.

Remark 3.6. Let $\gamma \in (0, 1)$.

(i) The radius of almost starlikeness of order γ of the class S is $\tanh(\pi/4)$, when $0 < \gamma \leq e^{-\pi/2}$ and $(1 - \gamma)/(1 + \gamma)$, when $e^{-\pi/2} < \gamma < 1$.

(ii) The radius of almost starlikeness of order γ of the class S^* is $(1 - \gamma)/(1 + \gamma)$.

Assuming $|1 - a| < b \leq a$, we refer to the $S^*(a, b)$ radius of class $\Phi_{n,\alpha,\beta}(S)$ (respectively $\Phi_{n,\alpha,\beta}(S^*)$) as the radius $r \in (0, 1]$ of the largest ball $\mathbf{B}_r^n$ such that every mapping $F \in \Phi_{n,\alpha,\beta}(S)$ (respectively $F \in \Phi_{n,\alpha,\beta}(S^*)$) is a member of the family $S^*(a, b, \mathbf{B}_r^n)$.

Theorem 3.7. *If $\alpha \in [0, 1]$, $\beta \in [0, 1/2]$ and $\alpha + \beta \leq 1$ then the $S^*(a, b)$ radius of $\Phi_{n,\alpha,\beta}(S)$ is $\rho^*(a, b)$, where $\rho^*(a, b)$ is given by* (3.2).

Proof. Let $f \in S$. Then $f \in S^*(a, b, U_{\rho^*(a,b)})$. We denote $F_{\alpha,\beta} = \Phi_{n,\alpha,\beta}(f)$. In view of Corollary 2.3 and Remark 3.1(ii), we have that $F_{\alpha,\beta} \in S^*(a, b, \mathbf{B}^n_{\rho^*(a,b)})$. This shows that the $S^*(a, b)$ radius of $\Phi_{n,\alpha,\beta}(S)$ is greater than or equal to $\rho^*(a, b)$. From the proof of Theorem 3.3, we know that the relation (3.1) may not hold when $|z| \geq \rho^*(a, b)$. From Remark 3.1(i), the mapping $F_{\alpha,\beta}$ may fail to be a Janowski starlike mapping on $\mathbf{B}_R^n$ with $R > \rho^*(a, b)$. Hence, we conclude that $\rho^*(a, b)$ is the biggest radius r for which every $F_{\alpha,\beta} = \Phi_{n,\alpha,\beta}(f)$ is Janowski starlike on $\mathbf{B}_r^n$. □

With arguments similar to those used in the proof of Theorem 3.7, the following results hold.

Theorem 3.8. *Let $\alpha, \beta \in [0, 1]$ with $\beta \leq 1/2$ and $\alpha + \beta \leq 1$.*

(i) *The $S^*(a, b)$ radius of $\Phi_{n,\alpha,\beta}(S^*)$ is $\rho_*(a, b)$, where $\rho_*(a, b)$ is given by* (3.4).

(ii) *Let $\gamma \in (0, 1)$. The radius of almost starlikeness of order γ of $\Phi_{n,\alpha,\beta}(S)$ is* $\tanh(\pi/4)$ *for $0 < \gamma \leq e^{-\pi/2}$ and $(1 - \gamma)/(1 + \gamma)$ for $e^{-\pi/2} < \gamma < 1$.*

Also, the radius of almost starlikeness of order γ of $\Phi_{n,\alpha,\beta}(S^*)$ is $(1 - \gamma)/(1 + \gamma)$.

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References

[1] T. Chirilă, *An extension operator associated with certain g-Loewner chains*, Taiwanese J. Math. **17** (2013), no. 5, 1819–1837.

[2] ———, *Subclasses of biholomorphic mappings associated with g-Loewner chains on the unit ball in* $\mathbb{C}^n$, Complex Var. Elliptic Equ. **59** (2014), no. 10, 1456–1474.

[3] P. Curt, *A Marx-Strohhäcker theorem in several complex variables*, Mathematica **39(62)** (1997), no. 1, 59–70.

[4] ———, *Janowski starlikeness in several complex variables and complex Hilbert spaces*, Taiwanese J. Math. **18** (2014), no. 4, 1171–1184.

[5] P. Duren, I. Graham, H. Hamada and G. Kohr, *Solutions for the generalized Loewner differential equation in several complex variables*, Math. Ann. **347** (2010), no. 2, 411–435.

[6] S. X. Feng, *Some Classes of Holomorphic Mappings in Several Complex Variables*, University of Science and Technology of China, Doctor Thesis, 2004.

[7] I. Graham, H. Hamada and G. Kohr, *Parametric representation of univalent mappings in several complex variables*, Canad. J. Math. **54** (2002), no. 2, 324–351.

[8] I. Graham, H. Hamada, G. Kohr and T. J. Suffridge, *Extension operators for locally univalent mappings*, Michigan Math. J. **50** (2002), no. 1, 37–55.

[9] I. Graham and G. Kohr, *An extension theorem and subclasses of univalent mappings in several complex variables*, Complex Var. Theory Appl. **47** (2002), no. 1, 59–72.

[10] ———, *Geometric Function Theory in One and Higher Dimensions*, Monographs and Textbooks in Pure and Applied Mathematics **255**, Marcel Dekker, New York, 2003.

[11] I. Graham, G. Kohr and M. Kohr, *Loewner chains and the Roper-Suffridge extension operator*, J. Math. Anal. Appl. **247** (2000), no. 2, 448–465.

[12] ———, *Loewner chains and parametric representation in several complex variables*, J. Math. Anal. Appl. **281** (2003), no. 2, 425–438.

[13] H. Hamada and T. Honda, *Sharp growth theorems and coefficient bounds for starlike mappings in several complex variables*, Chin. Ann. Math. Ser. B **29** (2008), no. 4, 353–368.

[14] H. Hamada and G. Kohr, *Subordination chains and the growth theorem of spirallike mappings*, Mathematica **42(65)** (2000), no. 2, 153–161.

[15] H. Hamada, G. Kohr and M. Kohr, *Parametric representation and extension operators for biholomorphic mappings on some Reinhardt domains*, Complex Var. Theory Appl. **50** (2005), no. 7-11, 507–519.

[16] W. Janowski, *Some extremal problems for certain families of analytic functions I*, Ann. Polon. Math. **28** (1973), 297–326.

[17] G. Kohr, *Certain partial differential inequalities and applications for holomorphic mappings defined on the unit ball of* $\mathbb{C}^n$, Ann. Univ. Mariae Curie-Skłodowska Sect. A **50** (1996), 87–94.

[18] ______, *On some sufficient conditions of almost starlikeness of order* $1/2$ *in* C^n, Studia Univ. Babeş-Bolyai Math. **41** (1996), no. 3, 51–55.

[19] X. Liu, *The generalized Roper-Suffridge extension operator for some biholomorphic mappings*, J. Math. Anal. Appl. **324** (2006), no. 1, 604–614.

[20] X. S. Liu and T. S. Liu, *The generalized Roper-Suffridge extension operator for spirallike mappings of type* β *and order* α, Chinese Ann. Math. Ser. A **27** (2006), no. 6, 789–798.

[21] J. R. Muir, Jr., *Extension of convex mappings of order* α *of the unit disk in* $\mathbb{C}$ *to convex mappings of the unit ball in* $\mathbb{C}^n$, J. Math. Anal. Appl. **356** (2009), no. 1, 369–377.

[22] M. M. Nargesi, R. M. Ali and V. Ravichandran, *Radius constants for analytic functions with fixed second coefficient*, The Scientific World Journal **2014** (2014), Art. ID 898614, 6 pp.

[23] J. A. Pfaltzgraff, *Subordination chains and univalence of holomorphic mappings in* C^n, Math. Ann. **210** (1974), 55–68.

[24] C. Pommerenke, *Univalent Functions*, Studia Mathematica/Mathematische Lehrbücher, Band XXV, Vandenhoeck & Ruprecht, Göttingen, 1975.

[25] K. A. Roper and T. J. Suffridge, *Convex mappings on the unit ball of* C^n, J. Anal. Math. **65** (1995), 333–347.

[26] H. Silverman, *Subclasses of starlike functions*, Rev. Roumaine Math. Pures Appl. **23** (1978), no. 7, 1093–1099.

[27] H. Silverman and E. M. Silvia, *Subclasses of starlike functions subordinate to convex functions*, Canad. J. Math. **37** (1985), no. 1, 48–61.

[28] T. J. Suffridge, *Starlikeness, convexity and other geometric properties of holomorphic maps in higher dimensions*, Complex analysis (Proc. Conf., Univ. Kentucky, Lexington, Ky., 1976), 146–159, Lecture Notes in Mathematics **599**, Springer, Berlin, 1977.

[29] M. I. Voda, *Loewner Theory in Several Complex Variables and Related Problems*, Thesis (Ph.D.)–University of Toronto (Canada), 2011.

[30] C. Wang, Y. Cui and H. Liu, *New subclasses of biholomorphic mappings and the modified Roper-Suffridge operator*, Chin. Ann. Math. Ser. B **37** (2016), no. 5, 691–704.

[31] Q.-H. Xu and T.-S. Liu, *Löwner chains and a subclass of biholomorphic mappings*, J. Math. Anal. Appl. **334** (2007), no. 2, 1096–1105.

[32] Y.-C. Zhu and M.-S. Liu, *The generalized Roper-Suffridge extension operator on Reinhardt domain* D_p, Taiwanese J. Math. **14** (2010), no. 2, 359–372.

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