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Ground State Solutions for Kirchhoff-type Problems with Critical Nonlinearity

Yiwei Ye

Abstract. In this paper, we study the Kirchhoff-type equation with critical exponent

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,dx\right)\Delta u+V(x)u=a(x)f(u)+u^5 \quad \text{in } \mathbb{R}^3,$$

where $a,b>0$ are constants, $V\in C(\mathbb{R}^3,\mathbb{R})$, $\lim_{|x|\to\infty}V(x)=V_\infty>0$ and $V(x)\le V_\infty+C_1e^{-b|x|}$ for some $C_1>0$ and $|x|$ large enough. Via variational methods, we prove the existence of ground state solution.

1. Introduction and main results

Consider the following Kirchhoff type problem with the critical exponent

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,dx\right)\Delta u+V(x)u=a(x)f(u)+u^5 \quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where $a,b>0$ are constants.

Problem (1.1) is related to the stationary version of the equation

$$\begin{cases} u_{tt}-\left(a+b\int_\Omega|\nabla_x u|^2\right)\Delta_x u=f(x,u) & x\in\Omega,\ t>0, \\ u(\,\cdot\,,t)|_{\partial\Omega}=0 & t\ge 0, \end{cases} \tag{1.2}$$

proposed by Kirchhoff [7] in 1883 as an extension of the classical d'Alembert's wave equation for free vibration of elastic strings. Kirchhoff's model takes into account the changes in the length of the string produced by transverse vibrations. In (1.2), u denotes the displacement, $f(x,u)$ the external force and b the initial tension while a is related to the intrinsic properties of the string (such as Young's modulus). We have to point out that Kirchhoff type problems also appear in other fields such as biological systems, where u describes a process which depends on the average of itself (for example, population density). Some early classical studies of Kirchhoff equations were those of Pohozaev [16] and Lions [10, 11].

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Variational methods have been widely used in the last ten more years in studying the Kirchhoff type problems, see, for instance, [4,6,14,15,20,21,26] for results concerning bounded domain, and [1,3,9,12,13,19,22,24] for unbounded domain. Figueiredo [4] studies Kirchhoff type problem on bounded domain with the following version

$$-M\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta u = \lambda f(x,u) + |u|^{2^*-2}u \quad \text{in } \Omega$$

and $u = 0$ on $\partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^N$. Applying truncation argument and priori estimates, they prove the existence of positive solutions and their asymptotic behavior depending on λ. Naimen [14] generalizes Brezis-Nirenberg's result to the Kirchhoff type equation

$$-\left(a + b\int_\Omega |\nabla u|^2\,dx\right)\Delta u = \mu f(x,u) + |u|^{2^*-2}u \quad \text{in } \Omega,$$

where $a, b \geq 0$ and $a + b > 0$. Some existence results as well as nonexistence results are obtained. See also [6, 15, 20, 26] for related results. For Kirchhoff type equations on the whole space $\mathbb{R}^3$, a few results are known. One difficulty is due to the lack of compactness of the embedding from $H^1(R^3)$ into $L^s(R^3)$ $(2 < s < 6)$. Alves and Figueiredo [1] study the following periodic Kirchhoff equation with critical growth

$$M\left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \int_{\mathbb{R}^N} V(x)|u|^2\,dx\right)[-\Delta u + V(x)u] = \lambda f(u) + u^\tau \quad \text{in } \mathbb{R}^N,$$

where $\tau = 5$ for $N = 3$ and $\tau \in (1, +\infty)$ for $N = 1, 2$. Under suitable assumptions on M, V and f, they construct the existence of positive solutions for $\lambda > 0$ large enough. The Kirchhoff type equation with critical nonlinearity of the form

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2\,dx\right)\Delta u + u = a(x)|u|^{p-2}u + b(x)|u|^{q-2}u + u^5 \quad \text{in } \mathbb{R}^3,$$

has been studied in Zhang [24]. Besides some other conditions, they assume that $a, b \in C(\mathbb{R}^3, \mathbb{R})$, $\lim_{|x|\to\infty} a(x) = a_\infty$, $\lim_{|x|\to\infty} b(x) = 0$ and $a(x) \geq a_\infty - Ce^{-a_0|x|}$ for some $a_0 > 0$ and $x \in \mathbb{R}^3$ and prove the existence of one ground state solution for $p, q \in (4, 6)$ and each $\lambda > 0$. It is also proven the existence of two nontrivial solutions for $\lambda > 0$ small. Wang et al. [17] discuss the existence, multiplicity and concentration behavior of positive solutions for the following nonlinear Kirchhoff type problem

$$-\left(\varepsilon^2 a + \varepsilon b\int_{\mathbb{R}^3} |\nabla u|^2\,dx\right)\Delta u + B(x)u = \lambda f(u) + u^5 \quad \text{in } \mathbb{R}^3. \tag{1.3}$$

As $\varepsilon \to 0$, they establish the number of solutions with the topology of the set where B attains its minimum in the case that f is superlinear at zero and subcritical at infinity.

For more results on the semiclassical limit of (1.3), we refer to [5,19,25] and the references therein.

Motivated by the works mentioned above, in this paper, we consider the existence of ground state solution of problem (1.1) in the critical case. We make the following hypotheses:

(V_1) $V \in C(\mathbb{R}^3, \mathbb{R})$, $V \geq 0$ on $\mathbb{R}^3$ and $\lim_{|x|\to\infty} V(x) = V_\infty > 0$.

(V_2) There exist constants $C_1 > 0$, $R_1 > 0$ and $b > 0$ such that $V(x) \leq V_\infty + C_1 e^{-b|x|}$ for $|x| \geq R_1$.

(a_1) $a \in C(\mathbb{R}^3, \mathbb{R})$, $a \geq 0$ on $\mathbb{R}^3$ and $\lim_{|x|\to\infty} a(x) = a_\infty > 0$.

(a_2) There exist constants $C_2 > 0$, $R_2 > 0$ and $d > 0$ such that $a(x) \geq a_\infty + C_2 e^{-d|x|}$ for $|x| \geq R_2$.

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$ is odd and $\lim_{u\to 0^+} \frac{f(u)}{u} = \lim_{u\to+\infty} \frac{f(u)}{u^5} = 0$.

(f_2) There exist $C_3 > 0$ and $p \in (4, 6)$ such that $f(u) \geq C_3 u^{p-1}$ for all $u \geq 0$.

(f_3) $f(u)/u^3$ is increasing on $(0, +\infty)$.

Theorem 1.1. *Assume that* (V_1)–(V_2), (a_1)–(a_2) *and* (f_1)–(f_3) *hold with* $0 < d < b < 2$. *Then problem* (1.1) *possesses a ground state solution.*

The proof is based on variational method. The main difficulties lie in the appearance of the nonlocal term $\left(\int_{\mathbb{R}^3} |\nabla u|^2\,dx\right)\Delta u$, the lack of compactness due to the unboundedness of the domain $\mathbb{R}^3$, and the nonlinearity with the critical Sobolev growth. Moreover, since $V(x)$ and $a(x)$ are non-radially symmetric, we cannot restrict the problem in the radially symmetric Sobolev space $H_r^1(\mathbb{R}^3)$, where the embedding $H_r^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ $(2 < s < 6)$ is compact. To overcome these difficulties, we have to develop some techniques. In particular, for a sequence $(u_n) \subset E$ with $u_n \rightharpoonup u$ weakly in E, we will analyze carefully the difference between $\left(\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx\right)^2$ and $\left(\int_{\mathbb{R}^3} |\nabla u|^2\,dx\right)^2$ to prove the $(PS)_c$ condition holds for a suitable range of c.

The paper is organized as follows. In Section 2, we give some preliminaries and prove the local Palais-Smale condition of the functional corresponding to (1.1). In Section 3, we will analyze the term $\int_{\mathbb{R}^3} V(x)u^2\,dx$ delicately and its competing with the nonlinearity $\int_{\mathbb{R}^3} a(x)F(u)\,dx$, where $F(u) = \int_0^u f(s)\,ds$, and prove Theorem 1.1.

Notations

- For any $1 \leq s \leq +\infty$, we denote by $\|\cdot\|_s$ the usual norm of the Lebesgue space $L^s(\mathbb{R}^3)$.

- $H^1(\mathbb{R}^3)$ is the usual Hilbert space endowed with the norm $\|u\|_{H^1}^2 = \int_{\mathbb{R}^3}(|\nabla u|^2 + u^2)\,dx$.
- $\mathcal{D}^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|_{\mathcal{D}^{1,2}}^2 := \int_{\mathbb{R}^3} |\nabla u|^2\,dx$.
- S denotes the best Sobolev constant
$$S := \inf_{u\in\mathcal{D}^{1,2}\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}.$$
- C and C_i denote various positive constants, which may vary from line to line.

2. Preliminary results

In this section, we assume (V_1), (a_1) and (f_1)–(f_3) hold. Let

$$E = \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2\,dx < +\infty \right\}.$$

It is easy to recognize that E is the Hilbert space with the inner product and norm

$$(u,v) = \int_{\mathbb{R}^3} (a\nabla u\cdot\nabla v + V(x)uv)\,dx, \quad \|u\| = (u,u)^{1/2}.$$

From (V_1) and (a_1), there exists a constant $M > 0$ such that

$$|V(x)| \le M, \quad |a(x)| \le M, \quad \forall\, x \in \mathbb{R}^3, \tag{2.1}$$

and there exists $R_0 > 0$ such that $V(x) \ge V_\infty/2$ for $|x| \ge R_0$. Hence we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)\,dx \\
&\le \int_{|x|\le R_0} |\nabla u|^2\,dx + \int_{|x|\le R_0} u^2\,dx + \int_{|x|\ge R_0} \left(|\nabla u|^2 + \frac{2}{V_\infty}V(x)u^2\right)dx \\
&\le \int_{|x|\le R_0} |\nabla u|^2\,dx + \left(\int_{|x|\le R_0} u^6\,dx\right)^{1/3} \left|\int_{|x|\le R_0} 1\,dx\right|^{2/3} \\
&\quad + \left(\frac{1}{a} + \frac{2}{V_\infty}\right)\int_{|x|\ge R_0} (a|\nabla u|^2 + V(x)u^2)\,dx \\
&\le \max\left\{\frac{1}{a} + (aS)^{-1}\left|\int_{|x|\le R_0} 1\,dx\right|^{2/3}, \frac{1}{a} + \frac{2}{V_\infty}\right\}\|u\|^2 \\
&\le \left(\frac{1}{a} + (aS)^{-1}\left|\int_{|x|\le R_0} 1\,dx\right|^{2/3} + \frac{2}{V_\infty}\right)\|u\|^2,
\end{aligned} \tag{2.2}$$

which implies that the embedding $E \hookrightarrow H^1(\mathbb{R}^3)$ is continuous. Consider the functional $I\colon E \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\,dx\right)^2 - \int_{\mathbb{R}^3}\left(a(x)F(u) + \frac{1}{6}u^6\right)dx.$$

It is easy to check that $I \in C^1(E,\mathbb{R})$ and $u \in E$ is a solution of problem (1.1) if and only if u is a critical point of I. Define the functional

$$I^\infty(u) = \frac{1}{2}\|u\|^2_{V_\infty} + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\,dx\right)^2 - \int_{\mathbb{R}^3}\left(a_\infty F(u) + \frac{1}{6}|u|^6\right)dx, \quad \forall\, u \in E,$$

where $\|u\|^2_{V_\infty} := \int_{\mathbb{R}^3}(a|\nabla u|^2 + V_\infty|u|^2)\,dx$. Take

$$m_\infty = \inf\{I^\infty(u) : u \in M_\infty\}$$

with $M_\infty = \{u \in E \setminus \{0\} : I^{\infty\prime}(u) = 0\}$. According to [12, Theorem 1.3], m_∞ is obtained by a positive function u_∞ and $I^{\infty\prime}(u_\infty) = 0$. Similar arguments as in [23, Lemma 2.5], we can deduce that for any $\delta > 0$, there exists $C_\delta > 0$ such that

$$u_\infty(x) \le C_\delta e^{-(1-\delta)\sqrt{V_\infty}|x|}, \quad \forall\, x \in \mathbb{R}^3. \tag{2.3}$$

By (f_1), we deduce that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\max\{|F(u)|, |f(u)u|\} \le \varepsilon|u|^2 + C_\varepsilon|u|^6, \quad \forall\, u \in \mathbb{R} \tag{2.4}$$

and

$$\max\{|F(u)|, |f(u)u|\} \le \varepsilon(|u|^2 + |u|^6) + C_\varepsilon|u|^s, \quad \forall\, u \in \mathbb{R}, \tag{2.5}$$

where $s \in (2,6)$. From (f_3), we derive that

$$\frac{1}{4}f(u)u - F(u) \ge 0, \quad \forall\, u \in \mathbb{R} \tag{2.6}$$

and $\frac{1}{4}f(u)u - F(u)$ is increasing for $u \ge 0$.

For $\varepsilon > 0$, let

$$v_\varepsilon(x) = \frac{\psi(x)\varepsilon^{1/4}}{(\varepsilon + |x|^2)^{1/2}},$$

where $\psi \in C_0^\infty(\mathbb{R}^3, [0,1])$ such that $\psi(x) = 1$ for $|x| \le r_0$ and $\psi(x) = 0$ for $|x| \ge 2r_0$. It is well known that S is attained by the function $\varepsilon^{1/4}/(\varepsilon + |x|^2)^{1/2}$. Direct calculation shows that (see [18])

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2\,dx &= \int_{\mathbb{R}^3} \frac{|x|^2}{(1+|x|^2)^3}\,dx + O(\varepsilon^{1/2}) := K_1 + O(\varepsilon^{1/2}),\\ \int_{\mathbb{R}^3} |v_\varepsilon|^6\,dx &= \int_{\mathbb{R}^3} \frac{1}{(1+|x|^2)^3}\,dx := K_2 + O(\varepsilon^{3/2})\end{aligned} \tag{2.7}$$

and

$$\int_{\mathbb{R}^3} |v_\varepsilon|^s\,dx = \begin{cases} O(\varepsilon^{(6-s)/4}) & s \in (3,6), \\ O(\varepsilon^{3/4}|\ln\varepsilon|) & s = 3, \\ O(\varepsilon^{s/4}) & s \in [2,3), \end{cases} \tag{2.8}$$

where K_1, K_2 are positive constants and $S = K_1/K_2^{1/3}$.

Lemma 2.1. $m_\infty < c^* := \dfrac{a}{3}\left\{\dfrac{bS^3 + \sqrt{(bS^3)^2 + 4aS^3}}{2}\right\} + \dfrac{b}{12}\left\{\dfrac{bS^3 + \sqrt{(bS^3)^2 + 4aS^3}}{2}\right\}^2.$

Proof. Let $c_\infty = \inf_{\gamma\in\Gamma_\infty}\max_{t\in[0,1]} I^\infty(\gamma(t))$, where $\Gamma_\infty = \{\gamma \in C([0,1],E) : \gamma(0) = 0, I^\infty(\gamma(1)) < 0\}$. Similar to [18, Theorems 4.1 and 4.2], we have

$$m_\infty = c_\infty = \inf_{u\in E\setminus\{0\}}\max_{t\geq 0} I^\infty(tu) > 0.$$

Hence $m_\infty \leq \sup_{t\geq0} I^\infty(tv_\varepsilon)$. To obtain $m_\infty < c^*$, it suffices to prove that $\sup_{t\geq0} I^\infty(tv_\varepsilon) < c^*$.

It follows from (2.7) and (2.8) that there exists $\varepsilon_1 > 0$ such that for $\varepsilon \in (0,\varepsilon_1)$,

$$\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2\,dx < \frac{3K_1}{2}, \quad \int_{\mathbb{R}^3} |v_\varepsilon|^2\,dx \leq \frac{1}{2}, \quad \int_{\mathbb{R}^3} |v_\varepsilon|^6\,dx \geq \frac{K_2}{2}.$$

Thus, for $\varepsilon \in (0,\varepsilon_1)$,

$$\begin{aligned} I^\infty(tv_\varepsilon) &\leq \frac{t^2}{2}\|v_\varepsilon\|^2_{V_\infty} + \frac{bt^4}{4}\left(\int_{\mathbb{R}^3}|\nabla v_\varepsilon|^2\,dx\right)^2 - \frac{t^6}{6}\int_{\mathbb{R}^3}|v_\varepsilon|^6\,dx \\ &\leq \frac{(3aK_1 + V_\infty)t^2}{4} + \frac{9bK_1^2t^4}{16} - \frac{K_2t^6}{12}, \end{aligned}$$

which implies that there exist $t_1 > 0$ small and $t_2 > 0$ large (independent of ε) such that

$$\sup_{t\in[0,t_1]\cup[t_2,+\infty]} I^\infty(tv_\varepsilon) < c^*. \tag{2.9}$$

Take

$$A_\varepsilon = \frac{\int_{\mathbb{R}^3} a|\nabla v_\varepsilon|^2\,dx}{\left(\int_{\mathbb{R}^3}|v_\varepsilon|^6\,dx\right)^{1/3}} \quad \text{and} \quad B_\varepsilon = \frac{b\left(\int_{\mathbb{R}^3}|\nabla v_\varepsilon|^2\,dx\right)^2}{\left(\int_{\mathbb{R}^3}|v_\varepsilon|^6\,dx\right)^{2/3}}.$$

From (2.7), one has $A_\varepsilon = aS + O(\varepsilon^{1/2})$ and $B_\varepsilon = bS^2 + O(\varepsilon^{1/2})$. Then a direct calculation shows that

(2.10)

$$\begin{aligned} &\sup_{t\geq0}\left[\frac{t^2}{2}\int_{\mathbb{R}^3} a|\nabla v_\varepsilon|^2\,dx + \frac{bt^4}{4}\left(\int_{\mathbb{R}^3}|\nabla v_\varepsilon|^2\,dx\right)^2 - \frac{t^6}{6}\int_{\mathbb{R}^3}|v_\varepsilon|^6\,dx\right] \\ &= \frac{1}{4}A_\varepsilon\left(B_\varepsilon + \sqrt{B_\varepsilon^2 + 4A_\varepsilon}\right) + \frac{1}{16}B_\varepsilon\left(B_\varepsilon + \sqrt{B_\varepsilon^2 + 4A_\varepsilon}\right)^2 - \frac{1}{48}\left(B_\varepsilon + \sqrt{B_\varepsilon^2 + 4A_\varepsilon}\right)^3 \\ &= \frac{1}{6}A_\varepsilon\left(B_\varepsilon + \sqrt{B_\varepsilon^2 + 4A_\varepsilon}\right) + \frac{1}{48}B_\varepsilon\left(B_\varepsilon + \sqrt{B_\varepsilon^2 + 4A_\varepsilon}\right)^2 \\ &= c^* + O(\varepsilon^{1/2}). \end{aligned}$$

Hence, combining (2.10), (2.8) and (f_2), we obtain

$$\begin{aligned}\sup_{t\in[t_1,t_2]} I^\infty(tv_\varepsilon) &\le \sup_{t\ge 0}\left[\frac{t^2}{2}\int_{\mathbb{R}^3} a|\nabla v_\varepsilon|^2\,dx + \frac{bt^4}{4}\left(\int_{\mathbb{R}^3}|\nabla v_\varepsilon|^2\,dx\right)^2 - \frac{t^6}{6}\int_{\mathbb{R}^3}|v_\varepsilon|^6\,dx\right] \\ &\quad + \frac{V_\infty t_2^2}{2}\int_{\mathbb{R}^3}|v_\varepsilon|^2\,dx - \frac{C_3 a_\infty t_1^p}{p}\int_{\mathbb{R}^3}|v_\varepsilon|^p\,dx \\ &= c^* + O(\varepsilon^{1/2}) - O(\varepsilon^{(6-p)/4}).\end{aligned}$$

Observing $(6-p)/4 < 1/2$, choosing $\varepsilon \in (0, \varepsilon_1)$ sufficiently small, we get $\sup_{t\in[t_1,t_2]} I^\infty(tv_\varepsilon) < c^*$, which, jointly with (2.9), shows that $\sup_{t\ge 0} I^\infty(tv_\varepsilon) < c^*$. This completes the proof. □

Recall that, for $c \in \mathbb{R}$, a $(PS)_c$-sequence for the functional I is referred to a sequence $(u_n) \subset E$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$ in E^* (the dual space of E). I is said to satisfy $(PS)_c$ condition if any $(PS)_c$ sequence has a convergent subsequence.

Lemma 2.2. *Suppose that (u_n) be a $(PS)_c$-sequence of I with $c \in (0, m_\infty)$. Then (u_n) possesses a strongly convergent subsequence.*

Proof. It follows from (a_1) and (2.6) that

$$c + o(1) + o(1)\|u_n\| = I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle \ge \frac{1}{4}\|u_n\|^2,$$

which implies that $(u_n)_{n\in\mathbb{N}}$ is bounded. Going if necessary to a subsequence, still denoted by (u_n), we may assume that there is $u \in E$ such that for each bounded domain $\Omega \subset \mathbb{R}^3$,

$$\begin{aligned} u_n &\rightharpoonup u \ \text{ in } E, \quad \nabla u_n \rightharpoonup \nabla u \ \text{ in } L^2(\mathbb{R}^3), \\ u_n &\to u \ \text{ in } L^s(\Omega) \quad (2 < s < 6), \\ u_n(x) &\to u(x) \ \text{ a.e. } x \in \mathbb{R}^3. \end{aligned} \tag{2.11}$$

Set $A = \lim_{n\to\infty}\int_{\mathbb{R}^3}|\nabla u_n|^2\,dx$. We define the functionals J, H, Φ, Ψ on E by

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 + \frac{bA}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,dx - \int_{\mathbb{R}^3}\left(a(x)F(u) + \frac{1}{6}u^6\right)dx, \\ H(u) &= \frac{1}{2}\|u\|^2 + \frac{bA}{4}\int_{\mathbb{R}^3}|\nabla u|^2\,dx - \int_{\mathbb{R}^3}\left(a(x)F(u) + \frac{1}{6}u^6\right)dx, \\ \Phi(u) &= \frac{1}{2}\|u\|^2_{V_\infty} + \frac{bA}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,dx - \int_{\mathbb{R}^3}\left(a_\infty F(u) + \frac{1}{6}u^6\right)dx, \\ \Psi(u) &= \frac{1}{2}\|u\|^2_{V_\infty} + \frac{bA}{4}\int_{\mathbb{R}^3}|\nabla u|^2\,dx - \int_{\mathbb{R}^3}\left(a_\infty F(u) + \frac{1}{6}u^6\right)dx. \end{aligned}$$

We claim that $J'(u) = 0$, i.e., $\langle J'(u), \varphi\rangle = 0$ for any $\varphi \in C_0^\infty(\mathbb{R}^3)$. Assume that $1 \le p, q, r, s < +\infty$, Ω is a bounded domain and $h \in C(\Omega \times \mathbb{R})$ satisfying $|h(x,u)| \le C(|u|^{p/r} + |u|^{q/s})$, then, according to [18, Theorem A.4], the operator

$$A : L^p(\Omega) \cap L^q(\Omega) \to L^r(\Omega) + L^s(\Omega) : u \to h(x,u)$$

is continuous, where $L^p(\Omega) \cap L^q(\Omega)$ is the space endowed with the norm $|u|_{p\wedge q} = \|u\|_{L^p(\Omega)} + \|u\|_{L^q(\Omega)}$ and $L^r(\Omega) + L^s(\Omega)$ endowed with the norm

$$|u|_{r\vee s} = \inf\left\{\|v\|_{L^r(\Omega)} + \|w\|_{L^s(\Omega)} : u = v + w, v \in L^r(\Omega), w \in L^s(\Omega)\right\}.$$

Now set $p = r = 2$, $q \in (5,6)$, $s = q/5$ and $h(x,u) = a(x)f(u) + u^5$. By (a_1) and (f_1), we have

$$|h(x,u)| \le C(|u|^{2/2} + |u|^{q/s}), \quad \forall\, (x,u) \in \mathbb{R}^3 \times \mathbb{R}.$$

Since $\varphi \in C_0^\infty(\mathbb{R}^3)$ has a compact support Ω_0. $u_n \rightharpoonup u$ in E implies that $u_n \to u$ in $L^2(\Omega_0) \cap L^q(\Omega_0)$. So, by virtue of [18, Theorem A.4],

$$h(x,u_n) \to h(x,u) \quad \text{in } L^2(\Omega_0) + L^s(\Omega_0).$$

Hence

$$\begin{aligned}\int_{\mathbb{R}^3} |(h(x,u_n) - h(x,u))\varphi|\, dx &= \int_{\Omega_0} |(h(x,u_n) - h(x,u))\varphi|\, dx \\ &\le |h(x,u_n) - h(x,u)|_{2\vee s} |\varphi|_{2\wedge s'} \xrightarrow{n} 0,\end{aligned}$$

where $1/s + 1/s' = 1$. Combining this and (2.11), we get that $o(1) = \langle I'(u_n), \varphi\rangle = \langle J'(u), \varphi\rangle + o(1)$ for any $\varphi \in C_0^\infty(\mathbb{R}^3)$. Thus $J'(u) = 0$.

Let $v_n := u_n - u$. It follows from [27, Lemma 2.2] and Brezis-Lieb lemma that

$$\int_{\mathbb{R}^3} a(x)(F(u_n) - F(u) - F(v_n))\, dx \to 0 \tag{2.12}$$

and

$$\int_{\mathbb{R}^3} (u_n^6 - u^6 - v_n^6)\, dx \to 0 \tag{2.13}$$

as $n \to \infty$. Since $v_n \rightharpoonup 0$ in E, by (V_1), (a_1) and (f_1), we obtain that

$$\int_{\mathbb{R}^3} (V(x) - V_\infty) v_n^2\, dx \to 0, \quad \int_{\mathbb{R}^3} (a(x) - a_\infty) F(v_n)\, dx \to 0 \tag{2.14}$$

and

$$\int_{\mathbb{R}^3} (V(x) - V_\infty) v_n \xi\, dx \to 0, \quad \int_{\mathbb{R}^3} (a(x) - a_\infty) f(v_n) \xi\, dx \to 0, \quad \forall\, \xi \in E \tag{2.15}$$

as $n \to \infty$. Hence, using (2.14), (2.13) and (2.12), we deduce that

$$\begin{aligned} c + o(1) &= I(u_n) \\ &= \frac{1}{2}(\|u\|^2 + \|v_n\|^2) + \frac{bA}{4}\int_{\mathbb{R}^3}(|\nabla u|^2 + |\nabla v_n|^2)\,dx \\ &\quad - \int_{\mathbb{R}^3}(a(x)F(u) + a(x)F(v_n))\,dx - \int_{\mathbb{R}^3}\frac{1}{6}(u^6 + v_n^6)\,dx + o(1) \\ &= \frac{1}{2}(\|u\|^2 + \|v_n\|^2_{V_\infty}) + \frac{bA}{4}\int_{\mathbb{R}^3}(|\nabla u|^2 + |\nabla v_n|^2)\,dx \\ &\quad - \int_{\mathbb{R}^3}(a(x)F(u) + a_\infty F(v_n))\,dx - \int_{\mathbb{R}^3}\frac{1}{6}(u^6 + v_n^6)\,dx + o(1) \\ &= H(u) + \Psi(v_n) + o(1). \end{aligned} \tag{2.16}$$

Moreover, noticing $J'(u) = 0$, by (a_1) and (2.6) we have

$$\begin{aligned} H(u) &= H(u) - \frac{1}{4}\langle J'(u), u\rangle \\ &= \frac{1}{4}\|u\|^2 + \int_{\mathbb{R}^3}\left[a(x)\left(\frac{1}{4}f(u)u - F(u)\right) + \frac{1}{12}u^6\right]dx \geq 0. \end{aligned} \tag{2.17}$$

From [18, Lemma 8.9], we get

$$\left|\int_{\mathbb{R}^3}(u_n^5 - u^5 - v_n^5)\xi\,dx\right| = o(1)\|\xi\|, \quad \forall \xi \in E. \tag{2.18}$$

Similar to [18, Lemma 8.1], one has

$$\left|\int_{\mathbb{R}^3} a(x)(f(u_n) - f(u) - f(v_n))\xi\,dx\right| = o(1)\|\xi\|, \quad \forall \xi \in E. \tag{2.19}$$

Combining (2.19), (2.18) and (2.15) and using the fact $J'(u) = 0$, we obtain

$$\begin{aligned} o(1) &= \langle I'(u_n), \xi\rangle - \langle J'(u), \xi\rangle \\ &= (v_n, \xi) + bA\int_{\mathbb{R}^3}\nabla v_n \nabla\xi\,dx - \int_{\mathbb{R}^3}(a(x)f(v_n)\xi + v_n^5\xi)\,dx + o(1) \\ &= \int_{\mathbb{R}^3}(a\nabla v_n\nabla\xi + V_\infty v_n\xi)\,dx + bA\int_{\mathbb{R}^3}\nabla v_n\nabla\xi\,dx - \int_{\mathbb{R}^3}(a_\infty f(v_n)\xi + v_n^5\xi)\,dx + o(1) \\ &= \langle \Phi'(v_n), \xi\rangle + o(1), \quad \forall \xi \in E, \end{aligned}$$

which implies that

$$\Phi'(v_n) = o(1). \tag{2.20}$$

Next we prove that $v_n \to 0$ in E. Let $\rho_n(x) = |v_n(x)|^2$. Then by [8, Lemma 2.1], for some subsequence of $\{\rho_n(x)\}$, either "vanishing" or "nonvanishing" holds. If "nonvanishing" occurs, there exist $\sigma > 0$, $r > 0$ and $(y_n) \subset \mathbb{R}^3$ such that

$$\liminf_{n\to\infty}\int_{B_r(y_n)}|v_n(x)|^2\,dx \geq \sigma > 0. \tag{2.21}$$

Let $\widetilde{v}_n(x) = v_n(x + y_n)$. We claim that

$$\Phi'(\widetilde{v}_n) = o(1). \tag{2.22}$$

Indeed, for all $\xi \in E$, set $\xi_n(x) = \xi(x - y_n)$. It is easy to see that $\|\xi_n\|_{H^1} = \|\xi\|_{H^1}$, therefore, by (2.20) and (2.1),

$$\begin{aligned}
|\langle \Phi'(\widetilde{v}_n), \xi\rangle| &= \left| \int_{\mathbb{R}^3} \left(a\nabla \widetilde{v}_n \cdot \nabla \xi + V_\infty \widetilde{v}_n \xi + bA\nabla \widetilde{v}_n \cdot \nabla \xi - a_\infty f(\widetilde{v}_n)\xi + \widetilde{v}_n^5 \xi \right) dx \right| \\
&= \left| \int_{\mathbb{R}^3} \left(a\nabla v_n \cdot \nabla \xi_n + V_\infty v_n \xi_n + bA\nabla v_n \cdot \nabla \xi_n - a_\infty f(v_n)\xi_n + v_n^5 \xi_n \right) dx \right| \\
&= |\langle \Phi'(v_n), \xi_n\rangle| \le \|\Phi'(v_n)\| \|\xi_n\| \le C\|\Phi'(v_n)\| \|\xi_n\|_{H^1} \\
&\le C\|\Phi'(v_n)\| \|\xi\|_{H^1} \xrightarrow{n} 0,
\end{aligned}$$

and (2.22) is proved. Since (v_n) is bounded in E, $(\widetilde{v}_n)$ is also bounded in E. Then we may assume that $\widetilde{v}_n \rightharpoonup \widetilde{v}$ in E and $\widetilde{v} \neq 0$ by (2.21). Observing Φ' is weakly continuous, it follows that $\Phi'(\widetilde{v}) = 0$, so

$$\|\widetilde{v}\|_{V_\infty}^2 + bA \int_{\mathbb{R}^3} |\nabla \widetilde{v}|^2 \, dx = \int_{\mathbb{R}^3} (a_\infty f(\widetilde{v})\widetilde{v} + \widetilde{v}^6) \, dx. \tag{2.23}$$

For $\widetilde{v} \in E \setminus \{0\}$, there exists a unique $t > 0$ such that $t\widetilde{v} \in M_\infty$, i.e.,

$$t^2 \|\widetilde{v}\|_{V_\infty}^2 + bt^4 \left(\int_{\mathbb{R}^3} |\nabla \widetilde{v}|^2 \, dx \right)^2 = \int_{\mathbb{R}^3} \left(a_\infty f(t\widetilde{v}) t\widetilde{v} + t^6 \widetilde{v}^6 \right) dx. \tag{2.24}$$

We claim that $t \le 1$. If $t > 1$, by (2.24), (2.23) and (f_3) and using the fact $A \ge \int_{\mathbb{R}^3} |\nabla \widetilde{v}|^2 \, dx$, we deduce that

$$\begin{aligned}
&t^4 \int_{\mathbb{R}^3} \left(a_\infty f(\widetilde{v})\widetilde{v} + \widetilde{v}^6 \right) dx \\
&\le \int_{\mathbb{R}^3} \left(t^4 a_\infty f(\widetilde{v})\widetilde{v} + t^6 \widetilde{v}^6 \right) dx \le \int_{\mathbb{R}^3} \left(a_\infty \frac{f(t\widetilde{v})}{(t\widetilde{v})^3} t^4 \widetilde{v}^4 + t^6 \widetilde{v}^6 \right) dx \\
&= t^2 \|\widetilde{v}\|_{V_\infty}^2 + bt^4 \left(\int_{\mathbb{R}^3} |\nabla \widetilde{v}|^2 \, dx \right)^2 < t^4 \left(\|\widetilde{v}\|_{V_\infty} + b \left(\int_{\mathbb{R}^3} |\nabla \widetilde{v}|^2 \, dx \right)^2 \right) \\
&\le t^4 \int_{\mathbb{R}^3} \left(a_\infty f(\widetilde{v})\widetilde{v} + \widetilde{v}^6 \right) dx,
\end{aligned}$$

a contradiction. Thus $t \le 1$. Combining this with (2.22), (2.17), (2.16) and using Fatou's lemma, we obtain

$$\begin{aligned}
c+o(1) &\geq \Psi(\widetilde{v}_n)-\frac{1}{4}\langle\Phi'(\widetilde{v}_n),\widetilde{v}_n\rangle\\
&=\frac{1}{4}\|\widetilde{v}_n\|^2_{V_\infty}+\int_{\mathbb{R}^3}a_\infty\left(\frac{1}{4}f(\widetilde{v}_n)\widetilde{v}_n-F(\widetilde{v}_n)\right)dx+\frac{1}{12}\int_{\mathbb{R}^3}|\widetilde{v}_n|^6\,dx\\
&\geq \frac{1}{4}\|\widetilde{v}\|^2_{V_\infty}+\int_{\mathbb{R}^3}a_\infty\left(\frac{1}{4}f(\widetilde{v})\widetilde{v}-F(\widetilde{v})\right)dx+\frac{1}{12}\int_{\mathbb{R}^3}|\widetilde{v}|^6\,dx+o(1)\\
&\geq \frac{1}{4}\|t\widetilde{v}\|^2_{V_\infty}+\int_{\mathbb{R}^3}a_\infty\left(\frac{1}{4}f(t\widetilde{v})t\widetilde{v}-F(t\widetilde{v})\right)dx+\frac{1}{12}\int_{\mathbb{R}^3}|t\widetilde{v}|^6\,dx+o(1)\\
&= I^\infty(t\widetilde{v})-\frac{1}{4}\langle I^{\infty\prime}(t\widetilde{v}),t\widetilde{v}\rangle+o(1)=I^\infty(t\widetilde{v})+o(1)\geq m_\infty+o(1),
\end{aligned}$$

a contradiction with the fact $c\in(0,m_\infty)$.

Now we consider the "vanishing" case. In this case, $v_n\to 0$ in $L^s(\mathbb{R}^3)$ $(2<s<6)$, and then, by (2.5), we see that

$$\int_{\mathbb{R}^3}a(x)F(v_n)\,dx\to 0 \quad\text{and}\quad \int_{\mathbb{R}^3}a(x)f(v_n)v_n\,dx\to 0$$

as $n\to\infty$. Combining this and (2.20), (2.17), (2.16), we obtain

$$c+o(1)\geq\Psi(v_n)=\frac{1}{2}\|v_n\|^2_{V_\infty}+\frac{bA}{4}\int_{\mathbb{R}^3}|v_n|^2\,dx-\frac{1}{6}\int_{\mathbb{R}^3}|v_n|^6\,dx, \tag{2.25}$$

$$o(1)=\langle\Phi'(v_n),v_n\rangle=\|v_n\|^2_{V_\infty}+bA\int_{\mathbb{R}^3}|\nabla v_n|^2\,dx-\int_{\mathbb{R}^3}|v_n|^6\,dx. \tag{2.26}$$

Set $l=\lim_{n\to\infty}\int_{\mathbb{R}^3}|\nabla v_n|^2\,dx$. If $l>0$, then by (2.26), we have

$$\begin{aligned}
\int_{\mathbb{R}^3}a|\nabla v_n|^2\,dx+b\left(\int_{\mathbb{R}^3}|\nabla v_n|^2\,dx\right)^2 &\leq \int_{\mathbb{R}^3}|v_n|^6\,dx+o(1)\\
&\leq S^{-3}\left(\int_{\mathbb{R}^3}|\nabla v_n|^2\,dx\right)^3+o(1),
\end{aligned}$$

which implies that $l\geq\big(bS^3+\sqrt{(bS^3)^2+4aS^3}\big)/2$. Combining this and (2.25), (2.26), we deduce that

$$\begin{aligned}
c+o(1)&\geq\frac{1}{3}\|v_n\|^2_{V_\infty}+\frac{b}{12}\left(\int_{\mathbb{R}^3}|\nabla v_n|^2\,dx\right)^2\\
&\geq\frac{a}{3}\int_{\mathbb{R}^3}|\nabla v_n|^2\,dx+\frac{b}{12}\left(\int_{\mathbb{R}^3}|\nabla v_n|^2\,dx\right)^2\\
&\geq\frac{a}{3}\left\{\frac{bS^3+\sqrt{(bS^3)^2+4aS^3}}{2}\right\}+\frac{b}{12}\left\{\frac{bS^3+\sqrt{(bS^3)^2+4aS^3}}{2}\right\}^2+o(1)\\
&=c^*+o(1),
\end{aligned}$$

which contradicts $c<m_\infty<c^*$ (by Lemma 2.1). Thus $l=0$, which, jointly with (2.26), shows that $v_n\to 0$ in E. Therefore $u_n\to u$ in E. The proof is complete. □

3. Proof of Theorem 1.1

Proof of Theorem 1.1. The proof will be divided into three steps.

Step 1*: I possesses a mountain pass geometry.* For

$$\varepsilon \in \left(0, \frac{1}{4}\left(\frac{1}{a} + \frac{1}{aS}\left|\int_{|x|\le R_0} 1\,dx\right|^{2/3} + \frac{2}{V_\infty}\right)^{-1}\right),$$

by (2.4) and (2.1), we have

$$\begin{aligned} I(u) &\ge \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^3} \left(\varepsilon|u|^2 + C_\varepsilon|u|^6\right) dx - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6\,dx \\ &\ge \frac{1}{4}\|u\|^2 - \left(C_\varepsilon + \frac{1}{6}\right)(aM)^{-3}\|u\|^6. \end{aligned}$$

Thus there exist $\alpha, \rho > 0$ such that $I|_{\|u\|=\rho} \ge \alpha$. Choose $v \in E\setminus\{0\}$ with $v \ge 0$. By (a_1), we achieve that

$$I(tv) \le \frac{t^2}{2}\|v\|^2 + \frac{bt^4}{4}\left(\int_{\mathbb{R}^3} |\nabla v|^2\,dx\right)^2 - \frac{t^6}{6}\|v\|_6^6 \to -\infty$$

as $t \to +\infty$. So there exists $t_0 > 0$ large enough such that $I(t_0 v) < 0$ and $\|t_0 v\| \ge \rho$. By virtue of the mountain pass theorem (see [2]), there exists a sequence $(u_n) \subset E$ such that $I(u_n) \to c > 0$ and $I'(u_n) \to 0$ as $n \to \infty$, where

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t))$$

with $\Gamma = \{\gamma \in C([0,1],E) : \gamma(0) = 0, I(\gamma(1)) < 0\}$.

Step 2*: $c < m_\infty$.* Let $R > 0$ and $\beta = (1,0,0)$. It follows from the definition of c that $c \le \sup_{t\ge0} I(tu_\infty(x - R\beta))$. We claim that for $R > 0$ large enough,

$$\sup_{t\ge0} I(tu_\infty(x - R\beta)) < m_\infty. \tag{3.1}$$

It follows from (2.1) and (a_1) that

$$\begin{aligned} I(tu_\infty(x - R\beta)) \le \frac{t^2}{2}&\int_{\mathbb{R}^3} \left(a|\nabla u_\infty|^2 + V_M|u_\infty|^2\right) dx \\ &+ \frac{bt^4}{4}\left(\int_{\mathbb{R}^3} |\nabla u_\infty|^2\,dx\right)^2 - \frac{t^6}{6}\int_{\mathbb{R}^3} |u_\infty|^6\,dx. \end{aligned}$$

Thus there exist $t_3 > 0$ small and $t_4 > 0$ large such that

$$\sup_{t\in[0,t_3]\cup[t_4,+\infty]} I(tu_\infty(x - R\beta)) < m_\infty. \tag{3.2}$$

Observe that

$$I(tu_\infty) = I^\infty(tu_\infty) + \frac{t^2}{2}\int_{\mathbb{R}^3}(V(x)-V_\infty)|u_\infty|^2\,dx - \int_{\mathbb{R}^3}(a(x)-a_\infty)F(tu_\infty)\,dx. \tag{3.3}$$

Next we estimate the terms on the right-hand side. Choosing $\delta \in \big(0, 1-b/(2\sqrt{V_\infty})\big)$, it follows from (2.3) and assumption (V_2) that

$$\begin{aligned}
&\int_{|x|\geq R_1}(V(x)-V_\infty)|u_\infty(x-R\beta)|^2\,dx\\
&\leq \int_{|x|\geq R_1} C_1C_\delta^2 e^{-b|x+R\beta|}|u_\infty(x)|^2\,dx \leq \int_{\mathbb{R}^3} C_1C_\delta^2 e^{-b|x+R\beta|}e^{-2(1-\delta)\sqrt{V_\infty}|x|}\,dx\\
&\leq C_1C_\delta^2 e^{-bR}\int_{\mathbb{R}^3} e^{(b-2(1-\delta)\sqrt{V_\infty})|x|}\,dx \leq C_4e^{-bR}.
\end{aligned}\tag{3.4}$$

Furthermore, by (2.1) and (2.3), we have

$$\begin{aligned}
&\int_{|x|\leq R_1}(V(x)-V_\infty)|u_\infty(x-R\beta)|^2\,dx\\
&\leq 2MC_\delta^2\int_{|x|\leq R_1} e^{-2(1-\delta)\sqrt{V_\infty}|x-R\beta|}\,dx\\
&\leq 2MC_\delta^2 e^{-2(1-\delta)\sqrt{V_\infty}R}\int_{|x|\leq R_1} e^{2(1-\delta)\sqrt{V_\infty}|x|}\,dx \leq C_5e^{-2(1-\delta)\sqrt{V_\infty}R}.
\end{aligned}\tag{3.5}$$

Choose $\widetilde{R}>0$ such that $\int_{|x|\leq\widetilde{R}}|u_\infty|^p\,dx>0$. Set $R>R_2+\widetilde{R}$. By (a_2) and (f_2), we obtain

$$\begin{aligned}
&\int_{|x|\geq R_2}(a(x)-a_\infty)F(tu_\infty(x-R\beta))\,dx\\
&\geq \frac{C_2C_3t_3^p}{p}\int_{|x|\geq R_2} e^{-d|x|}|u_\infty(x-R\beta)|^p\,dx \geq C\int_{|x-R\beta|\leq\widetilde{R}} e^{-d|x|}|u_\infty(x-R\beta)|^p\,dx\\
&= C\int_{|x|\leq\widetilde{R}} e^{-d|x+R\beta|}|u_\infty(x)|^p\,dx \geq C_6e^{-dR}
\end{aligned}\tag{3.6}$$

for $t\in[t_3,t_4]$. Noting $|F(u)|\leq C(|u|^2+|u|^6)$ for all $u\in\mathbb{R}$, we derive that for $t\in[t_3,t_4]$,

$$\begin{aligned}
&\left|\int_{|x|\leq R_2}(a(x)-a_\infty)F(tu_\infty(x-R\beta))\,dx\right|\\
&\leq CC_\delta^2t_4^2\int_{|x|\leq R_2} e^{-2(1-\delta)\sqrt{V_\infty}|x-R\beta|}\,dx + CC_\delta^6t_4^6\int_{|x|\leq R_2} e^{-6(1-\delta)\sqrt{V_\infty}|x-R\beta|}\,dx\\
&\leq CC_\delta^2e^{-2(1-\delta)\sqrt{V_\infty}R}\int_{|x|\leq R_2} e^{2(1-\delta)\sqrt{V_\infty}|x|}\,dx + CC_\delta^6e^{-6(1-\delta)\sqrt{V_\infty}R}\int_{|x|\leq R_2} e^{6(1-\delta)\sqrt{V_\infty}|x|}\,dx\\
&\leq C_7e^{-2(1-\delta)\sqrt{V_\infty}R}.
\end{aligned}\tag{3.7}$$

For $t > 0$, set

$$\begin{aligned} g(t) &= I^{\infty}(tu_{\infty}) \\ &= \frac{t^2}{2}\int_{\mathbb{R}^3}\left(a|\nabla u_{\infty}|^2 + V_{\infty}u_{\infty}^2\right)dx + \frac{bt^4}{4}\left(\int_{\mathbb{R}^3}|\nabla u_{\infty}|^2\,dx\right)^2 \\ &\quad - \int_{\mathbb{R}^3}\left(a_{\infty}F(tu_{\infty}) + \frac{t^6}{6}u_{\infty}^6\right)dx. \end{aligned}$$

Since

$$\begin{aligned} g'(t) &= t\left[\int_{\mathbb{R}^3}\left(a|\nabla u_{\infty}|^2 + V_{\infty}u_{\infty}^2\right)dx + t^2 b\left(\int_{\mathbb{R}^3}|\nabla u_{\infty}|^2\,dx\right)^2\right] \\ &\quad - t^3\left[\int_{\mathbb{R}^3} a_{\infty}\frac{f(tu_{\infty})}{(tu_{\infty})^3}u_{\infty}^4\,dx + t^2\int_{\mathbb{R}^3}|u_{\infty}|^6\,dx\right], \end{aligned}$$

and $f(tu_{\infty})/(tu_{\infty})^3$ is increasing for $t > 0$, we deduce that $g(t)$ admits a unique critical point corresponding to its maximum. Noticing $g'(1) = \langle I^{\infty\prime}(u_{\infty}), u_{\infty}\rangle = 0$, one has

$$\sup_{t\geq 0} g(t) = g(1) = I^{\infty}(u_{\infty}) = m_{\infty}. \tag{3.8}$$

Hence, combining (3.3)–(3.8), we obtain that

$$\begin{aligned} \sup_{t\in[t_3,t_4]} I(tu_{\infty}(x - R\beta)) &\leq \sup_{t\geq 0} I^{\infty}(tu_{\infty}(x - R\beta)) + \frac{t_4^2}{2}\left(C_4 e^{-bR} + C_5 e^{-2(1-\delta)\sqrt{V_{\infty}}R}\right) \\ &\quad - C_6 e^{-dR} - C_7 e^{-2(1-\delta)\sqrt{V_{\infty}}R} \\ &\leq m_{\infty} + \widetilde{C}_4 e^{-bR} + \widetilde{C}_5 e^{-2(1-\delta)\sqrt{V_{\infty}}R} - C_6 e^{-dR}. \end{aligned}$$

Since $0 < d < b < 2\sqrt{V_{\infty}}$, we can find $R_3 > R_2 + \widetilde{R}$ sufficiently large such that

$$\sup_{t\in[t_3,t_4]} I(tu_{\infty}(x - R_3\beta)) < m_{\infty},$$

which, jointly with (3.2), shows that (3.1) holds. Therefore, $c \leq \sup_{t\geq 0} I(tu_{\infty}(x - R\beta)) < m_{\infty}$. In view of Lemma 2.2, $u_n \to u$ in E, $I(u) = c$ and $I'(u) = 0$.

Step 3. Let

$$m = \inf\{I(v) : v \in M\}$$

with $M = \{v \in E \setminus \{0\} : I'(v) = 0\}$. From Step 2, one has $u \in M$ and $m \leq I(u) < m_{\infty}$. For every $v \in M$ and $\varepsilon \in \left(0, \frac{1}{4}\left(\frac{1}{a} + \frac{1}{aS}\left|\int_{|x|\leq R_0} 1\,dx\right|^{2/3} + \frac{2}{V_{\infty}}\right)^{-1}\right)$, we have

$$\begin{aligned} \|v\|^2 &\leq \int_{\mathbb{R}^3}(a(x)f(v)v + v^6)\,dx \leq \varepsilon\int_{\mathbb{R}^3} v^2\,dx + (C_{\varepsilon} + 1)\int_{\mathbb{R}^3}|v|^6\,dx \\ &\leq \frac{1}{4}\|v\|^2 + (C_{\varepsilon} + 1)(aS)^{-3}\|v\|^6 \end{aligned}$$

by (2.4) and (2.2), which implies that $\|v\| \geq C_0$ for some $C_0 > 0$. Then $m > 0$, i.e., $m \in (0, m_\infty)$. By the definition of m, there exist $(v_n) \in E \setminus \{0\}$ such that $I(v_n) \to m$ and $I'(v_n) = 0$. Applying Lemma 2.2, we deduce that $v_n \to v$ in E, $I(v) = m$ and $I'(v) = 0$. This completes the proof. □

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Yiwei Ye
School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, P. R. China
E-mail address: yeyiwei2011@126.com