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Proof of a Conjecture of Farkas and Kra

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Abstract. In this paper we prove a conjecture of Farkas and Kra, which is a modular equation involving a half sum of certain modular form of weight 1 for congruence subgroup $\Gamma_1(k)$ with any prime k. We prove that their conjecture holds for all odd integers $k \geq 3$. A new modular equation of Farkas and Kra type is also established.

1. Introduction and statement of results

In this paper, we let $z \in \mathbb{C}$, $\tau \in \mathbb{C}$ with $\Im(\tau) > 0$ and $q = e^{2\pi i\tau}$. The theta function with characteristic $\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right] \in \mathbb{R}^2$ is defined by

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(z,\tau) = \sum_{n\in\mathbb{Z}} \exp\left\{2\pi i\left(\frac{1}{2}\left(n+\frac{\epsilon}{2}\right)^2\tau + \left(n+\frac{\epsilon}{2}\right)\left(z+\frac{\epsilon'}{2}\right)\right)\right\}, \tag{1.1}$$

which is a generalization of the Jacobi theta functions. The theory of above theta function was systematically studied by Farkas and Kra [2], which plays an important role in combinatorial number theory, algebraic geometry and physics.

In [2, Chapter 4], Farkas and Kra treated the theta function (1.1) with $\epsilon, \epsilon' \in \mathbb{Q}$ and $z = 0$, that is, the theta constants with rational characteristics. Their derived many interesting results, one of them is the following (see [2, Theorem 9.8, p. 318] and [3]):

Theorem 1.1. *For each odd prime k and all $\tau \in \mathbb{C}$ with $\Im(\tau) > 0$,*

$$\frac{d}{d\tau}\log\left(\frac{\eta(k\tau)}{\eta(\tau)}\right) + \frac{1}{2\pi i(k-2)}\sum_{0\leq \ell\leq (k-3)/2}\left(\frac{\theta'\left[\begin{smallmatrix}1\\ (1+2\ell)/k\end{smallmatrix}\right](0,\tau)}{\theta\left[\begin{smallmatrix}1\\ (1+2\ell)/k\end{smallmatrix}\right](0,\tau)}\right)^2 \tag{1.2}$$

is a cusp 1-form (cusp form of weight 1) for the Hecke congruence subgroup $\Gamma_0(k)$. This form is identically zero provided $k \leq 19$. Here $\eta(\tau) = q^{1/24}\prod_{n\geq 1}(1-q^n)$ is the Dedekind eta function and

$$\theta'\begin{bmatrix}1\\ (1+2\ell)/k\end{bmatrix}(0,\tau) = \frac{\partial}{\partial z}\theta\begin{bmatrix}1\\ (1+2\ell)/k\end{bmatrix}(z,\tau)\bigg|_{z=0}.$$

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They then in [2, Conjecture 9.10, p. 320] (see also [3]) conjectured that (1.2) is identically zero for each odd prime k and all $\tau \in \mathbb{C}$ with $\Im(\tau) > 0$.

Remark 1.2. We remark that for odd integers k, ℓ with $k \geq 3$,

$$\left[\frac{\partial}{\partial z}\log\left(\theta\begin{bmatrix}1\\ \ell/k\end{bmatrix}(0,\tau)\right)\right]^2$$

is a modular 1-form (modular form of wight 1) for the group:

$$G(k) = \Gamma_1(k) := \left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix}a & b\\ c & d\end{pmatrix} \equiv \begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix} \pmod k\right\}.$$

This fact and more related results can be found in [2, 3].

The aim of this paper is to give a proof of the conjecture of Farkas and Kra of above. For the simplicity of the proof, we shall introduce the Jacobi theta function $\theta_2(z,q)$, which is defined by (see for example [4])

$$\theta_2(z,q) = \sum_{n\in\mathbb{Z}} q^{(2n+1)^2/8} e^{\mathrm{i}(2n+1)z}. \tag{1.3}$$

Hence it is clear that

$$\theta\begin{bmatrix}1\\ \epsilon'\end{bmatrix}(z,\tau) = \theta_2\left(\pi z + \frac{\epsilon'\pi}{2}, q\right)$$

and the conjecture of our concern is equivalent to the following.

Conjecture 1.3. *For each odd prime k and all $\tau \in \mathbb{C}$ with $\Im(\tau) > 0$,*

$$4(k-2)q\frac{d}{dq}\log\left(\frac{\eta(k\tau)}{\eta(\tau)}\right) - \sum_{\substack{0\leq \ell < k\\ \ell \equiv 1 \,(\mathrm{mod}\, 2)}} \left[\frac{\partial}{\partial z}\log\theta_2\left(\frac{\ell}{2k}\pi, q\right)\right]^2 = 0.$$

We shall prove a more general result than Conjecture 1.3. To state our main result, we define the following half sum

$$S_\delta(k) := \sum_{\substack{0\leq \ell < k\\ \ell - k \equiv \delta \,(\mathrm{mod}\, 2)}} \left[\frac{\partial}{\partial z}\log\theta_2\left(\frac{\ell}{2k}\pi, q\right)\right]^2 \tag{1.4}$$

for each integer $k \geq 2$ and each $\delta \in \{0, 1\}$. Our main result is the following modular equations.

Theorem 1.4. *For all $\tau \in \mathbb{C}$ with $\Im(\tau) > 0$, we have if $\delta = 0$ then*

$$S_\delta(k) = 4(k-2)q\frac{d}{dq}\log\left(\frac{\eta(k\tau)}{\eta(\tau)}\right),$$

and if $\delta = 1$ then

$$S_\delta(k) = 4q\frac{d}{dq}\log\left(\frac{\eta(2k\tau)^{2k-2}}{\eta(\tau)^k\eta(k\tau)^{k-2}}\right).$$

We immediately obtain the proof of Conjecture 1.3 by setting $k \in 2\mathbb{Z}_+ + 1$ and $\delta = 0$ in Theorem 1.4.

Corollary 1.5. *Conjecture* 1.3 *holds for all odd integers* $k \geq 3$. *In particular, Conjecture* 1.3 *is true.*

We shall give some consequences of Theorem 1.4. For this purpose we first use Lemma 2.2 below to deduce a proposition as follows.

Proposition 1.6. *We have*

$$S_\delta(k) = \sum_{\substack{0 \leq \ell < k \\ \ell - k \equiv \delta \pmod 2}} \left[\tan\left(\frac{\ell\pi}{2k}\right) - 4\sum_{h=1}^{2k}(-1)^h \sin\left(\frac{\ell h\pi}{k}\right) \sum_{n\geq 1} \frac{q^{hn}}{1-q^{2kn}} \right]^2.$$

By setting $q = 0$ in Theorem 1.4, applying Proposition 1.6 and (2.3) below we obtain the following trigonometric identity, which has been proved in [2, 3] by using the theory of modular form.

Corollary 1.7. *For each integer* $k \geq 2$,

$$\sum_{\substack{0 \leq \ell < k \\ \ell - k \equiv \delta \pmod 2}} \left[\tan\left(\frac{\ell\pi}{2k}\right) \right]^2 = \begin{cases} (k-1)(k-2)/6 & \textit{if } \delta = 0, \\ k(k-1)/6 & \textit{if } \delta = 1. \end{cases}$$

From Theorem 1.4, Proposition 1.6 and (2.3), by choosing different pair (k, δ) one can obtain many Lambert series identities. For example, if we set $(k, \delta) = (3, 1)$, then it is easy to see

Corollary 1.8. *We have*

$$\left(1 + 2\sum_{n\geq 1} \frac{q^n + q^{2n} - q^{4n} - q^{5n}}{1-q^{6n}}\right)^2 = 1 + 4\sum_{n\geq 1}\left(\frac{nq^n}{1-q^n} + \frac{nq^{3n}}{1-q^{3n}} - \frac{8nq^{6n}}{1-q^{6n}}\right).$$

Our proof of the main theorem is based on the series expansion (1.3) of $\theta_2(z, q)$ and the Jacobi triple product identity. The rest of this paper is organized as follows. In the next section, we first establish some primary results for $\theta_2(z, q)$. In Section 3, we prove Theorem 1.4.

2. Primaries

We shall need the following primary results, which will be used to prove main results of this paper.

Proposition 2.1. *We have*

$$\left(\frac{\partial}{\partial z}\log\theta_2(z,q)\right)^2=\mathrm{T}_{z,q}(\log\theta_2(z,q)),$$

where and throughout, $\mathrm{T}_{z,q}$ *is a linear operator defined as*

$$\mathrm{T}_{z,q}=-8q\frac{\partial}{\partial q}-\frac{\partial^2}{\partial z^2}.$$

Proof. By (1.3) it is clear that

$$\left(8q\frac{\partial}{\partial q}+\frac{\partial^2}{\partial z^2}\right)\theta_2(z,q)=0,$$

which means that

$$\frac{1}{\theta_2(z,q)}\frac{\partial^2}{\partial z^2}\theta_2(z,q)=-8q\frac{\partial}{\partial q}\log\theta_2(z,q).$$

Then from the basic fact that

$$\frac{\partial^2}{\partial z^2}\log\theta_2(z,q)=\frac{1}{\theta_2(z,q)}\frac{\partial^2}{\partial z^2}\theta_2(z,q)-\left(\frac{\partial}{\partial z}\log\theta_2(z,q)\right)^2$$

we complete the proof of the proposition. □

We need the Jacobi triple product identity for $\theta_2(z,q)$ (see for example [1,4]),

$$\theta_2(z,q)=q^{1/8}e^{-\mathrm{i}z}\prod_{n\geq 1}(1-q^n)(1+e^{-2\mathrm{i}z}q^n)(1+e^{2\mathrm{i}z}q^{n-1}). \tag{2.1}$$

Lemma 2.2. *For each* $\ell,k\in\mathbb{Z}$ *with* $\ell\neq k$ *and* $k>0$*,*

$$-\frac{\partial}{\partial z}\log\theta_2\left(\frac{\ell}{2k}\pi,q\right)=\tan\left(\frac{\ell\pi}{2k}\right)-4\sum_{h=1}^{2k}(-1)^h\sin\left(\frac{\ell h\pi}{k}\right)\sum_{n\geq 1}\frac{q^{hn}}{1-q^{2kn}}.$$

Proof. Taking the logarithmic derivative of $\theta_2(z,q)$ respect to z by (2.1), we have the well known Fourier expansion

$$\frac{\partial}{\partial z}\log\theta_2(z,q)=-\tan(z)+4\sum_{n\geq 1}\frac{(-1)^nq^n}{1-q^n}\sin(2nz). \tag{2.2}$$

Noticing that

$$\begin{aligned}\sum_{n\geq 1}\frac{(-1)^nq^n}{1-q^n}\sin\left(2n\frac{\ell\pi}{2k}\right)&=\sum_{h=1}^{2k}\sum_{n\geq 0}\frac{(-1)^hq^{2nk+h}}{1-q^{2nk+h}}\sin\left(\frac{\ell h\pi}{k}\right)\\&=\sum_{h=1}^{2k}(-1)^h\sin\left(\frac{\ell h\pi}{k}\right)\sum_{n\geq 0}\sum_{\ell\geq 1}q^{(2nk+h)\ell}\end{aligned}$$

and (2.2) we immediately obtain that

$$\frac{\partial}{\partial z}\log\theta_2\left(\frac{\ell}{2k}\pi,q\right)=-\tan\left(\frac{\ell\pi}{2k}\right)+4\sum_{h=1}^{2k}(-1)^h\sin\left(\frac{\ell h\pi}{k}\right)\sum_{n\geq 1}\frac{q^{hn}}{1-q^{2kn}}.$$

This completes the proof of the lemma. □

The following lemma will be used to prove Theorem 1.4 in the next section.

Lemma 2.3. *For each $k\in\mathbb{Z}_+$,*

$$\mathrm{T}_{z,q}(\log\theta_2(kz,q^k))\big|_{z=0}=8(k-1)q\frac{d}{dq}\log\left(\frac{\eta(2k\tau)^2}{\eta(k\tau)}\right)$$

and

$$\mathrm{T}_{z,q}\left(\log\left(\frac{\theta_2(kz-\pi/2,q^k)}{\theta_2(z-\pi/2,q)}\right)\right)\bigg|_{z=0}=8q\frac{d}{dq}\log(\eta(k\tau)^{k-3}\eta(\tau)^2).$$

Proof. By (2.2) we have

$$\frac{\partial^2}{\partial z^2}\log\theta_2(z,q)=-\tan^2(z)-1+8\sum_{n\geq 1}\frac{(-1)^n nq^n}{1-q^n}\cos(2nz)$$

and

$$\frac{\partial^2}{\partial z^2}\log\theta_2(z-\pi/2,q)=-\cot^2(z)-1+8\sum_{n\geq 1}\frac{nq^n}{1-q^n}\cos(2nz).$$

Hence we obtain that

$$\begin{aligned}\frac{\partial^2}{\partial z^2}\log\theta_2(z,q)\bigg|_{z=0}&=-1+8\sum_{n\geq 1}\frac{(-1)^n nq^n}{1-q^n}\\&=-1+16\sum_{n\geq 1}\frac{2nq^{2n}}{1-q^{2n}}-8\sum_{n\geq 1}\frac{nq^n}{1-q^n}\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial^2}{\partial z^2}\log\left(\frac{\theta_2(kz-\pi/2,q^k)}{\theta_2(z-\pi/2,q)}\right)\bigg|_{z=0}\\&=\lim_{z\to 0}\left(\cot^2(z)+1-k^2(\cot^2(kz)+1)\right)+8\sum_{n\geq 1}\left(\frac{k^2nq^{kn}}{1-q^{kn}}-\frac{nq^n}{1-q^n}\right)\\&=\frac{1-k^2}{3}+8k^2\sum_{n\geq 1}\frac{nq^{kn}}{1-q^{kn}}-8\sum_{n\geq 1}\frac{nq^n}{1-q^n}.\end{aligned}$$

Using the fact that

$$q\frac{d}{dq}\log\eta(\alpha\tau)=\frac{\alpha}{24}-\sum_{n\geq 1}\frac{\alpha nq^{\alpha n}}{1-q^{\alpha n}},\quad \alpha\in\mathbb{R}_+, \tag{2.3}$$

and the above we obtain

$$\frac{\partial^2}{\partial z^2}\log\theta_2(z,q)\Big|_{z=0}=8q\frac{d}{dq}\log\left(\frac{\eta(\tau)}{\eta(2\tau)^2}\right) \tag{2.4}$$

and

$$\frac{\partial^2}{\partial z^2}\log\left(\frac{\theta_2(kz-\pi/2,q^k)}{\theta_2(z-\pi/2,q)}\right)\Big|_{z=0}=8q\frac{d}{dq}\log\left(\frac{\eta(\tau)}{\eta(k\tau)^k}\right). \tag{2.5}$$

Moreover, by (2.1) and the definition of $\eta(\tau)$, it is easy to see that

$$\theta_2(0,q)=2\frac{\eta(2\tau)^2}{\eta(\tau)} \tag{2.6}$$

and

$$\lim_{z\to 0}\frac{\theta_2(z-\pi/2,q)}{z}=2\eta(\tau)^3. \tag{2.7}$$

Thus for integer $k\geq 1$, application of (2.4) and (2.6) imply that

$$\begin{aligned}
\mathrm{T}_{z,q}(\log\theta_2(kz,q^k))\big|_{z=0}&=-8q\frac{d}{dq}\log\theta_2(0,q^k)-\frac{\partial^2}{\partial z^2}\log\theta_2(kz,q^k)\Big|_{z=0}\\
&=-8q\frac{d}{dq}\log\left(\frac{\eta(2k\tau)^2}{\eta(k\tau)}\right)+k^2\left(-8q^k\frac{d}{dq^k}\log\left(\frac{\eta(k\tau)}{\eta(2k\tau)^2}\right)\right)\\
&=8(k-1)q\frac{d}{dq}\log\left(\frac{\eta(2k\tau)^2}{\eta(k\tau)}\right),
\end{aligned}$$

and application of (2.5) and (2.7) imply that

$$\begin{aligned}
&\mathrm{T}_{z,q}\left(\log\left(\frac{\theta_2(kz-\pi/2,q^k)}{\theta_2(z-\pi/2,q)}\right)\right)\Big|_{z=0}\\
&=-8q\frac{d}{dq}\log\left(\frac{\eta(k\tau)^3}{\eta(\tau)^3}\right)-\frac{\partial^2}{\partial z^2}\log\left(\frac{\theta_2(kz-\pi/2,q^k)}{\theta_2(z-\pi/2,q)}\right)\Big|_{z=0}\\
&=-24q\frac{d}{dq}\log\left(\frac{\eta(k\tau)}{\eta(\tau)}\right)-8q\frac{d}{dq}\log\left(\frac{\eta(\tau)}{\eta(k\tau)^k}\right)\\
&=8q\frac{d}{dq}\log(\eta(k\tau)^{k-3}\eta(\tau)^2),
\end{aligned}$$

which completes the proof of the lemma. □

We need the following half product formula for Jacobi theta function θ_2, which will be used to prove Theorem 1.4 in the next section.

Lemma 2.4. *For integer $k\geq 1$ and $\delta\in\{0,1\}$,*

$$\prod_{\substack{0\leq\ell<2k\\ \ell-k\equiv\delta\,(\mathrm{mod}\,2)}}\theta_2\left(z+\frac{\ell}{2k}\pi,q\right)=C_{k,\delta}\frac{\eta(\tau)^k}{\eta(k\tau)}\theta_2\left(kz+\frac{(\delta-1)\pi}{2},q^k\right),$$

where $C_{k,\delta}=e^{\frac{i\pi}{2}(\delta-k+\mathbf{1}_{k\not\equiv\delta\,(\mathrm{mod}\,2)})}$. Here and throughout, $\mathbf{1}_{condition}=1$ if the 'condition' is true, and equals to 0 if the 'condition' is false.

Proof. From (2.1) we have

$$\prod_{\substack{0\le \ell<2k\\ \ell-k\equiv\delta \,(\mathrm{mod}\,2)}} \theta_2\left(z+\frac{\ell\pi}{2k},q\right) = \prod_{\substack{0\le \ell<2k\\ \ell-k\equiv\delta \,(\mathrm{mod}\,2)}} \left(q^{1/8}e^{-\mathrm{i}(z+\frac{\ell}{2k}\pi)}\prod_{n\ge1}(1-q^n)\right)$$
$$\times\prod_{n\ge1}\prod_{\substack{0\le \ell<2k\\ \ell-k\equiv\delta \,(\mathrm{mod}\,2)}} \left(1+q^n e^{-2\mathrm{i}z-\frac{\ell\pi\mathrm{i}}{k}}\right)\left(1+q^{n-1}e^{2\mathrm{i}z+\frac{\ell\pi\mathrm{i}}{k}}\right).$$

It is easy to check that

$$\prod_{\substack{0\le \ell<2k\\ \ell-k\equiv\delta \,(\mathrm{mod}\,2)}} (1+xe^{\pm\ell\pi\mathrm{i}/k}) = 1-e^{\delta\pi\mathrm{i}}x^k \quad\text{and}\quad \sum_{\substack{0\le \ell<2k\\ \ell-k\equiv\delta \,(\mathrm{mod}\,2)}} \ell = k(k-1)+k\mathbf{1}_{k\not\equiv\delta\,(\mathrm{mod}\,2)}.$$

Thus we obtain that

$$\begin{aligned}\prod_{\substack{0\le \ell<2k\\ \ell-k\equiv\delta \,(\mathrm{mod}\,2)}} \theta_2\left(z+\frac{\ell}{2k}\pi,q\right) &= q^{k/12}\eta(\tau)^k e^{-\mathrm{i}kz}e^{-\frac{\mathrm{i}\pi}{2}(k-1+\mathbf{1}_{k\not\equiv\delta\,(\mathrm{mod}\,2)})}\\ &\quad\times\prod_{n\ge1}\left(1-e^{-2\mathrm{i}kz-\delta\pi\mathrm{i}}q^{kn}\right)\left(1-e^{2\mathrm{i}kz+\delta\pi\mathrm{i}}q^{k(n-1)}\right)\\ &= C_{k,\delta}\theta_2(kz+(\delta-1)\pi/2,q^k)\frac{\eta(\tau)^k}{\eta(k\tau)}\end{aligned}$$

with

$$C_{k,\delta} = e^{\frac{\mathrm{i}\pi(\delta-1)}{2}-\frac{\mathrm{i}\pi(k-1+\mathbf{1}_{k\not\equiv\delta\,(\mathrm{mod}\,2)})}{2}} = e^{\frac{\mathrm{i}\pi}{2}(\delta-k+\mathbf{1}_{k\not\equiv\delta\,(\mathrm{mod}\,2)})},$$

which completes the proof of the lemma. □

3. Proof of Theorem 1.4

First of all, we define

$$G_{\delta,k}(z,q) := \sum_{\substack{0\le \ell<k\\ \ell-k\equiv\delta \,(\mathrm{mod}\,2)}} \left[\frac{\partial}{\partial z}\log\theta_2\left(z+\frac{\ell}{2k}\pi,q\right)\right]^2,$$

then from (1.4) we have $S_\delta(k)=G_{\delta,k}(0,q)$. By Proposition 2.1 we get

$$\begin{aligned}G_{\delta,k}(z,q) &= \sum_{\substack{0\le \ell<k\\ \ell-k\equiv\delta \,(\mathrm{mod}\,2)}} \mathrm{T}_{z,q}\left(\log\theta_2\left(z+\frac{\ell}{2k}\pi,q\right)\right)\\ &= \mathrm{T}_{z,q}\left(\sum_{\substack{0\le \ell<k\\ \ell-k\equiv\delta \,(\mathrm{mod}\,2)}} \log\theta_2\left(z+\frac{\ell}{2k}\pi,q\right)\right).\end{aligned}$$

We shall define the auxiliary function as follows:

$$A_{\delta,k}(z,q)=\mathrm{T}_{z,q}\left(\sum_{\substack{0\le\ell\le 2k\\ \ell-k\equiv\delta\ (\mathrm{mod}\ 2)}}\log\theta_2\left(z+\frac{\ell\pi}{2k},q\right)-\mathbf{1}_{\delta=0}\log\theta_2\left(z+\frac{\pi}{2},q\right)\right).$$

It is clear that $A_{\delta,k}(0,q):=\lim_{z\to 0}A_{\delta,k}(z,q)$ exists. We claim that

$$G_{\delta,k}(0,q)=\frac{1}{2}A_{\delta,k}(0,q). \tag{3.1}$$

In fact, by $\theta_2(z+\pi,q)=-\theta_2(z,q)$ and $\theta_2(z,q)=\theta_2(-z,q)$ we have

$$\begin{aligned}A_{\delta,k}(z,q)&=\sum_{\substack{0\le\ell<k\\ \ell-k\equiv\delta\ (\mathrm{mod}\ 2)}}\mathrm{T}_{z,q}\left(\log\theta_2\left(z+\frac{\ell\pi}{2k},q\right)+\log\theta_2\left(z+\frac{(2k-\ell)\pi}{2k},q\right)\right)\\&=\sum_{\substack{0\le\ell<k\\ \ell-k\equiv\delta\ (\mathrm{mod}\ 2)}}\mathrm{T}_{z,q}\left(\log\theta_2\left(z+\frac{\ell}{2k}\pi,q\right)+\log\left(-\theta_2\left(-z+\frac{\ell}{2k}\pi,q\right)\right)\right).\end{aligned}$$

Further, by the definition of $\mathrm{T}_{z,q}$, we see that

$$\begin{aligned}&\mathrm{T}_{z,q}\left(\log\left(-\theta_2\left(-z+\frac{\ell}{2k}\pi,q\right)\right)\right)\Big|_{z=0}\\&=-8q\frac{\partial}{\partial q}\left(\log\theta_2\left(\frac{\ell}{2k}\pi,q\right)\right)-\frac{\partial^2}{\partial z^2}\left(\log\theta_2\left(-z+\frac{\ell}{2k}\pi,q\right)\right)\Big|_{z=0}\\&=-8q\frac{\partial}{\partial q}\left(\log\theta_2\left(\frac{\ell}{2k}\pi,q\right)\right)-\frac{\partial^2}{\partial z^2}\left(\log\theta_2\left(z+\frac{\ell}{2k}\pi,q\right)\right)\Big|_{z=0}\\&=\mathrm{T}_{z,q}\left(\log\theta_2\left(z+\frac{\ell}{2k}\pi,q\right)\right)\Big|_{z=0},\end{aligned}$$

which immediately obtain the proof of (3.1).

On the other hand, from Lemma 2.4 we find that

$$\begin{aligned}\sum_{\substack{0\le\ell<2k\\ \ell-k\equiv\delta\ (\mathrm{mod}\ 2)}}\log\theta_2\left(z+\frac{\ell\pi}{2k},q\right)&=\log\prod_{\substack{0\le\ell<2k\\ \ell-k\equiv\delta\ (\mathrm{mod}\ 2)}}\theta_2\left(z+\frac{\ell}{2k}\pi,q\right)\\&=\log\left(C_{k,\delta}\frac{\eta(\tau)^k}{\eta(k\tau)}\theta_2\left(kz+\frac{(\delta-1)\pi}{2},q^k\right)\right),\end{aligned}$$

which implies that

$$\begin{aligned}A_{\delta,k}(z,q)=\mathrm{T}_{z,q}&\left(\log\theta_2\left(kz+\frac{(\delta-1)\pi}{2},q^k\right)+\log\left(C_{k,\delta}\frac{\eta(\tau)^k}{\eta(k\tau)}\right)\right)\\&+\mathrm{T}_{z,q}(\mathbf{1}_{k\equiv\delta\ (\mathrm{mod}\ 2)}\log\theta_2(z,q))-\mathrm{T}_{z,q}\left(\mathbf{1}_{\delta=0}\log\theta_2\left(z+\frac{\pi}{2},q\right)\right).\end{aligned}$$

Further by Lemma 2.3 we obtain that

$$\begin{aligned}
A_{0,k}(0,q) &= \mathrm{T}_{z,q}\left(\log\left(\frac{\theta_2(kz-\pi/2,q^k)}{\theta_2(z-\pi/2,q)}\right)\right)\Big|_{z=0} \\
&\quad - 8q\frac{\partial}{\partial q}\log\left(\frac{\eta(\tau)^k}{\eta(k\tau)}\right) + \mathbf{1}_{k\equiv 0\,(\mathrm{mod}\,2)}\,\mathrm{T}_{z,q}(\log\theta_2(z,q))\big|_{z=0} \\
&= 8q\frac{d}{dq}\log(\eta(k\tau)^{k-3}\eta(\tau)^2) - 8q\frac{d}{dq}\log\left(\frac{\eta(\tau)^k}{\eta(k\tau)}\right) \\
&= 8(k-2)q\frac{d}{dq}\log\left(\frac{\eta(k\tau)}{\eta(\tau)}\right)
\end{aligned}$$

and

$$\begin{aligned}
A_{1,k}(0,q) &= \mathrm{T}_{z,q}(\log\theta_2(kz,q^k))\big|_{z=0} \\
&\quad - 8q\frac{\partial}{\partial q}\log\left(\frac{\eta(\tau)^k}{\eta(k\tau)}\right) + \mathbf{1}_{k\equiv 1\,(\mathrm{mod}\,2)}\,\mathrm{T}_{z,q}(\log\theta_2(z,q))\big|_{z=0} \\
&= 8(k-1)q\frac{d}{dq}\log\left(\frac{\eta(2k\tau)^2}{\eta(k\tau)}\right) - 8q\frac{d}{dq}\log\left(\frac{\eta(\tau)^k}{\eta(k\tau)}\right) \\
&= 8q\frac{d}{dq}\log\left(\frac{\eta(2k\tau)^{2k-2}}{\eta(\tau)^k\eta(k\tau)^{k-2}}\right),
\end{aligned}$$

which completes the proof of Theorem 1.4 by noting that $S_\delta(k) = G_{\delta,k}(0,q) = \frac{1}{2}A_{\delta,k}(0,q)$.

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References

[1] G. E. Andrews, *A simple proof of Jacobi's triple product identity*, Proc. Amer. Math. Soc. **16** (1965), 333–334.

[2] H. M. Farkas and I. Kra, *Theta Constants, Riemann Surfaces and the Modular Group: An introduction with applications to uniformization theorems, partition identities and combinatorial number theory*, Graduate Studies in Mathematics **37**, American Mathematical Society, Providence, RI, 2001.

[3] ———, *On theta constant identities and the evaluation of trigonometric sums*, in: *Complex Manifolds and Hyperbolic Geometry* (Guanajuato, 2001), 115–131, Contemp. Math. **311**, Amer. Math. Soc., Providence, RI, 2002.

[4] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis: An introduction to the general theory of infinite processes and of analytic functions: with an account of the principal transcendental functions*, Fourth edition, Cambridge University Press, New York, 1962.

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