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A Numerical Method Based on the Jacobi Polynomials to Reconstruct an Unknown Source Term in a Time Fractional Diffusion-wave Equation

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Abstract. In this paper, we consider an inverse problem of identifying an unknown time dependent source function in a time-fractional diffusion-wave equation. First, some basic properties of the shifted Jacobi polynomials (SJPs) are presented. Then, the analytical solution of the direct problem is given and used to obtain an approximation of the unknown source function in a series of SJPs. Due to ill-posedness of this inverse problem, the Tikhonov regularization method with Morozov's discrepancy principle criterion is applied to find a stable solution. After that, an error bound is obtained for the approximation of the unknown source function. Finally, some numerical examples are provided to show effectiveness and robustness of the proposed algorithm.

1. Introduction

Partial differential equations of fractional order have been considered highly during the recent decades [1, 14, 17, 19, 32]. Nonlocality and memory effects are some of the main features of fractional derivatives. In other words, the next state of a fractional system depends on its current and all previous states. Hence, many scientific researchers have used these types of equations to present some mathematical models for phenomena in real world. These models are valuable in better understanding of the behavior of natural systems. One of these natural phenomena is anomalous diffusion in fractal media. Anomalous diffusion is a diffusion process in which the mean square displacement of diffusing particle of the form

$$\langle \Delta x^2 \rangle = 2Dt^{\alpha},$$

grows faster or slower than that in a Gaussian process [10,13,16]. In this relation α is the anomalous diffusion exponent. The process is named superdiffusion in the case $1 < \alpha < 2$ and subdiffusion in the case $0 < \alpha < 1$. High-frequency financial data [20], electrical conductance in the membranes of cells of biological organisms [4], fractional order model

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of HIV infection [18], optimal multiple control problems [7], hydrologic processes in earth system dynamics [31] are some other examples of these applications.

In this paper, we consider an inverse source problem which consists a time-fractional diffusion-wave equation as

$$\frac{\partial^{\mu} u(x,t)}{\partial t^{\mu}} - \frac{\partial^2 u(x,t)}{\partial x^2} = s(x,t), \quad (x,t) \in \Omega := [0,L] \times [0,\tau], \tag{1.1}$$

with the initial conditions

$$u(x,0) = f_0(x), \quad u_t(x,0) = f_1(x), \tag{1.2}$$

and the boundary conditions

$$u(0,t) = u(L,t) = 0, \tag{1.3}$$

in which $0 < \mu \leq 2$ (it should be noted that the second initial condition, $u_t(x,0) = f_1(x)$, is only for $1 < \mu \leq 2$), and $s(x,t)$ is the source term in a separable form as

$$s(x,t) = f(x)g(t) \tag{1.4}$$

with unknown factor $g(t)$. The forward problem (1.1)–(1.4) with the known function $s(x,t)$ has been investigated by many researchers in literature [2, 3, 6, 22, 25]. For solvability of the inverse problem an additional condition shall be considered as

$$u(x_0,t) = q(t), \tag{1.5}$$

where x_0 is an interior point of the interval $(0,L)$. The existence and uniqueness of solution for the inverse problem (1.1)–(1.5) have been investigated in [28]. The inverse time dependent source problems for time fractional diffusion equations, have been concerned by some authors. They used several approaches to solve these types of problems, such as the Fourier regularization method [30], the quasi-reversibility regularization method [29], boundary element method combined with the Tikhonov regularization [27], the conjugate gradient method [26], a Tikhonov regularization method based on the superposition principle and the technique of finite-element interpolation [21]. To our knowledge, in the field of inverse problems for time fractional diffusion-wave equations, very few works have been presented. In [9] and [11] authors considered some inverse source problems for time-fractional mixed parabolic-hyperbolic equations. Also in [15] authors investigated an inverse problem of determining diffusion coefficient in the diffusion-wave equation. In this work, we employ a spectral method with Jacobi polynomials as the basis functions. The main advantage of using Jacobi polynomials is to reduce the considered inverse problem to a system of linear algebraic equations which can be solved easily using the existing well-developed methods. It should be noted that the Jacobi polynomials include a variant class of orthogonal polynomials by considering different values for Jacobi parameters.

This paper is organized as follows: In Section 2, we give some preliminaries and basic properties of the SJPs. The analytical solution of the direct problem is presented in Section 3. In Section 4, an approximation of the unknown source function is given by using the SJPs. Section 5 is devoted to giving an error bound for the approximation of the unknown source function. In Section 6 numerical examples are provided. Finally, conclusion is given in Section 7.

2. Some useful preliminaries

In this section, we give some useful definitions and preliminaries which will be used further in this paper.

The one commonly used definition of fractional calculus is definition of the Caputo derivative.

Definition 2.1. The Caputo derivative of a function $f(t)$ is defined by

$$D_t^{\mu} f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\mu)} \displaystyle\int_0^t (t-s)^{n-\mu-1} \frac{d^n}{ds^n} f(s)\, ds, & \text{if } n-1 < \mu < n, \\ f^{(n)}(t), & \text{if } \mu = n, \end{cases}$$

where $\Gamma(\mu)$ is the gamma function defined as

$$\Gamma(\mu) = \int_0^{\infty} t^{\mu-1} e^{-t}\, dt.$$

Definition 2.2. The Jacobi polynomials are defined by

$$P_i^{(\alpha,\beta)}(x) = \frac{(-1)^i}{2^i i!} (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^n}{dx^n} \left[(1-x)^{\alpha} (1+x)^{\beta} (1-x^2)^i \right],$$

for parameters $\alpha, \beta > -1$ and $i \geq 0$.

These polynomials are solutions to the Jacobi differential equation as

$$(1-x^2)y'' + [\beta - \alpha - (\alpha+\beta+2)x]y' + i(i+\alpha+\beta+1)y = 0.$$

In some special cases, the Jacobi polynomials give some other so-called polynomials. In the case $\alpha = \beta = 0$, the Legendre polynomials are given. For $\alpha = \beta = -1/2$, the first kind Chebyshev polynomials are obtained and we get the second kind Chebyshev polynomials if $\alpha = \beta = 1/2$.

The set of Jacobi polynomials, $\{P_i^{(\alpha,\beta)}(x)\}_{i=0}^{\infty}$, represents an orthogonal basis for the Hilbert space $L^2[-1,1]$ with respect to the weight function

$$w^{(\alpha,\beta)}(x) = (1-x)^{\alpha} (1+x)^{\beta},$$

in such a way that

$$\int_{-1}^{1} w^{(\alpha,\beta)}(x)P_i^{(\alpha,\beta)}(x)P_j^{(\alpha,\beta)}(x)\,dx = \delta_{ij}\frac{2^{\alpha+\beta+1}\Gamma(i+\alpha+1)\Gamma(i+\beta+1)}{(2i+\alpha+\beta+1)\Gamma(i+\alpha+\beta+1)\Gamma(i+1)},$$

where δ_{ij} is the Kronecker delta.

The well-known SJPs on $[0,L]$ are defined by

$$P_{L,i}^{(\alpha,\beta)}(x) = P_i^{(\alpha,\beta)}\left(\frac{2}{L}x-1\right), \quad i=0,1,2,\ldots,$$

and have the following explicit analytic form [2]

$$P_{L,i}^{(\alpha,\beta)}(x) = \sum_{k=0}^{i}(-1)^{i-k}\frac{\Gamma(i+\beta+1)\Gamma(i+k+\alpha+\beta+1)}{\Gamma(k+\beta+1)\Gamma(i+\alpha+\beta+1)(i-k)!k!L^k}x^k. \tag{2.1}$$

Taking into consideration (2.1), it is turned out that for all acceptable values of α and β, we have the following properties:

$$P_{L,0}^{(\alpha,\beta)}(x)=1, \quad P_{L,i}^{(\alpha,\beta)}(0) = (-1)^i\frac{\Gamma(i+\beta+1)}{\Gamma(\beta+1)i!}, \quad P_{L,i}^{(\alpha,\beta)}(L) = \frac{\Gamma(i+\alpha+1)}{\Gamma(\alpha+1)i!}.$$

Also, the orthogonality property is satisfied for these polynomials as follows

$$\int_0^L w_L^{(\alpha,\beta)}(x)P_{L,i}^{(\alpha,\beta)}(x)P_{L,j}^{(\alpha,\beta)}(x)\,dx = \delta_{ij}h_{L,i}^{(\alpha,\beta)},$$

where

$$w_L^{(\alpha,\beta)}(x) = w^{(\alpha,\beta)}\left(\frac{2}{L}x-1\right) \quad \text{and} \quad h_{L,i}^{(\alpha,\beta)} = \frac{L2^{\alpha+\beta}\Gamma(i+\alpha+1)\Gamma(i+\beta+1)}{(2i+\alpha+\beta+1)\Gamma(i+\alpha+\beta+1)\Gamma(i+1)}.$$

A function $f(t)$ in $L^2[0,\tau]$ (the space of all square integrable functions with respect to the shifted Jacobi weight function $w_\tau^{(\alpha,\beta)}(t)$) may be approximated in terms of the SJPs as

$$f(t) \simeq \sum_{i=0}^{N} f_i P_{\tau,i}^{(\alpha,\beta)}(t) = F^T\phi_\tau(t), \tag{2.2}$$

where

$$f = [f_0, f_1, \ldots, f_N]^T \quad \text{and} \quad \phi_\tau(t) = \left[P_{\tau,0}^{(\alpha,\beta)}(t), P_{\tau,1}^{(\alpha,\beta)}(t), \ldots, P_{\tau,N}^{(\alpha,\beta)}(t)\right]^T.$$

The coefficients f_i in (2.2) are given by

$$f_i = \frac{1}{h_{\tau,i}^{(\alpha,\beta)}}\int_0^\tau w_\tau^{(\alpha,\beta)}(t)f(t)P_{\tau,i}^{(\alpha,\beta)}(t)\,dt, \quad i=0,1,2,\ldots.$$

Definition 2.3. Two parameter Mittag-Leffler function is defined as [19]

$$E_{a,b}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(b+ak)}.$$

Lemma 2.4. *For the derivative of the Mittag-Leffler function $E_{a,b}(z)$, it holds* [19]

$$D_t^{\mu}\left(t^{ak+b-1}E_{a,b}^{(k)}(\lambda t^a)\right) = t^{ak+b-\mu-1}E_{a,b-\mu}^{(k)}(\lambda t^a),$$

where μ is any arbitrary real number and $E_{a,b}^{(k)}(z) = \frac{d^k}{dz^k}E_{a,b}(z)$.

3. Analytical solution of the direct problem

Consider the problem (1.1)–(1.3) with the known source function $g(t)$. In this section, we represent the solution of this direct problem by using the method of separation of variables [5]. Suppose that the formal solution of (1.1)–(1.3) is of the form

$$u(x,t) = \sum_{n=1}^{\infty} G_n(t)\sin\left(\frac{n\pi}{L}x\right). \tag{3.1}$$

Also, let

$$f(x) = \sum_{n=1}^{\infty} f_n \sin\left(\frac{n\pi}{L}x\right),$$

be the Fourier sine series of $f(x)$, where

$$f_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\,dx.$$

By substituting (3.1) into (1.1), we get

$$\sum_{n=1}^{\infty}\left(\frac{\partial^{\mu}G_n(t)}{\partial t^{\mu}} + \left(\frac{n\pi}{L}\right)^2 G_n(t)\right)\sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{\infty} f_n g(t)\sin\left(\frac{n\pi}{L}x\right). \tag{3.2}$$

To get the unknown coefficients $G_n(t)$ in (3.2), we consider the following non-homogeneous equation

$$\frac{\partial^{\mu}G_n(t)}{\partial t^{\mu}} + \left(\frac{n\pi}{L}\right)^2 G_n(t) = f_n g(t), \quad n = 1, 2, \ldots. \tag{3.3}$$

The general solution of (3.3) is as

$$G_n(t) = a_n y0_n(t) + b_n y1_n(t) + y_n(t), \tag{3.4}$$

where

$$y_n(t) = f_n \int_0^t s^{\mu-1}E_{\mu,\mu}\left(-\left(\frac{n\pi}{L}\right)^2 s^{\mu}\right) g(t-s)\,ds, \tag{3.5}$$

and

$$y0_n(t) = 1 - \left(\frac{n\pi}{L}\right)^2 t^\mu E_{\mu,\mu+1}\left(-\left(\frac{n\pi}{L}\right)^2 t^\mu\right),$$
$$y1_n(t) = t - \left(\frac{n\pi}{L}\right)^2 t^{\mu+1} E_{\mu,\mu+2}\left(-\left(\frac{n\pi}{L}\right)^2 t^\mu\right).$$

Also, the coefficients a_n and b_n in (3.4) can be determined according to the initial conditions (1.2). Thus, we get

$$a_n = \frac{2}{L}\int_0^L f_0(x)\sin\left(\frac{n\pi}{L}x\right)dx, \quad b_n = \frac{2}{L}\int_0^L f_1(x)\sin\left(\frac{n\pi}{L}x\right)dx.$$

4. Determination of the unknown source function

Suppose that the unknown source function $g(t)$ in (1.4) has an approximation in terms of the SJPs as follows

$$g(t) \simeq g_N(t) = \sum_{j=0}^{N} c_j P_{\tau,j}^{(\alpha,\beta)}(t) = X^T \phi_\tau(t), \tag{4.1}$$

where the coefficients c_j, $j = 0, 1, 2, \ldots, N$ are unknown and

$$X = [c_0, c_1, \ldots, c_N]^T.$$

We substitute (4.1) into (3.5) and obtain

$$\begin{aligned} y_n(t) &\simeq f_n \int_0^t s^{\mu-1} E_{\mu,\mu}\left(-\left(\frac{n\pi}{L}\right)^2 s^\mu\right) \sum_{j=0}^{N} c_j P_{\tau,j}^{(\alpha,\beta)}(t-s)\,ds \\ &= \sum_{j=0}^{N} c_j f_n \int_0^t s^{\mu-1} E_{\mu,\mu}\left(-\left(\frac{n\pi}{L}\right)^2 s^\mu\right) P_{\tau,j}^{(\alpha,\beta)}(t-s)\,ds. \end{aligned}$$

Let us define

$$r_{n,j}(t,s) = s^{\mu-1} E_{\mu,\mu}\left(-\left(\frac{n\pi}{L}\right)^2 s^\mu\right) P_{\tau,j-1}^{(\alpha,\beta)}(t-s),$$

then we have

$$y_n(t) \simeq \sum_{j=1}^{N+1} c_{j-1} f_n \int_0^t r_{n,j}(t,s)\,ds. \tag{4.2}$$

Using the additional condition (1.5) and equations (3.1) and (3.4), we get

$$\sum_{n=1}^{\infty} [a_n y0_n(t) + b_n y1_n(t) + y_n(t)] \sin\left(\frac{n\pi}{L}x_0\right) = q(t). \tag{4.3}$$

Substituting (4.2) into (4.3) yields

$$\begin{aligned}&\sum_{n=1}^{\infty}\left(\sum_{j=1}^{N+1} c_{j-1} f_n \int_0^t r_{n,j}(t,s)\,ds\right)\sin\left(\frac{n\pi}{L}x_0\right)\\ &\simeq q(t)-\sum_{n=1}^{\infty}[a_n y0_n(t)+b_n y1_n(t)]\sin\left(\frac{n\pi}{L}x_0\right).\end{aligned} \tag{4.4}$$

By collocating the equation (4.4) at $N+1$ points $t=t_i$ and replacing the infinite upper bound of the first summation with a finite positive integer number $k>1$ we obtain

$$\begin{aligned}&\sum_{n=1}^{k}\sum_{j=1}^{N+1} c_{j-1} f_n \int_0^{t_i} r_{n,j}(t_i,s)\,ds\ \sin\left(\frac{n\pi}{L}x_0\right)\\ &\simeq q(t_i)-\sum_{n=1}^{k}[a_n y0_n(t_i)+b_n y1_n(t_i)]\sin\left(\frac{n\pi}{L}x_0\right),\end{aligned} \tag{4.5}$$

where

$$t_i=\frac{i}{N+2}\tau,\quad i=1,2,\ldots,N+1.$$

Gauss-Legendre integration formula is used in order to compute the integral part of the equation (4.5), so we have

$$\begin{aligned}&\sum_{j=1}^{N+1} c_{j-1}\sum_{n=1}^{k} f_n\frac{t_i}{2}\sum_{l=1}^{m} w_l r_{n,j}\left(t_i,\frac{t_i}{2}(1+s_l)\right)\sin\left(\frac{n\pi}{L}x_0\right)\\ &= q(t_i)-\sum_{n=1}^{k}[a_n y0_n(t_i)+b_n y1_n(t_i)]\sin\left(\frac{n\pi}{L}x_0\right),\quad i=1,2,\ldots,N+1,\end{aligned} \tag{4.6}$$

where s_l are zeros of the Legendre polynomial of degree m and w_l are the corresponding weights. Finally, by considering

$$B=\begin{bmatrix} q(t_1)-\sum_{n=1}^{k}[a_n y0_n(t_1)+b_n y1_n(t_1)]\sin\left(\frac{n\pi}{L}x_0\right)\\ q(t_2)-\sum_{n=1}^{k}[a_n y0_n(t_2)+b_n y1_n(t_2)]\sin\left(\frac{n\pi}{L}x_0\right)\\ \vdots\\ q(t_{N+1})-\sum_{n=1}^{k}[a_n y0_n(t_{N+1})+b_n y1_n(t_{N+1})]\sin\left(\frac{n\pi}{L}x_0\right)\end{bmatrix},$$

and

$$A=[a_{ij}]_{(N+1)\times(N+1)}$$

with

$$a_{ij}=\sum_{n=1}^{k} f_n\frac{t_i}{2}\sum_{l=1}^{m} w_l r_{n,j}\left(t_i,\frac{t_i}{2}(1+s_l)\right)\sin\left(\frac{n\pi}{L}x_0\right),\quad i,j=1,2,\ldots N+1,$$

the equation (4.6) can be rewritten as

$$AX = B. \tag{4.7}$$

By solving this system, the approximate values of the unknown coefficients are obtained and therefore an approximation of the unknown source function, $g(t)$, is given. In our implementation, we have solved this system using the *Mathematica* function "*LinearSolve*".

The elements of the vector B in the system (4.7) come from the overspecified conditions (1.5). This condition is obtained from practical measurements that are inherently contaminated with random noise. On the other hand, due to the ill-posedness of this inverse problem, (4.7) is ill-conditioned. Hence, some special regularization methods are required to obtain an accurate approximation. Here, the Tikhonov regularization method is applied for finding the solution of this system. By this technique, we have a minimization problem [24] as

$$\min_{X\in R^s} \|AX - B\|^2 + \gamma\|X\|^2,$$

where $\gamma > 0$ is a regularization parameter. Different methods are presented by authors in the literature to determine the regularization parameter. We use the discrepancy principle [8, 12]. In this principle, we have

$$X = A^{-1}B^{\delta} \quad \text{and} \quad X_{\gamma} = (A^TA + \gamma I)^{-1}A^TB^{\delta},$$

where B^{δ} is perturbed vector and A^T is the transpose of the matrix A. The regularization parameter defined by discrepancy principle is

$$\gamma = \sup\{\gamma > 0 \mid \|AX_{\gamma} - B^{\delta}\| \le \tau\delta\}, \tag{4.8}$$

where $\tau > 1$ is a constant. For more details refer to [8]. In this work, we use the procedure described in [12] to choose an appropriate regularization parameter. The algorithm starts with some very small γ and increases it by multiplying with some constant, when the condition in the supremum of (4.8) is still valid. It repeats the step until the condition is no longer valid [12, 23].

5. Error bound

In this section, we are concerned with the error bound for the approximation of the unknown source term obtained by presented method in the previous section.

Suppose that $g(t)$ is a sufficiently smooth function on $[0, \tau]$ and $p_N(t)$ is the interpolating polynomial to g at points t_i, where t_i, $i = 0, 1, \ldots, N$, are the roots of the $N+1$-degree shifted first-kind Chebyshev polynomial in $[0, \tau]$, then we have

$$g(t) - p_N(t) = \frac{g^{(N+1)}(\eta)}{(N+1)!}\prod_{i=0}^{N}(t - t_i), \quad \eta \in [0, \tau].$$

So, we get

$$|g(t) - p_N(t)| \leq \frac{M_N(\tau)^{N+1}}{2^{2N+1}(N+1)!}, \tag{5.1}$$

with $M_N = \max_{0\leq t\leq \tau} |g^{(N+1)}(t)|$.

Now, we use (5.1) to obtain the following result.

Theorem 5.1. *Suppose that the unknown source function $g(t)$ in equation (1.1) is a real $(N+1)$-times continuously differentiable function on the bounded interval $[0,\tau]$ and $g_N(t) = \sum_{i=0}^{N} c_i P_{\tau,i}^{(\alpha,\beta)}(t)$ be the SJPs expansion of g. Let $\overline{g}_N(t) = \sum_{i=0}^{N} \overline{c}_i P_{\tau,i}^{(\alpha,\beta)}(t)$ be the approximate solution obtained by the method proposed in Section 4 and $M_N = \max_{0\leq t\leq \tau} |g^{(N+1)}(t)|$, then, there exist real numbers $K_\tau^{\alpha,\beta}$ and $\kappa_\tau^{\alpha,\beta}$ such that*

$$\|g(t) - \overline{g}_N(t)\|_2 \leq K_\tau^{\alpha,\beta} \frac{M_N(\tau)^{N+1}}{2^{2N+1}(N+1)!} + \kappa_\tau^{\alpha,\beta} \|C - \overline{C}\|_2, \tag{5.2}$$

where

$$C = [c_0, c_1, \ldots, c_N]^T, \quad \overline{C} = [\overline{c}_0, \overline{c}_1, \ldots, \overline{c}_N]^T,$$

and the norm on the right-hand side is the usual Euclidian norm for vectors.

Proof. Let $\mathbb{R}_N[t]$ be the space of all real-valued polynomials of degree $\leq N$. Using the definition, $g_N(t)$ and $\overline{g}_N(t)$ are in $\mathbb{R}_N[t]$. Also, $g_N(t)$ is the best approximation of $g(t)$ in $\mathbb{R}_N[t]$. We have

$$\|g(t) - \overline{g}_N(t)\|_2 \leq \|g(t) - g_N(t)\|_2 + \|g_N(t) - \overline{g}_N(t)\|_2. \tag{5.3}$$

Taking (5.1) into consideration, we obtain

$$\begin{aligned}
\|g(t) - g_N(t)\|_2 &= \left(\int_0^\tau w_\tau^{(\alpha,\beta)}(t) |g(t) - g_N(t)|^2 \, dt\right)^{1/2} \\
&\leq \left(\int_0^\tau w_\tau^{(\alpha,\beta)}(t) \left[\frac{M_N(\tau)^{N+1}}{2^{2N+1}(N+1)!}\right]^2 dt\right)^{1/2} \\
&= \frac{M_N(\tau)^{N+1}}{2^{2N+1}(N+1)!} \left(\int_0^\tau w_\tau^{(\alpha,\beta)}(t)\, dt\right)^{1/2} \\
&= \sqrt{2^{\alpha+\beta}\tau B(\alpha+1,\beta+1)}\, \frac{M_N(\tau)^{N+1}}{2^{2N+1}(N+1)!},
\end{aligned} \tag{5.4}$$

where $B(a,b)$ is the Beta function defined by

$$B(a,b) = \int_0^1 s^{a-1}(1-s)^{b-1}\, ds.$$

Furthermore, we get

$$
\begin{aligned}
\|g_N(t)-\overline{g}_N(t)\|_2 &= \left(\int_0^{\tau} w_{\tau}^{(\alpha,\beta)}(t)\left[\sum_{i=0}^{N}(c_i-\overline{c}_i)P_{\tau,i}^{(\alpha,\beta)}(t)\right]^2 dt\right)^{1/2}\\
&\le \left(\int_0^{\tau} w_{\tau}^{(\alpha,\beta)}(t)\left[\sum_{i=0}^{N}|c_i-\overline{c}_i|^2\right]\left[\sum_{i=0}^{N}|P_{\tau,i}^{(\alpha,\beta)}(t)|^2\right] dt\right)^{1/2}\\
&= \left(\sum_{i=0}^{N}|c_i-\overline{c}_i|^2\right)^{1/2}\left(\sum_{i=0}^{N}\int_0^{\tau} w_{\tau}^{(\alpha,\beta)}(t)|P_{\tau,i}^{(\alpha,\beta)}(t)|^2\, dt\right)^{1/2}\\
&= \|C-\overline{C}\|_2\left(\sum_{i=0}^{N}h_{\tau,i}^{(\alpha,\beta)}\right)^{1/2}.
\end{aligned}
\tag{5.5}
$$

Therefore, from (5.3)–(5.5) it is seen that (5.2) is valid with

$$
K_{\tau}^{\alpha,\beta}=\sqrt{2^{\alpha+\beta}\tau B(\alpha+1,\beta+1)},\quad \kappa_{\tau}^{\alpha,\beta}=\sqrt{\sum_{i=0}^{N}h_{\tau,i}^{(\alpha,\beta)}}. \qquad \square
$$

6. Numerical examples

In this section, three numerical examples are carried out to illustrate the applicability and accuracy of the proposed method. To simulate the data for the inverse problem some random noises are added to the additional data resulted from the function $q(t)$ in the overspecified condition (1.5). Suppose that δ indicates a relative noise level in the data functions. Then, for generating noisy data, we use the formula

$$
q^{\delta}(t_i)=q(t_i)(1+\delta\times \mathrm{rand}(i)),
$$

where $\mathrm{rand}(i)$ is a random number uniformly distributed in $[-1,1]$.

Example 6.1. Consider the equation (1.1) with $g(t)=\pi t\big(\pi t^{\mu}+\csc(\pi\mu)/\Gamma(-1-\mu)\big)$, $f(x)=\sin(\pi x)$ and zero initial and boundary conditions. By this assumptions, the problem (1.1)–(1.3) has the solution $u(x,t)=t^{\mu+1}\sin(\pi x)$. Also, suppose that $x^*=0.5$.

We have used the proposed method to approximate the function $g(t)$. Table 6.1 displays the condition number (CN) of the matrix A in (4.7) and the L^2 norm of the error in computing the unknown source function $g(t)$ for different values of N and $\mu=1.5$ when there is no noise in data and we do not use the regularization scheme. The results confirm the accuracy of the numerical approach in the absence of noise. Also, this table shows that the growth order of condition number A with respect to N is more than 2.

N	$\alpha=\beta=0$		$\alpha=\beta=0.5$		$\alpha=\beta=-0.5$	
	Error	CN	Error	CN	Error	CN
2	3.07×10^{-1}	18.3016	2.07×10^{-1}	12.2834	5.51×10^{-1}	35.2218
4	1.23×10^{-2}	198.817	7.34×10^{-3}	108.280	2.50×10^{-2}	465.121
6	3.31×10^{-3}	1513.71	1.92×10^{-3}	720.530	7.12×10^{-3}	3985.56
8	1.77×10^{-3}	9729.93	1.11×10^{-3}	4206.87	3.68×10^{-3}	27870.0

Table 6.1: L^2-norm errors and condition numbers for Example 6.1 when $\mu=1.5$.

Figure 6.1 shows the behaviour of numerical approximations to $g(t)$ with regularization and without regularization for $\alpha=\beta=0$ and various noise levels $\delta=1\%, 5\%, 10\%, 15\%$. Finally, the approximate values of $g(t)$ obtained based on various parameters of Jacobi polynomials are compared in Figure 6.2.

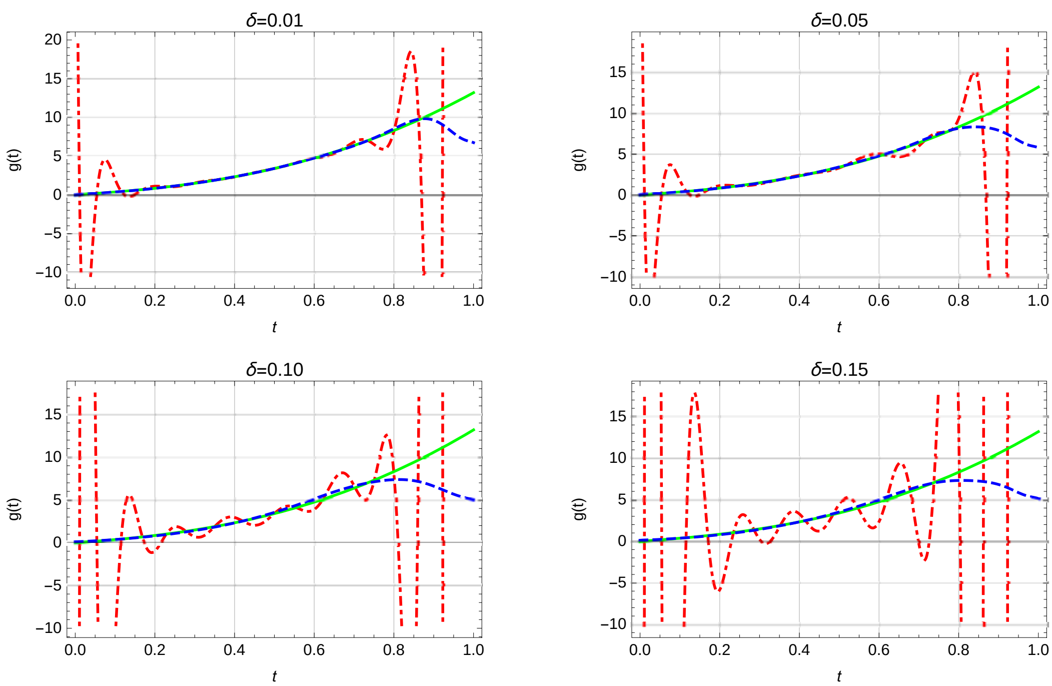

Figure 6.1: Plot of the function $g(t)$ (Green) and the numerical results for it when $\mu=1.5$, $\alpha=\beta=0$ and $N=16$ for Example 6.1: without regularization (Red) and with regularization (Blue).

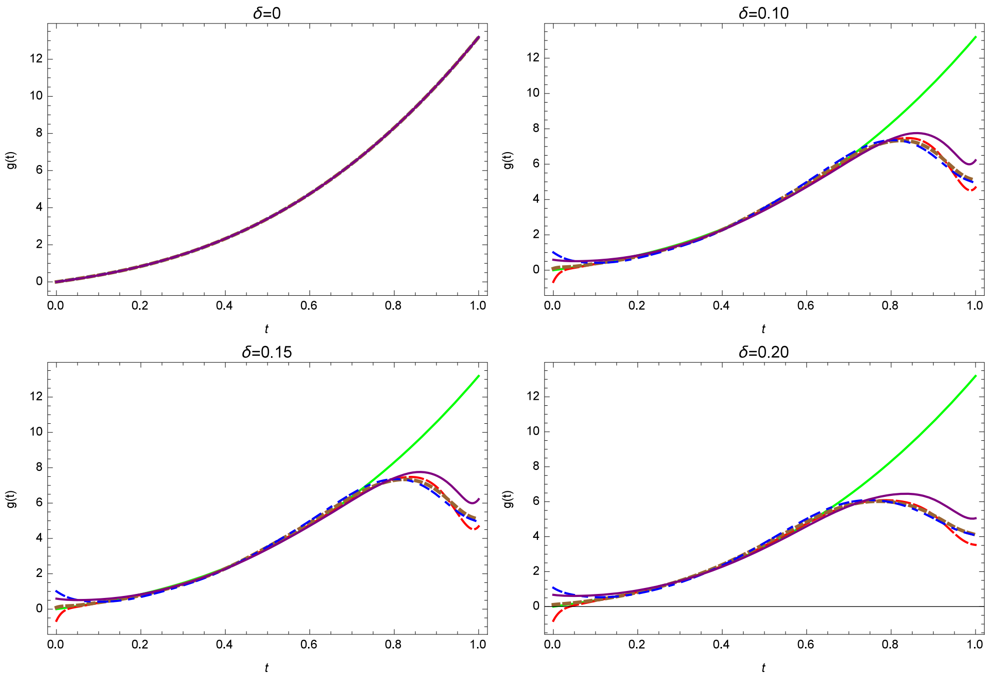

Figure 6.2: The exact and numerical values of the source function $g(t)$ in Example 6.1 when $\mu = 1.5$ and $N = 14$, for different values of the parameters α and β: $\alpha = \beta = 0.5$ (Red), $\alpha = \beta = 0$ (Brown), $\alpha = \beta = -0.5$ (Blue), $\alpha = 0.5$, $\beta = -0.5$ (Purple).

Example 6.2. In this example, we observe the following inverse problem:

$$\begin{aligned}
\frac{\partial^\mu u(x,t)}{\partial t^\mu} &= \frac{\partial^2 u(x,t)}{\partial x^2} + g(t)\sin(\pi x), && (x,t) \in \Omega = [0,1]\times[0,1],\\
u(x,0) &= \sin(\pi x), && 0 < x < 1,\\
u_t(x,0) &= \sin(\pi x), && 0 < x < 1,\\
u(0,t) &= 0, \quad u(1,t) = 0, && 0 < t < 1,\\
u(x^*,t) &= \sin(\pi x^*)e^t, && 0 < t < 1.
\end{aligned}$$

This problem has the exact solution $u(x,t) = \sin(\pi x)e^t$ and

$$g(t) = e^t\left(1 + \pi^2 - \frac{\Gamma(2-\mu,t)}{\Gamma(2-\mu)}\right),$$

in which $\Gamma(a,t)$ is the incomplete gamma function defined by

$$\Gamma(a,t) = \int_t^\infty s^{a-1}e^{-s}\,ds.$$

Let $x^* = 0.4$. We have applied the proposed method to this problem with different values of N and also with different noise levels. Figure 6.3 shows the instability of the numerical approximations to $g(t)$ for different noise levels, when no regularization scheme is applied to the algorithm. Figure 6.4 indicates the relative errors of the estimations to $g(t)$ according to various parameters of Jacobi polynomials when the noise level is 10% or 20%. Finally, the numerical approximations of the source term related to several values of the fractional order μ are depicted in Figure 6.5.

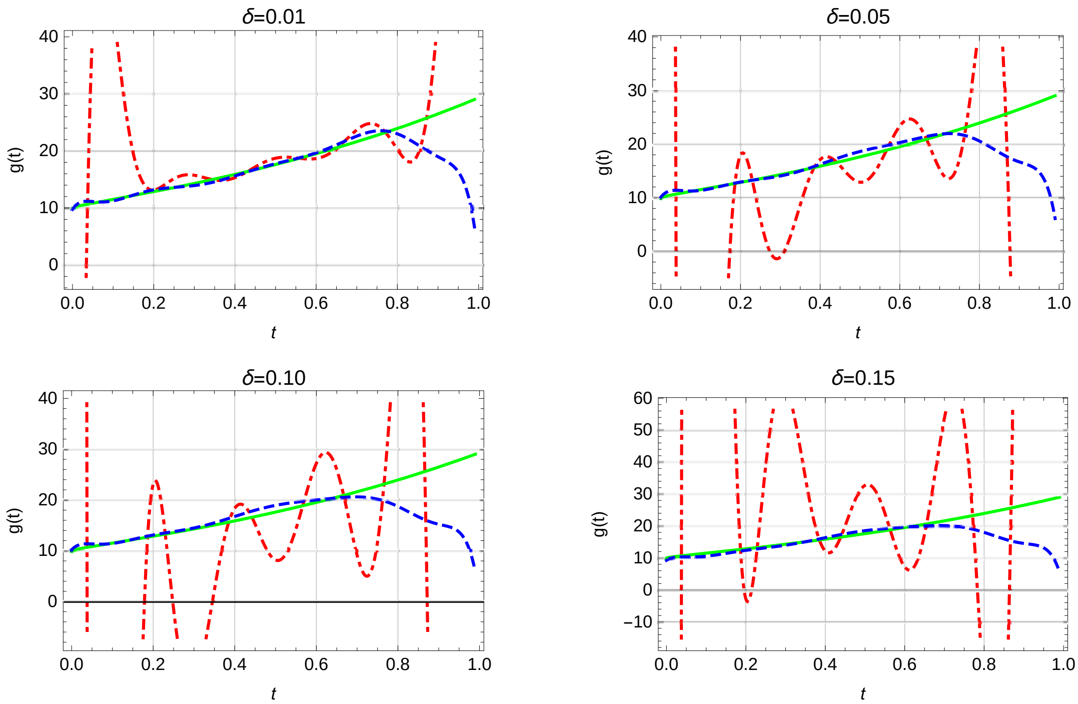

Figure 6.3: Plot of the function $g(t)$ (Green) and the numerical results for it when $\mu = 1.7$, $\alpha = \beta = 0$ and $N = 10$ for Example 6.2: without regularization (Red) and with regularization (Blue).

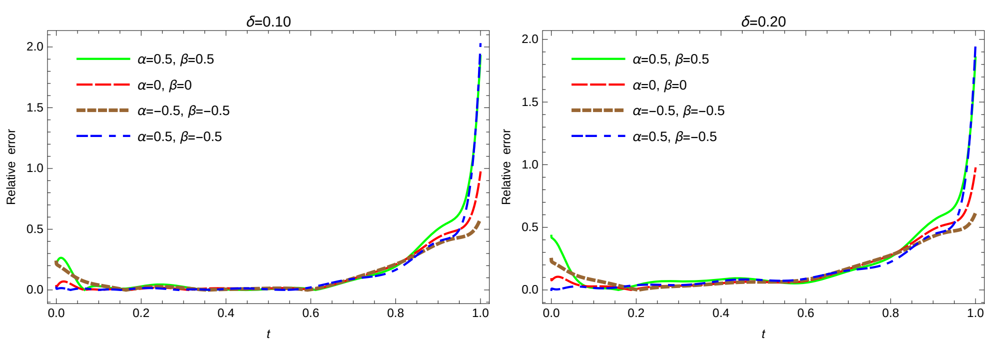

Figure 6.4: The relative errors of the estimated source function $g(t)$ in Example 6.2 when $\mu = 1.7$ and $N = 12$, for different values of the parameters α and β.

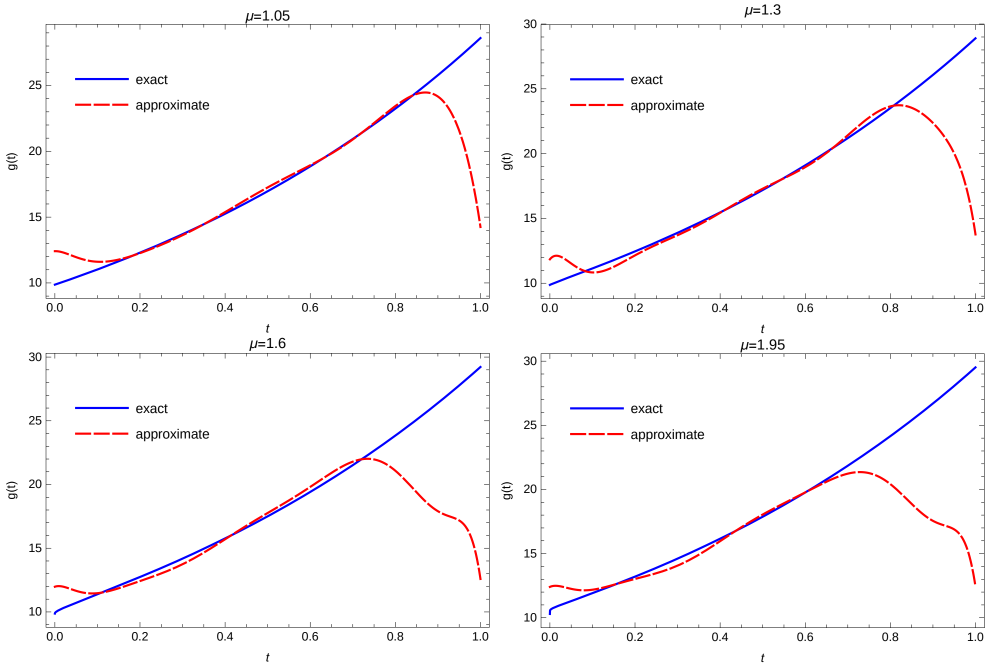

Figure 6.5: The numerical approximations to $g(t)$ for several values of μ in Example 6.2 obtained from the proposed method with the parameters $\alpha = \beta = -0.5$, $N = 10$ and $\delta = 0.05$.

Example 6.3. Let us consider the equation (1.1) with $L = \tau = 1$, $f(x) = \sin(\pi x)$, $g(t) = t^{-\mu}E_{\mu,1-\mu}(-t^{\mu}) + \pi^2 E_{\mu,1}(-t^{\mu}) - \big(t^{-\mu}/\Gamma(1-\mu)\big)$, and the initial functions $f_0(x) = \sin(\pi x)$ and $f_1(x) = 0$. Figure 6.6 displays the estimations of the unknown source term obtained based on Jacobi polynomials with the various values of α and β. Figure 6.7 shows the behaviour of numerical approximations to $g(t)$ with regularization and without regularization, for several percentage of noise levels. Also, Figure 6.8 indicates the relative errors of the estimations to $g(t)$ for numerous values of N when the noise level is 10%.

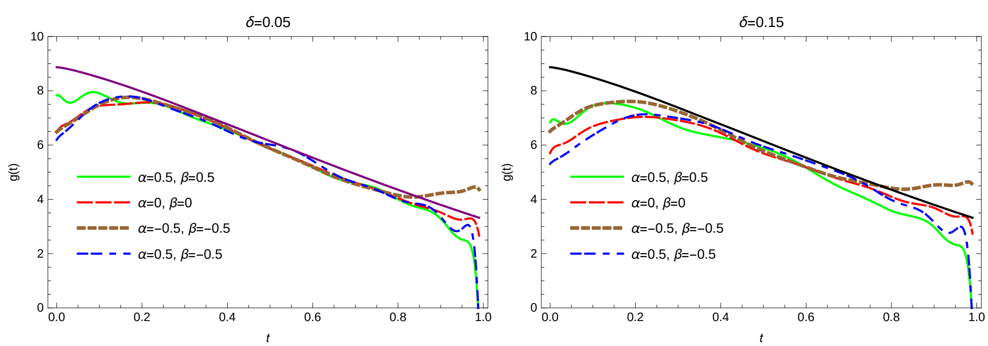

Figure 6.6: The estimated source function $g(t)$ in Example 6.3 when $\mu = 1.3$, $N = 16$ and $x^* = 0.7$, for different values of the parameters α and β.

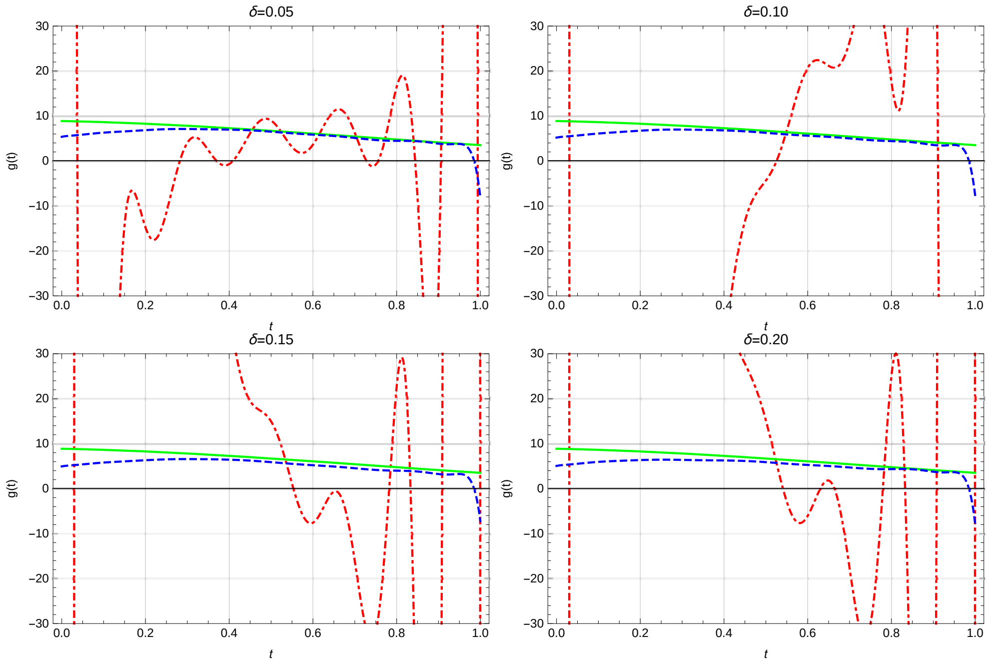

Figure 6.7: Plot of the function $g(t)$ (Green) and the numerical results for it when $x^* = 0.5$, $\mu = 1.5$, $\alpha = 0.5$, $\beta = -0.5$ and $N = 14$ for Example 6.3: without regularization (Red) and with regularization (Blue).

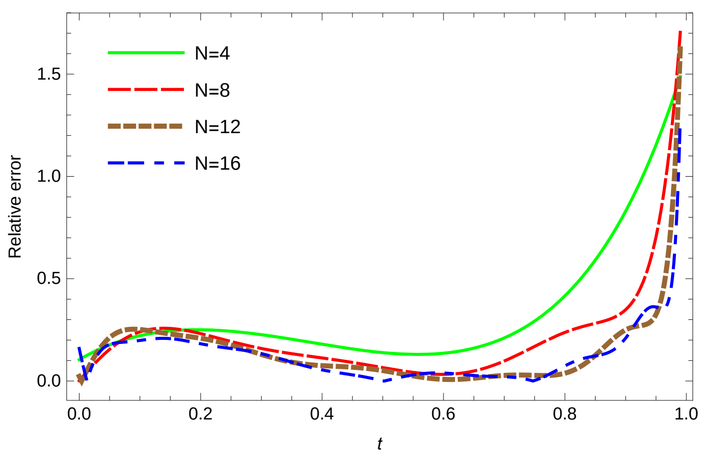

Figure 6.8: The relative errors of the estimated source function $g(t)$ in Example 6.3 when $\mu = 1.7$, $x^* = 0.3$ and $\alpha = \beta = 0.5$, for different values of N.

From the numerical experiments for Examples 6.1, 6.2 and 6.3, specially Figures 6.1, 6.3 and 6.7, it can be observed that the approximations to $g(t)$ without regularization have some oscillations. It illustrates ill-posedness of this type of time fractional inverse problems even in the presence of small noise in input data. Hence, to find stable solution for the problem, combining the presented spectral method based on Jacobi polynomials with the proposed regularization technique, is a useful idea. The numerical results verify that the obtained solutions are stable and accurate even up to 15% or 20% noise in the additional condition.

7. Conclusion

In this article, we applied a numerical method based on the Jacobi polynomials to find the unknown source function in a time-fractional diffusion-wave equation. First, the analytical solution of the direct problem has been reviewed, then we have proposed a method to find an approximation of the unknown source function by considering this function in the form of a linear combination of the shifted Jacobi polynomials. A system of linear equations was constructed to obtain the coefficients of this combination. Since this inverse problem is generally ill-posed, the Tikhonov regularization technique with Morozov's discrepancy principle was applied to find a stable solution of this system. This proposed method is quite different essentially from those methods in literature, as it approximates the unknown source function spectrally in terms of a series of the shifted Jacobi polynomials. As it was mentioned, Legendre polynomials and Chebyshev polynomials of the first and second kind can be viewed as special cases of the Jacobi polynomials. The main characteristic behind this approach is to reduce such inverse problems to those of solving systems of algebraic equations in the unknown expansion coefficients of the unknown source function. An error bound has been given for the approximation of the unknown source function. The numerical results show that the proposed method in this paper is a reliable method to find an approximation of the unknown source function.

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