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Minimal Ideals and Primitivity in Near-rings

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Abstract. We address and answer the question when a minimal ideal of a zero symmetric near-ring is a primitive near-ring. This implies that a minimal ideal of a zero symmetric near-ring is a simple near-ring in many natural situations.

1. Introduction

In what follows, we consider right near-rings, this means the right distributive law holds, but not necessarily the left distributive law. The notation is that of [7]. It is well known in ring theory that given a minimal ideal I of a ring, then $I^2 = \{0\}$ or I is a simple ring (a consequence of the so called "Andrunakievich Lemma" in rings, see for example [2, Theorem 5.7.1]). In near-ring theory an analogous result is not true. Minimal ideals in zero symmetric near-rings exist which are neither square zero nor simple near-rings. A first example of such an ideal was given by K. Kaarli (see [4]). Thus, the study of the structure of minimal ideals in near-rings is much more complicated as in the ring case and still incomplete.

First thorough studies on minimal ideals in near-rings were carried out by S. Scott (see [7] for references and detailed discussions on Scott's results) and K. Kaarli (see [3]) where it basically has been shown that under the presence of chain conditions in a zero symmetric near-ring a minimal ideal decomposes into a finite direct sum of minimal left ideals of the near-ring. The question if a non-nilpotent minimal ideal in a near-ring is a simple near-ring could be answered to the positive by K. Kaarli in [5] in case of distributively generated (d.g.) near-rings which satisfy the descending chain condition on N-subgroups contained in the near-ring N. In general, it is only known that minimal ideals in d.g. near-rings are either nilpotent or subdirectly irreducible, see [5]. This need not be true for arbitrary zero symmetric near-rings. In [10] there is an example of a non-nilpotent minimal ideal in a finite zero symmetric near-ring where the ideal is not subdirectly irreducible, another example will be given here in Section 5. So, the question

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concerning the simplicity or the subdirect irreducibility of a minimal ideal in a zero symmetric near-ring is a complicated one. However, if one changes the question from asking for simplicity to asking for primitivity of an ideal, considered as a subnear-ring, then we can derive powerful theorems as we will see in this paper. Since primitive near-rings are simple near-rings in case of suitable finiteness conditions, we are able to derive results concerning simplicity of ideals under this point of view.

2. Basic definitions and notation

In our discussion that will follow we have to deal with N-groups of type v, $v \in \{0, 1, 2\}$, of a near-ring and the concept of v-primitivity. Thus, we give a brief overview of these definitions and follow the notation of [7]. Let N be a zero symmetric near-ring, this means that $n * 0 = 0$ for all $n \in N$ where $*$ is the near-ring multiplication. Let Γ be an N-group of the near-ring N. An N-ideal I of Γ is a normal subgroup of the group $(\Gamma, +)$ such that $\forall\, n \in N$, $\forall\, \gamma \in \Gamma$, $\forall\, \delta \in I$: $n(\gamma + \delta) - n\gamma \in I$. A left ideal L of a near-ring N is an N-ideal of the natural N-group N and in case N is zero symmetric, a left ideal is also an N-group. The left ideal L is an ideal, if $LN \subseteq L$.

A subgroup S of Γ is called an N-subgroup if $NS \subseteq S$. In case of zero symmetric near-rings, N-ideals of an N-group Γ are also N-subgroups.

Given an N-group Γ and a non-empty subset $S \subseteq \Gamma$ then $(0 : S) = \{n \in N \mid \forall\, s \in S, ns = 0\}$ is called the annihilator of S. Such annihilators always are left ideals of the near-ring N. The annihilator of an N-subgroup is always an ideal of the near-ring N. We call an N-subgroup faithful, if $(0 : S) = \{0\}$.

Let Γ be a non-zero N-group. We define two sets, the set $\theta_1 := \{\gamma \in \Gamma \mid N\gamma = \Gamma\}$ and $\theta_0 := \{\gamma \in \Gamma \mid N\gamma \neq \Gamma\}$. θ_1 is called the set of generators of the N-group Γ and θ_0 is called the set of non-generators. Note that $\Gamma = \theta_1 \cup \theta_0$. We now introduce three different types of simplicity of N-groups.

A non-zero N-group Γ of the near-ring N is of type 0 if there is an element $\gamma \in \Gamma$ such that $N\gamma = \Gamma$, so $\theta_1 \neq \emptyset$ and there are no non-trivial N-ideals in Γ. A non-zero N-group Γ is of type 1 if it is of type 0 and N acts strongly monogenic on Γ. N acting strongly monogenic on Γ means that $N\gamma = \Gamma$ or $N\gamma = \{0\}$ for all $\gamma \in \Gamma$. The N-group Γ is called N-group of type 2 if $N\Gamma \neq \{0\}$ and there are no non-trivial N-subgroups in Γ. In case N has an identity element with respect to multiplication, an N-group is of type 1 if and only if it is of type 2 (see [7, Propositions 3.7 and 3.4]). In general we have for an N-group that type 2 implies type 1 and type 1 implies type 0, see [7, Proposition 3.7].

A near-ring is called v-primitive if it acts on a faithful N-group Γ of type v. The structure of 0-primitive near-rings is not completely known. However, the 2- and 1-primitive zero symmetric near-rings can be described in a satisfactory way, see [9] for references and methods using so called sandwich near-rings to describe 2- and 1-primitive near-rings.

The Jacobson radicals of type v of a near-ring N are defined as the intersection of the annihilators of the N-groups of type v of the near-ring, so $J_v(N) := \bigcap_{\Gamma \text{ of type } v}(0 : \Gamma)$. In our discussion we also need $J_{1/2}(N)$, which is the intersection of all 0-modular left ideals (see [7, Definition 3.28]) of N and we have that $J_0(N) \subseteq J_{1/2}(N) \subseteq J_1(N)$.

3. Minimal ideals and primitivity

Let N be a zero symmetric near-ring and I a minimal ideal of N. In [1] it is proved that in case $J_2(I) = \{0\}$, I is a 2-primitive near-ring and under suitable finiteness conditions $N = I \dot{+} (0 : I)$. The proofs in [1] make use of a powerful theorem on 2-primitive ideals of a subnear-ring of a near-ring, due to K. Kaarli and to be found in [6, Corollary 8.5], for example. It states that given a 2-primitive ideal T of a subnear-ring S of a near-ring N, then T is already an ideal of N. Kaarli's theorem is powerful and needed for the study of Jacobson radicals of type 2 of subnear-rings of a near-ring. For example, one can show with Kaarli's theorem that for an ideal I of a zero symmetric near-ring we have $J_2(I) = I \cap J_2(N)$. For more details in this line of discussion we refer to [6] or [7]. Here we quickly outline how it is proved in [1] that $J_2(I) = \{0\}$, I a minimal ideal in a zero symmetric near-ring N, implies I being 2-primitive. $J_2(I) = \{0\}$ implies that I is a subdirect product of 2-primitive near-rings. The kernel of each projection mapping from this subdirect product is then a 2-primitive ideal of I, T say. By Kaarli's Theorem, T is an ideal of N and by minimality of I as an ideal of N we have that the subdirect product decomposition of I consists exactly of one term. This means that I is itself a 2-primitive near-ring. For more details consult [1].

Of course it is interesting what happens if we consider minimal ideals not sitting in $J_1(N)$ of a near-ring N, which means that $I \cap J_1(N) = \{0\}$ by minimality of I. It turns out that we can transfer the results of [1] accordingly. We want to emphasize that the results presented here do not simply carry over from [1] because Kaarli's powerful theorem on 2-primitive ideals does not extend to 1-primitive ideals, as the examples in Section 5 show.

First we prove some facts about strongly monogenic N-groups. Parts of the following discussion can be found in [8] and is included here for completeness and self containment.

Lemma 3.1. *Let N be a zero symmetric near-ring with a strongly monogenic N-group Γ. Then there exists a greatest proper N-ideal in Γ.*

Proof. Remember that $\theta_1 := \{\gamma \in \Gamma \mid N\gamma = \Gamma\}$ and $\theta_0 := \{\gamma \in \Gamma \mid N\gamma \neq \Gamma\}$. N acting strongly monogenic on Γ means that $N\theta_0 = \{0\}$. Let L be a proper N-ideal of Γ. Since

N is zero symmetric, L is an N-subgroup of Γ. Hence $NL \subseteq L \neq \Gamma$. Consequently, $L \cap \theta_1 = \varnothing$ so each proper N-ideal L of Γ is contained in θ_0 and so $NL = \{0\}$.

Note that the sum of N-ideals is again an N-ideal (see [7, Corollary 2.3]). We now show that a finite sum $\sum_{i=1}^{n} L_i$ of proper N-ideals is again a proper N-ideal. We use induction on the natural number n of N-ideals appearing in the sum $\sum_{i=1}^{n} L_i$, L_i a proper N-ideal of Γ.

The case $n = 1$ is clear since we only consider proper N-ideals. So suppose that $n > 1$ and each sum of $n-1$ proper N-ideals is again a proper N-ideal of Γ. We have to show that the sum of n proper N-ideals is a proper N-ideal. Let $\sum_{k=1}^{n} L_k$ be a sum of n proper N-ideals. Let $l_1 + \cdots + l_n \in \sum_{k=1}^{n} L_k$. Then, for all $m \in N$, $m(l_1 + (l_2 + \cdots + l_n)) - ml_1 \in \sum_{k=2}^{n} L_k \subseteq \theta_0$, by induction hypothesis. Since $ml_1 = 0$, $m(l_1 + l_2 + \cdots + l_n) \in \theta_0$ for all $m \in N$. Thus, $N(l_1 + l_2 + \cdots + l_n) \in \theta_0$ and we see that we cannot have $(l_1 + \cdots + l_n) \in \theta_1$. Hence, $(l_1 + \cdots + l_n) \in \theta_0$ and $\sum_{k=1}^{n} L_k$ is a proper N-ideal.

Now let S be the sum of all proper N-ideals of Γ. If $s \in S$, then s can be written as a finite sum of elements of some proper N-ideals. Therefore, $s \in \theta_0$ as we have seen. So, $S \subseteq \theta_0$ and hence, S is the greatest proper N-ideal. This finishes our proof. □

Lemma 3.2. *Let N be a zero symmetric near-ring which has a strongly monogenic N-group Γ. Let $\triangle$ be the greatest proper N-ideal in Γ, existing by Lemma 3.1. Then, $\Gamma/\triangle$ is an N-group of type 1.*

Proof. $\Gamma/\triangle$ is again an N-group by defining $n(\gamma + \triangle) := n\gamma + \triangle$ for all $n \in N$ and $\gamma \in \Gamma$. If $\gamma \in \theta_1$, then $N(\gamma + \triangle) = \Gamma/\triangle$ and if $\gamma \in \theta_0$ then $N(\gamma + \triangle) = \{0 + \triangle\}$. So, $\Gamma/\triangle$ is a strongly monogenic N-group. The fact that $\triangle$ is the greatest proper N-ideal of Γ implies that $\Gamma/\triangle$ contains no non-trivial N-ideals. □

We now prove our main tool for studying minimal ideals in this paper.

Lemma 3.3. *Let N be a zero symmetric near-ring which has a faithful and strongly monogenic N-group Γ. Then $J_1(N) = J_0(N)$ and $N/J_1(N) = N/J_0(N)$ is a 1-primitive near-ring, acting 1-primitively on $\Gamma/\triangle$, $\triangle$ the greatest proper N-ideal in Γ existing by Lemma 3.1. Furthermore, $NJ_1(N) = NJ_0(N) = \{0\}$ and $J_0(N) = J_1(N) = (0 : \Gamma/\triangle)$.*

Proof. By Lemma 3.2, $\Gamma/\triangle$ is an N-group of type 1, so we have $J_1(N) \subseteq (0 : \Gamma/\triangle)$.

We now prove that the annihilator ideal $(0 : \Gamma/\triangle)$ is a nilpotent ideal. We consider the natural action of $(0 : \Gamma/\triangle)$ on Γ. We have $(0 : \Gamma/\triangle)\Gamma \subseteq \triangle \subseteq \theta_0$. Thus, $N(0 : \Gamma/\triangle)\Gamma \subseteq N\theta_0 = \{0\}$. Faithfulness of Γ implies that $N(0 : \Gamma/\triangle) = \{0\}$ and therefore also $(0 : \Gamma/\triangle)^2 = \{0\}$. Thus, $(0 : \Gamma/\triangle)$ is nilpotent. So, $(0 : \Gamma/\triangle) \subseteq J_1(N)$ by [7, Theorem 5.37]. So we finally have $J_1(N) = (0 : \Gamma/\triangle)$ and following from that, because

$N(0 : \Gamma/\triangle) = \{0\}$, $NJ_1(N) = \{0\}$. Thus, $J_1(N)$ is a nilpotent ideal and we have that $J_1(N) \subseteq J_0(N)$ by [7, Theorem 5.37]. Therefore, $J_0(N) = J_1(N)$.

By [7, Proposition 3.14], $\Gamma/\triangle$ is a faithful N-group of type 1 of the near-ring $N/(0 : \Gamma/\triangle)$. Consequently, the near-ring $N/J_1(N)$ is a 1-primitive near-ring. □

Note that if N fulfilles the assumptions of Lemma 3.3 and additionally has an identity element then N acts 2-primitively on Γ and so $J_2(N) = J_1(N) = J_0(N) = \{0\}$. This is because in that case we must have that $N\gamma \neq \{0\}$ for each $\gamma \in \Gamma \setminus \{0\}$, due to the identity element in N (see [7, Proposition 3.4]). Then, since N is strongly monogenic, we must have $N\gamma = \Gamma$ for each $\gamma \in \Gamma \setminus \{0\}$. So, Γ is of type 2 and N is 2-primitive. Thus, Lemma 3.3 is a generalisation of this easy observation to near-rings without an identity element.

We need another lemma which will be frequently used in this paper.

Lemma 3.4. *Let N be a zero symmetric near-ring and let I be an ideal of N. Let Γ be an N-group of type 0 such that $I \cap (0 : \Gamma) = \{0\}$. Let $\gamma \in \Gamma$ such that $N\gamma = \Gamma$. Then, $I\gamma = \Gamma$.*

Proof. Let $\gamma \in \Gamma$ such that $N\gamma = \Gamma$. Suppose that $I\gamma = \{0\}$. Then $I\Gamma = IN\gamma \subseteq I\gamma = \{0\}$ contradicting the fact that $I \cap (0 : \Gamma) = \{0\}$. Since $\Gamma = N\gamma$ we have that $I\gamma$ is an N-ideal in Γ (see [7, Proposition 3.4]). By assumption, Γ is an N-group of type 0 implying that $I\gamma = \Gamma$. □

We now present our main theorem of this section and one of our main theorems in this paper.

Theorem 3.5. *Let N be a zero symmetric near-ring. Let I be a minimal ideal of N and $I \not\subseteq J_1(N)$. Then, I is a 1-primitive near-ring.*

Proof. $I \not\subseteq J_1(N)$ implies that $I \cap J_1(N) = \{0\}$ due to minimality of I. Thus, $J_1(N) \neq N$ and since $I \not\subseteq J_1(N)$ there exists an N-group Γ of type 1 such that $I\Gamma \neq \{0\}$. Since $(0 : \Gamma)$ is an ideal, minimality of I implies that $I \cap (0 : \Gamma) = \{0\}$. Hence, I acts faithfully on Γ. Γ is an N-group of type 1, so N acts strongly monogenic on Γ. Thus, $N\theta_0 = \{0\}$ and so also $I\theta_0 = \{0\}$. From Lemma 3.4 we have that for $\gamma \in \theta_1$, $I\gamma = \Gamma$. This implies that also I acts strongly monogenic on Γ. Therefore we can apply Lemma 3.3 which shows that $I/J_1(I)$ is a 1-primitive near-ring. By [7, Theorem 5.33], $J_1(I) \subseteq J_1(N) \cap I = \{0\}$ which proves that I is a 1-primitive near-ring. □

We will prove that simplicity of a minimal ideal arises naturally when considering near-rings satisfying the descending chain condition on left ideals (abbreviated by DCCL) and minimal ideals sitting outside the Jacobson 1 radical of the near-ring. We need a technical lemma first. As usual, the symbol $\dot{+}$ means a direct composition of the near-rings in question.

Lemma 3.6. *Let N be a zero symmetric near-ring and suppose that $N = I\dot{+}(0 : I)$, I an ideal of the near-ring. Let L be a left ideal of the subnear-ring I. Then L is a left ideal of the near-ring N.*

Proof. Since $(L, +)$ is normal in $(I, +)$ and $(I, +)$ is a direct summand of $(N, +)$ it follows from standard group theory that $(L, +)$ is a normal subgroup of $(N, +)$. We now have to show the following: $\forall\, n, m \in N$, $\forall\, l \in L$, $n(m + l) - nm \in L$. Let $n, m \in N$. We use the direct sum decomposition to get $n = i + a$ and $m = j + b$ with $i, j \in I$ and $a, b \in (0 : I)$. Then we have that $n(m+l)-nm = (i+a)(j+b+l)-(i+a)(j+b)$. By right distributivity, this expression equals $i(j+b+l)+a(j+b+l)-a(j+b)-i(j+b)$. Since $j+l \in I$ and $j \in I$ we have $a(j+l) = 0 = aj$. Since $j \in I$ and $b \in (0 : I)$ we have that $j+b = b+j$ from the direct sum decomposition of $(N, +)$. Thus we have $a(j+b+l) = a(b+j+l) = a(b+(j+l))$, $j \in I$ and $l \in I$, so we can use that the sum $N = I\dot{+}(0 : I)$ is distributive (see [7, Theorem 2.30] or [7, Proposition 2.6]) to get $a(b+(j+l)) = ab+a(j+l) = ab$ and $a(j+b) = aj+ab = ab$. Thus, $a(j + b + l) - a(j + b) = ab - ab = 0$.

Therefore $i(j+b+l)+a(j+b+l)-a(j+b)-i(j+b) = i(j+b+l)-i(j+b)$. From $j+b = b+j$ and the distributivity of the sum $N = I\dot{+}(0 : I)$ we get $i(j+b+l) = i(b+(j+l)) = ib+i(j+l)$ and $i(j+b) = i(b+j) = ib+ij$. Thus, $i(j+b+l)-i(j+b) = ib+i(j+l)-ij-ib$. Now we use that L is a left ideal of I, $i \in I$ and $j \in I$ and we see that $i(j+l)-ij \in L$. Since $(L, +)$ is a normal subgroup of $(N, +)$ we finally have that $n(m+l)-nm = ib+i(j+l)-ij-ib \in L$. $\square$

We have the next theorem which shows simplicity of the minimal ideal in question, in case the near-ring has DCCL.

Theorem 3.7. *Let N be a zero symmetric near-ring with DCCL. Let I be a minimal ideal not contained in $J_1(N)$. Then, I is a 1-primitive near-ring and $N = I\dot{+}(0 : I)$. I is a simple subnear-ring of N and I is a direct sum of left ideals of N which are I-isomorphic as well as N-isomorphic and which are I-groups of type 1 as well as N-groups of type 1.*

Proof. That I is a 1-primitive near-ring follows from Theorem 3.5. $I \not\subseteq J_1(N)$ implies that there is an N-group Γ of type 1 such that $I\Gamma \neq \{0\}$. Since $(0 : \Gamma)$ is an ideal, minimality of I implies that $I \cap (0 : \Gamma) = \{0\}$. Now $N/(0 : \Gamma)$ is a 1-primitive near-ring with DCCL (see [7, Proposition 3.14 and Theorem 2.35]) and therefore simple by [7, Theorem 4.46]. Thus, $(0 : \Gamma)$ is a maximal ideal and this already implies $N = I\dot{+}(0 : \Gamma)$. Since $(0 : \Gamma)I \subseteq I \cap (0 : \Gamma) = \{0\}$ we see that $(0 : \Gamma) \subseteq (0 : I)$.

Due to the DCCL there is a minimal left ideal L contained in I. Then $I \cap (0 : \Gamma) = \{0\}$, so $L\Gamma \neq \{0\}$. As N acts strongly monogenic on Γ, there must be a generator γ of the

N-group Γ such that $L\gamma \neq \{0\}$. Thus, by [7, Proposition 3.10] $L \cong_N \Gamma$ and consequently, $(0 : L) = (0 : \Gamma)$ (see [7, Proposition 1.45]). Hence, $(0 : I) \subseteq (0 : L) = (0 : \Gamma)$ which proves $N = I \dot{+} (0 : I)$.

This decomposition shows that each left ideal of the subnear-ring I is a left ideal of N by Lemma 3.6. Consequently, the 1-primitive near-ring I satisfies the DCC on left ideals of I and by [7, Theorem 4.46], I is a simple near-ring and I decomposes into a finite direct sum of I-isomorphic left ideals which are I-groups of type 1. Using the fact that each element $n \in N$ decomposes as $n = i + a$ with $i \in I$ and $a \in (0 : I)$ it is a straightforward calculation to see that these left ideals are also N-isomorphic N-groups of type 1. □

Corollary 3.8. *Let N be a zero symmetric near-ring with $J_1(N)$ nilpotent. Let I be a minimal ideal, $I^2 \neq \{0\}$. Then I is a 1-primitive near-ring. In case N has DCCL, I is a simple near-ring and a direct summand as an ideal of the near-ring.*

Proof. From [7, Proposition 3.53] it follows that in a nilpotent minimal ideal I we already have $I^2 = \{0\}$. Thus, our assumption implies $I \not\subseteq J_1(N)$. So, the result follows from Theorems 3.5 and 3.7. □

Remark 3.9. In Example 5.3 we have an example of a near-ring N of order 32 which contains a non nilpotent minimal ideal K which is a 1-primitive and simple near-ring and $J_1(N) = K$. Moreover, K is not a direct summand as an ideal in the near-ring N.

We now state the corresponding version of Theorem 3.5 for minimal ideals not contained in J_2 of a near-ring. We get analogous results. Note that this basically has been proved in [1]. In [1] Kaarli's theorem on 2-primitive ideals was used to get the result. It turns out that we can re-prove this with much easier methods now, probably making the result more accessible.

Theorem 3.10. *Let N be a zero symmetric near-ring. Let I be a minimal ideal not contained in $J_2(N)$. Then I is a 2-primitive near-ring. If N has the DCCL, then $N = I \dot{+} (0 : I)$, I is a simple near-ring and I is a direct sum of left ideals of N which are I-isomorphic as well as N-isomorphic and which are I-groups of type 2 as well as N-groups of type 2.*

Proof. Let $I \not\subseteq J_2(N)$. This implies that there is an N-group Γ of type 2 such that that $I\Gamma \neq \{0\}$, which by minimality of I implies that $I \cap (0 : \Gamma) = \{0\}$. Let $\gamma \in \theta_1$. From Lemma 3.4 we get that $I\gamma = \Gamma$. Thus, each nontrivial I-subgroup of Γ must be contained in θ_0. N is acting strongly monogenic on the N-group Γ, so for each $\delta \in \theta_0$, $I\delta = \{0\}$. So I acts strongly monogenic on Γ, also. Since for each $\delta \in \theta_0$ we have $N\delta = \{0\}$, every non-trivial subgroup of θ_0 is an N-subgroup of N. Since Γ is of type 2, we see that θ_0 does not contain any non-trivial subgroup. Since each non-trivial I-subgroup must also be contained in θ_0, we see that Γ contains no non-trivial I-subgroups. Thus, Γ is an I-group of type 2 and I acts faithfully on Γ. Hence, I is a 2-primitive near-ring. That I is simple and $N = I \dot{+} (0 : I)$ in case N has DCCL follows from Theorem 3.5 because $I \not\subseteq J_2(N)$ implies $I \not\subseteq J_1(N)$. The fact that I is a direct sum of I- and N-isomorphic left ideals which are I- and N-groups of type 2 follows as in Theorem 3.5, in case N has DCCL. $\square$

As we will explicitly point out in Example 5.6, Theorems 3.5 and 3.10 cannot be extended accordingly to non-nilpotent minimal ideals having zero intersection with $J_{1/2}(N)$ (with $J_0(N)$, respectively).

As we will see in the following section, the situation for describing non-nilpotent minimal ideals sitting inside $J_1(N)$ of a zero symmetric near-ring N is not completely hopeless. To the contrary, such ideals have quite a clear structure, but they need not be primitive near-rings anymore.

4. Minimal ideals which are not necessarily primitive

In this section we first prove that non-zero ideals of a certain type always exist in a zero symmetric near-ring, provided we have a suitable chain condition which is more general than the DCCN (the descending chain condition on N-subgroups contained in N). This then leads to the result that certain minimal ideals I have faithfully and strongly monogenic I-groups which enables us to apply Lemma 3.3.

First we have to show that minimal left ideals which do not have zero multiplication and satisfy a generalized finiteness condition contain a multiplicative right identity when considered as subnear-ring. From this we then are able to derive the existence of certain non-zero ideals in a near-ring. To fix a notation, the symbol $\supset$ means a proper subset.

Proposition 4.1. *Let N be a zero symmetric near-ring. Let L be a minimal left ideal such that L satisfies the DCC on N-subgroups contained in L. Suppose $M \subseteq L$ is an N-subgroup such that $M \neq L$. Then, L and M cannot be N-isomorphic.*

Proof. Suppose to the contrary that M and L are N-isomorphic. Then, due to N-isomorphism also M contains a proper subgroup M_1 which is N-isomorphic to M. Then also M_1 does, and so on. Hence we get an infinite decreasing chain of N-isomorphic N-subgroups $L \supset M \supset M_1 \supset \cdots$. $\square$

In particular, Proposition 4.1 applies to zero symmetric near-rings with DCCN.

We now can show that minimal left ideals which do not have zero multiplication and satisfy a generalized finiteness condition are N-groups of type 0. The following two Lemmas 4.2 and 4.3 were proved in [10] (see [10, Lemmas 2.2 and 2.5]). Since they are central for our main results in this section, we include their proofs for self containment of the paper.

Lemma 4.2. *Let N be a zero symmetric near-ring and let L be a minimal left ideal such that $L^2 \neq \{0\}$. Suppose that L does not contain N-subgroups properly contained in L and being N-isomorphic to L. Then L contains a multiplicative right identity e when considered as subnear-ring of N. Furthermore, L is an N-group of type 0.*

Proof. We first show that L has a generator as an L-group. Since we have $L^2 \neq \{0\}$, there is an element $l \in L$ such that $Ll \neq \{0\}$. By minimality of L as a left ideal, this implies $L \cap (0 : l) = \{0\}$. Consequently, the map $\psi_l \colon L \to Ll$, $j \mapsto jl$ is injective. Certainly, ψ_l is a surjective N-homomorphism and thus L and Ll are N-isomorphic. By assumption this implies $L = Ll$. So, we see that L has the generator l.

What is more, L contains an idempotent e which is a right identity in L. To see this, let $e \in L$ such that $el = l$. Such an e exists since $Ll = L$. Thus, $e^2 l = el$ and consequently, $(e^2 - e)l = 0$. So, $e^2 - e \in L \cap (0 : l) = \{0\}$ and we see that $e = e^2$. Let $j \in L$. Then, $je = je^2$, so $(j - je)e = 0$. Hence, $j - je \in L \cap (0 : e)$. Since $e \in Le$ by idempotence of e we see that $Le \neq \{0\}$ and so, by minimality of L we have that $L \cap (0 : e) = \{0\}$. Hence, $j = je$ and e is a multiplicative right identity in L.

Consequently, we have a Peirce decomposition of N as $N = (0 : e)\dot{+}Ne = (0 : e)\dot{+}L$. Suppose that $I \subseteq L$ is an N-ideal contained in L. Consequently, $(I, +)$ is a normal subgroup of $(L, +)$. From the direct decomposition $(0 : e)\dot{+}L = N$ as a group we have from standard group theory (or an easy verification) that this implies that $(I, +)$ is also normal in $(N, +)$.

Let $n, m \in N$. So, there is an element $l \in L$ and $a \in (0 : e)$ such that $m = a + l$. Let $i \in I$. Then,

$$n(m + i) - nm = n((a + l) + i) - n(a + l).$$

By [7, Proposition 2.29] the sum $N = (0 : e)\dot{+}L$ is distributive, so

$$n((a + l) + i) = n(a + (l + i)), \quad n(a + (l + i)) = na + n(l + i) \quad \text{and} \quad n(a + l) = na + nl.$$

Consequently,

$$n(m + i) - nm = na + n(l + i) - nl - na.$$

By assumption, $n(l + i) - nl \in I$ and since $(I, +)$ is a normal subgroup of $(N, +)$ we have that

$$na + n(l + i) - nl - na \in I.$$

This shows that $I \subseteq L$ is a left ideal of N. By minimality of L as a left ideal we either have $I = \{0\}$ or $I = L$. Thus, L contains no non-trivial N-ideals and this proves that L is an N-group of type 0. $\square$

In particular the results of Lemma 4.2 apply to near-rings with DCCN. Lemma 4.2 can be used to derive powerful structure results for 0-primitive near-rings, see [10].

The next lemma guarantees the existence of certain non-zero ideals provided we have minimal left ideals which do not have zero multiplication in a zero symmetric near-ring N. In a zero symmetric near-ring N a left ideal L always is an N-group. We use the notation $\theta_0^L := \{l \in L \mid Nl \neq L\}$ and $\theta_1^L := \{l \in L \mid Nl = L\}$ for the set of non-generators, generators respectively, of this N-group L.

Lemma 4.3. *Let N be a zero symmetric near-ring and let L be a minimal left ideal such that $L^2 \neq \{0\}$. Suppose that L does not contain N-subgroups properly contained in L and being N-isomorphic to L. Then $(0 : \theta_0^L)$ is a non-zero ideal of N containing L.*

Proof. As an annihilator, $(0 : \theta_0^L)$ is a left ideal of N. Let $l \in \theta_0^L$ and $n \in N$. Then $N(nl) \subseteq Nl \neq L$, so $nl \in \theta_0^L$. Let $a \in (0 : \theta_0^L)$, $n \in N$ and $l \in \theta_0^L$. Then $(an)l = a(nl) = 0$ because $nl \in \theta_0^L$. So, we have shown that $(0 : \theta_0^L)$ is an ideal of N.

It remains to show that the non-zero left ideal L is contained in $(0 : \theta_0^L)$. Let $l \in \theta_0^L$, so $Nl \neq L$.

Let $m \in Nl$. Then $(0 : m) \cap L$ is a left ideal contained in L and so, by minimality of L either $(0 : m) \cap L = L$ which gives $Lm = \{0\}$ or $(0 : m) \cap L = \{0\}$. Suppose that $(0 : m) \cap L = \{0\}$. Then, the map $\psi_m \colon L \to Lm$, $l \mapsto lm$, is injective. ψ_m clearly is a surjective N-homomorphism between L and Lm and so we have that L and Lm are N-isomorphic. Since $m \in Nl \subseteq L$ we have $Lm \subseteq L$ and it follows from our assumption that $L = Lm$. Also we have $Lm \subseteq LNl \subseteq Nl$ and therefore, $L = Lm \subseteq Nl \subseteq L$, which contradicts the fact that $Nl \neq L$. Thus, for all $m \in Nl$, $Lm = \{0\}$ or in other words, $LNl = \{0\}$.

By Lemma 4.2, L contains a multiplicative right identity e and therefore, $LNl = \{0\}$ implies $Lel = Ll = \{0\}$. So we have shown that for $l \in \theta_0^L$, $Ll = \{0\}$. This finally shows that $L \subseteq (0 : \theta_0^L)$. □

Now we are in a position to prove that under the finiteness conditions we used in the previous lemmas, a minimal ideal I in a zero symmetric near-ring hosts a faithful and strongly monogenic I-group.

Proposition 4.4. *Let I be a minimal ideal of a zero symmetric near-ring N. Let $L \subseteq I$ be a minimal left ideal such that $L^2 \neq \{0\}$. Suppose that L does not contain N-subgroups properly contained in L and being N-isomorphic to L. Then I acts faithfully and strongly monogenic on L and $I \subseteq (0 : \theta_0^L)$.*

Proof. By Lemma 4.3, $(0 : \theta_0^L)$ is a non-zero ideal of N, containing the left ideal L. Thus, we must have $L \subseteq I \cap (0 : \theta_0^L)$. Minimality of I implies that $I \subseteq (0 : \theta_0^L)$. On the other hand, by Lemma 4.2, L is an N-group of type 0. Due to the fact that $L^2 \neq \{0\}$ and $L \subseteq I$ we must have $I \cap (0 : L) = \{0\}$, by minimality of I as an ideal. Hence, I acts faithfully on L and we can apply Lemma 3.4 to see that $Il = L$ for $l \in \theta_1^L$. Thus, I acts strongly monogenic on L and the proposition is proved. □

The next theorem is the main theorem concerning the structure of non-nilpotent minimal ideals I which are contained in $J_1(N)$ of a zero symmetric near-ring N. Note that the containement $I \subseteq J_1(N)$ is not an assumption in the theorem but of course, in case $I \not\subseteq J_1(N)$ we can apply the much stronger Theorems 3.5 and 3.10. The examples given in Section 5 show non-nilpotent minimal ideals sitting inside the J_1-radical of a zero symmetric near-ring.

Theorem 4.5. *Let I be a minimal ideal of a zero symmetric near-ring N. Let $L \subseteq I$ be a minimal left ideal such that $L^2 \neq \{0\}$. Suppose that L does not contain N-subgroups properly contained in L and being N-isomorphic to L. Then $I/J_0(I)$ is a 1-primitive near-ring, $J_1(I) = J_0(I)$ and $IJ_0(I) = \{0\}$.*

Proof. That $I/J_0(I)$ is 1-primitive, $J_1(I) = J_0(I)$ and $IJ_0(I) = \{0\}$ follows from the fact that I acts faithfully and strongly monogenic on the minimal left ideal L as shown in Proposition 4.4 and then applying Lemma 3.3. □

Remember Kaarli's result on 2-primitive ideals in a near-ring we discussed in the introduction of the paper. According to Theorem 4.5, $J_0(I)$ is a 1-primitive ideal of the minimal ideal I. But as we will see by examples in Section 5 there are non-nilpotent minimal ideals I with $J_0(I) \neq \{0\}$. So Kaarli's result indeed does not extend to the case of 1-primitive ideals in a near-ring. A 1-primitive ideal of an ideal I of a near-ring N need not be an ideal of the near-ring N. But we can use Kaarli's result to decide when the near-ring $I/J_0(I)$ occurring in Theorem 4.5 is not only 1-primitive but even 2-primitive. As we will see this will not be the case, as long as we restrict to the interesting case (in the light of Theorem 3.10) where $I \subseteq J_1(N)$. We also collect some additional information concerning the Jacobson radicals of the subnear-ring I.

Corollary 4.6. *Let I be a minimal ideal of a zero symmetric near-ring N. Let $L \subseteq I$ be a minimal left ideal such that $L^2 \neq \{0\}$. Suppose that L does not contain N-subgroups properly contained in L and being N-isomorphic to L. Suppose that $I \subseteq J_1(N)$. Then $I/J_0(I)$ is a 1-primitive near-ring which is a 2-radical near-ring, $J_2(I) = I$ and $J_1(I) = J_0(I) \neq I$ and $IJ_0(I) = \{0\}$.*

Proof. By assumption, $I \subseteq J_1(N) \subseteq J_2(N)$, so $I = I \cap J_2(N)$ and we have from [7, Theorem 5.21] $J_2(I) = I \cap J_2(N) = I$. Since the map which assigns to each near-ring its Jacobson 2 radical is a radical map, we have $(J_2(I) + J_0(I))/J_0(I) \subseteq J_2(I/J_0(I))$

(see [7, Proposition 5.15]). Using the fact that $J_2(I) = I$, this results in $I/J_0(I) = (I+J_0(I))/J_0(I) = (J_2(I)+J_0(I))/J_0(I) \subseteq J_2(I/J_0(I))$. Thus, $J_2(I/J_0(I)) = I/J_0(I)$ and $I/J_0(I)$ is a 2-radical near-ring. The rest of the statements follow from Theorem 4.5. □

We now will add some additional information to Theorem 4.5, namely we want to describe the subideal structure of the minimal ideal I and give a condition when I has to be simple. Our proof requires that the minimal ideal I in question decomposes into a direct sum of minimal left ideals of the near-ring N. This is known to be the case when the near-ring has DCCN and is available in the standard literature on near-rings (see [7, Theorem 3.54] for example). Therefore, we give the proof of the following corollary in the setting of DCCN near-rings. Often such direct sum compositions of minimal ideals are available even if the near-ring itself does not satisfy the DCCN, see for example [10]. So the results in Corollary 4.8 could be generalized to these more general situations.

First we need another technical lemma.

Lemma 4.7. *Let N be a zero symmetric near-ring with DCCN and I a minimal ideal, $I^2 \neq \{0\}$. Then, each minimal left ideal $L \subseteq I$ has the property that $L^2 \neq \{0\}$.*

Proof. Suppose there is a left ideal $L \subseteq I$ such that $L^2 = \{0\}$. Then by [7, Corollary 3.55], I is nilpotent. From [7, Proposition 3.53] we have that $I^2 = \{0\}$, contradicting our assumptions. The DCCN guarantees the existence of minimal left ideals in I. □

Corollary 4.8. *Let I be a minimal ideal of a zero symmetric near-ring N satisfying the DCCN, $I^2 \neq \{0\}$. Then $J_0(I)$ is the greatest proper ideal of I and for each proper ideal A of the subnear-ring I we have $IA = \{0\}$. Furthermore, I is a simple near-ring if and only if $J_0(I) = \{0\}$. In case I is simple it is a 1-primitive near-ring.*

Proof. From Lemma 4.7 we have the existence of a minimal left ideal $L \subseteq I$ such that $L^2 \neq \{0\}$. The DCCN of N implies that L does not contain proper N-subgroups being N-isomorphic to L (see Proposition 4.1). Thus we have Theorem 4.5 and Proposition 4.4 at hand. By [7, Theorem 3.54], I (considered as an N-group) decomposes into a finite direct sum of minimal left ideals of N, which are all N-isomorphic. We write $I = \sum_{i \in S} L_i$, S a suitable finite index set, where for $i, j \in S$ we have that L_i and L_j are N-isomorphic, each L_i is a minimal left ideal of the near-ring N and $L_i^2 \neq \{0\}$, by Lemma 4.7. Thus by Lemma 4.2, each L_i, $i \in S$, is an N-group of type 0. Proposition 4.4 shows that I acts strongly monogenic on L_i, $i \in S$.

Let A be a non-trivial ideal of I. Since A is non-trivial we have $A \neq I$ which implies the existence of $i \in S$ such that $L_i \not\subseteq A$. Let $\theta_0 := \{l \in L_i \mid Il = \{0\}\}$ and $\theta_1 := \{l \in L_i \mid Il = L_i\}$. By Proposition 4.4, I acts faithfully and strongly monogenic on L_i. Thus, $L_i = \theta_0 \cup \theta_1$. Note that by Lemma 4.2, L_i contains a right identity element, so $\theta_1 \neq \emptyset$. Let

$l \in \theta_1$ and suppose that $Al = L_i$. Since A is an ideal of I we must have $L_i = Al \subseteq A$ which contradicts $L_i \nsubseteq A$. Thus, $Al \neq L_i$. Since $L_i = Il$ we have that Al is a proper I-ideal of L_i (see [7, Proposition 3.4]). Since I acts strongly monogenic on L_i, Al being proper in L_i implies $Al \subseteq \theta_0$. Thus we have $I(Al) \subseteq I\theta_0 = \{0\}$. Since l was chosen arbitrarily from θ_1, $IA\theta_1 = \{0\}$. Clearly, $A\theta_0 = \{0\}$, so we also have $IA\theta_0 = \{0\}$. From $L_i = \theta_0 \cup \theta_1$ we now have $IAL_i = \{0\}$. The action of I on L_i is faithful by Proposition 4.4, so $IA = \{0\}$. This shows also that $A^2 = \{0\}$. So, by [7, Theorem 5.37], $A \subseteq J_0(I)$. By Theorem 4.5 we have $IJ_0(I) = \{0\}$. Thus $J_0(I) \neq I$ and we have shown that $J_0(I)$ is the greatest proper ideal of I. That I is a simple near-ring if and only if $J_0(I) = \{0\}$ now follows immediately from the fact that $J_0(I)$ is the greatest proper ideal in I. From Theorem 4.5 we have that I is 1-primitive in case I is simple. □

Note that in general, without any finiteness condition, the result of Corollary 4.8 is not true! Let N be a zero symmetric near-ring with identity and N being 2-primitive. Thus, $J_2(N) = J_1(N) = J_0(N) = \{0\}$. But $J_0(N)$ cannot be the greatest proper ideal in general because there exist 2-primitive near-rings (even primitive rings) which are not simple near-rings!

When we have a zero symmetric near-ring N and an ideal I, then we have that $J_2(I) \subseteq J_2(N)$ and $J_1(I) \subseteq J_1(N)$ (see [7]). As we have seen in Theorems 3.5 and 3.10 this leads to I being 2-primitive, 1-primitive respectively when I is a non-nilpotent minimal ideal not contained in $J_2(N)$, $J_1(N)$ respectively. For near-rings in general we do not have that $J_0(I) \subseteq J_0(N)$, I an ideal in the near-ring N (for examples of that kind, see Section 5). This is the deeper reason why non-nilpotent minimal ideals in zero symmetric near-rings with DCCN are not necessarily simple near-rings, as we will point out in the following.

Corollary 4.9. *Let N be a zero symmetric near-ring with DCCN. Let I be a minimal ideal, $I^2 \neq \{0\}$. Then I is a simple near-ring if and only if $J_0(I) \subseteq J_0(N)$.*

Proof. Suppose that I is a simple near-ring. Application of Corollary 4.8 shows that $J_0(I) = \{0\}$ in this case and thus, $J_0(I) \subseteq J_0(N)$.

Let I be a minimal ideal, $I^2 \neq \{0\}$ and $J_0(I) \subseteq J_0(N)$. In case of a near-ring with DCCN we know that $J_0(N)$ is nilpotent (see [7, Theorem 5.40]). By minimality of I we must have $J_0(N) \cap I = I$ or $\{0\}$. If $J_0(N) \cap I = I$, this would imply I being nilpotent. It follows from [7, Proposition 3.53] that $I^2 = \{0\}$, contradicting our assumption. Thus we must have $I \cap J_0(N) = \{0\}$. From our assumption we have that $J_0(I) \subseteq J_0(N)$. This implies $J_0(I) = \{0\}$ and simplicity by Corollary 4.8. □

Distributively generated near-rings are examples of near-rings where we have $J_0(K) \subseteq J_0(N)$ for each ideal K of the near-ring, see [7, Theorem 6.34]. It is known that a non-nilpotent minimal ideal in a d.g. near-ring with DCCN is a simple near-ring (cf. [5]). Here we have obtained it from another point of view.

Remark 4.10. Note that if we have a zero symmetric near-ring N which is a simple near-ring, satisfies DCCN and $N^2 \neq \{0\}$, then N is a 1-primitive near-ring. This result has been originally proved in [3]. We can also quickly deduce this fact now: By simplicity of N we have that N is a minimal ideal. Proposition 4.1 allows us to apply Lemma 4.7 and then Theorem 4.5, from which we have that $N/J_0(N)$ is a 1-primitive near-ring and $NJ_0(N) = \{0\}$. Thus, $J_0(N) \neq N$. Simplicity of N now forces $J_0(N) = \{0\}$ and we have that N is 1-primitive.

5. Examples

In this section the results obtained in this paper will be illustrated by numerous examples. The examples are chosen such that one can verify the calculations by easy computations with the help of the theory developed in this paper. To help our presentation, we introduce a well known notation. Let Γ be a (additively written) group with zero 0. Then, the set of all zero preserving functions $M_0(\Gamma) := \{f : \Gamma \to \Gamma \mid f(0) = 0\}$ is a near-ring with respect to pointwise addition of functions and function composition.

Of course, the statements of Theorem 4.5 and its corollaries are only of interest if the minimal ideal I sits inside the Jacobson radical $J_1(N)$ of the near-ring N. Otherwise the much stronger Theorems 3.5 and 3.10 apply. Can we find minimal ideals which sit inside $J_1(N)$ and which are not nilpotent at all? Does this happen frequently? If N is a ring and satisfies the DCCL it will not happen of course, since the Jacobson radical of an artinian ring is nilpotent.

Proposition 5.1. *Let N be a zero symmetric near-ring containing a minimal ideal H. Suppose that N is* 0*-primitive on the N-group Γ. Then N is a subdirectly irreducible near-ring, H is the unique non-trivial minimal ideal and $H^2 \neq \{0\}$.*

Proof. Suppose there is a non-zero ideal I such that $H \not\subseteq I$. By minimality of H, this implies $H \cap I = \{0\}$ and therefore, $HI = \{0\}$. Let γ be a generator of the N-group Γ. Since $(0 : \Gamma) = \{0\}$, $I \cap (0 : \Gamma) = \{0\}$ and $H \cap (0 : \Gamma) = \{0\}$, so Lemma 3.4 shows that $I\gamma = \Gamma = H\gamma$. Consequently, there is an $i \in I$ such that $\gamma = i\gamma$. Thus, $H\gamma = Hi\gamma = \{0\}$, a contradiction. Thus, N is subdirectly irreducible. $H^2 \neq \{0\}$ because $J_0(N) = \{0\}$ (see [7, Theorem 5.37]). $\square$

Following the notation used in [1, 2] we call the unique non-trivial minimal ideal of a subdirectly irreducible near-ring N the heart of N. The following corollary to Proposition 5.1 shows that we can find minimal ideals of interest as in Theorem 4.5 and Corollary 4.8 when looking at 0-primitive near-rings with DCCN, for example.

Corollary 5.2. *Let N be a zero symmetric 0-primitive near-ring with DCCN which is not a 1-primitive near-ring. Then N has a unique non-trivial minimal ideal H with $H \subseteq J_1(N)$ and H satisfying the assumptions of Theorem* 4.5.

Proof. From Proposition 5.1 we have that N is subdirectly irreducible with heart H. Suppose that $J_1(N) = \{0\}$. Then, N is a subdirect product of 1-primitive near-rings by [7, Theorem 5.29]. By subdirect irreducibility, we get that N itself must be 1-primitive, contradicting our assumptions. Thus, $J_1(N) \neq \{0\}$ and therefore $H \subseteq J_1(N)$. By 0-primitivity of N we have that $J_0(N) = \{0\}$ and therefore, by [7, Theorem 5.37], N contains no nilpotent ideals. Thus, H is not nilpotent. From Lemma 4.7 we see that each minimal left ideal L in H is non-nilpotent. From Proposition 4.1 we have that L does not contain properly N-groups which are N-isomorphic to L, so H is a minimal ideal as described in Theorem 4.5. □

From [2, Proposition 5.7.3] we have that once we have given a minimal ideal I with $I^2 \neq \{0\}$ in a zero symmetric near-ring N, then $N/(0 : I)$ is a subdirectly irreducible and 0-prime near-ring whose heart is isomorphic to I (this result was originally proved in [1]). 0-prime near-rings with DCCN are known to be 0-primitive, see [7, Theorem 5.40]. Thus, in case of zero symmetric near-rings N with DCCN it suffices to study the unique minimal ideal of 0-primitive near-rings to get results about non nilpotent minimal ideals up to isomorphism. This justifies that we focus on 0-primitive near-rings in the following examples.

The first example in this section presents a non-nilpotent minimal ideal I in a zero symmetric near-ring, where I is simple as a subnear-ring. This example is discussed for other purposes than ours in detail in [7, Examples 5.11 and 5.19], so we omit the details.

Example 5.3. Let $\Gamma := \mathbb{Z}_4$, the cyclic group of order 4. Let $S := \{0, 2\}$ the subgroup of order 2 in $\mathbb{Z}_4$. Consider the near-ring $N := \{f \in M_0(\mathbb{Z}_4) \mid f(S) \subseteq S\}$. It is shown in [7, Example 5.11] that N is 0-primitive but not 1-primitive on Γ and $J_2(N) = J_1(N) = (0 : S)$. From [7, Example 5.19] we have that $J_1(N)$ is 1-primitive on Γ, so by [7, Theorem 4.46] it is a simple near-ring. Thus, $J_1(N)$ is a non-nilpotent minimal ideal of the near-ring N which is a simple subnear-ring of N. Note that N is subdirectly irreducible by Proposition 5.1, so $J_1(N)$ is not a direct summand as an ideal, see Remark 3.9.

Examples of a non-nilpotent minimal ideal I in a zero symmetric near-ring where I is not simple as a subnear-ring seem to be rare in the near-ring literature. The first example of that kind was given by K. Kaarli and is presented in [2, Example 5.7.2], for example. We do not discuss the details here, only that it is the near-ring $N := \{f \in M_0(\mathbb{Z}_8) \mid f(\{0, 2, 4, 6\}) \subseteq \{0, 2, 4, 6\}$ and $\forall\, x \in \mathbb{Z}_8, f(5x) = 5f(x)\}$. N is 0-primitive on $\mathbb{Z}_8$ and has the non-nilpotent minimal ideal $I = (0 : \{0, 2, 4, 6\})$ which is not simple as a subnear-ring. Kaarli's example is not only the first one which has been presented in this line of discussion, it is also of interest because the near-ring has an identity element and abelian addition.

Our first example of a non-simple non-nilpotent minimal ideal in a zero symmetric near-ring is the example discussed in [7, Remark 4.50] and thus being available in the standard literature on near-rings (though not observed as an example of such a type in [7, Remark 4.50]). It will also show that Theorem 3.5 cannot be extended to the Jacobson radicals of type 1/2 and type 0 (since $J_0(N) \subseteq J_{1/2}(N)$) of a near-ring N. In [7, Remark 4.50] it is discussed as an example of a zero symmetric near-ring N which is 0-primitive and contains an ideal I which is not 0-primitive and hence not 1-primitive. So, according to Theorem 4.5 and Corollary 4.8, I is not simple. Of course, this is a situation we exactly need when obtaining an example of a non-nilpotent minimal ideal which is not simple as a subnear-ring (according to the results of Corollary 4.8). We only need to show that I is indeed a minimal ideal (this question was not considered in [7, Remark 4.50] and also not in [6, Example 6.42] where we find a discussion of Pilz's example, also). This will be done with the help of the following lemmas.

Lemma 5.4. [7, Lemma 1 of Theorem 3.54] *Let N be a zero symmetric near-ring with DCCN and Γ a faithful N-group. Let K be a minimal N-ideal of Γ. Let $\{0\} \neq L \subseteq (K : \Gamma)$ be a left ideal such that $\forall\, \gamma \in \Gamma$, $N\gamma = \Gamma$ or $L\gamma = \{0\}$. Then L is a finite direct sum of N-isomorphic minimal left ideals of N.*

Lemma 5.4 is one of the key lemmas in Scott's proof that a minimal ideal in a zero symmetric near-ring N with DCCN decomposes as a direct sum of minimal left ideals. A proof of this result can be found in [7].

The following lemma now will help us identifying when a given ideal I is indeed a minimal ideal.

Lemma 5.5. *Let N be a zero symmetric and 0-primitive near-ring with DCCN which is acting 0-primitively on Γ. Let $\theta_0 := \{\delta \in \Gamma \mid N\delta \neq \Gamma\}$. Then, $H := (0 : \theta_0)$ is the unique minimal ideal of N; H acts strongly monogenic and faithfully on Γ; H is a finite direct sum of minimal left ideals which are all N-isomorphic to Γ; Γ can be considered as an H-group containing a greatest proper H-ideal $\triangle_H$; and $J_0(H) = (0 : \Gamma/\triangle_H)$.*

Proof. For each $\delta \in \theta_0$ we have $N\delta \neq \Gamma$. Let $n \in N$ and $\delta \in \theta_0$. Then $N(n\delta) \subseteq N\delta \neq \Gamma$, so $n\delta \in \theta_0$. The annihilator $(0 : \theta_0)$ is a left ideal in N. Let $a \in (0 : \theta_0)$, $n \in N$ and $\delta \in \theta_0$. Then $(an)\delta = a(n\delta) = 0$ because $n\delta \in \theta_0$. This shows that $(0 : \theta_0)$ is an ideal in N.

From Proposition 5.1 we know that N is subdirectly irreducible with heart H, $H^2 \neq \{0\}$. From [7, Theorem 3.54] we know that H is the direct sum of minimal left ideals of

N. From Lemma 3.4 we know that for $\gamma \in \theta_1$, $H\gamma = \Gamma$. Thus, there is a minimal left ideal L in the sum decomposition of H, such that $L\gamma \neq \{0\}$. By [7, Proposition 3.10] we have $L \cong_N \Gamma$. Faithfulness of Γ (or Lemma 4.7) implies $L^2 \neq \{0\}$. Since N has DCCN, Proposition 4.1 applies and from Proposition 4.4 we know that $H \subseteq (0 : \theta_0^L)$. A straightforward calculation shoes that $L \cong_N \Gamma$ implies $H \subseteq (0 : \theta_0)$, in particular $(0 : \theta_0)$ is non-zero.

We now apply Lemma 5.4. By 0-primitivity, Γ is a faithful N-group and Γ is minimal as an N-ideal. We let $K := \Gamma$ and so, $(K : \Gamma) = N$. Hence, $\{0\} \neq (0 : \theta_0) \subseteq (K : \Gamma)$. Let $\gamma \in \theta_1$. Then, $N\gamma = \Gamma$. If $\gamma \in \theta_0$, then $(0 : \theta_0)\gamma = \{0\}$. So, Lemma 5.4 applies and we get that $(0 : \theta_0)$ is a finite direct sum of N-isomorphic minimal left ideals of N. Let $(0 : \theta_0) = \sum_{i=1}^{s} L_i$, $s \in \mathbb{N}$. Let $j \in \{1, \ldots, s\}$. By faithfulness of Γ and the fact that $L_j\theta_0 = \{0\}$, there is an element $\gamma_j \in \theta_1$ such that $L_j \not\subseteq (0 : \gamma_j)$. By [7, Proposition 3.10], we get $L_j \cong_N \Gamma$.

Suppose that H is properly contained in $(0 : \theta_0)$. Since we know that $(0 : \theta_0) = \sum_{i=1}^{s} L_i$, $s \in \mathbb{N}$ and the L_i being minimal left ideals of N, there must be a $j \in \{1, \ldots, s\}$ such that $L_j \not\subseteq H$. By minimality of L_j we get $L_j \cap H = \{0\}$ and hence $HL_j \in L_j \cap H = \{0\}$. By N-isomorphism of L_j and Γ this now implies $H\Gamma = \{0\}$, contradicting the faithfulness of Γ. Hence, $H = (0 : \theta_0)$ and $(0 : \theta_0)$ is the unique minimal ideal in N. Note that H acts faithfully and as a consequence of Lemma 3.4 strongly monogenic on Γ. Application of Lemma 3.3 shows that $H/J_0(H)$ is a 1-primitive near-ring and $J_0(H) = (0 : \Gamma/\triangle_H)$, where $\triangle_H$ is the greatest proper H-ideal contained in Γ. □

Note that $\triangle_H$ as in Lemma 5.5 may be $\{0\}$. Then we have that $J_0(H) = (0 : \Gamma) = \{0\}$ and H is 1 primitive (see Theorem 4.5), simple as a subnear-ring, respectively (see Corollary 4.8).

Example 5.6. Let $\Gamma := \mathbb{Z}_8$, the cyclic group of order 8. Let $S_2 := \{0, 2, 4, 6\}$ and $S_1 := \{0, 4\}$, the subgroups of order 4 and 2 in $\mathbb{Z}_8$. Consider the near-ring $N := \{f \in M_0(\mathbb{Z}_8) \mid f(S_2) \subseteq S_2$ and $f(5) = f(1)$ and $f(7) = f(3)\}$. It is shown in [7, Remark 4.50] that N acts 0-primitively on Γ with set of generators $\{1, 3, 5, 7\}$ and S_2 the set of non-generators. $H := (0 : S_2)$ has the H-ideal S_1 in Γ. In fact, these results can be calculated in a straightforward way, also. Since S_2 is the set of non-generators of the N-group Γ, we have from Lemma 5.5 that H is a minimal ideal, in fact the unique minimal ideal. Note that H acts faithfully and strongly monogenic on Γ.

Since H has the H-ideal S_1 in Γ it follows from Lemma 3.1 that there exists a greatest proper and non-zero H-ideal $\triangle_H$ in Γ. Lemma 5.5 shows that $J_0(H) = (0 : \Gamma/\triangle_H)$. In fact, S_1 is the greatest proper H-ideal in Γ, because the other possible candidate S_2 is not an H-ideal. This can be seen by taking the function $f_1 \colon \Gamma \to \Gamma$, $f(5) = f(1) = 1$,

$f(\gamma) = 0$ else. $f_1 \in (0 : S_2)$ and $f_1(3+2) - f_1(3) = 1 \notin S_2$. Since $2 \in S_2$, this shows that S_2 is not an H-ideal.

Consider the function $h\colon \Gamma \to \Gamma$, $h(\delta) = 0$, $\delta \in \theta_0$, $h(\gamma) = 4 \in S_1$, $\gamma \in \theta_1$. Then $h \in H$ and since $S_1 = \triangle_H$, $h \in (0 : \Gamma/\triangle_H)$. So we have $J_0(H) \neq \{0\}$ and H is not a simple near-ring.

It is also shown in [7, Remark 4.50] that $N = (0:1) \cap (0:3) \dot{+} H$. Since Γ is an N-group of type 0, $(0:1)$ and $(0:3)$ are 0-modular left ideals and therefore, $J_{1/2}(N) \subseteq (0:1) \cap (0:3)$. In fact, for a 0-primitive near-ring with finiteness condition we always have such a decomposition into two left ideals, one annihilating all the generators of the N-group of type 0 and the other annihilating all the non-generators, see [10]. Thus, H is not simple as an ideal and has zero intersection with $J_{1/2}(N)$. This shows that Theorem 3.5 cannot be extended to the Jacobson radicals of type 1/2 and type 0 (since $J_0(N) \subseteq J_{1/2}(N)$) of a near-ring N.

We want to give another example of a minimal ideal I in a 0-primitive near-ring N where I is non-nilpotent and not simple as a subnear-ring. Here we will have an N-group Γ of type 0 where the set of non-generators is not a subgroup of the N-group Γ. This has not been the case in Example 5.6 and also not in Kaarli's example.

Example 5.7. Let $\Gamma := \mathbb{Z}_{16}$, the cyclic group of order 16. Let $\theta_0 := \{0, 1, 4, 5, 8, 9, 12, 13\}$, $S_2 := \{0, 4, 8, 12\}$ and $S_1 := \{0, 8\}$, the subgroups of order 4 and 2 in $\mathbb{Z}_{16}$. Consider the near-ring $N := \{f \in M_0(\mathbb{Z}_{16}) \mid f(\theta_0) \subseteq S_2 \text{ and } f(2) = f(10) \text{ and } f(3) = f(11) \text{ and } f(6) = f(14) \text{ and } f(7) = f(15)\}$. Since for $f \in N$, $f(\theta_0)$ is contained in the subgroup S_2 of Γ, N is additively closed and is a near-ring. θ_0 is not a group, but a union of cosets with respect to S_1. Note that $\theta_1 = \{2, 3, 6, 7, 10, 11, 14, 15\}$ is the set of generators of the N-group Γ. Always two generators in the same coset with respect to S_1 will have the same function values by functions in N.

Suppose that S_2 is an N-ideal of Γ. Then, for each $f \in N$, $f(2+4) - f(2) = f(6) - f(2) \in S_2$. But 6 and 2 are both generators of the N-group which can be mapped independently by functions in N and so we can define the function $f_1\colon \Gamma \to \Gamma$, $f_1(14) = f_1(6) = 1$, $f_1(\gamma) = 0$ else and we see that $f_1 \in N$ and $f_1(6) - f_1(2) = 1 \notin S_2$. So, S_2 is not an N-ideal of Γ.

Suppose that S_1 is an N-ideal. Then, for each $f \in N$, we must have $f(4+8) - f(4) = f(12) - f(4) \in S_1$. By definition of N we only have to observe that $f(12) \in S_2$ and $f(4) \in S_2$, with no further restrictions on the function f. So, we can define the function $f_2\colon \Gamma \to \Gamma$, $f_2(12) = 12$, $f_2(\gamma) = 0$ else and we see that $f_2 \in N$ and $f_2(4+8) - f_2(4) = f_2(12) - f_2(4) = 12 \notin S_1$. So S_1 is not an N-ideal also. There are no other possible candidates for N-ideals in the faithful N-group Γ, so N acts 0-primitively on Γ. From Lemma 5.5 we know that $H := (0 : \theta_0)$ is a minimal ideal. Note that the function

$f_1 \in (0 : \theta_0)$, so we see that S_2 is not an H-ideal, also.

But S_1 is an H-ideal of Γ. To see this, we need to show that $h(\gamma+8) - h(\gamma) \in S_1$, $\gamma \in \Gamma$, $h \in H$. Let $h \in H$. In case $\gamma \in \theta_1$ we have by definition of N, that $h(\gamma+8) = h(\gamma)$ and we see that $h(\gamma+8) - h(\gamma) = 0 \in S_1$. In case $\gamma \in \theta_0$ we have that $\gamma + 8 \in \theta_0$ (θ_0 was taken to be a union of cosets with respect to S_1) and so we have that $h(\gamma+8) = 0$ and $h(\gamma) = 0$ and so again, $h(\gamma+8) - h(\gamma) \in S_1$. So, S_1 is an H-ideal.

Since H has the H-ideal S_1 in Γ it follows from Lemma 3.1 that there exists a greatest proper and non-zero H-ideal $\triangle_H$ in Γ, which is S_1 itself (S_2 is not an H-ideal as we have seen and the set θ_0 does not contain further non-trivial subgroups). Lemma 5.5 shows that $J_0(H) = (0 : \Gamma/\triangle_H) = (0 : \Gamma/S_1)$.

Consider the function $h \colon \Gamma \to \Gamma$, $h(\delta) = 0$, $\delta \in \theta_0$, $h(\gamma) = 8 \in S_1$, $\gamma \in \theta_1$. Then $h \in H$ and $h \in (0 : \Gamma/S_1)$. So we have $J_0(H) \neq \{0\}$ and H is not a simple near-ring.

Now we present an example of a near-ring which has non-abelian addition and some other interesting property we will outline during our discussion.

Example 5.8. Let Γ be a non-abelian group containing a normal subgroup S_1 which is contained in a subgroup S_2 of Γ which is not normal and such that S_1 is the unique non-trivial normal subgroup of Γ contained in S_2. An example of such a group is the dihedral group D_6 of order 12. $D_6 = \{a^i b^j \mid 0 \leq i \leq 5, 0 \leq j \leq 1, b^2 = e = a^6, ab = ba^5\}$, e denoting the neutral element of the group. D_6 has a normal subgroup of order 2, namely the group $S_1 = \{e, a^3\}$ and a subgroup $S_2 = \{e, a^3, b, ba^3\}$ of order 4 containing S_1. S_2 is not a normal subgroup of D_6 (for example, $a^2 b (a^2)^{-1} = a^2 b a^4 = ba^2 \notin S_2$).

Let $N := \{f \in M_0(\Gamma) \mid f(S_2) \subseteq S_2 \text{ and } \forall \gamma, \delta \in \Gamma \setminus S_2, f(\gamma) = f(\delta)\}$. N is a zero symmetric near-ring which operates on the N-group Γ, the set of generators θ_1 of this N-group being $\Gamma \setminus S_2$. Each generator must be mapped to the same element, this element can be chosen arbitrarily, by a function in N. Clearly, Γ is a faithful N-group and each proper N-ideal must be contained in S_2. Now S_2 itself is an N-subgroup but not an N-ideal of Γ, because S_2 is not normal in Γ. S_1 is not an N-ideal in Γ. To see this, let $s_1 \in S_1 \setminus \{0\}$ and take an element $s_2 \in S_2 \setminus S_1$. Then we can define the function $f \colon \Gamma \to \Gamma$, $f(s_1) = s_2$, $f(\gamma) = 0$, else. $f \in N$ but $f(s_1) \notin S_1$ and S_1 is not an N-subgroup and therefore not an N-ideal.

Thus, N acts 0-primitively on Γ. Lemma 5.5 shows that $H := (0 : S_2)$ is a minimal ideal of N. H is acting faithfully and strongly monogenic on Γ.

But now, S_1 is an H-ideal of Γ. First, S_1 is a normal subgroup of Γ. Then, let $h \in H$ and $s_2 \in S_2$ and $s_1 \in S_1$. Thus, $s_2 + s_1 \in S_2$ and so, $h(s_2 + s_1) - h(s_2) = 0 - 0 = 0 \in S_1$. Let $h \in H$, $\gamma \in \Gamma \setminus S_2$ and $s_1 \in S_1$. Then, $\gamma + s_1 \in \Gamma \setminus S_2$. Thus, $h(\gamma + s_1) = h(\gamma)$ by definition of functions in N. So we see that $h(\gamma + s_1) - h(\gamma) = 0 \in S_1$ and we have that S_1 is an H-ideal.

Since H has the H-ideal S_1 in Γ it follows from Lemma 3.1 that there exists a greatest proper and non-zero H-ideal $\triangle_H$ in Γ (which is the H-ideal S_1 itself, since it is the only normal subgroup contained in S_2). Lemma 5.5 shows that $J_0(H) = (0 : \Gamma/\triangle_H)$. Let $s_1 \in S_1 \setminus \{0\}$. Consider the function $h \colon \Gamma \to \Gamma$, $h(\delta) = 0$, $\delta \in \theta_0$, $h(\gamma) = s_1$, $\gamma \in \theta_1$. Then $h \in H$ and since $S_1 \subseteq \triangle_H$, $h \in (0 : \Gamma/\triangle_H)$. So we have $J_0(H) \neq \{0\}$ and H is not a simple near-ring.

In particular, H is an ideal in a 0-primitive near-ring which is not 0-primitive as a subnear-ring. In [7, Remark 4.50] it says that seemingly the near-ring in [7, Remark 4.50] which is discussed in Example 5.6 is the smallest 0-primitive near-ring in size which hosts a proper ideal which is not a 0-primitive near-ring. The Example 5.6 has 4096 elements. Here we see that there is a smaller 0-primitive near-ring N containing an ideal which is not 0-primitive. Take $\Gamma = D_6$, $S_2 = \{e, a^3, b, ba^3\}$ and $S_1 = \{e, a^3\}$. Then, N has order $12 \cdot 4^3 = 768$. Moreover, N has non-abelian addition. Note that the order of $H = (0 : S_2)$ is only 12 and $H \cong_N \Gamma$ (see Lemma 5.5). So we have found an example of a small minimal ideal which is non-nilpotent and not simple. If this is the smallest possible order of an ideal of such a kind is not known to the author.

Finally, we want to give an example of a minimal ideal in a zero symmetric near-ring N which as a subnear-ring is not subdirectly irreducible. The first example of such a kind was given in [10]. It was a long standing open question in the pure structure theory of near-rings whether such examples exist at all (see [10] for references). Here we give another example of such a type.

Example 5.9. Let $\Gamma := \mathbb{Z}_{36}$, θ_0 the subgroup of order 18 consisting of the multiples of 2, and $S_1 := \{0, 18\}$ and $S_2 := \{0, 12, 24\}$. Let $\triangle := \{0, 6, 12, 18, 24, 30\}$. Note that $\triangle = S_1 + S_2$. Consider the near-ring $N := \{f \in M_0(\mathbb{Z}_{36}) \mid f(\theta_0) \subseteq \theta_0 \text{ and } \forall z \in \{1, 3, 5\}, \forall \delta \in \triangle, f(z) = f(z + \delta)\}$. N is additively closed because θ_0 is a subgroup of $\mathbb{Z}_{36}$, so N is a subnear-ring. Note that the odd numbers in $\mathbb{Z}_{36}$ is the set of generators θ_1 of the N-group Γ, N acts on faithfully. Note that the elements $1, 7, 13, 19, 25, 31$, which is the coset of 1 with respect to $\triangle$, are always mapped to the same element in $\mathbb{Z}_{36}$ by functions in N and the same is the case with the elements $3, 9, 15, 21, 27, 33$, the coset of 3 with respect to $\triangle$ and $5, 11, 17, 23, 29, 35$, the coset of 5 with respect to $\triangle$.

θ_0 is an N-subgroup of Γ. Suppose it is an N-ideal also. Then, for all $f \in N$ we must have $f(1+2) - f(1) = f(3) - f(1) \in \theta_0$. 1 and 3 are elements in θ_1 which can be mapped independently by elements in N and we see that θ_0 is not an N-ideal.

As we will point out in the following, the non-trivial proper subgroups of θ_0 are no N-subgroups and so, by zero symmetry of the near-ring, they cannot be N-ideals. First, S_1 is not an N-subgroup. This is because for a given element $\delta \in \theta_0$ and $\delta_1 \in \theta_0 \setminus S_1$ we can define the function $f \in N$ with $f(\delta) = \delta_1$ and $f(\gamma) = 0$, else. Then we have $f(S_1) \not\subseteq S_1$.

Similarly we see that S_2 and $\triangle$ are not N-subgroups and also not the 9-element group of the multiplies of 4 contained in θ_0. This proves that N acts 0-primitively on Γ.

Let $H := (0 : \theta_0)$. Application of Lemma 5.5 now shows that $(0 : \theta_0)$ is a minimal ideal. We now show that S_1 and S_2 are H-ideals and thus, so is $\triangle$.

We first show that S_1 is an H-ideal of Γ. Let $h \in H$. We need to show that $h(\gamma + 18) - h(\gamma) \in H$, $\gamma \in \Gamma$. In case $\gamma \in \theta_1$ we have by definition of N, that $h(\gamma + 18) = h(\gamma)$ and we see that $h(\gamma + 8) - h(\gamma) = 0 \in S_1$. In case $\gamma \in \theta_0$ we have that $\gamma + 18 \in \theta_0$ and so we have that $h(\gamma + 18) = 0$ and $h(\gamma) = 0$ and so again, $h(\gamma + 18) - h(\gamma) \in S_1$. Thus, S_1 is an H-ideal.

We now show that S_2 is an H-ideal of Γ. Let $h \in H$. We need to show that $h(\gamma + 12) - h(\gamma) \in H$, $\gamma \in \Gamma$ and $h(\gamma + 24) - h(\gamma) \in H$, $\gamma \in \Gamma$. In case $\gamma \in \theta_1$ we have by definition of N, that $h(\gamma + 12) = h(\gamma)$ and we see that $h(\gamma + 12) - h(\gamma) = 0 \in S_2$. In case $\gamma \in \theta_0$ we have that $\gamma + 12 \in \theta_0$ and so we have that $h(\gamma + 12) = 0$ and $h(\gamma) = 0$ and so again, $h(\gamma + 12) - h(\gamma) \in S_2$. In the same way we see that $h(\gamma + 24) - h(\gamma) \in S_2$, $\gamma \in \Gamma$. Thus, S_2 is an H-ideal.

So, also $S_1 + S_2 = \triangle$ is an H-ideal.

But now we can also show that there are two non-zero ideals I_1 and I_2 of the subnear-ring H with $I_1 \cap I_2 = \{0\}$, proving that H is not a subdirectly irreducible near-ring. We consider Γ as an H-group. Note that, as S_1 is an H-ideal, we have that $I_1 := (0 : \Gamma/S_1)$ is an ideal of H, Γ/S_1 being an H-group. Also, $I_2 := (0 : \Gamma/S_2)$ is an ideal of H. We need to show that these are non-zero ideals of H. Let $s_1 \in S_1 \setminus \{0\}$. Consider the function $i_1 : \Gamma \to \Gamma$, $i_1(\delta) = 0$, $\delta \in \theta_0$, $i_1(\gamma) = s_1$, $\gamma \in \theta_1$. Then $i_1 \in H$ and $i_1 \in (0 : \Gamma/S_1)$. In the same way we can construct a function $i_2 \in H$ and $i_2 \in (0 : \Gamma/S_2)$. So, I_1 and I_2 are two non-zero ideals of the subnear-ring H. Let $i \in I_1 \cap I_2$. Thus, $i(\Gamma) \subseteq S_1 \cap S_2 = \{0\}$. By faithfulness of Γ, $i = 0$. Therefore, $I_1 \cap I_2 = \{0\}$ and H is not a subdirectly irreducible near-ring.

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