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Schur Product with Operator-valued Entries

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Abstract. In this paper we characterize Toeplitz matrices with entries in the space of bounded operators on Hilbert spaces $\mathcal{B}(H)$ which define bounded operators acting on $\ell^2(H)$ and use it to get the description of the right Schur multipliers acting on $\ell^2(H)$ in terms of certain operator-valued measures.

1. Introduction

Throughout the paper X, Y and E are complex Banach spaces and H denotes a separable complex Hilbert space with orthonormal basis (e_n). We write $\mathcal{L}(X,Y)$ for the space of bounded linear operators, X^* for the dual space and denote $\mathcal{B}(X) = \mathcal{L}(X,X)$. We also use the notations $\ell^2(E)$, $C(\mathbb{T},E)$, $L^p(\mathbb{T},E)$ or $M(\mathbb{T},E)$ for the space of sequences $\mathbf{z} = (z_n)$ in E such that $\|\mathbf{z}\|_{\ell^2(E)} = \big(\sum_{n=1}^{\infty} \|z_n\|^2\big)^{1/2} < \infty$, the space of E-valued continuous functions, the space of strongly measurable functions from the measure space $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ into E with $\|f\|_{L^p(\mathbb{T},E)} = \big(\int_0^{2\pi} \|f(e^{it})\|^p \frac{dt}{2\pi}\big)^{1/p} < \infty$ for $1 \leq p \leq \infty$ (with the usual modification for $p = \infty$) and the space of regular vector-valued measures of bounded variation respectively. As usual, for $E = \mathbb{C}$ we simply write ℓ^2, $C(\mathbb{T})$, $L^p(\mathbb{T})$ and $M(\mathbb{T})$.

Given two matrices $A = (\alpha_{kj})$ and $B = (\beta_{kj})$ with complex entries, their Schur product is defined by $A * B = (\alpha_{kj}\beta_{kj})$. This operation endows the space $\mathcal{B}(\ell^2)$ with a structure of Banach algebra. A proof of the next result, due to J. Schur, can be found in [2, Proposition 2.1] or [10, Theorem 2.20].

Theorem 1.1. (Schur, [12]) *If $A = (\alpha_{kj}) \in \mathcal{B}(\ell^2)$ and $B = (\beta_{kj}) \in \mathcal{B}(\ell^2)$ then $A * B \in \mathcal{B}(\ell^2)$. Moreover $\|A * B\|_{\mathcal{B}(\ell^2)} \leq \|A\|_{\mathcal{B}(\ell^2)} \|B\|_{\mathcal{B}(\ell^2)}$.*

More generally, a matrix $A = (\alpha_{kj})$ is said to be a Schur multiplier, to be denoted by $A \in \mathcal{M}(\ell^2)$, whenever $A * B \in \mathcal{B}(\ell^2)$ for any $B \in \mathcal{B}(\ell^2)$. For the study of Schur multipliers we refer the reader to [2, 10]. Recall that a Toeplitz matrix is a matrix $A = (\alpha_{kj})$ such that there exists a sequence of complex numbers $(\gamma_l)_{l\in\mathbb{Z}}$ with $\alpha_{kj} = \gamma_{j-k}$. The study of

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Toeplitz matrices which define bounded operators or Schur multipliers goes back to work of Toeplitz in [15]. The reader is referred to $[1,2,10]$ for recent proofs of the following results concerning Toeplitz matrices.

Theorem 1.2. (Toeplitz, [15]) *Let* $A = (\alpha_{kj})$ *be a Toeplitz matrix. Then* $A \in \mathcal{B}(\ell^2)$ *if and only if there exists* $f \in L^\infty(\mathbb{T})$ *such that* $\alpha_{kj} = \widehat{f}(j-k)$ *for all* $k, j \in \mathbb{N}$. *Moreover* $\|A\| = \|f\|_{L^\infty(\mathbb{T})}$.

Theorem 1.3. (Bennet, [2]) *Let* $A = (\alpha_{kj})$ *be a Toeplitz matrix. Then* $A \in \mathcal{M}(\ell^2)$ *if and only if there exists* $\mu \in M(\mathbb{T})$ *such that* $\alpha_{kj} = \widehat{\mu}(j-k)$ *for all* $k, j \in \mathbb{N}$. *Moreover* $\|A\| = \|\mu\|_{M(\mathbb{T})}$.

It is known the recent interest for operator-valued functions (see [9]) and for the matricial analysis (see [10]) concerning their uses in different problems in Analysis. In this paper, we would like to formulate the analogues of the theorems above in the context of matrices $\mathbf{A} = (T_{kj})$ with entries $T_{kj} \in \mathcal{B}(H)$. For such a purpose, we are led to consider operator-valued measures. We shall make use of several notions and spaces from the theory of vector-valued measures and the reader is referred to classical books $[6,7]$ or to [3] for some new results in connection with Fourier analysis.

In the sequel we write $\langle \cdot , \cdot \rangle$ and $\langle\langle \cdot , \cdot \rangle\rangle$ for the scalar products in H and $\ell^2(H)$ respectively, where $\langle\langle \mathbf{x}, \mathbf{y} \rangle\rangle = \sum_{j=1}^\infty \langle x_j, y_j \rangle$ and we use the notation $x\mathbf{e}_j = (0, \dots, 0, x, 0, \dots)$ for the element in $\ell^2(H)$ in which $x \in H$ is placed in the j-coordinate for $j \in \mathbb{N}$. As usual, $c_{00}(H) = \operatorname{span}\{x\mathbf{e}_j : x \in H, j \in \mathbb{N}\}$.

Definition 1.4. Given a matrix $\mathbf{A} = (T_{kj})$ with entries $T_{kj} \in \mathcal{B}(H)$ and $\mathbf{x} \in c_{00}(H)$ we write $\mathbf{A}(\mathbf{x})$ for the sequence $\left(\sum_{j=1}^\infty T_{kj}(x_j)\right)_k$. We say that $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ if the map $\mathbf{x} \to \mathbf{A}(\mathbf{x})$ extends to a bounded linear operator in $\ell^2(H)$, that is

$$\left(\sum_{k=1}^\infty \left\| \sum_{j=1}^\infty T_{kj}(x_j) \right\|^2 \right)^{1/2} \leq C \left(\sum_{j=1}^\infty \|x_j\|^2 \right)^{1/2}.$$

We shall write

$$\|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} = \inf\{C \geq 0 : \|\mathbf{A}\mathbf{x}\|_{\ell^2(H)} \leq C \|\mathbf{x}\|_{\ell^2(H)}\}.$$

Definition 1.5. Given two matrices $\mathbf{A} = (T_{kj})$ and $\mathbf{B} = (S_{kj})$ with entries $T_{kj}, S_{kj} \in \mathcal{B}(H)$ we define the Schur product $\mathbf{A} * \mathbf{B} = (T_{kj} S_{kj})$ where $T_{kj} S_{kj}$ stands for the composition of the operators T_{kj} and S_{kj}.

Contrary to the scalar-valued case this product is not commutative.

Definition 1.6. Given a matrix $\mathbf{A} = (T_{kj})$, we say that $\mathbf{A}$ is a right Schur multiplier (respectively left Schur multiplier), to be denoted by $\mathbf{A} \in \mathcal{M}_r(\ell^2(H))$ (respectively $\mathbf{A} \in \mathcal{M}_l(\ell^2(H))$), whenever $\mathbf{B} * \mathbf{A} \in \mathcal{B}(\ell^2(H))$ (respectively $\mathbf{A} * \mathbf{B} \in \mathcal{B}(\ell^2(H))$) for any $\mathbf{B} \in \mathcal{B}(\ell^2(H))$. We shall write

$$\|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))} = \inf\{C \geq 0 : \|\mathbf{B} * \mathbf{A}\|_{\mathcal{B}(\ell^2(H))} \leq C\|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}\}$$

and

$$\|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))} = \inf\{C \geq 0 : \|\mathbf{A} * \mathbf{B}\|_{\mathcal{B}(\ell^2(H))} \leq C\|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}\}.$$

Denoting by $\mathbf{A}^*$ the adjoint matrix given by $S_{kj} = T^*_{jk}$ for all $k, j \in \mathbb{N}$, one easily sees that $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ if and only if $\mathbf{A}^* \in \mathcal{B}(\ell^2(H))$ with $\|\mathbf{A}\| = \|\mathbf{A}^*\|$ and also that $\mathbf{A} \in \mathcal{M}_l(\ell^2(H))$ if and only if $\mathbf{A}^* \in \mathcal{M}_r(\ell^2(H))$ and $\|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))} = \|\mathbf{A}^*\|_{\mathcal{M}_r(\ell^2(H))}$.

If X and Y are Banach spaces we write $X\widehat{\otimes}Y$ for the projective tensor product. We refer the reader to [6, Chapter 8], [11, Chapter 2] or [4] for all possible results needed in the paper. We recall that $(X\widehat{\otimes}Y)^* = \mathcal{L}(X, Y^*)$ and to avoid misunderstandings, for each $T \in \mathcal{L}(X, Y^*)$, we write $\mathcal{J}T$ when T is seen as an element in $(X\widehat{\otimes}Y)^*$. In other words, we write $\mathcal{J}\colon \mathcal{L}(X, Y^*) \to (X\widehat{\otimes}Y)^*$ for the isometry given by $\mathcal{J}T(x \otimes y) = T(x)(y)$ for any $T \in \mathcal{L}(X, Y^*)$, $x \in X$ and $y \in Y$. Also, given $x^* \in X^*$ and $y^* \in Y^*$, we write $\widetilde{x^* \otimes y^*}$ for the operator in $\mathcal{L}(X, Y^*)$ given by $\widetilde{x^* \otimes y^*}(z) = x^*(z)y^*$ for each $z \in X$. In the paper we shall restrict ourselves to the case $\mathcal{L}(X, Y^*) = \mathcal{B}(H)$, that is $X = Y^* = H$. Using the Riesz theorem we identify $Y = Y^* = H$. Hence, for $T, S \in \mathcal{B}(H)$ and $x, y \in H$, we shall use the following formulae

$$\langle T(x), y\rangle = \mathcal{J}T(x \otimes y),$$

$$(\widetilde{x \otimes y})(z) = \langle z, x\rangle y, \quad z \in H,$$

$$T(\widetilde{x \otimes y}) = (\widetilde{x \otimes (Ty)}), \quad (\widetilde{x \otimes y})T = (\widetilde{T^*x) \otimes y},$$

$$\mathcal{J}(TS)(x \otimes y) = \mathcal{J}T(Sx \otimes y) = \mathcal{J}S(x \otimes T^*y).$$

The paper is divided into four sections. The first section is of a preliminary character and we recall the basic notions on vector-valued sequences and functions to be used in the sequel. Next section contains several results on regular operator-valued measures which are the main ingredients for the remaining proofs in the paper. In Section 4 we are concerned with several necessary and sufficient conditions for a matrix $\mathbf{A}$ to belong to $\mathcal{B}(\ell^2(H))$ and we show that the Schur product endows $\mathcal{B}(\ell^2(H))$ with a Banach algebra structure also in this case. The final section deals with Toeplitz matrices $\mathbf{A}$ with entries in $\mathcal{B}(H)$, that is those matrices for which there exists a sequence $(T_l)_{l\in\mathbb{Z}} \subset \mathcal{B}(H)$ so that $T_{kj} = T_{j-k}$. We shall write $\mathcal{T}$ the family of such Toeplitz matrices and we characterize $\mathcal{T} \cap \mathcal{B}(\ell^2(H))$ as those matrices where $T_{kj} = \widehat{\mu}(j - k)$ for a certain regular operator-valued vector measure

μ belonging to $V^\infty(\mathbb{T}, \mathcal{B}(H))$ (see Definition 3.6 below). Concerning the analogue of Theorem 1.3 we shall show that $M(\mathbb{T}, \mathcal{B}(H)) \subseteq \mathcal{M}_r(\ell^2(H)) \subseteq M_{\mathrm{SOT}}(\mathbb{T}, \mathcal{B}(H))$ where $M(\mathbb{T}, \mathcal{B}(H))$ stands for the space of regular operator-valued measures and $M_{\mathrm{SOT}}(\mathbb{T}, \mathcal{B}(H))$ is defined, using the strong operator topology, as the space of vector measures μ such that $\mu_x \in M(\mathbb{T}, H)$ where $\mu_x(A) = \mu(A)(x)$ for any $x \in H$.

2. Preliminaries on operator-valued sequences and functions

Write $\ell^2_{\mathrm{weak}}(\mathbb{N}, \mathcal{B}(H))$ and $\ell^2_{\mathrm{weak}}(\mathbb{N}^2, \mathcal{B}(H))$ for the space of sequences $\mathbf{T} = (T_n) \subset \mathcal{B}(H)$ and matrices $\mathbf{A} = (T_{kj}) \subset \mathcal{B}(H)$ such that

$$\|\mathbf{T}\|_{\ell^2_{\mathrm{weak}}(\mathbb{N}, \mathcal{B}(H))} = \sup_{\|x\|=1, \|y\|=1} \left(\sum_{n=1}^\infty |\langle T_n(x), y\rangle|^2 \right)^{1/2} < \infty$$

and

$$\|\mathbf{A}\|_{\ell^2_{\mathrm{weak}}(\mathbb{N}^2, \mathcal{B}(H))} = \sup_{\|x\|=1, \|y\|=1} \left(\sum_{k=1}^\infty \sum_{j=1}^\infty |\langle T_{kj}(x), y\rangle|^2 \right)^{1/2} < \infty.$$

The reader can see that these spaces actually coincide with the ones appearing using notation in [5]. Of course $\ell^2(E) \subsetneq \ell^2_{\mathrm{weak}}(E)$. In the case $\mathcal{B}(H)$ we can actually introduce certain spaces between $\ell^2(E)$ and $\ell^2_{\mathrm{weak}}(E)$.

Definition 2.1. Given a sequence $\mathbf{T} = (T_n)$ and a matrix $\mathbf{A} = (T_{kj})$ of operators in $\mathcal{B}(H)$, we write

$$\|\mathbf{T}\|_{\ell^2_{\mathrm{SOT}}(\mathbb{N}, \mathcal{B}(H))} = \sup_{\|x\|=1} \left(\sum_{n=1}^\infty \|T_n(x)\|^2 \right)^{1/2}$$

and

$$\|\mathbf{A}\|_{\ell^2_{\mathrm{SOT}}(\mathbb{N}^2, \mathcal{B}(H))} = \sup_{\|x\|=1} \left(\sum_{j=1}^\infty \sum_{k=1}^\infty \|T_{kj}(x)\|^2 \right)^{1/2}.$$

We set $\ell^2_{\mathrm{SOT}}(\mathbb{N}, \mathcal{B}(H))$ and $\ell^2_{\mathrm{SOT}}(\mathbb{N}^2, \mathcal{B}(H))$ for the spaces of sequences and operators with $\|\mathbf{T}\|_{\ell^2_{\mathrm{SOT}}(\mathbb{N}, \mathcal{B}(H))} < \infty$ and $\|\mathbf{A}\|_{\ell^2_{\mathrm{SOT}}(\mathbb{N}^2, \mathcal{B}(H))} < \infty$ respectively.

Remark 2.2. It is easy to show that

$$\ell^2(\mathbb{N}^2, \mathcal{B}(H)) \subsetneq \ell^2(\mathbb{N}, \ell^2_{\mathrm{SOT}}(\mathbb{N}, \mathcal{B}(H)) \subsetneq \ell^2_{\mathrm{SOT}}(\mathbb{N}^2, \mathcal{B}(H)).$$

As usual, we denote $\varphi_k(t) = e^{ikt}$ for $k \in \mathbb{Z}$, and, given a complex Banach space E, we write $\mathcal{P}(\mathbb{T}, E) = \mathrm{span}\{e\varphi_j : j \in \mathbb{Z}, e \in E\}$ for the E-valued trigonometric polynomials, $\mathcal{P}_a(\mathbb{T}, E) = \mathrm{span}\{e\varphi_j : j \in \mathbb{N}, e \in E\}$ for the E-valued analytic polynomials. It is well-known that $\mathcal{P}(\mathbb{T}, E)$ is dense in $C(\mathbb{T}, E)$ and $L^p(\mathbb{T}, E)$ for $1 \leq p < \infty$. Also, we shall use $H^2_0(\mathbb{T},E)=\{f\in L^2(\mathbb{T},E):\widehat{f}(k)=0, k\le 0\}$, where $\widehat{f}(k)=\int_0^{2\pi} f(t)\overline{\varphi_k(t)}\,\frac{dt}{2\pi}$ for $k\in\mathbb{Z}$. Recall that $H^2_0(\mathbb{T},E)$ coincides with the closure of $\mathcal{P}_a(\mathbb{T},E)$ with the norm in $L^2(\mathbb{T},E)$. Similarly $H^2_0(\mathbb{T}^2,E)=\{f\in L^2(\mathbb{T}^2,E):\widehat{f}(k,j)=0, k,j\le 0\}$, where $\widehat{f}(k,j)=\int_0^{2\pi}\int_0^{2\pi} f(t,s)\overline{\varphi_k(t)\varphi_j(s)}\,\frac{dt}{2\pi}\frac{ds}{2\pi}$ for $k,j\in\mathbb{Z}$.

Let us now introduce some new spaces that we shall need later on.

Definition 2.3. Let $\mathbf{T}=(T_n)\subset\mathcal{B}(H)$ and $\mathbf{A}=(T_{kj})\subset\mathcal{B}(H)$. We say that $\mathbf{T}\in\widetilde{H}^2(\mathbb{T},\mathcal{B}(H))$ whenever

$$\|\mathbf{T}\|_{\widetilde{H}^2(\mathbb{T},\mathcal{B}(H))}=\sup_N\left(\int_0^{2\pi}\left\|\sum_{j=1}^N T_j\varphi_j(t)\right\|^2\frac{dt}{2\pi}\right)^{1/2}<\infty.$$

We say that $\mathbf{A}\in\widetilde{H}^2(\mathbb{T}^2,\mathcal{B}(H))$ whenever

$$\|\mathbf{A}\|_{\widetilde{H}^2(\mathbb{T}^2,\mathcal{B}(H))}=\sup_{N,M}\left(\int_0^{2\pi}\int_0^{2\pi}\left\|\sum_{j=1}^N\sum_{k=1}^M T_{kj}\varphi_j(t)\varphi_k(s)\right\|^2\frac{dt}{2\pi}\frac{ds}{2\pi}\right)^{1/2}<\infty.$$

Remark 2.4. $\widetilde{H}^2(\mathbb{T},\mathcal{B}(H))\nsubseteq H^2_0(\mathbb{T},\mathcal{B}(H))$.

Consider $T_j=\widetilde{e_j\otimes e_j}$. Then for any $t\in[0,2\pi)$ and $N\in\mathbb{N}$,

$$\left\|\sum_{j=1}^N(\widetilde{e_j\otimes e_j})\varphi_j(t)\right\|_{\mathcal{B}(H)}=\sup_{\|x\|=1}\left\|\sum_{j=1}^N\langle x,e_j\rangle\varphi_j(t)e_j\right\|_H=1.$$

Hence we have $\mathbf{T}=(e_j\otimes e_j)_j\in\widetilde{H}^2(\mathbb{T},\mathcal{B}(H))$. On the other hand, since $\|T_j\|=1$ for all j, we have $\lim_{j\to\infty}\|T_j\|=1\neq 0$, which implies that $\mathbf{T}\notin L^1(\mathbb{T},\mathcal{B}(H))$ and so $\mathbf{T}\notin H^2_0(\mathbb{T},\mathcal{B}(H))$, as desired.

Proposition 2.5. (i) $\widetilde{H}^2(\mathbb{T},\mathcal{B}(H))\subsetneq\ell^2_{\mathrm{SOT}}(\mathbb{N},\mathcal{B}(H))$ *and* $\widetilde{H}^2(\mathbb{T}^2,\mathcal{B}(H))\subsetneq\ell^2_{\mathrm{SOT}}(\mathbb{N}^2,\mathcal{B}(H))$.

(ii) $\widetilde{H}^2(\mathbb{T},\mathcal{B}(H))\nsubseteq\ell^2(\mathbb{N},\mathcal{B}(H))$ *and* $\ell^2(\mathbb{N},\mathcal{B}(H))\nsubseteq\widetilde{H}^2(\mathbb{T},\mathcal{B}(H))$.

Proof. (i) Both inclusions are immediate from Plancherel's theorem (which holds for Hilbert-valued functions). It suffices to see that there exists $\mathbf{T}\in\ell^2_{\mathrm{SOT}}(\mathbb{N},\mathcal{B}(H))\setminus\widetilde{H}^2(\mathbb{T},\mathcal{B}(H))$ because choosing matrices with a single row we obtain also a counterexample for the other inclusion. Now selecting $T_n=\widetilde{e_n\otimes x}\in\mathcal{B}(H)$ for a given $x\in H$ we clearly have $\mathbf{T}=(\widetilde{e_n\otimes x})_n\in\ell^2_{\mathrm{SOT}}(\mathbb{N},\mathcal{B}(H))$ with $\|\mathbf{T}\|_{\ell^2_{\mathrm{SOT}}(\mathbb{N},\mathcal{B}(H))}=\|x\|$. However, for any $t\in[0,2\pi)$ and $N\in\mathbb{N}$,

$$\left\|\sum_{n=1}^N(\widetilde{e_n\otimes x})\varphi_n(t)\right\|_{B(H)}=\left\|\widetilde{\left(\sum_{n=1}^N e_n\varphi_n(t)\right)\otimes x}\right\|_{\mathcal{B}(H)}=\|x\|\sqrt{N},$$

showing that $\mathbf{T} \notin \widetilde{H}^2(\mathbb{T}, \mathcal{B}(H))$.

(ii) The example in Remark 2.4 shows that $\widetilde{H}^2(\mathbb{T}, \mathcal{B}(H)) \nsubseteq \ell^2(\mathbb{N}, \mathcal{B}(H))$. Let us now find $\mathbf{T} \in \ell^2(\mathbb{N}, \mathcal{B}(H)) \setminus \widetilde{H}^2(\mathbb{T}, \mathcal{B}(H))$. Consider $H = L^2(\mathbb{T})$ and $\mathbf{T} = (T_j)$ where $T_j \colon L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is given by $T_j(f) = \frac{\varphi_j}{j} f$.

Clearly $\mathbf{T} \in \ell^2(\mathbb{N}, \mathcal{B}(H))$ since $\|T_j\| = 1/j$ for all $j \in \mathbb{N}$. On the other hand, for each $t \in [0, 2\pi)$ and $N \in \mathbb{N}$ one has that $\big(\sum_{j=1}^N T_j \varphi_j(t)\big)(f) = \big(\sum_{j=1}^N \frac{\varphi_j(t)}{j} \varphi_j\big) f$ and therefore

$$\left\| \sum_{j=1}^N T_j \varphi_j(t) \right\|_{B(H)} = \left\| \sum_{j=1}^N \frac{\varphi_j(t)}{j} \varphi_j \right\|_{C(\mathbb{T})} = \sum_{j=1}^N \frac{1}{j}.$$

This shows that $\mathbf{T} \notin \widetilde{H}^2(\mathbb{T}, \mathcal{B}(H))$. □

3. Preliminaries on regular vector measures

We recall some facts for vector measures that can be found in [6, 7]. Let us consider the measure space $(\mathbb{T}, \mathfrak{B}(\mathbb{T}), m)$ where $\mathfrak{B}(\mathbb{T})$ stands for the Borel sets over $\mathbb{T}$ and m for the Lebesgue measure on $\mathbb{T}$. Given a vector measure $\mu \colon \mathfrak{B}(\mathbb{T}) \to E$ and $B \in \mathfrak{B}(\mathbb{T})$, we shall denote $|\mu|(B)$ and $\|\mu\|(B)$ the variation and semi-variation of μ of the set B given by

$$|\mu|(B) = \sup \left\{ \sum_{A \in \pi} \|\mu(A)\|, A \in \mathfrak{B}(\mathbb{T}), \pi \text{ is a finite partition of } B \right\}$$

and

$$\|\mu\|(B) = \sup\{ |\langle e^*, \mu \rangle|(B) : e^* \in E^*, \|e^*\| = 1 \},$$

where $\langle e^*, \mu \rangle (A) = e^*(\mu(A))$ for all $A \in \mathfrak{B}(\mathbb{T})$. Of course $|\mu|(\cdot)$ becomes a positive measure on $\mathfrak{B}(\mathbb{T})$, while $\|\mu\|(\cdot)$ is only sub-additive in general. We shall denote $|\mu| = |\mu|(\mathbb{T})$ and $\|\mu\| = \|\mu\|(\mathbb{T})$. For dual spaces $E = F^*$ it is easy to see that $\|\mu\| = \sup\{ |\langle \mu, f \rangle| : f \in F, \|f\| = 1 \}$ where $\langle \mu, f \rangle (A) = \mu(A)(f)$.

In what follows we shall consider regular vector measures, that is to say vector measures $\mu \colon \mathfrak{B}(\mathbb{T}) \to E$ such that for each $\varepsilon > 0$ and $B \in \mathfrak{B}(\mathbb{T})$ there exists a compact set K, an open set O such that $K \subset B \subset O$ with $\|\mu\|(O \setminus K) < \varepsilon$. Let us denote by $\mathfrak{M}(\mathbb{T}, E)$ and $M(\mathbb{T}, E)$ the spaces of regular Borel measures with values in E endowed with the norm given the semi-variation and variation respectively. Of course $M(\mathbb{T}, E) \subsetneq \mathfrak{M}(\mathbb{T}, E)$ when E is infinite dimensional.

It is well known that the space $\mathfrak{M}(\mathbb{T}, E)$ can be identified with the space of weakly compact linear operators $T_\mu \colon C(\mathbb{T}) \to E$ and that $\|T_\mu\| = \|\mu\|$ (see [6, Chapter 6]). Hence, for each $\mu \in \mathfrak{M}(\mathbb{T}, E)$ and $k \in \mathbb{Z}$ we can define (see [3]) the k-Fourier coefficient by

$$\widehat{\mu}(k) = T_\mu(\varphi_{-k}).$$

Also, the description of measures in $M(\mathbb{T},E)$ can be done using absolutely summing operators (see [5]) and the variation can be described as the norm in such space (see [6]) but we shall not follow this approach. On the other hand, since we deal with either $E=\mathcal{B}(H)$ or $E=H$ we have at our disposal Singer's theorem (see for instance [8,13,14]), which in the case of dual spaces $E=F^*$ asserts that $M(\mathbb{T},E)=C(\mathbb{T},F)^*$. In other words there exists a bounded linear map $\Psi_\mu\colon C(\mathbb{T},F)\to\mathbb{C}$ with $\|\Psi_\mu\|=|\mu|$ such that

$$\Psi_\mu(y\phi)=T_\mu(\phi)(y),\quad \phi\in C(\mathbb{T}),\ y\in F.$$

In particular, for $k\in\mathbb{Z}$ one has $\widehat{\mu}(-k)(y)=\Psi_\mu(y\varphi_k)$ for each $y\in F$.

As mentioned above since $M(\mathbb{T},\mathcal{L}(X,Y^*))=C(\mathbb{T},X\widehat{\otimes}Y)^*$, for each $\mu\in M(\mathbb{T},\mathcal{L}(X,Y^*))$ we can associate two operators T_μ and Ψ_μ. Of course the connection between them is given by the formula

$$T_\mu(\phi)(x)(y)=\Psi_\mu((x\otimes y)\phi),\quad \phi\in C(\mathbb{T}),\ x\in X,\ y\in Y.$$

There is still one more possibility to be considered using the strong operator topology, namely $\Phi_\mu\colon C(\mathbb{T},X)\to Y^*$ defined by

$$\Phi_\mu(f)(y)=\Psi_\mu(f\otimes y),\quad f\in C(\mathbb{T},X),\ y\in Y,$$

where $f\otimes y(t)=f(t)\otimes y$.

Therefore, given $\mu\in\mathfrak{M}(\mathbb{T},\mathcal{L}(X,Y^*))$, we have three different linear operators defined on the corresponding spaces of polynomials: $T_\mu\colon\mathcal{P}(\mathbb{T})\to\mathcal{L}(X,Y^*)$, $\Psi_\mu\colon\mathcal{P}(\mathbb{T},X\widehat{\otimes}Y)\to\mathbb{C}$ and $\Phi_\mu\colon\mathcal{P}(\mathbb{T},X)\to Y^*$ defined by the formulae

$$T_\mu\left(\sum_{j=-M}^{N}\alpha_j\varphi_j\right)=\sum_{j=-M}^{N}\alpha_j\widehat{\mu}(-j),\quad N,M\in\mathbb{N},\ \alpha_j\in\mathbb{C},$$

$$\Psi_\mu\left(\sum_{j=-M}^{N}\left(\sum_{n=1}^{n_j}x_{jn}\right)\otimes\left(\sum_{m=1}^{m_j}y_{jm}\right)\varphi_j\right)=\sum_{j=-M}^{N}\left(\sum_{n=1}^{n_j}\sum_{m=1}^{m_j}\widehat{\mu}(-j)(x_{jn})(y_{jm})\right),$$

$$\Phi_\mu\left(\sum_{j=-M}^{N}x_j\varphi_j\right)=\sum_{j=-M}^{N}\widehat{\mu}(-j)(x_j),\quad N,M\in\mathbb{N},\ x_j\in X.$$

When restricting to the case $Y^*=H$ we obtain the following connection between them:

$$\mathcal{J}T_\mu(\psi)(x\otimes y)=\Psi_\mu((x\otimes y)\psi)=\langle\Phi_\mu(x\psi),y\rangle,\quad \psi\in\mathcal{P}(\mathbb{T}),\ x,y\in H.$$

Given $\mu\in\mathfrak{M}(\mathbb{T},\mathcal{L}(X,Y^*))$ and $x\in X$, let us denote by μ_x the Y^*-valued measure given by

$$\mu_x(A)=\mu(A)(x),\quad A\in\mathfrak{B}(\mathbb{T}).$$

It is elementary to see that μ_x is a regular measure because one can associate the weakly compact operator $T_{\mu_x} = \delta_x \circ T_\mu \colon C(\mathbb{T}) \to Y^*$ where δ_x stands for the operator $\delta_x \colon \mathcal{L}(X, Y^*) \to Y^*$ given by $\delta_x(T) = T(x)$ for $T \in \mathcal{L}(X, Y^*)$.

If $\mu \in \mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$, $k \in \mathbb{Z}$ and $x, y \in H$ then $\mu_x \in \mathfrak{M}(\mathbb{T}, H)$,

$$\langle \mu_x(A), y\rangle = \mathcal{J}\mu(A)(x \otimes y), \quad A \in \mathfrak{B}(\mathbb{T})$$

and

$$\langle \widehat{\mu}(k)(x), y\rangle = \langle \widehat{\mu}_x(k), y\rangle = \mathcal{J}\widehat{\mu}(k)(x \otimes y).$$

Let us introduce a new space of measures appearing in the case $E = \mathcal{B}(H)$.

Definition 3.1. Let $\mu \in \mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$. We say that $\mu \in M_{\mathrm{SOT}}(\mathbb{T}, \mathcal{B}(H))$ if $\mu_x \in M(\mathbb{T}, H)$ for any $x \in H$. We write

$$\|\mu\|_{\mathrm{SOT}} = \sup\{|\mu_x| : x \in H, \|x\| = 1\}.$$

Proposition 3.2. $M(\mathbb{T}, \mathcal{B}(H)) \subsetneq M_{\mathrm{SOT}}(\mathbb{T}, \mathcal{B}(H)) \subsetneq \mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$.

Proof. The inclusions between the spaces follow from the inequalities

$$|\langle \mu(A)(x), y\rangle| \le \|\mu(A)(x)\|\|y\| \le \|\mu(A)\|\|x\|\|y\|$$

which leads to

$$|\langle \mu_x, y\rangle| \le |\mu_x|\|y\| \le |\mu|\|x\|\|y\|$$

and the corresponding embeddings with norm 1 trivially follow.

Let $H = \ell^2$. We shall find measures $\mu_1 \in M_{\mathrm{SOT}}(\mathbb{T}, \mathcal{B}(H)) \setminus M(\mathbb{T}, \mathcal{B}(H))$ and $\mu_2 \in \mathfrak{M}(\mathbb{T}, \mathcal{B}(H)) \setminus M_{\mathrm{SOT}}(\mathbb{T}, \mathcal{B}(H))$. Both can be constructed relying on a similar argument. Let $y_0 \in H$ with $\|y_0\| = 1$ and select a Hilbert-valued regular measure ν with $|\nu| = \infty$ (for instance take a Pettis integrable, but not Bochner integrable function $f \colon \mathbb{T} \to H$ given by $t \to (f_n(t))_n$ and $\nu(A) = \big(\int_A f_n(t)\,\frac{dt}{2\pi}\big)_n$ for $A \in \mathfrak{B}(\mathbb{T})$). Denote $T_\nu \colon C(\mathbb{T}) \to H$ the corresponding bounded (and hence weakly compact) operator associated to ν with $\|T_\nu\| = \|\nu\|$.

Define

$$\mu_1(A)(x) = \langle x, \nu(A)\rangle y_0, \quad A \in \mathfrak{B}(\mathbb{T})$$

and

$$\mu_2(A)(x) = \langle x, y_0\rangle \nu(A), \quad A \in \mathfrak{B}(\mathbb{T}).$$

In other words, if $J_y \colon H \to \mathcal{B}(H)$ and $I_y \colon H \to \mathcal{B}(H)$ stand for the operators

$$J_y(x)(z) = \langle z, x\rangle y, \quad I_y(x)(z) = \langle x, y\rangle z, \quad x, y, z \in H,$$

then we have that $T_{\mu_1} = J_{y_0}T_\nu$ and $T_{\mu_2} = I_{y_0}T_\nu$ are weakly compact. Hence $\mu_1, \mu_2 \in \mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$.

Note that $|(\mu_1)_x| = |\langle x, \nu\rangle|$ and $|(\mu_2)_x| = |\langle x, y_0\rangle||\nu|$, $x \in H$. Hence

$$\|\mu_1\|_{\mathrm{SOT}} = \|\nu\|, \quad \|\mu_2\|_{\mathrm{SOT}} = |\nu|.$$

Also notice that $\|\mu_1(A)\|_{\mathcal{B}(H)} = \|\nu(A)\|_H$, and therefore $|\mu_1| = |\nu|$, which gives the desired results. □

Definition 3.3. Let $\mu\colon \mathfrak{B}(\mathbb{T}) \to \mathcal{L}(X, Y^*)$ be a vector measure. We define "the adjoint measure" $\mu^*\colon \mathfrak{B}(\mathbb{T}) \to \mathcal{L}(Y, X^*)$ by the formula

$$\mu^*(A)(y)(x) = \mu_x(A)(y), \quad A \in \mathfrak{B}(\mathbb{T}),\ x \in X,\ y \in Y.$$

In the case that $\mu \in \mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$ with the identification $Y^* = H$, one clearly has that

$$\langle x, \mu^*(A)(y)\rangle = \langle \mu(A)(x), y\rangle, \quad A \in \mathfrak{B}(\mathbb{T}),\ x, y \in H.$$

Remark 3.4. μ^* belongs to $\mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$ (resp. $M(\mathbb{T}, \mathcal{B}(H))$) if and only if μ belongs to $\mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$ (resp. $M(\mathbb{T}, \mathcal{B}(H))$). Moreover $\|\mu\| = \|\mu^*\|$ (resp. $|\mu| = |\mu^*|$).

The results follow using that $T_{\mu^*}(\phi) = (T_\mu(\phi))^*$ for any $\phi \in C(\mathbb{T})$ and $\|\mu(A)\| = \|\mu^*(A)\|$ for any $A \in \mathfrak{B}(\mathbb{T})$.

Let us describe the norm in $M_{\mathrm{SOT}}(\mathbb{T}, \mathcal{B}(H)$ using the adjoint measure.

Proposition 3.5. *Let $\mu \in \mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$. Then $\mu \in M_{\mathrm{SOT}}(\mathbb{T}, \mathcal{B}(H))$ if and only if $\Phi_{\mu^*} \in \mathcal{L}(C(\mathbb{T}, H), H)$. Moreover $\|\mu\|_{\mathrm{SOT}} = \|\Phi_{\mu^*}\|$.*

Proof. By definition, $\mu \in M_{\mathrm{SOT}}(\mathbb{T}, \mathcal{B}(H))$ if and only if the operator $S_\mu(x) = \mu_x$ is well defined and belongs to $\mathcal{L}(H, M(\mathbb{T}, H))$. Moreover, $\|\mu\|_{\mathrm{SOT}} = \|S_\mu\|$. The result follows if we show that S_μ is the adjoint of Φ_{μ^*}. Recall that, identifying $H = H^*$, we have $\mu^* \in \mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$. Hence $\Phi_{\mu^*}\colon \mathcal{P}(\mathbb{T}, H) \to H$ is generated by linearity using

$$\Phi_{\mu^*}(x\varphi_k) = \widehat{\mu^*}(-k)(x) = \widehat{\mu}(-k)^*(x), \quad x \in H,\ k \in \mathbb{Z}.$$

Therefore, if $k \in \mathbb{Z}$, $x, y \in H$, since $M(\mathbb{T}, H) = (C(\mathbb{T}, H))^*$, we have

$$S_\mu(y)(x\varphi_k) = \Psi_{\mu_y}(x\varphi_k) = \langle \widehat{\mu_y}(-k), x\rangle = \langle \widehat{\mu}(-k)(y), x\rangle = \langle y, \Phi_{\mu^*}(x\varphi_k)\rangle.$$

By linearity we extend to $\langle y, \Phi_{\mu^*}(x\phi)\rangle = S_\mu(y)(x\phi)$ for any polynomial ϕ and since $\mathcal{P}(\mathbb{T}, H)$ is dense in $C(\mathbb{T}, H)$ we obtain the result. This completes the proof. □

Let us consider the following subspace of regular measures which plays an important role in what follows.

Definition 3.6. Let us write $V^\infty(\mathbb{T},E)$ for the subspace of those measures $\mu \in \mathfrak{M}(\mathbb{T},E)$ such that there exists $C>0$ with

$$\|\mu(A)\| \le Cm(A), \quad A \in \mathfrak{B}(\mathbb{T}).$$

We define

$$\|\mu\|_\infty = \sup\left\{ \frac{\|\mu(A)\|}{m(A)} : m(A) > 0 \right\}.$$

It is clear that any $\mu \in V^\infty(\mathbb{T},\mathcal{B}(H))$ also belongs to $M(\mathbb{T},\mathcal{B}(H))$ and it is absolutely continuous with respect to m.

Let us point out two more possible descriptions of $V^\infty(\mathbb{T},E)$. One option is to look at $V^\infty(\mathbb{T},E) = \mathcal{L}(L^1(\mathbb{T}),E)$ (see [7, page 261]), that is to say that T_μ has a bounded extension to $L^1(\mathbb{T})$. Hence a measure $\mu \in \mathfrak{M}(\mathbb{T},E)$ belongs to $V^\infty(\mathbb{T},E)$ if and only if

$$\|T_\mu(\psi)\| \le C\|\psi\|_{L^1(\mathbb{T})}, \quad \psi \in C(\mathbb{T}).$$

Moreover $\|T_\mu\|_{L^1(\mathbb{T})\to E} = \|\mu\|_\infty$.

In the case that $E = F^*$ also one has that $V^\infty(\mathbb{T},E) = L^1(\mathbb{T},F)^*$, that is the dual of the space of Bochner integrable functions. In this case a measure $\mu \in V^\infty(\mathbb{T},E)$ if and only if Ψ_μ has a bounded extension to $L^1(\mathbb{T},F)^*$, that is

$$\|\Psi_\mu(p)\| \le C\|p\|_{L^1(\mathbb{T},F)}, \quad p \in \mathcal{P}(\mathbb{T},F).$$

Moreover $\|\Psi_\mu\|_{L^1(\mathbb{T},F)^*} = \|\mu\|_\infty$.

Although measures in $V^\infty(\mathbb{T},\mathcal{B}(H))$ are absolutely continuous with respect to m, the reader should be aware that they might not have a Radon-Nikodym derivative in $L^1(\mathbb{T},E)$ (see [6, Chapter 3]).

For the sake of completeness we give an example for $E = \mathcal{B}(H)$ of such a situation.

Proposition 3.7. *Let $H = \ell^2$ and $\mu \in \mathfrak{M}(\mathbb{T},\mathcal{B}(H))$ such that $T_\mu \in \mathcal{L}(C(\mathbb{T}),\mathcal{B}(H))$ is given by*

$$T_\mu(\phi) = \sum_{n=1}^\infty \widehat{\phi}(n)\widetilde{e_n \otimes e_n}.$$

Then $\mu \in V^\infty(\mathbb{T},\mathcal{B}(H))$ with $\|\mu\|_\infty = 1$,

$$\widehat{\mu}(k) = \begin{cases} \widetilde{e_k \otimes e_k} & \text{if } k \ge 1, \\ 0 & \text{if } k \le 0, \end{cases}$$

but it does not have a Radon-Nikodym derivative in $L^1(\mathbb{T},\mathcal{B}(H))$.

Proof. Let us show that T_μ defines a continuous operator from $L^1(\mathbb{T})$ to $\mathcal{B}(H)$ with norm 1. In such a case, using that the inclusion $C(\mathbb{T}) \to L^1(\mathbb{T})$ is weakly compact, one automatically has that $\mu \in \mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$. For $x = (\alpha_n) \in H$ and $y = (\beta_n) \in H$ one has

$$|\langle T_\mu(\phi)(x), y\rangle| = \left|\sum_{n=1}^{\infty} \widehat{\phi}(n)\alpha_n\beta_n\right| \leq \sup_{n\geq 1} |\widehat{\phi}(n)|\|x\|\|y\| \leq \|\phi\|_{L^1(\mathbb{T})}\|x\|\|y\|.$$

This gives that $\mu \in V^\infty(\mathbb{T}, \mathcal{B}(H))$ and $\|\mu\|_\infty \leq 1$. Using that $T_\mu(\varphi_j) = \widetilde{e_j \otimes e_j}$ and $\|\widetilde{e_j \otimes e_j}\|_{\mathcal{B}(H)} = 1$ we get the equality of norms.

The result on Fourier coefficients is obvious. To show that μ does not have a Bochner integrable Radon-Nikodym derivative follows now using that otherwise $\widehat{\mu}(k) = \widehat{f}(k)$ for some $f \in L^1(\mathbb{T}, \mathcal{B}(H))$ which implies that $\|\widehat{f}(k)\| \to 0$ as $k \to \infty$ while $\|\widehat{\mu}(k)\| = 1$ for $k \geq 1$. This completes the proof. □

We finish this section with a known characterization of measures in $M(\mathbb{T}, F^*)$ to be used later on, that we include for sake of completeness.

Lemma 3.8. *Let $E = F^*$ be a dual Banach space and $\mu \in \mathfrak{M}(\mathbb{T}, E)$. For each $0 < r < 1$ we define*

$$P_r * \mu(t) = \sum_{k\in\mathbb{Z}} \widehat{\mu}(k) r^{|k|}\varphi_k(t), \quad t \in [0, 2\pi). \tag{3.1}$$

Then

(i) $P_r * \mu \in C(\mathbb{T}, E)$ *and* $\|P_r * \mu\|_{C(\mathbb{T},E)} \leq \|\mu\|\frac{1+r}{1-r}$ *for any* $0 < r < 1$.

(ii) $\mu \in M(\mathbb{T}, E)$ *if and only if* $\sup_{0<r<1} \|P_r * \mu\|_{L^1(\mathbb{T},E)} < \infty$. *Moreover*

$$|\mu| = \sup_{0<r<1} \|P_r * \mu\|_{L^1(\mathbb{T},E)}.$$

Proof. (i) Observe that

$$\sum_{k\in\mathbb{Z}} |\widehat{\mu}(k)|r^{|k|}\|\varphi_k\|_{C(\mathbb{T})} \leq \|T_\mu\| \left(1 + 2\sum_{k=1}^{\infty} r^k\right) = \|\mu\|\frac{1+r}{1-r}.$$

This shows that the series in (3.1) is absolutely convergent in $C(\mathbb{T}, E)$ and we obtain (i).

(ii) Assume that $\mu \in M(\mathbb{T}, E)$. In particular $|\mu| \in M(\mathbb{T})$ and

$$\int_0^{2\pi} \|P_r * \mu(t)\| \frac{dt}{2\pi} \leq \int_0^{2\pi} P_r * |\mu|(t) \frac{dt}{2\pi}.$$

Hence, using the scalar-valued result, we have

$$\sup_{0<r<1} \|P_r * \mu\|_{L^1(\mathbb{T},E)} \leq \sup_{0<r<1} \|P_r * |\mu|\|_{L^1(\mathbb{T})} \leq \sup_{0<r<1} |\mu|\|P_r\|_{L^1(\mathbb{T})} = |\mu|.$$

Conversely, assume that $\sup_{0<r<1}\|P_r * \mu\|_{L^1(\mathbb{T},E)} < \infty$. Since $L^1(\mathbb{T},E) \subseteq M(\mathbb{T},E) = C(\mathbb{T},F)^*$, from the Banach-Alaoglu theorem one can find a sequence r_n converging to 1 and a measure $\nu \in M(\mathbb{T},E)$ such that $P_{r_n} * \mu \to \nu$ in the w^*-topology. Selecting now functions in $C(\mathbb{T},F)$ given by $y\varphi_{-k}$ for all $y \in F$ and $k \in \mathbb{Z}$ one shows that $\widehat{\nu}(k) = \widehat{\mu}(k)$. This gives that $\mu = \nu$ and therefore $\mu \in M(\mathbb{T},E)$. Finally, notice that

$$|\mu| = \sup\{|\Psi_\mu(p)| : p \in \mathcal{P}(\mathbb{T},F), \|p\|_{C(\mathbb{T},F)} = 1\}.$$

Given now $p = \sum_{k=-M}^{N} y_k\varphi_k$, one has $P_r * p = \sum_{k=-M}^{N} y_k r^{|k|}\varphi_k$ and

$$\Psi_\mu(P_r * p) = \sum_{k=-M}^{N} \widehat{\mu}(k)(y_k)r^{|k|} = \int_0^{2\pi} P_r * \mu(t)(p(t))\,\frac{dt}{2\pi}.$$

Finally, since $p = \lim_{r\to 1} P_r * p$ is in $C(\mathbb{T},F)$, we have

$$\begin{aligned}|\Psi_\mu(p)| &= \lim_{r\to 1}|\Psi_\mu(P_r * p)| \\ &\le \sup_{0<r<1}\left|\int_0^{2\pi} P_r * \mu(t)(p(t))\,\frac{dt}{2\pi}\right| \\ &\le \sup_{0<r<1}\|P_r * \mu\|_{L^1(\mathbb{T},E)}\|p\|_{C(\mathbb{T},F)}.\end{aligned}$$

This gives the inequality $|\mu| \le \sup_{0<r<1}\|P_r * \mu\|_{L^1(\mathbb{T},E)}$ and the proof is complete. □

4. Some results on matrices of operators

Throughout the rest of the paper, we write $\mathbf{A} = (T_{kj}) \subset \mathcal{B}(H)$, $\mathbf{R}_k$ and $\mathbf{C}_j$ the k-row respectively, that is

$$\mathbf{R}_k = (T_{kj})_{j=1}^{\infty}, \quad \mathbf{C}_j = (T_{kj})_{k=1}^{\infty}$$

and

$$\mathbf{A}_{N,M}(s,t) = \sum_{k=1}^{M}\sum_{j=1}^{N} T_{kj}\overline{\varphi_j(s)}\varphi_k(t), \quad 0 \le t,s < 2\pi,\ N,M \in \mathbb{N}.$$

For each $\mathbf{x} = (x_j) \in \ell^2(H)$ we consider the function $h_{\mathbf{x}}$ given by

$$h_{\mathbf{x}}(t) = \sum_{j=1}^{\infty} x_j\varphi_j(t), \quad t \in [0,2\pi).$$

Remark 4.1. Observe that $\mathbf{A} \in \widetilde{H}^2(\mathbb{T}^2,\mathcal{B}(H))$ if and only if

$$\sup_{N,M}\|\mathbf{A}_{N,M}\|_{L^2(\mathbb{T}^2,\mathcal{B}(H))} < \infty.$$

Note that $\mathbf{x} \in \ell^2(H)$ if and only if $h_{\mathbf{x}} \in H_0^2(\mathbb{T},H)$. Moreover

$$\|\mathbf{x}\|_{\ell^2(H)} = \|h_{\mathbf{x}}\|_{H^2(\mathbb{T},H)}.$$

Proposition 4.2. *Let* $\mathbf{A} = (T_{kj}) \subset \mathcal{B}(H)$*.*

(i) *If* $\mathbf{A} \in \ell^2_{\mathrm{SOT}}(\mathbb{N}^2, \mathcal{B}(H))$ *then* $\mathbf{R}_k, \mathbf{C}_j \in \ell^2_{\mathrm{SOT}}(\mathbb{N}, \mathcal{B}(H))$ *for all* $k, j \in \mathbb{N}$*.*

(ii) *If* $\mathbf{A} \in \widetilde{H}^2(\mathbb{T}^2, \mathcal{B}(H))$ *then* $\mathbf{C}_j, \mathbf{R}_k \in \widetilde{H}^2(\mathbb{T}, \mathcal{B}(H))$ *for all* $j, k \in \mathbb{N}$*.*

Proof. (i) follows trivially from the definitions.

(ii) Let $k' \in \mathbb{N}$, $M \in \mathbb{N}$ and $t \in [0, 2\pi)$. For $N \geq k'$ we have

$$\sum_{j=1}^{N} T_{k'j}\varphi_j(t) = \int_0^{2\pi} \left(\sum_{k=1}^{N} \sum_{j=1}^{M} T_{kj}\varphi_j(t)\varphi_k(s) \right) \overline{\varphi_{k'}(s)}\, \frac{ds}{2\pi}.$$

Therefore

$$\int_0^{2\pi} \left\| \sum_{j=1}^{N} T_{k'j}\varphi_j(t) \right\|^2 \frac{dt}{2\pi} \leq \int_0^{2\pi}\int_0^{2\pi} \left\| \sum_{k=1}^{N} \sum_{j=1}^{M} T_{kj}\varphi_j(t)\varphi_k(s) \right\|^2 \frac{ds}{2\pi}\frac{dt}{2\pi}.$$

Hence $\|\mathbf{R}_{k'}\|_{\widetilde{H}^2(\mathbb{T},\mathcal{B}(H))} \leq \|\mathbf{A}\|_{\widetilde{H}^2(\mathbb{T}^2,\mathcal{B}(H))}$. A similar argument shows that $\|\mathbf{C}_j\|_{\widetilde{H}^2(\mathbb{T},\mathcal{B}(H))} \leq \|\mathbf{A}\|_{\widetilde{H}^2(\mathbb{T}^2,\mathcal{B}(H))}$ and it is left to the reader. □

Definition 4.3. Let $\mathbf{A} = (T_{kj}) \subset \mathcal{B}(H)$. Define $B_{\mathbf{A}} \colon \mathcal{P}_a(\mathbb{T}, H) \times \mathcal{P}_a(\mathbb{T}, H) \to \mathbb{C}$ by

$$(h_{\mathbf{x}}, h_{\mathbf{y}}) \to \int_0^{2\pi}\int_0^{2\pi} \mathcal{J}\mathbf{A}_{N,M}(s,t)(h_{\mathbf{x}}(s) \otimes h_{\mathbf{y}}(t))\, \frac{ds}{2\pi}\frac{dt}{2\pi},$$

where $h_{\mathbf{x}} = \sum_{j=1}^{N} x_j\varphi_j$ and $h_{\mathbf{y}} = \sum_{k=1}^{M} y_k\varphi_k$ for $x_j, y_k \in H$.

We now give the characterization of bounded operators in $\mathcal{B}(\ell^2(H))$ in terms of bilinear maps.

Proposition 4.4. *If* $\mathbf{A} = (T_{kj}) \subset \mathcal{B}(H)$ *then*

$$\langle\!\langle \mathbf{A}(\mathbf{x}), \mathbf{y} \rangle\!\rangle = B_{\mathbf{A}}(h_{\mathbf{x}}, h_{\mathbf{y}}), \quad \mathbf{x}, \mathbf{y} \in c_{00}(H). \tag{4.1}$$

In particular, $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ *if and only if* $B_{\mathbf{A}}$ *extends to a bounded bilinear map on* $H^2_0(\mathbb{T}, H) \times H^2_0(\mathbb{T}, H)$*. Moreover* $\|\mathbf{A}\| = \|B_{\mathbf{A}}\|$*.*

Proof. To show (4.1) we observe that for $h_{\mathbf{x}} = \sum_{j=1}^{N} x_j\varphi_j$ and $h_{\mathbf{y}} = \sum_{k=1}^{M} y_k\varphi_k$ we have

$y_k = \int_0^{2\pi} h_{\mathbf{y}}(t)\overline{\varphi_k(t)}\,\frac{dt}{2\pi}$ and $x_j = \int_0^{2\pi} h_{\mathbf{x}}(t)\overline{\varphi_j(s)}\,\frac{ds}{2\pi}$. Hence

$$\begin{aligned}\sum_{k=1}^{M}\left\langle \sum_{j=1}^{N} T_{kj}x_j, y_k\right\rangle &= \int_0^{2\pi}\left\langle \sum_{k=1}^{M}\left(\sum_{j=1}^{N} T_{kj}x_j\right)\varphi_k(t), h_{\mathbf{y}}(t)\right\rangle \frac{dt}{2\pi}\\ &= \int_0^{2\pi}\left\langle \sum_{k=1}^{M}\left(\sum_{j=1}^{N} T_{kj}\varphi_k(t)\right)(x_j), h_{\mathbf{y}}(t)\right\rangle \frac{dt}{2\pi}\\ &= \int_0^{2\pi}\left\langle \int_0^{2\pi} \mathbf{A}_{N,M}(s,t)(h_{\mathbf{x}}(s))\,\frac{ds}{2\pi}, h_{\mathbf{y}}(t)\right\rangle \frac{dt}{2\pi}\\ &= \int_0^{2\pi}\int_0^{2\pi} \mathcal{J}\mathbf{A}_{N,M}(s,t)(h_{\mathbf{x}}(s)\otimes h_{\mathbf{y}}(t))\,\frac{ds}{2\pi}\frac{dt}{2\pi}.\end{aligned}$$

The equality of norms follows trivially. □

From Proposition 4.4 one can produce some sufficient conditions for $\mathbf{A}$ to belong to $\mathcal{B}(\ell^2(H))$.

Corollary 4.5. *If* $\mathbf{A} \in \widetilde{H}^2(\mathbb{T}^2, \mathcal{B}(H)) \cup \ell^2(\mathbb{N}^2, \mathcal{B}(H))$ *then* $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ *and* $\|\mathbf{A}\| \le \min\{\|\mathbf{A}\|_{\widetilde{H}^2(\mathbb{T}^2,\mathcal{B}(H))}, \|\mathbf{A}\|_{\ell^2(\mathbb{N}^2,\mathcal{B}(H))}\}$.

Proof. Assume first $\mathbf{A} \in \ell^2(\mathbb{N}^2, \mathcal{B}(H))$. Then

$$|\langle\langle \mathbf{A}(\mathbf{x}), \mathbf{y}\rangle\rangle| \le \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \|T_{kj}\|\|x_j\|\|y_k\|$$

and therefore, using Cauchy-Schwarz's inequality in $\ell^2(\mathbb{N}^2)$,

$$\begin{aligned}|\langle\langle \mathbf{A}(\mathbf{x}), \mathbf{y}\rangle\rangle| &\le \|\mathbf{A}\|_{\ell^2(\mathbb{N}^2,\mathcal{B}(H))}\|(\|x_j\|\|y_k\|)\|_{\ell^2(\mathbb{N}^2)}\\ &= \|\mathbf{A}\|_{\ell^2(\mathbb{N}^2,\mathcal{B}(H))}\|\mathbf{x}\|\|\mathbf{y}\|.\end{aligned}$$

Assume now $\mathbf{A} \in \widetilde{H}^2(\mathbb{T}^2, \mathcal{B}(H))$ and apply Cauchy-Schwarz in $L^2(\mathbb{T}^2)$

$$\begin{aligned}&\left|\int_0^{2\pi}\int_0^{2\pi} \mathcal{J}\mathbf{A}_{N,M}(s,t)(h_{\mathbf{x}}(s)\otimes h_{\mathbf{y}}(t))\,\frac{ds}{2\pi}\frac{dt}{2\pi}\right|\\ &\le \|\mathbf{A}_{N,M}\|_{H_0^2(\mathbb{T}^2,\mathcal{B}(H))}\|h_{\mathbf{x}}\|_{H_0^2(\mathbb{T},H)}\|h_{\mathbf{y}}\|_{H_0^2(\mathbb{T},H)}.\end{aligned}$$

Now the result follows from Proposition 4.4. □

Actually a sufficient condition better than $\mathbf{A} \in \ell^2(\mathbb{N}^2, \mathcal{B}(H))$ is given in the following result.

Proposition 4.6. *Let* $\mathbf{A} = (T_{kj}) \subset \mathcal{B}(H)$ *such that* $\mathbf{C}_j$ *for all* $j \in \mathbb{N}$ *or* $\mathbf{R}_k^* \in \ell^2_{\mathrm{SOT}}(\mathbb{N}, \mathcal{B}(H))$ *for all* $k \in \mathbb{N}$ *and satisfy*

$$\min\{\|(\mathbf{C}_j)\|_{\ell^2(\mathbb{N},\ell^2_{\mathrm{SOT}}(\mathbb{N},\mathcal{B}(H)))}, \|(\mathbf{R}_k^*)\|_{\ell^2(\mathbb{N},\ell^2_{\mathrm{SOT}}(\mathbb{N},\mathcal{B}(H)))}\} = M < \infty.$$

Then $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ *and* $\|\mathbf{A}\| \le M$.

Proof. Let $\mathbf{x}, \mathbf{y} \in \ell^2(H)$, we have

$$\begin{aligned}
|\langle\langle \mathbf{A}(\mathbf{x}), \mathbf{y}\rangle\rangle| &\leq \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \|y_k\| \left\| T_{kj}\left(\frac{x_j}{\|x_j\|}\right)\right\| \|x_j\| \\
&\leq \left(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \left\| T_{kj}\left(\frac{x_j}{\|x_j\|}\right)\right\|^2\right)^{1/2} \left(\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \|y_k\|^2\|x_j\|^2\right)^{1/2} \\
&\leq \|\mathbf{x}\|_{\ell^2(H)}\|\mathbf{y}\|_{\ell^2(H)} \left(\sum_{j=1}^{\infty} \|\mathbf{C}_j\|^2_{\ell^2_{\mathrm{SOT}}(\mathbb{N},\mathcal{B}(H))}\right)^{1/2}.
\end{aligned}$$

Similar argument works with $\mathbf{R}_k^*$, which completes the proof. □

Let us now present some necessary conditions for $\mathbf{A} \in \mathcal{B}(\ell^2(H))$. Since $\langle\langle \mathbf{A}(x\mathbf{e}_j), y\mathbf{e}_k\rangle\rangle = \langle T_{kj}(x), y\rangle$, we have that if $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ then $\mathbf{A} \in \ell^\infty(\mathbb{N}^2, \mathcal{B}(H))$ and $\sup_{k,j} \|T_{kj}\| \leq \|\mathbf{A}\|$.

Lemma 4.7. *Let* $\mathbf{A} = (T_{kj}) \in \mathcal{B}(\ell^2(H))$. *Then* $(\mathbf{C}_j)_j, (\mathbf{R}_k)_k, (\mathbf{C}_j^*)_j, (\mathbf{R}_k^*)_k \in \ell^\infty(\mathbb{N}, \ell^2_{\mathrm{SOT}}(\mathbb{N}, \mathcal{B}(H)))$.

Proof. Since for each $\mathbf{y} \in \ell^2(H)$, $x, y \in H$ and $k, j \in \mathbb{N}$ we have

$$\langle\langle \mathbf{A}(x\mathbf{e}_k), \mathbf{y}\rangle\rangle = \langle\langle \mathbf{R}_k(x), \mathbf{y}\rangle\rangle$$

and

$$\langle\langle \mathbf{A}(\mathbf{x}), y\mathbf{e}_j\rangle\rangle = \langle\langle \mathbf{x}, \mathbf{C}_j(y)\rangle\rangle,$$

we clearly have

$$\|\mathbf{R}_k\|_{\ell^2_{\mathrm{SOT}}(\mathbb{N},\mathcal{B}(H))} = \sup_{\|x\|=1} \sup_{\|\mathbf{y}\|_{\ell^2(H)}=1} |\langle\langle \mathbf{A}(x\mathbf{e}_k), \mathbf{y}\rangle\rangle| \leq \|\mathbf{A}\|.$$

A similar argument allows to obtain $\|\mathbf{C}_j\|_{\ell^2_{\mathrm{SOT}}(\mathbb{N},\mathcal{B}(H))} \leq \|\mathbf{A}\|$. Now, since $\|T_{kj}\| = \|T_{kj}^*\|$, applying the fact that rows in $\mathbf{A}^*$ correspond with the adjoint operators in the columns in $\mathbf{A}$ we obtain the other cases. □

Let us give another necessary condition for boundedness to be used later on.

Proposition 4.8. *Let* $\mathbf{A} = (T_{kj}) \in \mathcal{B}(\ell^2(H))$. *Then*

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \|T_{kj}x_j\|^2 \leq \|\mathbf{A}\|^2 \sum_{j=1}^{\infty} \|x_j\|^2.$$

Proof. Let $\mathbf{x} \in \ell^2(H)$ and assume that $\sum_{j=1}^{\infty} \|x_j\|^2 = 1$. Denote by $F_{\mathbf{x}} \colon [0, 2\pi] \to \ell^2(H)$ the continuous function given by $F_{\mathbf{x}}(s) = (x_j \varphi_j(s))$. Trivially, we have $\|\mathbf{x}\| = \|F_{\mathbf{x}}\|_{C(\mathbb{T},\ell^2(H))}$. Then

$$\begin{aligned}
\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \|T_{kj}x_j\|^2 &= \sum_{k=1}^{\infty} \int_0^{2\pi} \left\| \sum_{j=1}^{\infty} T_{kj}x_j\varphi_j(s) \right\|^2 \frac{ds}{2\pi} \\
&= \int_0^{2\pi} \sum_{k=1}^{\infty} \left\| \sum_{j=1}^{\infty} T_{kj}x_j\varphi_j(s) \right\|^2 \frac{ds}{2\pi} \\
&= \int_0^{2\pi} \|\mathbf{A}(F_{\mathbf{x}}(s))\|^2 \frac{ds}{2\pi} \\
&\leq \|\mathbf{A}\|^2 \int_0^{2\pi} \|F_{\mathbf{x}}(s)\|^2 \frac{ds}{2\pi} = \|\mathbf{A}\|^2.
\end{aligned}$$

This concludes the result. □

From Proposition 4.8 we can get an extension of Schur theorem to matrices whose entries are operators in $\mathcal{B}(H)$.

Theorem 4.9. *Let* $\mathbf{A} = (T_{kj})$ *and* $\mathbf{B} = (S_{kj})$. *If* $\mathbf{A}, \mathbf{B} \in \mathcal{B}(\ell^2(H))$ *then* $\mathbf{A} * \mathbf{B} \in \mathcal{B}(\ell^2(H))$. *Moreover*

$$\|\mathbf{A} * \mathbf{B}\|_{\mathcal{B}(\ell^2(H))} \leq \|\mathbf{A}\|_{\mathcal{B}(\ell^2(H))} \|\mathbf{B}\|_{\mathcal{B}(\ell^2(H))}.$$

Proof. It suffices to show that if $\mathbf{x}, \mathbf{y} \in c_{00}(H)$ then

$$|\langle\langle \mathbf{A} * \mathbf{B}(\mathbf{x}), \mathbf{y} \rangle\rangle| \leq \|\mathbf{A}\| \|\mathbf{B}\| \|\mathbf{x}\| \|\mathbf{y}\|. \tag{4.2}$$

Notice that

$$\begin{aligned}
|\langle\langle \mathbf{A} * \mathbf{B}(\mathbf{x}), \mathbf{y} \rangle\rangle| &= \left| \sum_{k=1}^{\infty} \left\langle \sum_{j=1}^{\infty} T_{kj}S_{kj}(x_j), y_k \right\rangle \right| \\
&= \left| \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \langle S_{kj}(x_j), T_{kj}^*(y_k) \rangle \right| \\
&\leq \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \|T_{kj}^*(y_k)\| \|S_{kj}(x_j)\| \\
&\leq \left(\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \|T_{kj}^*(y_k)\|^2 \right)^{1/2} \left(\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \|S_{kj}(x_j)\|^2 \right)^{1/2}.
\end{aligned}$$

Using the estimate above and applying Proposition 4.8 to $\mathbf{B}$ and $\mathbf{A}^*$, we obtain (4.2) immediately since $\|\mathbf{A}\| = \|\mathbf{A}^*\|$. The proof is then complete. □

Given $S \subset \mathbb{N} \times \mathbb{N}$ and $\mathbf{A} = (T_{kj})$, we write $P_S\mathbf{A} = (T_{kj}\chi_S)$, that is the matrix with entries T_{kj} if $(k,j) \in S$ and 0 otherwise. In particular, matrices with a single row, column or diagonal correspond to $S = \{k\} \times \mathbb{N}$, $S = \mathbb{N} \times \{j\}$ and $D_l = \{(k, k+l) : k \in \mathbb{N}\}$ for $l \in \mathbb{Z}$ respectively. Also, the case of finite or upper (or lower) triangular matrices coincides with $P_S\mathbf{A}$ for $S = [1,N] \times [1,M] = \{(k,j) : 1 \le k \le N, 1 \le j \le M\}$ or $S = \Delta = \{(k,j) : j \ge k\}$ (or $S = \{(k,j) : j \le k\}$) respectively.

It is well known that the mapping $\mathbf{A} \to P_S\mathbf{A}$ is not continuous in $\mathcal{B}(H)$ for all sets S (for instance, the reader is referred to [10, Chapter 2, Theorem 2.19] to see that $S = \Delta$ the triangle projection is unbounded) but there are cases where this holds true. Clearly we have that $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ if and only if $\|\mathbf{A}\| = \sup_{N,M} \|P_{[1,N]\times[1,M]}\mathbf{A}\| < \infty$. This easily follows noticing that

$$\langle\langle P_{[1,N]\times[1,M]}\mathbf{A}(\mathbf{x}), \mathbf{y}\rangle\rangle = \langle\langle \mathbf{A}(P_N\mathbf{x}), P_M\mathbf{y}\rangle\rangle,$$

where $P_N\mathbf{x}$ stands for the projection on the N-first coordinates of $\mathbf{x}$.

In general it is rather difficult to compute the norm of the matrix $\mathbf{A}$. Let us consider some trivial cases.

Corollary 4.10. *Let* $\mathbf{A} = (T_{kj}) \subset \mathcal{B}(H)$. *Then*

(i) $\|P_{\mathbb{N}\times\{j\}}\mathbf{A}\| = \|\mathbf{C}_j\|_{\ell^2_{\mathrm{SOT}}(\mathbb{N},\mathcal{B}(H))}$ *for each* $j \in \mathbb{N}$.

(ii) $\|P_{\{k\}\times\mathbb{N}}\mathbf{A}\| = \|\mathbf{R}_k\|_{\ell^2_{\mathrm{SOT}}(\mathbb{N},\mathcal{B}(H))}$ *for each* $k \in \mathbb{N}$.

(iii) $\|P_{D_l}\mathbf{A}\| = \sup_k \|T_{k,k+l}\|$ *for each* $l \in \mathbb{Z}$ *(where* $T_{k,k+l} = 0$ *whenever* $k + l \le 0$*).*

Proof. (i) and (ii) follow trivially from Lemma 4.7.

To see (iii), note that $(P_{D_l}\mathbf{A}(\mathbf{x}))_k = (T_{k,k+l}x_{k+l})_k$. Hence $\|P_{D_l}\mathbf{A}(\mathbf{x})\| \le \big(\sup_k \|T_{k,k+l}\|\big)\|\mathbf{x}\|$. Since the other inequality always holds, the proof is complete. □

5. Toeplitz multipliers on operator-valued matrices

In this section we shall achieve the operator-valued analogues to the Toeplitz and Bennet theorems presented in the introduction.

Theorem 5.1. *Let* $\mathbf{A} = (T_{kj}) \in \mathcal{T}$. *Then* $A \in \mathcal{B}(\ell^2(H))$ *if and only if there exists* $\mu \in V^\infty(\mathbb{T}, \mathcal{B}(H))$ *such that* $T_{kj} = \widehat{\mu}(j-k)$ *for all* $k, j \in \mathbb{N}$. *Moreover,* $\|\mathbf{A}\| = \|\mu\|_\infty$.

Proof. Assume that $\mu \in V^\infty(\mathbb{T}, \mathcal{B}(H))$ and $T_{kj} = \widehat{\mu}(j-k)$ for all $k, j \in \mathbb{N}$. Then, for

$\mathbf{x}, \mathbf{y} \in c_{00}(H)$, we have

$$\begin{aligned}\langle\langle \mathbf{A}(\mathbf{x}), \mathbf{y}\rangle\rangle &= \sum_{k=1}^{M}\sum_{j=1}^{N}\langle T_{kj}(x_j), y_k\rangle = \sum_{k=1}^{M}\sum_{j=1}^{N}\langle T_\mu(\overline{\varphi_j}\varphi_k)(x_j), y_k\rangle \\ &= \sum_{k=1}^{M}\sum_{j=1}^{N}\Psi_\mu(\overline{\varphi}_j x_j \otimes \overline{\varphi_k} y_k) = \Psi_\mu\left(\sum_{k=1}^{M}\sum_{j=1}^{N}\overline{\varphi}_j x_j \otimes \overline{\varphi}_k y_k\right) \\ &= \Psi_\mu\left(\left(\sum_{j=1}^{N}\overline{\varphi}_j x_j\right)\otimes\left(\sum_{k=1}^{M}\overline{\varphi}_k y_k\right)\right).\end{aligned}$$

Therefore

$$\begin{aligned}|\langle\langle \mathbf{A}(\mathbf{x}), \mathbf{y}\rangle\rangle| &\le \|\Psi_\mu\|_{L^1(\mathbb{T}, H\widehat{\otimes}H)^*}\int_0^{2\pi}\|h_{\mathbf{x}}(-t)\otimes h_{\mathbf{y}}(-t))\|_{H\widehat{\otimes}H}\frac{dt}{2\pi} \\ &= \|\mu\|_\infty\int_0^{2\pi}\|h_{\mathbf{x}}(-t)\|\|h_{\mathbf{y}}(-t)\|\frac{dt}{2\pi} \\ &\le \|\mu\|_\infty\left(\int_0^{2\pi}\|h_{\mathbf{x}}(-t)\|^2\frac{dt}{2\pi}\right)^{1/2}\left(\int_0^{2\pi}\|h_{\mathbf{y}}(-t))\|^2\frac{dt}{2\pi}\right)^{1/2} \\ &\le \|\mu\|_\infty\|\mathbf{x}\|_{\ell^2(H)}\|\mathbf{y}\|_{\ell^2(H)}.\end{aligned}$$

Hence, $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ and $\|\mathbf{A}\| \le \|\mu\|_\infty$.

Conversely, let us assume that $\mathbf{A} \in \mathcal{B}(\ell^2(H))$ and $T_{kj} = T_{j-k}$ for a given sequence $\mathbf{T} = (T_n)_{n\in\mathbb{Z}}$ of operators in $\mathcal{B}(H)$. We define

$$T\left(\sum_{n=-M}^{N}\alpha_n\varphi_n\right) = \alpha_0 T_{1,1} + \sum_{n=1}^{M}\alpha_{-n}T_{n+1,1} + \sum_{n=1}^{N}\alpha_n T_{1,n+1}.$$

We are going to show that $T \in \mathcal{L}(L^1(\mathbb{T}), \mathcal{B}(H))$. Since $L^1(\mathbb{T}) = \overline{\mathrm{span}\{\varphi_k : k \in \mathbb{Z}\}}^{\|\cdot\|_1}$, it suffices to show that

$$\left\|T\left(\sum_{n=-M}^{N}\alpha_n\varphi_n\right)\right\| \le \|\mathbf{A}\|\int_0^{2\pi}\left|\sum_{n=-M}^{N}\alpha_n\varphi_n(t)\right|\frac{dt}{2\pi}. \tag{5.1}$$

Let $x, y \in H$ and notice that

$$\left\langle T\left(\sum_{n=-M}^{N}\alpha_n\varphi_n\right)(x), y\right\rangle = \sum_{n=-M}^{N}\alpha_n\beta_n(x,y),$$

where $\beta_n(x,y) = \langle T_n(x), y\rangle$. Now taking into account that $A_{x,y} = (\langle T_{kj}(x), y\rangle)$ is a Toeplitz matrix and defines a bounded operator $A_{x,y} \in \mathcal{B}(\ell^2)$ with $\|A_{x,y}\| \le \|\mathbf{A}\|\|x\|\|y\|$ we obtain, due to Theorem 1.2, that

$$\psi_{x,y} = \sum_{n\in\mathbb{Z}}\beta_n(x,y)\varphi_n \in L^\infty(\mathbb{T})$$

with $\|\psi_{x,y}\|_{L^\infty(\mathbb{T})} \le \|\mathbf{A}\|\|x\|\|y\|$. Finally, we have

$$\left|\left\langle T\left(\sum_{n=-M}^{N}\alpha_n\varphi_n\right)(x), y\right\rangle\right| = \left|\int_0^{2\pi}\left(\sum_{n=-M}^{N}\alpha_n\varphi_n(t)\right)\overline{\psi_{x,y}(t)}\,\frac{dt}{2\pi}\right|$$
$$\le \left\|\sum_{n=-M}^{N}\alpha_n\varphi_n(t)\right\|_{L^1(\mathbb{T})}\|\mathbf{A}\|\|x\|\|y\|.$$

This shows (5.1) which gives $\|T\|_{L^1(\mathbb{T})\to\mathcal{B}(H)} \le \|\mathbf{A}\|$. Finally, from the embedding $C(\mathbb{T}) \to L^1(\mathbb{T})$ we have that there exists $\mu \in V^\infty(\mathbb{T}, \mathcal{B}(H))$ such that $T_\mu = T$ and $\|\mu\|_\infty \le \|A\|$. The proof is then complete. □

To prove the analogue of Bennet't theorem on Schur multipliers we shall need the following lemmas.

Lemma 5.2. *Let* $\mathbf{A} = (T_{kj}) \in \mathcal{M}_l(\ell^2(H)) \cup \mathcal{M}_r(\ell^2(H))$ *and* $x_0, y_0 \in H$ *with* $\|x_0\| = \|y_0\| = 1$. *Denote by* $A_{x_0,y_0} = (\gamma_{kj})$ *the matrix with entries*

$$\gamma_{kj} = \langle T_{kj}(x_0), y_0\rangle, \quad k, j \in \mathbb{N}.$$

Then $A_{x_0,y_0} \in \mathcal{M}(\ell^2)$ *and* $\|A_{x_0,y_0}\|_{\mathcal{M}(\ell^2)} \le \min\{\|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))}, \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}\}$.

Proof. Let $z_0 \in H$ and $\|z_0\| = 1$ and consider the bounded operators $\pi_{z_0}\colon \ell^2(H) \to \ell^2$ and $i_{z_0}\colon \ell^2 \to \ell^2(H)$ given by

$$\pi_{z_0}((x_j)) = (\langle x_j, z_0\rangle)_j, \quad i_{z_0}((\alpha_k)) = (\alpha_k z_0)_k.$$

Now, given $B = (\beta_{kj}) \in \mathcal{B}(\ell^2)$ with $\|B\| = 1$, we define $\mathbf{B} = i_{z_0} B\, \pi_{z_0}$.

Hence $\mathbf{B} \in \mathcal{B}(\ell^2(H))$. Moreover $\|\mathbf{B}\| = \|B\|$ because $\|i_{z_0}\| = \|\pi_{z_0}\| = 1$ and $B((\alpha_j))z_0 = \mathbf{B}((\alpha_j z_0))$ for any $(\alpha_j) \in \ell^2$.

Let us write $\mathbf{B} = (S_{kj})$ and observe that $S_{kj} = \beta_{kj}\widetilde{z_0 \otimes z_0}$. Indeed,

$$\langle S_{kj}(x), y\rangle = \langle\langle \mathbf{B}(x\mathbf{e}_j), y\mathbf{e}_k\rangle\rangle = \langle\langle (\langle x, z_0\rangle\beta_{kj}z_0)_k, y\mathbf{e}_k\rangle\rangle = \beta_{kj}\langle x, z_0\rangle\langle z_0, y\rangle.$$

Recall that $T(\widetilde{x \otimes y}) = \widetilde{x \otimes T(y)}$ and $(\widetilde{x \otimes y})T = \widetilde{T^*x \otimes y}$ for any $T \in \mathcal{B}(H)$ and $x, y \in H$. In particular we obtain

$$\langle (T_{kj}S_{kj})(x_0), y_0\rangle = \beta_{kj}\langle T_{kj}(z_0), y_0\rangle\langle x_0, z_0\rangle$$

and

$$\langle (S_{kj}T_{kj})(x_0), y_0\rangle = \beta_{kj}\langle T_{kj}(x_0), z_0\rangle\langle z_0, y_0\rangle.$$

Therefore, choosing $z_0 = x_0$ and $\mathbf{C} = \mathbf{A} * \mathbf{B}$ one has $C_{x_0,y_0} = A_{x_0,y_0} * B$, and using that $\|C_{x_0,y_0}\| \le \|\mathbf{C}\|$ we obtain

$$\|A_{x_0,y_0} * B\|_{\mathcal{B}(\ell^2)} \le \|\mathbf{A} * \mathbf{B}\|_{\mathcal{B}(\ell^2(H))} \le \|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))}.$$

Similarly, choosing $z_0 = y_0$ and $\mathbf{C} = \mathbf{B} * \mathbf{A}$ one obtains

$$\|B * A_{x_0,y_0}\|_{\mathcal{B}(\ell^2)} \le \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}.$$

This completes the proof. □

Lemma 5.3. *Let* $\mu \in \mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$, $\mathbf{A} = (T_{kj}) \in \mathcal{T}$ *with* $T_{kj} = \widehat{\mu}(j-k)$ *for* $k, j \in \mathbb{N}$, $\mathbf{B} = (S_{kj}) \subset \mathcal{B}(H)$ *and* $\mathbf{x}, \mathbf{y} \in c_{00}(H)$. *Then*

$$\langle\!\langle \mathbf{A} * \mathbf{B}(\mathbf{x}), \mathbf{y} \rangle\!\rangle = \Psi_\mu \left(\int_0^{2\pi} \int_0^{2\pi} \mathbf{B}_{N,M}(s - \cdot\,, t - \cdot\,)(h_{\mathbf{x}}(s)) \otimes h_{\mathbf{y}}(t) \frac{ds}{2\pi} \frac{dt}{2\pi} \right).$$

Proof. Let $\mathbf{x}, \mathbf{y} \in c_{00}(H)$, say $h_{\mathbf{x}} = \sum_{j=1}^N x_j \varphi_j$ and $h_{\mathbf{y}} = \sum_{k=1}^M y_k \varphi_k$. Recall that $x_j = \int_0^{2\pi} h_{\mathbf{x}}(s) \overline{\varphi_j(s)} \frac{ds}{2\pi}$ and $y_k = \int_0^{2\pi} h_{\mathbf{y}}(t) \overline{\varphi_k(t)} \frac{dt}{2\pi}$. Then

$$\begin{aligned}
&\langle\!\langle \mathbf{A} * \mathbf{B}(\mathbf{x}), \mathbf{y} \rangle\!\rangle \\
&= \sum_{k=1}^M \sum_{j=1}^N \langle \widehat{\mu}(j-k) S_{kj}(x_j), y_k \rangle \\
&= \int_0^{2\pi} \left\langle \sum_{k=1}^M \left(\sum_{j=1}^N \widehat{\mu}(j-k) S_{kj}(x_j) \right) \varphi_k(t), h_{\mathbf{y}}(t) \right\rangle \frac{dt}{2\pi} \\
&= \int_0^{2\pi} \left\langle \sum_{l=-M}^N \widehat{\mu}(l) \left(\sum_{j-k=l} S_{kj}(x_j) \varphi_k(t) \right), h_{\mathbf{y}}(t) \right\rangle \frac{dt}{2\pi} \\
&= \int_0^{2\pi} \int_0^{2\pi} \left\langle \sum_{l=-M}^N \widehat{\mu}(l) \left(\sum_{j-k=l} S_{kj} \overline{\varphi_j(s)} \varphi_k(t)(h_{\mathbf{x}}(s)) \right), h_{\mathbf{y}}(t) \right\rangle \frac{ds}{2\pi} \frac{dt}{2\pi} \\
&= \int_0^{2\pi} \int_0^{2\pi} \sum_{l=-M}^N \mathcal{J}\mu(l) \left(\left(\sum_{j-k=l} S_{kj} \overline{\varphi_j(s)} \varphi_k(t) \right) (h_{\mathbf{x}}(s)) \otimes h_{\mathbf{y}}(t) \right) \frac{ds}{2\pi} \frac{dt}{2\pi} \\
&= \sum_{l=-M}^N \mathcal{J}\mu(l) \left(\int_0^{2\pi} \int_0^{2\pi} \left(\sum_{j-k=l} S_{kj} \overline{\varphi_j(s)} \varphi_k(t) \right) (h_{\mathbf{x}}(s)) \otimes h_{\mathbf{y}}(t) \frac{ds}{2\pi} \frac{dt}{2\pi} \right) \\
&= \Psi_\mu \left(\sum_{l=-M}^N \left(\int_0^{2\pi} \int_0^{2\pi} \left(\sum_{j-k=l} S_{kj} \overline{\varphi_j(s)} \varphi_k(t) \right) (h_{\mathbf{x}}(s)) \otimes h_{\mathbf{y}}(t) \frac{ds}{2\pi} \frac{dt}{2\pi} \right) \varphi_l \right) \\
&= \Psi_\mu \left(\int_0^{2\pi} \int_0^{2\pi} \left(\sum_{k=1}^M \sum_{j=l}^N S_{kj} \overline{\varphi_j(s)} \varphi_k(t) \varphi_j \varphi_{-k} \right) (h_{\mathbf{x}}(s)) \otimes h_{\mathbf{y}}(t) \frac{ds}{2\pi} \frac{dt}{2\pi} \right) \\
&= \Psi_\mu \left(\int_0^{2\pi} \int_0^{2\pi} \mathbf{B}_{N,M}(s - \cdot\,, t - \cdot\,)(h_{\mathbf{x}}(s)) \otimes h_{\mathbf{y}}(t) \frac{dt}{2\pi} \frac{ds}{2\pi} \right).
\end{aligned}$$

The proof is complete. □

Theorem 5.4. *If* $\mu \in M(\mathbb{T}, \mathcal{B}(H))$ *and* $\mathbf{A} = (T_{kj}) \in \mathcal{T}$ *with* $T_{kj} = \widehat{\mu}(j-k)$ *for* $k, j \in \mathbb{N}$ *then* $\mathbf{A} \in \mathcal{M}_l(\ell^2(H)) \cap \mathcal{M}_r(\ell^2(H))$ *and*

$$\max\{\|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))}, \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}\} \leq |\mu|.$$

Proof. Since $\|\mathbf{A}\|_{\mathcal{M}_l(\ell^2(H))} = \|\mathbf{A}^*\|_{\mathcal{M}_l(\ell^2(H))}$ and $|\mu| = |\mu^*|$ then it suffices to show the case of left Schur multipliers. Let $\mathbf{x}, \mathbf{y} \in c_{00}(H)$ and $\mathbf{B} = (S_{kj}) \subset \mathcal{B}(H)$ such that $\mathbf{B} \in \mathcal{B}(\ell^2(H))$. Define

$$G(u) = \int_0^{2\pi}\int_0^{2\pi} \mathbf{B}_{N,M}(s-u, t-u)(h_{\mathbf{x}}(s)) \otimes h_{\mathbf{y}}(t)\, \frac{dt}{2\pi}\frac{ds}{2\pi}.$$

Hence we can rewrite, since $(\lambda x) \otimes y = x \otimes \overline{\lambda} y$,

$$G(u) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} S_{kj}(x_j\varphi_j(u)) \otimes y_k\varphi_k(u).$$

In particular,

$$\begin{aligned}\|G(u)\|_{H\widehat{\otimes}H} &\leq \sum_{k=1}^{\infty}\left\|\sum_{j=1}^{\infty} S_{kj}(x_j\varphi_j(u))\right\| \|y_k\varphi_k(u)\| \leq \left(\sum_{k=1}^{\infty}\left\|\sum_{j=1}^{\infty} S_{kj}(x_j\varphi_j(u))\right\|^2\right)^{1/2} \|\mathbf{y}\| \\ &\leq \|\mathbf{B}\|\|\mathbf{x}\|\|\mathbf{y}\|.\end{aligned}$$

From Lemma 5.3, we have

$$|\langle\langle \mathbf{A} * \mathbf{B}(\mathbf{x}), \mathbf{y}\rangle\rangle| \leq \|\Psi_\mu\|_{C(\mathbb{T}, H\widehat{\otimes}H)^*} \sup_{0\leq u<2\pi} \|G(u)\|_{H\widehat{\otimes}H} = |\mu|\|\mathbf{B}\|\|\mathbf{x}\|\|\mathbf{y}\|.$$

This finishes the proof. □

Lemma 5.5. *Let* $\mu, \nu \in \mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$, $\mathbf{A} = (T_{kj}) \in \mathcal{T}$ *with* $T_{kj} = \widehat{\mu}(j-k)$, $\mathbf{B} = (S_{kj}) \in \mathcal{T}$ *with* $S_{kj} = \widehat{\nu}(j-k)$ *for* $k, j \in \mathbb{N}$ *and* $\mathbf{x}, \mathbf{y} \in c_{00}(H)$. *Then*

$$\langle\langle \mathbf{A} * \mathbf{B}(\mathbf{x}), \mathbf{y}\rangle\rangle = \Psi_\mu\left(\sum_{k=1}^{M}\left(\sum_{j=1}^{N} \widehat{\nu}(j-k)(x_j)\overline{\varphi_j}\right) \otimes y_k\varphi_k\right).$$

Proof. Denote $h_{\mathbf{x}} = \sum_{k=1}^{M} y_k\varphi_k$ and $h_{\mathbf{y}} = \sum_{j=1}^{N} x_j\varphi_j$. Then

$$\begin{aligned}\langle\langle \mathbf{A} * \mathbf{B}(\mathbf{x}), \mathbf{y}\rangle\rangle &= \sum_{k=1}^{M}\sum_{j=1}^{N}\langle \widehat{\mu}(j-k)\widehat{\nu}(j-k)(x_j), y_k\rangle = \sum_{l=-M}^{N}\sum_{k=1}^{M}\langle \widehat{\mu}(l)\widehat{\nu}(l)(x_{k+l}), y_k\rangle \\ &= \sum_{l=-M}^{N}\sum_{k=1}^{M} \mathcal{J}\widehat{\mu}(l)(\nu(l)(x_{k+l}) \otimes y_k) = \sum_{l=-M}^{N} \mathcal{J}\widehat{\mu}(l)\left(\sum_{k=1}^{M} \widehat{\nu}(l)(x_{k+l}) \otimes y_k\right)\end{aligned}$$

$$= \Psi_\mu \left(\sum_{l=-M}^{N} \left(\sum_{k=1}^{M} \widehat{\nu}(l)(x_{k+l}) \otimes y_k \right) \varphi_{-l} \right)$$
$$= \Psi_\mu \left(\sum_{k=1}^{M} \left(\sum_{j=1}^{N} \widehat{\nu}(j-k)(x_j)\overline{\varphi_j} \right) \otimes y_k \overline{\varphi_k} \right).$$

The proof is complete. □

Corollary 5.6. *Let $\mathbf{A} = (S_{kj}) \in \mathcal{T}$ such that $S_{kj} = \widehat{\nu}(j-k)$ for some $\nu \in \mathfrak{M}(T, \mathcal{B}(H))$. For each $\mathbf{x}, \mathbf{y} \in c_{00}(H)$ we denote*

$$F_{\mathbf{x},\mathbf{y},\mathbf{A}}(t) = \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} \widehat{\nu}(j-k)(x_j)\overline{\varphi_j}(t) \right) \otimes y_k \overline{\varphi_k}(t).$$

If $\mathbf{A} \in \mathcal{M}_r(\ell^2(H))$ then

$$\|F_{\mathbf{x},\mathbf{y},\mathbf{A}}\|_{L^1(\mathbb{T}, H\widehat{\otimes}H)} \le \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))} \|\mathbf{x}\|_{\ell^2(H)} \|\mathbf{y}\|_{\ell^2(H)}.$$

Proof. If $\mathbf{A} \in \mathcal{M}_r(\ell^2(H))$ then $\mathbf{B} * \mathbf{A} \in \mathcal{B}(\ell^2(H))$ for any $\mathbf{B} \in \mathcal{B}(\ell^2(H)) \cap \mathcal{T}$. In particular for any $\mathbf{B} = (T_{kj})$ with $T_{kj} = \widehat{\mu}(j-k)$ for some $\mu \in V^\infty(\mathbb{T}, \mathcal{B}(H))$ with $\|\mu\|_\infty = \|\mathbf{B}\|$. Since $L^1(\mathbb{T}, H\widehat{\otimes}H) \subseteq (V^\infty(\mathbb{T}, B(H)))^*$ isometrically, we can use Lemma 5.5 to obtain

$$\begin{aligned} \|F_{\mathbf{x},\mathbf{y},\mathbf{A}}\|_{L^1(\mathbb{T}, H\widehat{\otimes}H)} &= \sup\{|\Psi_\mu(F_{\mathbf{x},\mathbf{y},\mathbf{A}})| : \|\mu\|_\infty = 1\} \\ &= \sup\{|\langle\langle \mathbf{B} * \mathbf{A}(\mathbf{x}), \mathbf{y} \rangle\rangle| : \|\mathbf{B}\| = 1\} \\ &\le \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))} \|\mathbf{x}\|_{\ell^2(H)} \|\mathbf{y}\|_{\ell^2(H)}. \end{aligned}$$

This completes the proof. □

Theorem 5.7. *Let $\mathbf{A} = (T_{kj}) \in \mathcal{T} \cap \mathcal{M}_r(\ell^2(H))$. Then there exists $\mu \in M_{\mathrm{SOT}}(\mathbb{T}, \mathcal{B}(H))$ such that $T_{kj} = \widehat{\mu}(j-k)$ for all $k, j \in \mathbb{N}$. Moreover, $\|\mu\|_{\mathrm{SOT}} \le \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}$.*

Proof. Let $\mathbf{A} \in \mathcal{M}_r(\ell^2(H))$. For each $x_0, y_0 \in H$, as above we consider the scalar-valued Toeplitz matrix $A_{x_0,y_0} = (\langle T_{kj}(x_0), y_0 \rangle)$. Using Lemma 5.2, we have that $A_{x_0,y_0} \in \mathcal{M}(\ell^2)$ and $\|A_{x_0,y_0}\|_{\mathcal{M}(\ell^2)} \le \|\mathbf{A}\|_{\mathcal{M}(\ell^2(H))}$. This guarantees, invoking Theorem 1.3, that there exists $\eta_{x_0,y_0} \in M(\mathbb{T})$ such that $\langle T_{kj}(x_0), y_0 \rangle = \widehat{\eta_{x_0,y_0}}(j-k)$ for all $j, k \in \mathbb{N}$ and $|\eta_{x_0,y_0}| = \|A_{x_0,y_0}\|_{\mathcal{M}_r(\ell^2)}$.

Now define $\mu(A) \in \mathcal{B}(H)$ given by

$$\langle \mu(A)(x), y \rangle = \eta_{x,y}(A), \quad x, y \in H.$$

Let us show that $\mu \in M_{\mathrm{SOT}}(\mathbb{T}, \mathcal{B}(H))$ and $\|\mu\|_{\mathrm{SOT}} \le \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}$.

First we need to show that $\mu(A) \in \mathcal{B}(H)$ for any $A \in \mathfrak{B}(\mathbb{T})$. This follows using that

$$\widehat{\eta_{\lambda x+\beta x',y}}(l) = \lambda\widehat{\eta_{x,y}}(l) + \beta\widehat{\eta_{x',y}}(l), \quad l \in \mathbb{Z}$$

for any $\lambda, \beta \in \mathbb{C}$ and $x, x', y \in H$. This guarantees that $\eta_{\lambda x+\beta x',y} = \lambda\eta_{x,y} + \beta\eta_{x',y}$ and hence $\mu(A)\colon H \to H$ is a linear map. The continuity follows from the estimate $|\eta_{x,y}| \leq \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}\|x\|\|y\|$. To show that it is a regular measure, select $\{x_n : n \in \mathbb{N}\}$ dense in H. Hence, for any $S \in \mathcal{B}(H)$ we have

$$\|S\| = \sup\{\langle S(x_n), x_m\rangle : n, m \in \mathbb{N}\}.$$

Denoting by $\eta_{n,m} = \eta_{x_n,x_m}$ we have that for each $B \in \mathfrak{B}(\mathbb{T})$, given $(n, m) \in \mathbb{N} \times \mathbb{N}$ and $\varepsilon > 0$, there exists $K_{n,m} \subset B \subset O_{n,m}$ which are compact and open respectively so that

$$|\eta_{n,m}|(O_{n,m} \setminus K_{n,m}) < \varepsilon.$$

Now selecting $K = \overline{\bigcup_{n,m} K_{n,m}}$ and $O = \left(\bigcap_{n,m} O_{n,m}\right)^\circ$ we conclude that

$$\|\mu\|(O \setminus K) < \varepsilon.$$

This shows that $\mu \in \mathfrak{M}(\mathbb{T}, \mathcal{B}(H))$.

Using now that

$$\langle T_\mu(\phi)(x), y\rangle = T_{\eta_{x,y}}(\phi)$$

for each $\phi \in C(\mathbb{T})$, where $T_{\eta_{x,y}} \in \mathcal{L}(C(\mathbb{T}), \mathbb{C})$ denotes the operator associated to $\eta_{x,y} \in M(\mathbb{T})$, we clearly have that $T_{kj} = \widehat{\mu}(j-k)$ for all $j, k \in \mathbb{N}$.

Select $y_k = y\beta_k$ for some $\beta_k \in \mathbb{C}$ and $\|y\| = 1$. From Corollary 5.6 we obtain that

$$\begin{aligned}
&\int_0^{2\pi} \left\| \left(\sum_{k=1}^{M}\sum_{j=1}^{N} \widehat{\mu}(j-k)(x_j)\beta_k\overline{\varphi}_j(t)\varphi_k(t) \right) \otimes y \right\|_{H\widehat{\otimes}H} \frac{dt}{2\pi} \\
&= \int_0^{2\pi} \left\| \sum_{l=-M}^{N} \widehat{\mu}(l) \left(\sum_{k=1}^{M} x_{k+l}\beta_k \right) \varphi_{-l}(t) \right\| \frac{dt}{2\pi} \\
&\leq \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}\|\mathbf{x}\|_{\ell^2(H)} \left(\sum_{k=1}^{M} |\beta_k|^2 \right)^{1/2}.
\end{aligned}$$

Taking $x_j = x\alpha_j$ such that $\|x\| = 1$, we get

$$\begin{aligned}
&\int_0^{2\pi} \left\| \sum_{l=-M}^{N} \widehat{\mu}(l)(x) \left(\sum_{j-k=l} \alpha_j\overline{\varphi}_j(t)\beta_k\varphi_k(t) \right) \right\| \frac{dt}{2\pi} \\
&\leq \|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))} \left(\sum_{j=1}^{N} |\alpha_j|^2 \right)^{1/2} \left(\sum_{k=1}^{M} |\beta_k|^2 \right)^{1/2}.
\end{aligned}$$

Using now

$$\gamma(s)=\sum_{l=-M}^{N}\left(\sum_{j-k=l}\beta_k\alpha_j\right)\varphi_l(s).$$

Now recall that $\widehat{\mu}(l)(x)=\widehat{\mu}_x(l)$ and

$$\sum_{l=-M}^{N}\widehat{\mu}_x(l)\left(\sum_{j-k=l}\alpha_j\overline{\varphi}_j(t)\beta_k\varphi_k(t)\right)=\int_0^{2\pi}\left(\sum_{l=-M}^{N}\widehat{\mu}_x(l)\varphi_l(s)\right)\gamma(-t-s)\,\frac{ds}{2\pi}.$$

Therefore, if $\alpha=\sum_{j=1}^{\infty}\alpha_j\varphi_j$ and $\beta=\sum_{k=1}^{\infty}\beta_k\varphi_k$ belong to $L^2(\mathbb{T})$, we have that $\gamma(t)=\alpha(t)\beta(-t)$ and

$$\int_0^{2\pi}\|\mu_x*\gamma(-t)\|\,\frac{dt}{2\pi}\le\|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}\|\alpha\|_{L^2(\mathbb{T})}\|\beta\|_{L^2(\mathbb{T})}. \tag{5.2}$$

To show that $\mu_x\in M(\mathbb{T},H)$, due to Lemma 3.8, it suffices to prove that

$$\sup_{0<r<1}\|\mu_x*P_r\|_{L^1(\mathbb{T},H)}<\infty. \tag{5.3}$$

Choosing $\beta(t)=\alpha(t)=\sqrt{1-r^2}/|1-re^{it}|$ we obtain that $\gamma(t)=P_r(t)$ and from (5.2) we get (5.3) and the estimate $\|\mu_x\|_{M(\mathbb{T},H)}\le\|\mathbf{A}\|_{\mathcal{M}_r(\ell^2(H))}$. This finishes the proof. □

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