

Gap Theorems on Critical Point Equation of the Total Scalar Curvature with Divergence-free Bach Tensor

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Abstract. On a compact n -dimensional manifold, it is well known that a critical metric of the total scalar curvature, restricted to the space of metrics with unit volume is Einstein. It has been conjectured that a critical metric of the total scalar curvature, restricted to the space of metrics with constant scalar curvature of unit volume, will be Einstein. This conjecture, proposed in 1987 by Besse, has not been resolved except when M has harmonic curvature or the metric is Bach flat. In this paper, we prove some gap properties under divergence-free Bach tensor condition for $n \geq 5$, and a similar condition for $n = 4$.

1. Introduction

Let M be an n -dimensional compact manifold, and let \mathcal{M}_1 be the set of all smooth Riemannian structures of unit volume on M . The total scalar curvature \mathcal{S} on \mathcal{M}_1 is given by

$$\mathcal{S}(g) = \int_M s_g dv_g,$$

where s_g is the scalar curvature of $g \in \mathcal{M}_1$. Hilbert showed that critical points of \mathcal{S} on \mathcal{M}_1 are Einstein. In [6], Koiso introduced the space \mathcal{C} of constant scalar curvature metrics of unit volume. The Euler-Lagrange equation of \mathcal{S} restricted to \mathcal{C} may be written in the form of the following critical point equation

$$(1.1) \quad z_g = s_g'^*(f).$$

Here, z_g is the traceless Ricci tensor corresponding to g , and the operator $s_g'^*$ is the L^2 adjoint of the linearization s_g' of the scalar curvature, given by

$$s_g'^*(f) = D_g df - (\Delta_g f)g - fr_g,$$

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where $D_g d$ and Δ_g denote the Hessian and the (negative) Laplacian, respectively, and r_g is the Ricci curvature of g . If $f = 0$ in (1.1), then g is clearly Einstein. By taking the trace of (1.1), we obtain

$$\Delta_g f = -\frac{s_g}{n-1} f.$$

Thus, if $s_g/(n-1)$ is not in the spectrum of Δ_g , then the critical metric g is again Einstein. For example, if $s_g \leq 0$, then g is Einstein. Note that if a non-trivial solution (g, f) of (1.1) is Einstein, then (1.1) is reduced to the Obata equation, and so (M, g) should be isometric to a standard n -sphere (see [7]).

We remark that the existence of a non-trivial solution is a strong condition. The only known case satisfying this is that of the standard sphere. It was conjectured in [2] that this is the only possible case.

Besse Conjecture. *Let (g, f) be a solution of (1.1) on an n -dimensional compact manifold M . Then, (M, g) is Einstein.*

There are some partial answers to this conjecture. For example, it was proved that the Besse conjecture holds if M has harmonic curvature (see [10, Theorem 1.2] and also [12]). A Riemannian manifold (M, g) is said to have harmonic curvature if $\operatorname{div} R = 0$, where R is the full Riemann tensor, and δ is the negative divergence operator. In particular, a locally conformally flat non-trivial solution (g, f) of (1.1) with $s_g > 0$ is clearly isometric to a standard sphere. Note that when s_g is constant, $\operatorname{div} R = 0$ if and only if $\operatorname{div} \mathcal{W} = 0$ (cf. (2.1) below), where \mathcal{W} is the Weyl tensor. Qing and Yuan showed in [9] that the Besse conjecture holds if g is Bach-flat, i.e., $B = 0$, where B is the n -dimensional Bach tensor (see Section 2 for its definition). In fact, they proved that Bach-flatness implies harmonic curvature.

On the other hand, in [4] Catino, Mastrolia, and Monticelli classified gradient Ricci solitons satisfying a fourth-order vanishing condition on the Weyl tensor. This condition on the Weyl tensor is clearly weaker than $\operatorname{div} \mathcal{W} = 0$. Similarly, it is natural to consider the divergence-free Bach tensor condition in the critical point equation (1.1) as a way to generalize Bach-flat condition. It turns out that δB vanishes automatically when $n = 4$, and $\delta B = 0$ if and only if $\langle i_X C, z_g \rangle = 0$ for any vector X when $n \geq 5$ (see Proposition 2.2 below). Here, C is the Cotton tensor defined by (2.2) below.

In this paper, we will prove some gap properties under the assumption $\delta B = 0$. For a non-trivial solution (g, f) of (1.1), we define μ by

$$\mu = \max_M |1 + f|.$$

If f satisfies $f \geq -1$, we can easily show that (M, g) is Einstein. In fact, one can show from (1.1) that $\operatorname{div}(z_g(\nabla f, \cdot)) = (1 + f)|z_g|^2$, and so rigidity follows from the divergence

theorem. Note that $\mu \geq 1$, because we have

$$0 = \int_M \Delta f = -\frac{s}{n-1} \int_M f,$$

which implies that there exists a point $p \in M$ satisfying $f(p) = 0$. In fact, $\mu > 1$ unless f is trivial.

Our first main result for gap property on the critical point equation is the following.

Theorem 1.1. *Let (g, f) be a non-trivial solution of (1.1) on an n -dimensional compact manifold M . Assume that $\langle i_X C, z_g \rangle = 0$ for any vector X and $n \geq 4$. If $|z_g|^2 \leq s_g^2/[4n(n-1)\mu^2]$, then (M, g) is isometric to a standard n -sphere.*

As mentioned above, when $n \geq 5$, the condition that the Bach tensor is divergence-free implies the first hypothesis in Theorem 1.1. Thus, for $n \geq 5$ we have the following result.

Corollary 1.2. *Let $n \geq 5$ and (g, f) be a non-trivial solution of (1.1) on an n -dimensional compact manifold M having divergence-free Bach tensor. If $|z_g|^2 \leq s_g^2/[4n(n-1)\mu^2]$, then (M, g) is isometric to a standard n -sphere.*

In [10, 12], we proved the Besse conjecture is true when (M, g) has harmonic curvature. In this case, the traceless Ricci tensor z_g can be decomposed into ∇f -direction and its orthogonal complement. In other words, for a vector X orthogonal to ∇f , we have $z_g(\nabla f, X) = 0$, and so z_g can be controlled by $z_g(N, \cdot) = i_N z_g$ with $N = \nabla f/|\nabla f|$ on each hypersurface given by a level set of f . Related to $i_N z_g$, we have the following gap property.

Theorem 1.3. *Let (g, f) be a non-trivial solution of (1.1) on an n -dimensional compact manifold M . Assume that $\langle i_X C, z_g \rangle = 0$ for any vector X and $n \geq 4$. If*

$$|z_g|^2 \leq \min \left\{ 2|i_N z_g|^2, \frac{s_g^2}{4n(n-1)} \right\},$$

then (M, g) is isometric to a standard sphere.

It is comparable with Theorem 2 of [1], which states that a non-trivial solution (g, f) of (1.1) has zero radial Weyl curvature with

$$|z_g|^2 \leq \frac{s_g^2}{n(n-1)},$$

then (M, g) is isometric to a standard sphere. We say that g has zero radial Weyl curvature if $\tilde{i}_{\nabla f} \mathcal{W} = 0$, where \tilde{i}_X is defined in (3.1).

This paper is organized as follows. In Section 2, we give some properties of Bach tensor and Cotton tensor with their divergences. In particular, we include the fact that

the divergence of Bach tensor is given by the inner product of the Cotton tensor with traceless Ricci tensor (Proposition 2.2). In Section 3, we introduce a covariant 3-tensor and derive some properties of it to handle the critical point equation. In Sections 4 and 5, we prove our main results, Theorems 1.1 and 1.3.

2. Divergences of a Bach tensor

Let (M, g) be an n -dimensional Riemannian manifold. For convenience, we denote s_g, r_g and z_g by s, r and z , respectively, if there is no ambiguity. Throughout the paper, we will assume that the dimension $n \geq 4$.

Let D be the Levi-Civita connection on (M, g) and let us denote by $C^\infty(S^2M)$ the space of sections of symmetric 2-tensors on (M, g) . Then, the differential operator d^D from $C^\infty(S^2M)$ to $C^\infty(\Lambda^2M \otimes T^*M)$ is defined by

$$d^D\eta(X, Y, Z) = (D_X\eta)(Y, Z) - (D_Y\eta)(X, Z)$$

for $\eta \in C^\infty(S^2M)$ and vectors X, Y , and Z . In particular, the following result is well known (see [2, p. 435]): under the identification of $C^\infty(T^*M \otimes \Lambda^2M)$ with $C^\infty(\Lambda^2M \otimes T^*M)$,

$$(2.1) \quad \operatorname{div} R = d^D r.$$

For a function $\varphi \in C^\infty(M)$ and $\eta \in C^\infty(S^2M)$, $d\varphi \wedge \eta$ is defined by

$$(d\varphi \wedge \eta)(X, Y, Z) = d\varphi(X)\eta(Y, Z) - d\varphi(Y)\eta(X, Z).$$

Here, $d\varphi$ denotes the usual total differential of φ .

The Cotton tensor $C \in \Gamma(\Lambda^2M \otimes T^*M)$ is defined by

$$(2.2) \quad C = d^D r - \frac{1}{2(n-1)} ds \wedge g$$

and the n -dimensional Bach tensor B is defined by

$$B = -\frac{1}{n-3} \delta^D \operatorname{div} \mathcal{W} + \frac{1}{n-2} \mathring{W}z.$$

Here, δ^D is the L^2 adjoint operator of d^D , and

$$\mathring{W}z(X, Y) = \sum_{i=1}^n z(\mathcal{W}(X, E_i)Y, E_i)$$

for an orthonormal frame $\{E_i\}_{i=1}^n$. $\mathring{R}r$ is defined similarly. From now on, we will omit the summation notation, as we employ the Einstein convention.

Because

$$\operatorname{div} \mathcal{W} = \frac{n-3}{n-2} d^D \left(r - \frac{s}{2(n-1)} g \right),$$

we have

$$(2.3) \quad C = \frac{n-2}{n-3} \operatorname{div} \mathcal{W} \quad \text{and} \quad \operatorname{div} C = -\frac{n-2}{n-3} \delta^D \operatorname{div} \mathcal{W}.$$

As a consequence, we have

$$(2.4) \quad (n-2)B = \operatorname{div} C + \overset{\circ}{\mathcal{W}}z.$$

Proposition 2.1. [2, Corollary 1.22] *For any tensor h , we have*

$$D_{X,Y}^2 h - D_{Y,X}^2 h = -R(X, Y)h$$

and

$$D_{X,Y,Z}^3 h - D_{Y,X,Z}^3 h = -R(X, Y)D_Z h + D_{R(X,Y)}Z h.$$

Recall that the Schouten tensor A is defined by

$$A = r - \frac{s}{2(n-1)} g$$

so that $C = d^D A$. The following is Lemma 5.1 of [3]. We include the proof for the sake of completeness.

Proposition 2.2. *For any vector field X we have*

$$(n-2)\delta B(X) = -\frac{n-4}{n-2} \langle i_X C, z \rangle.$$

Here,

$$i_X C(Y, Z) = C(X, Y, Z), \quad \langle i_X C, z \rangle = \sum_{i,j=1}^n i_X C(E_i, E_j) z(E_i, E_j),$$

and a 2-tensor $z \circ z$ is defined by

$$z \circ z(X, Y) = \sum_{i=1}^n z(X, E_i) z(E_i, Y)$$

for any orthonormal frame $\{E_i\}_{i=1}^n$ and vector fields X, Y , and Z .

Proof. Let $\{E_i\}_{i=1}^n$ be a geodesic frame. Denoting $z_{ij} = z(E_i, E_j)$ and $r_{ij} = r(E_i, E_j)$, it follows from (2.3) and (2.4) that

$$\begin{aligned} (n-2)\delta B(X) &= -\operatorname{div}^2(C)(X) - \operatorname{div}(\overset{\circ}{\mathcal{W}}z)(X) \\ &= -\operatorname{div}^2(C)(X) - \frac{n-3}{n-2} C(X, E_i, E_j) z_{ij} \\ &\quad + \frac{1}{2} \mathcal{W}(E_i, E_j, E_k, X) C(E_i, E_j, E_k). \end{aligned}$$

Note that

$$\begin{aligned} \operatorname{div}^2(C)(X) &= \operatorname{div}^2(d^D A)(X) \\ &= D_{E_i} D_{E_k} (D_{E_k} A(E_i, X) - D_{E_i} A(E_k, X)) \\ &= (D_{E_i} D_{E_k} - D_{E_k} D_{E_i}) D_{E_k} A(E_i, X). \end{aligned}$$

Thus, by Proposition 2.1 we have

$$\begin{aligned} \operatorname{div}^2(C)(X) &= R(E_k, E_i) D_{E_k} A(E_i, X) - D_{R(E_k, E_i) E_k} A(E_i, X) \\ &= -D_{E_k} A(R(E_k, E_i) E_i, X) - D_{E_k} A(E_i, R(E_k, E_i) X) - r_{is} D_{E_s} A(E_i, X) \\ &= r_{kj} D_{E_k} A(E_j, X) + \langle R(E_k, E_i) E_s, X \rangle D_{E_k} A(E_i, E_s) - r_{ik} D_{E_k} A(E_i, X) \\ &= \frac{1}{2} \langle R(E_k, E_i) E_s, X \rangle C(E_k, E_i, E_s). \end{aligned}$$

Hence,

$$(n - 2)\delta B(X) = -\frac{n - 3}{n - 2} C(X, E_i, E_j) z_{ij} + \frac{1}{2} (\mathcal{W} - R)(E_i, E_j, E_k, X) C(E_i, E_j, E_k).$$

From the decomposition of Riemann tensor, it follows

$$\mathcal{W}_{ijkl} = R_{ijkl} - \frac{1}{n - 2} (g_{ik} A_{jl} + g_{jl} A_{ik} - g_{jk} A_{il} - g_{il} A_{jk}),$$

and so

$$(\mathcal{W} - R)(E_i, E_j, E_k, E_l) C(E_i, E_j, E_k) = -\frac{2}{n - 2} (A_{jl} C_{iji} + A_{ik} C_{ilk}) = -\frac{2}{n - 2} r_{ik} C_{ilk}.$$

Here, $C_{ijk} = C(E_i, E_j, E_k)$ and we have used the fact that $\sum_i C(E_i, Y, E_i) = 0$ for any Y . By substituting these results, we obtain the desired equation

$$(n - 2)\delta B(X) = -\frac{n - 4}{n - 2} C(X, E_j, E_k) z_{jk}. \quad \square$$

By Proposition 2.2, it is clear that $\delta B = 0$ when $n = 4$, and $\langle i_X C, z \rangle = 0$ for any vector field X if and only if $\delta B = 0$ when $n \geq 5$. In particular, when s_g is constant, it follows from $\delta z = 0$ and $C = d^D z$ that

$$\delta(z \circ z)(X) = -D_{E_i} (z \circ z)(E_i, X) = \delta z(E_j) z(E_j, X) - z_{ij} D_{E_i} z(E_j, X),$$

implying that

$$0 = \langle i_X C, z \rangle = D_X z(E_i, E_j) z_{ij} - D_{E_i} z(X, E_j) z_{ij} = \frac{1}{2} d|z|^2(X) + \delta(z \circ z)(X).$$

Thus, if s_g is constant, for $n \geq 5$ we have

$$(2.5) \quad \frac{1}{2} \Delta |z|^2 = \delta \delta (z \circ z).$$

The following result holds when the scalar curvature is constant.

Proposition 2.3. [11, (10)] *Assume that s_g is constant. Then,*

$$\operatorname{div}(d^D r) = -D^* D z - \frac{n}{n-2} z \circ z - \frac{s}{n-1} z + \frac{1}{n-2} |z|^2 g + \mathring{W} z.$$

Proof. The proof follows from the identity (see [2, 4.71])

$$\operatorname{div}(d^D r) = -D^* D r - \frac{1}{2} D d s - r \circ r + \mathring{R} r,$$

and the relation in [11]

$$\mathring{R} r = \mathring{W} z + \frac{1}{n-2} |z|^2 g + \frac{(n-2)s}{n(n-1)} z - \frac{2}{n-2} z \circ z + \frac{s^2}{n^2} g,$$

which comes from the decomposition of the Riemann tensor. □

3. Critical metrics

In this section, we turn our attention to a non-trivial solution (g, f) of (1.1). To do this, we will introduce a covariant 3-tensor T defined by

$$T = \frac{1}{n-2} df \wedge z + \frac{1}{(n-1)(n-2)} i_{\nabla f} z \wedge g.$$

Also we define the interior product \tilde{i} to the final factor by

$$(3.1) \quad \tilde{i}_V \omega(X, Y, Z) = \omega(X, Y, Z, V)$$

for a $(4, 0)$ -tensor ω and a vector field V .

Now, from the critical metric equation (1.1) we have

$$(3.2) \quad (1 + f)z = Ddf + \frac{sf}{n(n-1)} g.$$

By applying d^D to both sides of this equation and using the Ricci identity

$$d^D Ddf(X, Y, Z) = R(X, Y, Z, \nabla f)$$

for any vector fields X, Y, Z on M , we obtain

$$(df \wedge z + (1 + f)d^D z)(X, Y, Z) = \tilde{i}_{\nabla f} R(X, Y, Z) + \frac{s}{n(n-1)} df \wedge g(X, Y, Z).$$

Since $C = d^D z$ when s is constant, we obtain

$$(3.3) \quad (1 + f)C = \tilde{i}_{\nabla f} R - df \wedge z + \frac{s}{n(n-1)} df \wedge g = \tilde{i}_{\nabla f} \mathcal{W} - (n-1)T.$$

Here, we used the fact that

$$\tilde{i}_{\nabla f}R = \tilde{i}_{\nabla f}\mathcal{W} - \frac{1}{n-2}i_{\nabla f}r \wedge g - \frac{1}{n-2}df \wedge r + \frac{s}{(n-1)(n-2)}df \wedge g,$$

which follows from the curvature decomposition (cf. [2, 1.116, p. 48])

$$\begin{aligned} R(X, Y, Z, W) &= \mathcal{W}(X, Y, Z, W) + \frac{1}{n-2}(g(X, Z)r(Y, W) + g(Y, W)r(X, Z) \\ &\quad - g(Y, Z)r(X, W) - g(X, W)r(Y, Z)) \\ &\quad - \frac{s}{(n-1)(n-2)}(g(X, Z)g(Y, W) - g(Y, Z)g(X, W)). \end{aligned}$$

From the definition of the Bach tensor (2.4) and (3.3), we have

$$(3.4) \quad (n-2)B = \operatorname{div} C + \mathring{W}z = \operatorname{div} \left(\frac{1}{1+f}\tilde{i}_{\nabla f}\mathcal{W} - (n-1)\frac{T}{1+f} \right) + \mathring{W}z.$$

Since

$$\operatorname{div}(\tilde{i}_{\nabla f}\mathcal{W})(X, Y) = \frac{n-3}{n-2}d^D r(Y, \nabla f, X) - (1+f)\mathring{W}z(X, Y),$$

we have

$$\begin{aligned} &\operatorname{div} \left(\frac{1}{1+f}\tilde{i}_{\nabla f}\mathcal{W} \right) (X, Y) \\ &= -\frac{1}{(1+f)^2}\mathcal{W}(\nabla f, X, Y, \nabla f) - \frac{1}{1+f}\delta\tilde{i}_{\nabla f}\mathcal{W}(X, Y) \\ &= \frac{1}{(1+f)^2}\mathcal{W}(X, \nabla f, Y, \nabla f) + \frac{n-3}{n-2}\frac{1}{1+f}d^D r(Y, \nabla f, X) - \mathring{W}z(X, Y). \end{aligned}$$

Therefore, it follows from (3.3) and (3.4) that

$$(3.5) \quad (n-2)(1+f)B(X, Y) = C(X, \nabla f, Y) + \frac{n-3}{n-2}C(Y, \nabla f, X) + (n-1)\delta T(X, Y).$$

On the other hand, by taking the divergence of T , we have

$$\begin{aligned} &(n-1)(n-2)\operatorname{div} T(X, Y) \\ (3.6) \quad &= -\frac{n-2}{n-1}sfz(X, Y) + (n-2)D_{\nabla f}z(X, Y) - C(Y, \nabla f, X) \\ &\quad - n(1+f)z \circ z(X, Y) + (1+f)|z|^2g(X, Y). \end{aligned}$$

By combining (3.5) and (3.6), we obtain the followings.

Proposition 3.1. *On M , we have*

$$\begin{aligned} (n-1)\langle \operatorname{div}(T), z \rangle &= -(n-2)(1+f)\langle B, z \rangle \\ &= -\frac{sf}{n-1}|z|^2 + \frac{1}{2}\nabla f(|z|^2) - \frac{n}{n-2}(1+f)\langle z \circ z, z \rangle. \end{aligned}$$

Also we have

Proposition 3.2. *On M , we have*

$$(n - 1) \operatorname{div}^2(T)(X) = \frac{n - 1}{n - 2} (1 + f) \langle i_X C, z \rangle + (n - 1) \langle i_X T, z \rangle.$$

Proof. By taking the derivative of (3.5), we have

$$\begin{aligned} (n - 2)B(\nabla f, X) &= (n - 2)(1 + f)\delta B(X) + \operatorname{div}(C)(\nabla f, X) \\ &\quad + \frac{n - 3}{n - 2} (1 + f) \langle i_X C, z \rangle - (n - 1) \operatorname{div}^2(T)(X). \end{aligned}$$

Thus, by (2.4) and Proposition 2.2 we have

$$\begin{aligned} (n - 1) \operatorname{div}^2(T)(X) &= -\dot{W}z(\nabla f, X) + \frac{1}{n - 2} (1 + f) \langle i_X C, z \rangle \\ &= \frac{n - 1}{n - 2} (1 + f) \langle i_X C, z \rangle + (n - 1) \langle i_X T, z \rangle, \end{aligned}$$

where the last equality comes from (3.3). □

We also have the following.

Lemma 3.3. *We have*

$$|T|^2 = \frac{2}{n - 2} \langle i_{\nabla f} T, z \rangle,$$

and

$$\frac{(n - 2)^2}{2} |T|^2 = |z|^2 |\nabla f|^2 - \frac{n}{n - 1} z \circ z(\nabla f, \nabla f).$$

Proof. It is a straightforward computation. From the definition of T ,

$$|T|^2 = \frac{1}{n - 2} \sum_{i,j,k} T(E_i, E_j, E_k) \left(df \wedge z + \frac{1}{n - 1} i_{\nabla f} z \wedge g(E_i, E_j, E_k) \right) = \frac{2}{n - 2} \langle i_{\nabla f} T, z \rangle.$$

Also

$$\begin{aligned} (n - 2)^2 |T|^2 &= \sum_{i,j,k} \left| df(E_i)z_{jk} - df(E_j)z_{ik} + \frac{1}{n - 1} (z(\nabla f, E_i)g_{jk} - z(\nabla f, E_j)g_{ik}) \right|^2 \\ &= 2|\nabla f|^2 |z|^2 - \frac{2n}{n - 1} z \circ z(\nabla f, \nabla f). \end{aligned} \quad \square$$

4. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Throughout the section and the next section, we assume that $\langle i_X C, z \rangle = 0$ for any vector X with $n \geq 4$. To prove Theorem 1.1, we first need the following.

Lemma 4.1. *Let (g, f) be a non-trivial solution of (1.1) on an n -dimensional compact manifold M , $n \geq 4$. Assume that $\langle i_X C, z \rangle = 0$. Then*

$$\int_M (1 + f) \langle z \circ z, z \rangle = \frac{(n - 2)s}{2n(n - 1)} \int_M |z|^2.$$

Proof. Note that

$$\frac{1}{2} \int_M (1 + f) \Delta |z|^2 = \frac{1}{2} \int_M |z|^2 \Delta f = -\frac{s}{2(n - 1)} \int_M f |z|^2.$$

Also, by (2.5) we have

$$\begin{aligned} \frac{1}{2} \int_M (1 + f) \Delta |z|^2 &= \int_M (1 + f) \delta \delta (z \circ z) = \int_M \delta (z \circ z) (\nabla f) = \int_M \langle z \circ z, Ddf \rangle \\ &= \int_M (1 + f) \langle z \circ z, z \rangle - \frac{s}{n(n - 1)} \int_M f |z|^2. \end{aligned}$$

Thus,

$$\int_M (1 + f) \langle z \circ z, z \rangle = -\frac{(n - 2)s}{2n(n - 1)} \int_M f |z|^2.$$

However, by (1.1) it is easy to see that

$$\operatorname{div}(i_{\nabla f} z) = (1 + f) |z|^2,$$

which implies that

$$(4.1) \quad \int_M f |z|^2 = - \int_M |z|^2. \quad \square$$

The following is the Okumura inequality which can be found in Lemma 2.6 of [8] (see also [5, Lemma 2.4]).

Lemma 4.2. *For any real numbers a_1, \dots, a_n with $\sum_{i=1}^n a_i = 0$, we have*

$$-\frac{n - 2}{\sqrt{n(n - 1)}} \left(\sum_{i=1}^n a_i^2 \right)^{3/2} \leq \sum_{i=1}^n a_i^3 \leq \frac{n - 2}{\sqrt{n(n - 1)}} \left(\sum_{i=1}^n a_i^2 \right)^{3/2},$$

and equality holds if and only if at least $n - 1$ of the a_i 's are all equal.

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Since $\langle z \circ z, z \rangle = \operatorname{tr}(z^3)$, by applying Lemma 4.2 to the traceless Ricci tensor z , we have

$$(1 + f) \langle z \circ z, z \rangle \leq \frac{n - 2}{\sqrt{n(n - 1)}} \mu |z|^3.$$

Recall that $\mu = \max_M |1 + f|$. Thus, by Lemma 4.1,

$$\frac{(n-2)s}{2n(n-1)} \int_M |z|^2 = \int_M (1+f) \langle z \circ z, z \rangle \leq \frac{(n-2)\mu}{\sqrt{n(n-1)}} \int_M |z|^3.$$

Consequently, we obtain

$$\frac{n-2}{\sqrt{n(n-1)}} \int_M \left(\frac{s}{2\sqrt{n(n-1)}} - \mu|z| \right) |z|^2 \leq 0.$$

From the assumption,

$$\frac{s}{2\sqrt{n(n-1)}} - \mu|z| \geq 0.$$

As a result, we have either $z = 0$, or

$$|z| = \frac{s}{2\mu\sqrt{n(n-1)}}.$$

From (4.1) and the fact that $\int_M f = 0$, the second case should be excluded; otherwise

$$0 = \int_M (1+f)|z|^2 = \frac{s^2}{4n(n-1)\mu^2} \int_M (1+f) = \frac{s^2}{4n(n-1)\mu^2},$$

which is a contradiction. □

5. Proof of Theorem 1.3

In this section, we prove Theorem 1.3. To do this, we first show the following integral identity.

Lemma 5.1. *We have*

$$(5.1) \quad \begin{aligned} \frac{(n-1)(n-2)}{2} \int_M |T|^2 &= \frac{s}{n} \int_M f^2 |z|^2 + \int_M z \circ z (\nabla f, \nabla f) \\ &+ \frac{2(n-1)}{n-2} \int_M f(1+f) \langle z \circ z, z \rangle. \end{aligned}$$

Proof. It follows from Proposition 3.2 and Lemma 3.3 together with the assumption $\langle i_X C, z \rangle = 0$ for any vector X that

$$\operatorname{div}^2(T)(\nabla f) = \langle i_{\nabla f} T, z \rangle = \frac{n-2}{2} |T|^2.$$

Thus,

$$(5.2) \quad \int_M f \operatorname{div}^3(T) = - \int_M \operatorname{div}^2(T)(\nabla f) = - \frac{n-2}{2} \int_M |T|^2.$$

By Proposition 3.2 again with $\langle i_X C, z \rangle = 0$, we have

$$(5.3) \quad \operatorname{div}^2(T)(X) = \langle i_X T, z \rangle.$$

From the definition of T ,

$$\langle T, C \rangle = \frac{1}{n-2} \langle df \wedge z, C \rangle = \frac{2}{n-2} \langle i_{\nabla f} C, z \rangle = 0,$$

and so, by taking the divergence of $\operatorname{div}^2(T)$, it follows from (5.3) that

$$\operatorname{div}^3(T) = \langle \operatorname{div}(T), z \rangle + \frac{1}{2} \langle T, C \rangle = \langle \operatorname{div}(T), z \rangle.$$

Thus, by Proposition 3.1,

$$(n-1) \operatorname{div}^3(T) = -\frac{sf}{n-1} |z|^2 + \frac{1}{2} \nabla f(|z|^2) - \frac{n}{n-2} (1+f) \langle z \circ z, z \rangle.$$

From this together with (5.2), we have

$$(5.4) \quad \begin{aligned} \frac{(n-1)(n-2)}{2} \int_M |T|^2 &= \frac{s}{n-1} \int_M f^2 |z|^2 - \frac{1}{2} \int_M f \nabla f(|z|^2) \\ &+ \frac{n}{n-2} \int_M f(1+f) \langle z \circ z, z \rangle. \end{aligned}$$

Next, by Proposition 2.2 with the assumption that $\langle i_X C, z \rangle = 0$, we have

$$\begin{aligned} \frac{1}{2} \int_M f \nabla f(|z|^2) &= - \int_M \delta(z \circ z)(f \nabla f) \\ &= - \int_M z \circ z(\nabla f, \nabla f) - \int_M f \langle z \circ z, Ddf \rangle \\ &= - \int_M z \circ z(\nabla f, \nabla f) - \int_M f(1+f) \langle z \circ z, z \rangle + \frac{s}{n(n-1)} \int_M f^2 |z|^2. \end{aligned}$$

Here, the last equality comes from (3.2). Substituting this into (5.4), we obtain (5.1). \square

Now, we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. By Lemma 3.3,

$$\frac{(n-1)(n-2)}{2} \int_M |T|^2 = \frac{n-1}{n-2} \int_M |z|^2 |\nabla f|^2 - \frac{n}{n-2} \int_M z \circ z(\nabla f, \nabla f).$$

Comparing this to (5.1), we have

$$\begin{aligned} &\frac{n-1}{n-2} \int_M |z|^2 |\nabla f|^2 - \frac{2(n-1)}{n-2} \int_M z \circ z(\nabla f, \nabla f) \\ &= \frac{s}{n} \int_M f^2 |z|^2 + \frac{2(n-1)}{n-2} \int_M f(1+f) \langle z \circ z, z \rangle. \end{aligned}$$

From the assumption

$$|z|^2 \leq \min \left\{ 2|i_N z|^2, \frac{s^2}{4n(n-1)} \right\},$$

we have

$$|\nabla f|^2 |z|^2 \leq 2|i_{\nabla f} z|^2 = 2z \circ z \langle \nabla f, \nabla f \rangle,$$

and so

$$\frac{s}{n} \int_M f^2 |z|^2 + \frac{2(n-1)}{n-2} \int_M f(1+f) \langle z \circ z, z \rangle \leq 0.$$

Note that, by Lemma 4.1,

$$\begin{aligned} \int_M f(1+f) \langle z \circ z, z \rangle &= \int_M f^2 \langle z \circ z, z \rangle + \int_M f \langle z \circ z, z \rangle \\ &= \int_M f^2 \langle z \circ z, z \rangle + \frac{(n-2)s}{2n(n-1)} \int_M |z|^2 - \int_M \langle z \circ z, z \rangle. \end{aligned}$$

Therefore, applying Lemma 4.2

$$\frac{s}{n} \int_M (1+f^2) |z|^2 \leq \frac{2(n-1)}{n-2} \int_M (1-f^2) \langle z \circ z, z \rangle \leq \frac{2\sqrt{n-1}}{\sqrt{n}} \int_M (1+f^2) |z|^3,$$

which implies

$$0 \leq \int_M (1+f^2) |z|^2 \left(\frac{s}{2\sqrt{n(n-1)}} - |z| \right) \leq 0,$$

where the first inequality follows from the assumption

$$|z| \leq \frac{s}{2\sqrt{n(n-1)}}.$$

Hence, we may conclude that $z = 0$ on all of M . If the equality holds,

$$|z| = \frac{s}{2\sqrt{n(n-1)}}$$

then we reach a contradiction as in the proof of Theorem 1.1. □

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