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Decay Solutions and Decay Rate for a Class of Retarded Abtract Semilinear Fractional Evolution Inclusions

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Abstract. In this paper, we prove the existence of decay integral solutions to a class of fractional differential inclusions with finite delays and estimate their decay rate. For these purposes, we have to construct a suitable regular measure of noncompactness on the space of solutions and then deploy the fixed point theory for condensing multivalued maps. An application to a class of fractional PDE with almost sectorial operator is also given.

1. Introduction

We are concerned with the following problem in a Banach space X

$$ {}^{C}D_0^{\alpha} u(t) - Au(t) \in F(t, u(t), u_t), \quad t \neq t_k,\ t_k \in (0, +\infty),\ k \in \Lambda, \tag{1.1}$$

$$\Delta u(t_k) = I_k(u(t_k)), \tag{1.2}$$

$$u(s) + g(u)(s) = \varphi(s), \quad s \in [-h, 0], \tag{1.3}$$

where ${}^{C}D_0^{\alpha}$ ($\alpha \in (0,1)$) is the fractional derivative in the Caputo sense, A is a closed linear operator in X which generates a strongly continuous semigroup $W(\cdot)$, $F\colon \mathbb{R}^+ \times X \times C([-h,0];X) \to \mathcal{P}(X)$ is a multivalued map, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, $k \in \Lambda \subset \mathbb{N}$, I_k and g are the functions which will be specified in Section 3. Here u_t stands for the history of the state function up to the time t, i.e., $u_t(s) = u(t+s)$, $s \in [-h, 0]$.

The system (1.1)–(1.3) is a generalized Cauchy problem which involves impulsive effect and nonlocal condition expressed by (1.2) and (1.3), respectively. Recently, several specific situations of the system (1.1)–(1.3) have been widely research. However, these researches only focused on the existence and the structure of the solution set, (see e.g., [4, 16, 17, 20]) or control problem, (see e.g., [10–14, 18, 19, 23]). Also, the stability for the generalized above problem has not studied yet.

In [7], we proved a stability result for (1.1)–(1.3) in the case when F is single-valued and Lipschitzian. And in [8], we study the weakly asymptotic stability for system (1.1)–(1.3) in the cases without nonlocal condition. Nevertheless, the technique used in [7]

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and [8] does not work to study the decay rate. In this paper, by constructing the suitable regular MNC and functions space, the existence of decay integral solutions in general cases is proved and in special cases, without impulsive effect, the polynomial decay rate of the solutions is certainly achieved.

The rest of our work is as follows. In the next section, we recall some notions and facts related to fractional calculus, including some properties of fractional resolvent operators. We also recall concept of measure of noncompactness and the fixed point theory for condensing multivalued maps. The focus of Section 3 is the process of proving the global solvability for (1.1)–(1.3) on interval $[-h, T]$ for each $T > 0$, under some regular conditions imposed on the nonlinearities F, I and g. In Section 4, we construct a regular MNC on $\mathcal{PC}_0$ (the space of piecewise-continuous which tend to zero functions) and give a sufficient condition to existence of decay solutions for (1.1)–(1.3). And then, in Section 5, the problem without impulsive condition is studied, in this cases, we prove the existence of decay integral solutions with a polynomial decay rate. In the last section, we apply the abstract results to a class of fractional functional partial differential equations with almost sectorial operator.

2. Preliminaries

2.1. Fractional calculus

Let $L^p(0,T;X)$ ($p \in (1,+\infty)$) be the space of X-valued functions u defined on $[0,T]$ such that the function $t \mapsto \|u(t)\|^p$ is integrable. The integrals appeared in this work will be understood in the Bochner sense. The notation $L^p(0,T)$ stands for $L^p(0,T;\mathbb{R})$. Now we recall some notions in fractional calculus (see e.g., [9, 21]).

Definition 2.1. The fractional integral of order $\alpha > 0$ of a function $f \in L^1(0,T;X)$ is defined by

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds,$$

where Γ is the Gamma function, provided the integral converges.

Definition 2.2. For a function $f \in C^N([0,T];X)$, the Caputo fractional derivative of order $\alpha \in (N-1, N)$ is defined by

$${}^C D_0^\alpha f(t) = \frac{1}{\Gamma(N-\alpha)} \int_0^t (t-s)^{N-\alpha-1} f^{(N)}(s)\, ds.$$

Consider the following problem

$$\begin{aligned} D_0^\alpha u(t) &= Au(t) + f(t), \quad t > 0,\ t \neq t_k \in (0, +\infty),\ k \in \Lambda, \\ \Delta u(t_k) &= I_k(u(t_k)), \\ u(s) &= \varphi(s) - g(u)(s), \quad s \in [-h, 0], \end{aligned}$$

where $f \in L^p(0, T; X)$. In this note we assume that the C_0-semigroup $W(\cdot)$ generated by A is globally bounded, i.e.,

$$\|W(t)x\| \le M_A \|x\|, \quad \forall\, t \ge 0,\ x \in X$$

for some $M_A \ge 1$. By the arguments in [7, 22], we have the following presentation

$$\begin{aligned} u(t) &= S_\alpha(t)[\varphi(0) - g(u)(0)] + \sum_{0<t_k<t} S_\alpha(t - t_k) I_k(u(t_k)) \\ &\quad + \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s)\, ds, \quad t > 0, \end{aligned}$$

where

$$\begin{aligned} S_\alpha(t)x &= \int_0^\infty \phi_\alpha(\theta) W(t^\alpha \theta) x\, d\theta, \\ P_\alpha(t)x &= \alpha \int_0^\infty \theta \phi_\alpha(\theta) W(t^\alpha \theta) x\, d\theta, \quad x \in X, \\ \phi_\alpha(\theta) &= \frac{1}{\pi} \sum_{n=1}^\infty \frac{(-\theta)^{n-1}}{(n-1)!} \Gamma(n\alpha) \sin(n\pi\alpha). \end{aligned}$$

Following [22], we have the following estimates

$$\|S_\alpha(t)x\| \le M_A \|x\| \quad \text{and} \quad \|P_\alpha(t)x\| \le \frac{M_A}{\Gamma(\alpha)} \|x\|, \quad \forall\, x \in X.$$

Lemma 2.3. *We have the following properties*

(1) *If the semigroup $\{W(\cdot)\}$ is norm continuous, that is, $t \mapsto W(t)$ is continuous for $t > 0$, then $S_\alpha(t)$ and $P_\alpha(t)$ are norm continuous as well;*

(2) *If $\{W(\cdot)\}$ is a compact semigroup then $S_\alpha(t)$ and $P_\alpha(t)$ are compact for $t > 0$.*

Proof. The proof is similar to that in [15]. □

Let $p > 1/\alpha$, we define the operator $Q_\alpha \colon L^p(0, T; X) \to C([0, T]; X)$ as follows:

$$Q_\alpha(f)(t) = \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s)\, ds. \tag{2.1}$$

It follows from [7] that Q_α has this following important property.

Proposition 2.4. [7] *Let $\{W(t)\}_{t \ge 0}$ be the C_0-semigroup generated by A. Then for each bounded set $\Omega \subset L^p(0, T; X)$, $Q_\alpha(\Omega)$ is an equicontinuous set in $C([0, T]; X)$ provided that $W(t)$ is norm continuous for $t > 0$.*

2.2. Fixed point theory for condensing operators

Let $\mathcal{E}$ be a Banach space. Denote

$$\begin{aligned}\mathcal{P}(\mathcal{E}) &= \{B \subset \mathcal{E} : B \neq \emptyset\},\\ \mathcal{B}(\mathcal{E}) &= \{B \in \mathcal{P}(\mathcal{E}) : B \text{ is bounded}\},\\ \mathcal{P}_c(\mathcal{E}) &= \{B \in \mathcal{P}(\mathcal{E}) : B \text{ is closed}\},\\ \mathcal{K}v(\mathcal{E}) &= \{B \in \mathcal{P}(\mathcal{E}) : B \text{ is convex and compact}\}.\end{aligned}$$

We will use the following definition of measure of noncompactness given in [6].

Definition 2.5. A function $\beta\colon \mathcal{B}(\mathcal{E}) \to \mathbb{R}^+$ is called *a measure of noncompactness* (MNC) in $\mathcal{E}$ if

$$\beta(\overline{\text{co}}\Omega) = \beta(\Omega) \quad \text{for every } \Omega \in \mathcal{B}(\mathcal{E}),$$

where $\overline{\text{co}}\Omega$ is the closure of the convex hull of Ω. An MNC β is called

(i) monotone if $\Omega_0, \Omega_1 \in \mathcal{B}(\mathcal{E})$, $\Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$;

(ii) nonsingular if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for any $a \in \mathcal{E}$, $\Omega \in \mathcal{B}(\mathcal{E})$;

(iii) invariant with respect to union with compact set if $\beta(K \cup \Omega) = \beta(\Omega)$ for every relatively compact set $K \subset \mathcal{E}$ and $\Omega \in \mathcal{B}(\mathcal{E})$;

(iv) algebraically semi-additive if $\beta(\Omega_0 + \Omega_1) \leq \beta(\Omega_0) + \beta(\Omega_1)$ for any $\Omega_0, \Omega_1 \in \mathcal{B}(\mathcal{E})$;

(v) regular if $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.

An important example of MNC is the *Hausdorff* MNC $\chi(\cdot)$, which is defined as follows

$$\chi(\Omega) = \inf\{\varepsilon : \Omega \text{ has a finite } \varepsilon\text{-net}\}.$$

This MNC satisfies all properties given in Definition 2.5.

We now give some basic estimates based on MNCs. We first recall the *sequential MNC* χ_0 defined by

$$\chi_0(\Omega) = \sup\{\chi(D) : D \in \Delta(\Omega)\},$$

where $\Delta(\Omega)$ is the collection of all at-most-countable subsets of Ω (see [1]). We know that

$$\frac{1}{2}\chi(\Omega) \leq \chi_0(\Omega) \leq \chi(\Omega)$$

for all bounded set $\Omega \subset \mathcal{E}$. Then we have the following estimate.

Proposition 2.6. *Let χ be the Hausdorff MNC in $\mathcal{E}$.*

(1) *If* $\Omega \subset \mathcal{E}$ *be a bounded set, then for every* $\epsilon > 0$*, there exists a sequence* $\{x_n\} \subset \Omega$ *such that*

$$\chi(\Omega) \leq 2\chi(\{x_n\}) + \epsilon.$$

(2) *If* $\{w_n\} \subset L^1(0, T; \mathcal{E})$ *such that* $\|w_n(t)\| \leq \nu(t)$*, for a.e.* $t \in [0, T]$*, for some* $\nu \in L^1(0, T)$*, then we have*

$$\chi\left(\left\{\int_0^t w_n(s)\, ds\right\}\right) \leq 2\int_0^t \chi(\{w_n(s)\})\, ds \quad \text{for } t \in [0, T].$$

(3) *If* $D \subset L^1(0, T; \mathcal{E})$ *such that*

   (a) $\|\xi(t)\| \leq \nu(t)$ *for all* $\xi \in D$ *and for a.e.* $t \in [0, T]$*,*

   (b) $\chi(D(t)) \leq q(t)$ *for a.e.* $t \in [0, T]$*,*

   where $\nu, q \in L^1(0, T; \mathbb{R})$*. Then*

$$\chi\left(\int_0^t D(s)\, ds\right) \leq 4\int_0^t q(s)\, ds,$$

   here $\int_0^t D(s)\, ds = \{\int_0^t \xi(s)\, ds : \xi \in D\}$*.*

Now, we recall the concept of χ-norm of a bounded linear operator $\mathcal{T}$ ($\mathcal{T} \in \mathcal{L}(X)$) as follows

$$\|\mathcal{T}\|_\chi = \inf\{\beta > 0 : \chi(\mathcal{T}(B)) \leq \beta\chi(B) \text{ for all bounded set } B \subset X\}.$$

It is noted that the χ-norm of $\mathcal{T}$ can be formulated by

$$\|\mathcal{T}\|_\chi = \chi(\mathcal{T}(\mathbf{B_1})) = \chi(\mathcal{T}(\mathbf{S_1})),$$

where $\mathbf{B_1}$ and $\mathbf{S_1}$ are a unit ball and a unit sphere in X, respectively. It is know that

$$\|\mathcal{T}\|_\chi \leq \|\mathcal{T}\|_{\mathcal{L}(X)},$$

where the last norm is understood as the operator norm in $\mathcal{L}(X)$. Obviously, $\mathcal{T}$ is a compact operator if and only if $\|\mathcal{T}\|_\chi = 0$.

We make use of some notions and facts of set-valued analysis. Let Y be a metric space.

Definition 2.7. A multivalued map (multimap) $\mathcal{F} \colon Y \to \mathcal{P}(\mathcal{E})$ is said to be:

(i) upper semicontinous (u.s.c.) if $\mathcal{F}^{-1}(V) = \{y \in Y : \mathcal{F}(y) \cap V \neq \emptyset\}$ is a closed subset of Y for every closed set $V \subset \mathcal{E}$;

(ii) weakly upper semicontinous (weakly u.s.c.) if $\mathcal{F}^{-1}(V)$ is a closed subset of Y for all weakly closed set $V \subset \mathcal{E}$;

(iii) closed if its graph $\Gamma_{\mathcal{F}} = \{(y,z) : z \in \mathcal{F}(y)\}$ is a closed subset of $Y \times \mathcal{E}$;

(iv) compact if $\mathcal{F}(Y)$ is relatively compact in $\mathcal{E}$;

(v) quasicompact if its restriction to any compact subset $A \subset Y$ is compact.

The following lemmas give criteria for checking a given multimap to be (weakly) u.s.c.

Lemma 2.8. [6, Theorem 1.1.12] *Let $G\colon Y \to \mathcal{P}(\mathcal{E})$ be a closed quasicompact multimap with compact values. Then G is u.s.c.*

Lemma 2.9. [3, Proposition 2] *Let $\mathcal{E}$ be a Banach space and Ω a nonempty subset of another Banach space. Assume that $G\colon \Omega \to \mathcal{P}(\mathcal{E})$ is a multimap with weakly compact, convex values. Then G is weakly u.s.c. if $\{x_n\} \subset \Omega$ with $x_n \to x_0$ and $y_n \in G(x_n)$ implies $y_n \rightharpoonup y_0 \in G(x_0)$, up to a subsequence.*

We now introduce the concept of condensing multimaps.

Definition 2.10. Let E be a Banach space. A continuous map $\mathcal{F}\colon Z \subseteq \mathcal{E} \to \mathcal{P}(\mathcal{E})$ is said to be condensing with respect to a MNC β (β-condensing) if for any bounded set $\Omega \subset Z$, the relation

$$\beta(\Omega) \leq \beta(\mathcal{F}(\Omega))$$

implies the relative compactness of Ω.

Let β be a monotone nonsingular MNC in $\mathcal{E}$. The application of the topological degree theory for condensing maps (see, e.g., [1]) yields the following fixed point principle.

Theorem 2.11. [6, Corollary 3.3.1] *If $\mathcal{M}$ is a bounded closed convex subset of a Banach space $\mathcal{E}$, and $\mathcal{G}\colon \mathcal{M} \to \mathcal{K}v(\mathcal{M})$ is a closed and β-condensing multimap, where β is a nonsingular MNC in $\mathcal{E}$, then the fixed points set* $\operatorname{Fix}\mathcal{G} = \{x : x \in \mathcal{G}(x)\}$ *is nonempty and compact.*

3. Existence result

Given $T > 0$, we denote by $\mathcal{PC}([-h,T];X)$ the space of functions $u\colon [-h,T] \to X$ such that u is continuous on $[-h,T] \setminus \{t_k : k \in \Lambda\}$ and for each $t_k \in [0,T]$, $k \in \Lambda$, there exist

$$u(t_k^-) = \lim_{t \to t_k^-} u(t), \quad u(t_k^+) = \lim_{t \to t_k^+} u(t)$$

and $u(t_k) = u(t_k^-)$. Then $\mathcal{PC}([-h,T];X)$ is a Banach space endowed with the norm

$$\|u\|_{\mathcal{PC}} := \sup_{t \in [-h,T]} \|u(t)\|.$$

Let χ be the Hausdorff MNC in X, χ_T the Hausdorff MNC in $\mathcal{PC}([-h,T];X)$.

We recall the following facts (see [5]), which will be used later: for each bounded set $D \subset \mathcal{PC}([-h,T];X)$, one has

- $\chi(D(t)) \le \chi_T(D)$ for all $t \in [-h,T]$, where $D(t) := \{x(t) : x \in D\}$.
- If D is an equicontinuous set on each interval $(t_k, t_{k+1}] \subset [-h,T]$, then

$$\chi_T(D) = \sup_{t\in[-h,T]} \chi(D(t)).$$

Let $\mathcal{C}_h = C([-h,0];X)$ and χ_h be the Hausdorff MNC in $\mathcal{C}_h$. $\mathcal{C}_h$ will be considered as a normed space of piecewise continuous functions with the norm $\|v\|_{\mathcal{C}_h} = \frac{1}{h}\int_{-h}^{0} \|v(\theta)\|\, d\theta$.

Concerning problem (1.1)–(1.3), we give the following assumptions:

(A) The C_0-semigroup $\{W(t)\}_{t\ge 0}$ generated by A is norm continuous for $t > 0$.

(F) The multivalued nonlinearity function $F\colon \mathbb{R}^+ \times X \times \mathcal{C}_h \to \mathcal{K}v(X)$ satisfies:

(F1) $F(\,\cdot\,, v, w)$ admits a strongly measurable selection for each $u \in X$ and $v \in \mathcal{C}_h$;

(F2) $F(t, \cdot\,, \cdot\,)$ is u.s.c. for each $t \in J$;

(F3) $\|F(t,v,w)\| = \sup\{\|\xi\| : \xi \in \mathcal{F}(t,v,w)\} \le m(t)(\|v\| + \|w\|_{\mathcal{C}_h})$ for all $v \in X$, $w \in \mathcal{C}_h$, where $m \in L^p_{\mathrm{loc}}(\mathbb{R}^+)$, $p > 1/\alpha$;

(F4) if $W(t)$ is noncompact, there exists a function $k \in L^p(J)$ such that

$$\chi(F(t,B,C)) \le k(t)\left[\chi(B) + \sup_{s\in[-h,0]} \chi(C(s))\right]$$

for a.e. $t, s \in [0,T]$, $t \ge s$.

(G) The nonlocal function $g\colon \mathcal{PC}([-h,T];X) \to \mathcal{C}_h$ obeys the following conditions:

(G1) g is continuous and

$$\|g(u)\|_{\mathcal{C}_h} \le \Psi_g(\|u\|_{\mathcal{PC}})$$

for all $u \in \mathcal{PC}([-h,T];X)$, where Ψ_g is a continuous and nondecreasing function on $\mathbb{R}^+$;

(G2) There exists $\eta \ge 0$ such that

$$\chi_h(g(D)) \le \eta \chi_T(D)$$

for all bounded set $D \subset \mathcal{PC}([-h,T];X)$.

(I) The operator $I_k\colon X \to X$ $(k \in \Lambda)$ is continuous and satisfies:

(I1) there exists a real-valued, continuous, nondecreasing function Ψ_I and a nonnegative sequence $\{l_k\}_{k\in\Lambda}$ such that

$$\|I_k(x)\| \le l_k \Psi_I(\|x\|) \quad \text{for all } x \in X,\ k \in \Lambda;$$

(I2) there exists a nonnegative sequence $\{\mu_k\}_{k\in\Lambda}$ such that

$$\chi(I_k(B)) \le \mu_k \chi(B)$$

for all bounded subset $B \subset X$;

(I3) the sequence $\{t_k : k \in \Lambda\}$ satisfies $\inf_{k\in\Lambda}(t_{k+1} - t_k) > 0$.

For $v \in \mathcal{PC}([-h, T]; X)$, putting

$$\mathcal{P}^p_F(v) = \{f \in L^p([0;T]; X) : f(t) \in F(t, v(t), v_t) \text{ for a.e. } t \in [0, T]\},$$

we have the following property.

Proposition 3.1. *Assume that* (A) *and* (F1)–(F3) *hold. Then* $\mathcal{P}^p_F(u) \neq \emptyset$ *for each* $u \in \mathcal{PC}([-h, T]; X)$. *In addition,* $\mathcal{P}^p_F : C(J; X) \to \mathcal{P}(L^1[J; X])$ *is weakly u.s.c. with weakly compact and convex values.*

Proof. The proof is similar to that in [3, Theorem 1]. □

Definition 3.2. A function $u \in \mathcal{PC}([-h, T]; X)$ is said to be an integral solution of problem (1.1)–(1.3) on the interval $[-h, T]$ if and only if $u(t)+g(u)(t) = \varphi(t)$ for $t \in [-h, 0]$, and

$$u(t) = S_\alpha(t)[\varphi(0) - g(u)(0)] + \sum_{0<t_k<t} S_\alpha(t - t_k)I_k(u(t_k)) + \int_0^t (t - s)^{\alpha-1}P_\alpha(t - s)f(s)\,ds$$

for any $t \in [0, T]$, where $f \in \mathcal{P}^p_F(u)$.

We defined the *solution operator* $\mathcal{F} : \mathcal{PC}([-h, T]; X) \to \mathcal{P}(\mathcal{PC}([-h, T]; X))$ as follows

$$\mathcal{F}(u)(t) = \begin{cases} \varphi(t) - g(u)(t) & t \in [-h, 0], \\ S_\alpha(t)[\varphi(0) - g(u)(0)] + \sum_{0<t_k<t} S_\alpha(t - t_k)I_k(u(t_k)) + Q_\alpha \circ \mathcal{P}^p_F(u)(t) & t \in [0, T], \end{cases}$$

where Q_α is defined by (2.1). It is obvious that u is a fixed point of $\mathcal{F}$ if and only if u is an integral solution of (1.1)–(1.3) on $[-h, T]$.

We are ready to give the condensing property of the solution map.

Lemma 3.3. *Let the hypotheses* (A), (F), (G) *and* (I) *hold. Then the solution operator* $\mathcal{F}$ *satisfies*

$$\chi_T(\mathcal{F}(D)) \le \left[\left(\eta + \sum_{t_k\in[0,T]} \mu_k\right) S_\alpha^T + 8 \sup_{t\in[0,T]} \int_0^t (t-s)^{\alpha-1} \|P_\alpha(t-s)\|_\chi k(t)\, ds\right] \chi_T(D)$$

for all bounded set $D \subset \mathcal{PC}([-h,T];X)$*, where* $S_\alpha^T = \sup_{t\in[0,T]} \|S_\alpha(t)\|$*.*

Proof. Let $D \subset \mathcal{PC}([-h,T];X)$ be a bounded set. Then we have

$$\mathcal{F}(D) = \mathcal{F}_1(D) + \mathcal{F}_2(D) + \mathcal{F}_3(D),$$

where

$$\mathcal{F}_1(u)(t) = \begin{cases} S_\alpha(t)[\varphi(0) - g(u)(0)] & t \in [0,T], \\ \varphi(t) - g(u)(t) & t \in [-h,0], \end{cases}$$

$$\mathcal{F}_2(u)(t) = \begin{cases} \sum_{0<t_k<t} S_\alpha(t-t_k) I_k(u(t_k)) & t \in [0,T], \\ 0 & t \in [-h,0], \end{cases}$$

$$\mathcal{F}_3(u)(t) = \begin{cases} Q_\alpha \circ \mathcal{P}_F^p(u)(t) & t \in [0,T], \\ 0 & t \in [-h,0]. \end{cases}$$

From the algebraically semi-additive property of χ_T, we have

$$\chi_T(\mathcal{F}(D)) \le \chi_T(\mathcal{F}_1(D)) + \chi_T(\mathcal{F}_2(D)) + \chi_T(\mathcal{F}_3(D)).$$

For $z_1, z_2 \in \mathcal{F}_1(D)$, there exist $u_1, u_2 \in D$ such that

$$z_1(t) = \begin{cases} S_\alpha(t)[\varphi(0) - g(u_1)(0)] & t \in [0,T], \\ \varphi(t) - g(u_1)(t) & t \in [-h,0], \end{cases}$$

$$z_2(t) = \begin{cases} S_\alpha(t)[\varphi(0) - g(u_2)(0)] & t \in [0,T], \\ \varphi(t) - g(u_2)(t) & t \in [-h,0]. \end{cases}$$

Then

$$\|z_1(t) - z_2(t)\| \le \begin{cases} \|S_\alpha(t)\| \|g(u_1) - g(u_2)\|_{\mathcal{C}_h} & t \in [0,T], \\ \|g(u_1) - g(u_2)\|_{\mathcal{C}_h} & t \in [-h,0]. \end{cases}$$

Therefore

$$\|z_1 - z_2\|_{\mathcal{P}C} \le S_\alpha^T \|g(u_1) - g(u_2)\|_{\mathcal{C}_h},$$

thanks to the fact that $S_\alpha^T \ge 1$. This implies

$$\chi_T(\mathcal{F}_1(D)) \le S_\alpha^T \chi_h(g(D)).$$

Employing (G2), we have

$$\chi_T(\mathcal{F}_1(D)) \le \eta S_\alpha^T \chi_T(D). \tag{3.1}$$

Now let $z_1, z_2 \in \mathcal{F}_2(D)$, one can find $u_1, u_2 \in D$ such that

$$\|z_1(t) - z_2(t)\| = \sum_{0<t_k<t} S_\alpha(t-t_k)[I_k(u_1(t_k)) - I_k(u_2(t_k))].$$

Hence

$$\|z_1 - z_2\|_{\mathcal{PC}} \le S_\alpha^T \sum_{t_k\in[0,T]} \|I_k(u_1(t_k)) - I_k(u_2(t_k))\|.$$

This inequality deduces that

$$\begin{aligned}\chi_T(\mathcal{F}_2(D)) &\le S_\alpha^T \sum_{t_k\in[0,T]} \chi(I_k(D(t_k)))\\ &\le S_\alpha^T \sum_{t_k\in[0,T]} \mu_k \chi(D(t_k))\\ &\le \left(S_\alpha^T \sum_{t_k\in[0,T]} \mu_k\right)\chi_T(D),\end{aligned} \tag{3.2}$$

thanks to (I2).

Regarding $\mathcal{F}_3(D)$, if $W(t)$ is compact, so is $P_\alpha(t)$ and then

$$\begin{aligned}\chi_T(\mathcal{F}_3(D)) &= \chi_T\left(\int_0^t (t-s)^{\alpha-1} P_\alpha(t-s)\mathcal{P}_F^p(D(s))\,ds\right)\\ &\le 4\int_0^t (t-s)^{\alpha-1}\chi_T(P_\alpha(t-s)\mathcal{P}_F^p(D(s)))\,ds\\ &= 0.\end{aligned}$$

In the opposite case, from Proposition 2.4, we have $\mathcal{F}_3(D)$ is an equicontinuous set in $C([0,T];X)$. This leads to

$$\begin{aligned}\chi_T(\mathcal{F}_3(D)) &= \sup_{t\in[0,T]} \chi(\mathcal{F}_3(D)(t))\\ &\le 4\sup_{t\in[0,T]}\int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|k(s)\left[\chi(D(s)) + \sup_{\tau\in[-h,0]}\chi(D(s+\tau))\right] ds\\ &\le 8\chi_T(D)\sup_{t\in[0,T]}\int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|k(s)\,ds,\end{aligned} \tag{3.3}$$

here we have used Proposition 2.6 and hypothesis (F4). Combining (3.1), (3.2) and (3.3), we arrive at

$$\chi_T(\mathcal{F}(D)) \le \left[\left(\eta + \sum_{t_k\in[0,T]}\mu_k\right)S_\alpha^T + 8\sup_{t\in[0,T]}\int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|_\chi k(s)\,ds\right]\chi_T(D).$$

The proof is complete. $\square$

Now, we prove the main result of this section.

Theorem 3.4. *Assume that the hypotheses of Lemma* 3.3 *hold. Then the problem* (1.1)–(1.3) *has at least one integral solution in* $\mathcal{PC}([-h,T];X)$*, provided that*

$$\left[\left(\eta + \sum_{t_k\in[0,T]} \mu_k\right) S_\alpha^T + 8 \sup_{t\in[0,T]} \int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|_\chi k(s)\,ds\right] < 1 \tag{3.4}$$

and

$$\begin{aligned}\liminf_{r\to\infty}\Bigg[\frac{1}{r}\left(\Psi_g(r) + \Psi_I(r)\sum_{t_k\in[0,T]} l_k\right) S_\alpha^T \\ + 2\sup_{t\in[0,T]}\int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|m(s)\,ds\Bigg] < 1,\end{aligned}$$

where S_α^T *is given in Lemma* 3.3.

Proof. By (3.4), we obtain the χ_T-condensing property for $\mathcal{F}$ thanks to Lemma 3.3. Now, we prove the closedness of $\mathcal{F}$. Let $\{x_k\} \subset \mathcal{PC}([-h,T];X)$ such that $x_k \to x^*$ and $y_k \in \mathcal{F}(x_k)$, $y_k \to y^*$. We will verify that $y^* \in \mathcal{F}(x^*)$. Taking $f_k \in \mathcal{P}_F^p(x_k)$ such that

$$\begin{aligned} y_k(t) &= \varphi(t) - g(x_k)(t), \quad t \in [-h,0], \\ y_k(t) &= S_\alpha(t)[\varphi(0) - g(x_k)(0)] \\ &\qquad + \sum_{0<t_i<t} S_\alpha(t-t_k)I_i(x_k(t_i)) + Q_\alpha(f_k)(t), \quad t\in[0,T]. \end{aligned} \tag{3.5}$$

By Proposition 3.1, we get that $f_n \rightharpoonup f^* \in L^p(0,T;X)$ and $f^* \in \mathcal{F}(x^*)$. In addition, $C(t) = \{f_n(t) : n \geq 1\}$ is relatively compact and then

$$\begin{aligned} \chi(\{Q_\alpha(f_n)(t)\}) &\leq \left(\left\{\int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)f_n(s)\,ds\right\}\right) \\ &\leq 2\int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|\chi(\{f_n(s)\})\,ds \\ &= 0. \end{aligned}$$

Due to Proposition 2.3, $\{Q_\alpha(f_n)\}$ is equicontinuous. Then by the Arzelà-Ascoli theorem, we have the relatively compactness of $\{Q_\alpha(f_n)\}$. Since $f_n(t) \to f^*(t)$ for a.e. $t \in (0,T)$, one has $Q_\alpha(f_n) \to Q_\alpha(f^*)$. Therefore, it follows from (3.5) that

$$y^*(t) = S_\alpha(t)[\varphi(0) - g(x^*)(0)] + \sum_{0<t_i<t} S_\alpha(t-t_k)I_k(x^*(t_i)) + Q_\alpha(f_k)(t), \quad \forall\, t\in[0,T],$$

where $f^* \in \mathcal{P}_F^p(x^*)$. Thus $y^* \in \mathcal{F}(x^*)$ and we have the closedness of $\mathcal{F}$.

In order to apply Theorem 2.11, it remains to show that $\mathcal{F}(B_R) \subset B_R$ for some $R > 0$, where B_R is the closed ball in $\mathcal{PC}([-h,T];X)$ centered at 0 with radius R.

Assume to the contrary that there exists a sequence $\{v_n\} \subset \mathcal{PC}([-h,T];X)$ such that $\|v_n\|_{\mathcal{PC}} \le n$ but $\exists\, z \in \mathcal{F}(v_n)$, $\|z\|_{\mathcal{PC}} > n$. For $z \in \mathcal{F}(v_n)$, $z(t) = S_\alpha(t)[\varphi(0) - g(v_n)(0)] + \sum_{0<t_k<t} S_\alpha(t-t_k)I_k(v_n(t_k)) + Q_\alpha(f)(t)$, $t \in [0,T]$, $f \in \mathcal{P}_F^p(v_n)$, we have

$$\begin{aligned}
\|z(t)\| &\le \sup_{t\in[0,T]} \|S_\alpha(t)\| \left(\|\varphi\|_{\mathcal{C}_h} + \Psi_g(\|v_n\|_{\mathcal{PC}}) + \sum_{0<t_k<t} \|I_k(v_n(t_k))\| \right) \\
&\quad + \int_0^t (t-s)^{\alpha-1} \|P_\alpha(t-s)\| \|f(s)\|\, ds \\
&\le S_\alpha^T \left(\|\varphi\|_{\mathcal{C}_h} + \Psi_g(n) + \sum_{t_k\in[0,T]} l_k \Psi_I(\|v_n(t_k)\|) \right) \\
&\quad + \int_0^t (t-s)^{\alpha-1} \|P_\alpha(t-s)\| m(s)(\|v_n(s)\| + \|(v_n)_s\|_{\mathcal{C}_h})\, ds \\
&\le S_\alpha^T \left(\|\varphi\|_{\mathcal{C}_h} + \Psi_g(n) + \Psi_I(n) \sum_{t_k\in[0,T]} l_k \right) + 2n \int_0^t (t-s)^{\alpha-1} \|P_\alpha(t-s)\| m(s)\, ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
n < \|\mathcal{F}(v_n)\|_{\mathcal{PC}} \le S_\alpha^T &\left(\|\varphi\|_{\mathcal{C}_h} + \Psi_g(n) + \Psi_I(n) \sum_{t_k\in[0,T]} l_k \right) \\
&+ 2n \sup_{t\in[0,T]} \int_0^t (t-s)^{\alpha-1} \|P_\alpha(t-s)\| m(s)\, ds.
\end{aligned}$$

Then

$$\begin{aligned}
1 < \frac{1}{n}\|\mathcal{F}(v_n)\|_{\mathcal{PC}} \le \frac{1}{n} S_\alpha^T &\left(\|\varphi\|_{\mathcal{C}_h} + \Psi_g(n) + \Psi_I(n) \sum_{t_k\in[0,T]} l_k \right) \\
&+ 2 \sup_{t\in[0,T]} \int_0^t (t-s)^{\alpha-1} \|P_\alpha(t-s)\| m(s)\, ds.
\end{aligned}$$

Passing the last inequality into limits, one gets a contradiction. The proof is complete. □

4. Existence of decay integral solutions

In order to study the stability results for problem (1.1)–(1.3), we consider the function space

$$\mathcal{PC}_0 = \left\{ u \in \mathcal{PC}([-h,+\infty);X) : \lim_{t\to\infty} u(t) = 0 \right\}$$

with the norm

$$\|u\|_\infty = \sup_{t\geq -h} \|u(t)\|,$$

where $\mathcal{PC}([-h,\infty);X)$ is defined similarly to $\mathcal{PC}([-h,T];X)$ as $T=+\infty$. Then $\mathcal{PC}_0$ is a Banach space.

Let π_T $(T>0)$ be the truncate function on $\mathcal{PC}_0$, i.e., for $D\subset \mathcal{PC}_0$, $\pi_T(D)$ is the restriction of D on interval $[-h,T]$. Then one can see that the MNC χ_∞ in $\mathcal{PC}_0$ defined by

$$\chi_\infty(D) = \sup_{T>0} \chi_T(\pi_T(D))$$

satisfies all properties given in Definition 2.5, but regularity. To build a regular MNC on $\mathcal{PC}_0$, we define

$$d_T(D) = \sup_{t\geq T}\sup_{x\in D} \|x(t)\|, \quad d_\infty(D) = \lim_{T\to\infty} d_T(D) \quad \text{and} \quad \chi^*(D) = \chi_\infty(D) + d_\infty(D).$$

It is easily seen that $\chi_\infty(\cdot)$ and $d_\infty(\cdot)$ are monotone and nonsingular MNCs, so is $\chi^*(\cdot)$. We will prove an important property of $\chi^*(\cdot)$ in the next lemma.

Lemma 4.1. *Let $\Omega\subset\mathcal{PC}_0$ be a bounded set such that $\chi^*(\Omega)=0$. Then Ω is relatively compact.*

Proof. The proof is similar to that in [2]. □

We now prove that $\mathcal{F}$ keeps $\mathcal{PC}_0$ invariant, i.e., $\mathcal{F}(\mathcal{PC}_0)\subset\mathcal{PC}_0$, and $\mathcal{F}$ is χ^*-condensing on $\mathcal{PC}_0$. To this end, we assume that (F), (G) and (I) satisfy for all $T>0$ and replace (A) by stronger one.

(Aa) The operator A satisfies (A) such that $\{S_\alpha(t);P_\alpha(t)\}_{t\geq 0}$ are stable, that is,

$$\lim_{t\to\infty} \|S_\alpha(t)\| = 0 \quad \text{and} \quad \lim_{t\to\infty} \|P_\alpha(t)\| = 0.$$

Lemma 4.2. *Let* (Aa) *hold and* (F), (G), (I) *hold for all $T>0$. Then $\mathcal{F}(\mathcal{PC}_0)\subset\mathcal{PC}_0$ provided that $q = 2\sup_{t\geq 0}\int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|m(s)\,ds<\infty$ and $\sum_{k\in\Lambda} l_k<\infty$.*

Proof. Here we recall that

$$\mathcal{F}(u)(t) = S_\alpha(t)[\varphi(0)-g(u)(0)] + \sum_{0<t_k<t} S_\alpha(t-t_k)I_k(u(t_k)) + Q_\alpha\circ\mathcal{P}_F^p(u)(t), \quad t>0.$$

Let $u\in\mathcal{PC}_0$ such that $R=\|u\|_\infty>0$. We prove that $\mathcal{F}(u)\subset\mathcal{PC}_0$, i.e., $\mathcal{F}(u)(t)\to 0$ as $t\to+\infty$.

Let $\epsilon > 0$ be given. Then there exists $T_1 > 0$ such that

$$\|u(t)\| \le \epsilon, \quad \forall\, t > T_1, \tag{4.1}$$

$$\|u_t\|_{\mathcal{C}_h} \le \sup_{\tau\in[-h,0]} \|u(t+\tau)\| \le \epsilon, \quad \forall\, t > T_1 + h. \tag{4.2}$$

On the other hand, from the assumption that $\sum_{k\in\Lambda} l_k < +\infty$, there exists $N_0 \in \mathbb{N}$ such that

$$\sum_{k>N_0} l_k \le \epsilon.$$

Let $z \in \mathcal{F}(u)$, then for $t > 0$,

$$\begin{aligned}
\|z(t)\| \le\ & \|S_\alpha(t)\|(\|\varphi\|_{\mathcal{C}_h} + \|g(u)\|_{\mathcal{C}_h}) \\
&+ \sum_{k\le N_0} \|S_\alpha(t-t_k)\|\|I_k(u(t_k))\| + \sum_{k>N_0} \|S_\alpha(t-t_k)\|\|I_k(u(t_k))\| \\
&+ \int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|\|f(s)\|\, ds \\
\le\ & \|S_\alpha(t)\|(\|\varphi\|_{\mathcal{C}_h} + \Psi_g(R)) + \sum_{k\le N_0} \|S_\alpha(t-t_k)\| l_k \Psi_I(R) + S_\alpha^\infty \sum_{k>N_0} l_k \Psi_I(R) \\
&+ \int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\| m(s)(\|u(s)\| + \|u_s\|_{\mathcal{C}_h})\, ds,
\end{aligned}$$

where $S_\alpha^\infty = \sup_{t\ge 0} \|S_\alpha(t)\|$. Thus we have

$$\|\mathcal{F}(u)(t)\| \le E_1(t) + E_2(t) + E_3(t),$$

where

$$\begin{aligned}
E_1(t) &= \|S_\alpha(t)\|(\|\varphi\|_{\mathcal{C}_h} + \Psi_g(R)), \\
E_2(t) &= \Psi_I(R)\left[\sum_{k\le N_0} \|S_\alpha(t-t_k)\| l_k + S_\alpha^\infty \sum_{k>N_0} l_k\right], \\
E_3(t) &= \int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\| m(s)(\|u(s)\| + \|u_s\|_{\mathcal{C}_h})\, ds.
\end{aligned}$$

Observing from (Aa), that there is $T_2 > 0$ such that

$$\|S_\alpha(t)\| \le \epsilon, \quad \|P_\alpha(t)\| \le \epsilon, \quad \forall\, t > T_2,$$

so

$$E_1(t) \le (\|\varphi\|_{\mathcal{C}_h} + \Psi_g(R))\epsilon, \quad \forall\, t > T_2. \tag{4.3}$$

In addition,

$$E_2(t) \le \Psi_I(R)\left(\sum_{k\le N_0} l_k + S_\alpha^\infty\right)\epsilon, \quad \forall\, t > T_2 + t_{N_0}. \tag{4.4}$$

Concerning $E_3(t)$, for $t > T_1 + h$ one has

$$\begin{aligned} E_3(t) &= \left(\int_0^{T_1+h} + \int_{T_1+h}^t\right)(t-s)^{\alpha-1}\|P_\alpha(t-s)\|m(s)(\|u(s)\| + \|u_s\|_{\mathcal{C}_h})\,ds \\ &\le 2R\int_0^{T_1+h}(t-s)^{\alpha-1}\|P_\alpha(t-s)\|m(s)\,ds + 2\epsilon\int_{T_1+h}^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|m(s)\,ds, \end{aligned}$$

thanks to (4.1) and (4.2). Therefore,

$$E_3(t) \le 2R\epsilon\int_0^{T_1+h}(t-s)^{\alpha-1}m(s)\,ds + 2\epsilon\int_{T_1+h}^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|m(s)\,ds$$

for all $t > T_2 + T_1 + h$. Then by the Hölder inequality we get

$$\begin{aligned} E_3(t) &\le 2R\epsilon\left(\int_0^{T_1+h}(t-s)^{(\alpha-1)p'}\,ds\right)^{1/p'}\left(\int_0^{T_1+h}(m(s))^p\,ds\right)^{1/p} \\ &\quad + 2\epsilon\int_{T_1+h}^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|m(s)\,ds \\ &\le \left(2RC_\alpha(t)\|m\|_{L^p(\mathbb{R}^+)} + q\right)\epsilon \end{aligned} \tag{4.5}$$

for all $t > T_2 + T_1 + h$, where $p' = p/(p-1)$ and

$$C_\alpha(t) = \left\{\frac{1}{(\alpha-1)p'+1}\left[t^{(\alpha-1)p'+1} - (t - T_1 - h)^{(\alpha-1)p'+1}\right]\right\}^{1/p'}.$$

Combining (4.3), (4.4) and (4.5) gives

$$\|\mathcal{F}(u)(t)\| \le C\epsilon$$

for all $t > \max\{T_2 + T_1 + h, T_2 + t_{N_0}\}$, where

$$\begin{aligned} C &= \|\varphi\| + \Psi_g(R) + \Psi_I(R)\left(\sum_{k\le N_0}\mu_k + S_\alpha^\infty\right) + 2RC_\alpha(t)\|m\|_{L^p(\mathbb{R}^+)} + q \\ &\le \|\varphi\| + \Psi_g(R) + \Psi_I(R)\left(\sum_{k\in\Lambda}\mu_k + S_\alpha^\infty\right) + 2RC_\alpha(t)\|m\|_{L^p(\mathbb{R}^+)} + q. \end{aligned}$$

Taking $C_\alpha(t)$ into account, since $p > 1/\alpha$, $p' < 1/(1-\alpha)$, we see that $0 < (\alpha-1)p'+1 < 1$. Hence

$$\begin{aligned} t^{(\alpha-1)p'+1} - (t-T_1-h)^{(\alpha-1)p'+1} &= t^{(\alpha-1)p'+1}\left[1 - \left(1 - \frac{T_1+h}{t}\right)^{(\alpha-1)p'+1}\right] \\ &\sim [(\alpha-1)p'+1](T_1+h)t^{(\alpha-1)p'} \quad \text{as } t \to \infty. \end{aligned}$$

Thus $C_\alpha(t)$ is bounded, and so is C. This derives the claim that $\mathcal{F}(\mathcal{PC}_0) \subset \mathcal{PC}_0$. $\square$

Now, we prove χ^*-condensing property for $\mathcal{F}$.

Lemma 4.3. *Assume that the hypotheses of Lemma* 4.2 *hold.* $\mathcal{F}$ *is* χ^**-condensing provided that* $q < 1$ *and*

$$\ell = \left(\eta + \sum_{k\in\Lambda} \mu_k\right) S_\alpha^\infty + 8 \sup_{t\geq 0} \int_0^t (t-s)^{\alpha-1} \|P_\alpha(t-s)\|_\chi k(s)\, ds < 1.$$

Proof. Let $D \subset \mathcal{PC}_0$ be a bounded set. Taking $r > 0$ such that $\|u\|_\infty \leq r$, $\forall\, u \in D$. By the same argument as in the proof of Lemma 3.3, we have

$$\chi_\infty(\mathcal{F}(D)) \leq \ell \chi_\infty(D).$$

It remains to estimate d_∞. For each $z \in \mathcal{F}(D)$, there exist $u \in D$ and $f \in \mathcal{P}_F^p(u)$ such that

$$z(t) = S_\alpha(t)[\varphi(0) - g(u)(0)] + \sum_{0<t_k<t} S_\alpha(t-t_k) I_k(u(t_k)) + \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s)\, ds$$

for all $t \geq 0$. Put $p' = p/(p-1)$, since $\alpha < 1$, $p > 1/\alpha$, $p' < 1/(1-\alpha)$, we see that $(\alpha-1)p'+1 > 0$ and $(\alpha-1)p' < 0$. Then there exists $\delta \in (0,1)$ such that $(\alpha-1)p'+\delta < 0$.

We have

$$\begin{aligned}\|z(t)\| &\leq \|S_\alpha(t)\|(\|\varphi\|_{\mathcal{C}_h} + \Psi_g(\|u\|_\infty)) + \sum_{0<t_k<t} \|S_\alpha(t-t_k)\| \|I_k(u(t_k))\| \\ &\quad + \int_0^t (t-s)^{\alpha-1} \|P_\alpha(t-s)\| m(s)(\|u(s)\| + \|u_s\|_{\mathcal{C}_h})\, ds.\end{aligned}$$

Thus

$$\begin{aligned}\|\mathcal{F}(D)(t)\| &\leq \|S_\alpha(t)\|(\|\varphi\|_{\mathcal{C}_h} + \Psi_g(r)) + \sum_{0<t_k<t} \|S_\alpha(t-t_k)\| l_k \Psi_I(r) \\ &\quad + 2\int_0^t (t-s)^{\alpha-1} \|P_\alpha(t-s)\| m(s) \|D_s\|_{\mathcal{C}_h}\, ds \\ &\leq \|S_\alpha(t)\|(\|\varphi\|_{\mathcal{C}_h} + \Psi_g(r)) + \Psi_I(r) \sum_{k\in\Lambda} l_k \|S_\alpha(t-t_k)\| \\ &\quad + 2\left(\int_0^{t^\delta} + \int_{t^\delta}^t\right) (t-s)^{\alpha-1} \|P_\alpha(t-s)\| m(s) \|D_s\|_{\mathcal{C}_h}\, ds \\ &= F_1(t) + F_2(t) + F_3(t) + F_4(t),\end{aligned}$$

where

$$F_1(t) = \|S_\alpha(t)\|(\|\varphi\|_{\mathcal{C}_h} + \Psi_g(r)),$$

$$F_2(t) = \Psi_I(r) \sum_{k\in\Lambda} l_k \|S_\alpha(t - t_k)\|,$$
$$F_3(t) = 2\int_0^{t^\delta} (t - s)^{\alpha-1} \|P_\alpha(t - s)\| m(s) \|D_s\|_{\mathcal{C}_h}\, ds,$$
$$F_4(t) = 2\int_{t^\delta}^{t} (t - s)^{\alpha-1} \|P_\alpha(t - s)\| m(s) \|D_s\|_{\mathcal{C}_h}\, ds.$$

Let $\epsilon > 0$ be given, there exists $N_0 \in \mathbb{N}$ such that $\sum_{k\geq N_0} l_k \leq \epsilon$. Moreover, from (Aa), there exists $T_2 > 0$ such that

$$\|S_\alpha(t)\| \leq \epsilon, \quad \|P_\alpha(t)\| \leq \epsilon, \quad \forall\, t \geq T_2.$$

By the same argument as in the proof of Lemma 4.2, we have

$$F_1(t) \leq (\|\varphi\|_{\mathcal{C}_h} + \Psi_g(r))\epsilon, \quad \forall\, t > T_2,$$
$$F_2(t) \leq \Psi_I(r)\left(\sum_{k\leq N_0} l_k + S_\alpha^\infty\right)\epsilon, \quad \forall\, t > T_2 + t_{N_0},$$
$$F_3(t) \leq 2rC_\alpha^*(t)\|m\|_{L^p(\mathbb{R}^+)}\,\epsilon, \quad \forall\, t > T_2 + t^\delta,$$

where

$$C_\alpha^*(t) = \left\{\frac{1}{(\alpha-1)p'+1}\left[t^{(\alpha-1)p'+1} - (t - t^\delta)^{(\alpha-1)p'+1}\right]\right\}^{1/p'}$$
$$\sim \left[t^{(\alpha-1)p'+\delta}\right]^{1/p'} \quad \text{as } t \to \infty$$

is bounded. Therefore

$$\lim_{T\to\infty}\sup_{t\geq T} F_1(t) = 0, \quad \lim_{T\to\infty}\sup_{t\geq T} F_2(t) = 0, \quad \lim_{T\to\infty}\sup_{t\geq T} F_3(t) = 0. \tag{4.6}$$

On the other hand, for $T + h < t^h < t$, one has

$$\begin{aligned}\sup_{t\geq T} F_4(t) &\leq 2\sup_{t\geq T}\int_{t^h}^{t} (t - s)^{\alpha-1}\|P_\alpha(t - s)\| m(s)\|D_s\|_{\mathcal{C}_h}\, ds \\ &\leq 2\sup_{t\geq T}\|D(t)\|\int_{t^h}^{t}(t - s)^{\alpha-1}\|P_\alpha(t - s)\| m(s)\, ds.\end{aligned}$$

Hence

$$\lim_{T\to\infty}\sup_{t\geq T} F_4(t) \leq 2\lim_{T\to\infty}\sup_{t\geq T}\|D(t)\|\int_0^t (t - s)^{\alpha-1}\|P_\alpha(t - s)\| m(s)\, ds. \tag{4.7}$$

Combining (4.6) and (4.7), we have

$$\lim_{T\to\infty}\sup_{t\geq T}\|\mathcal{F}(D)(t)\| \leq \lim_{T\to\infty}\sup_{t\geq T}\|D(t)\| 2\sup_{t\geq 0}\int_0^t (t - s)^{\alpha-1}\|P_\alpha(t - s)\| m(s)\, ds.$$

That is,
$$d_\infty(\mathcal{F}(D)) \le q d_\infty(D).$$
Now, it follows that
$$\begin{aligned}\chi^*(\mathcal{F}(D)) &= \chi_\infty(\mathcal{F}(D)) + d_\infty(\mathcal{F}(D))\\ &\le \ell\chi_\infty(D) + qd_\infty(D)\\ &\le \max\{\ell, q\}(\chi_\infty(D) + d_\infty(D))\\ &\le \max\{\ell, q\}\chi^*(D) < \chi^*(D).\end{aligned}$$
The proof is complete. □

The following theorem is our main result in this section.

Theorem 4.4. *Assume that the hypotheses of Lemma* 4.3 *hold. Then problem* (1.1)–(1.3) *has at least one integral solution* $u \in \mathcal{PC}_0$ *provided that*

$$\begin{aligned}\liminf_{r\to\infty}\Bigg[\frac{1}{r}\left(\Psi_g(r) + \Psi_I(r)\sum_{k\in\Lambda} l_k\right) S_\alpha^\infty \\ + 2\sup_{t>0}\int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|m(s)\,ds\Bigg] < 1.\end{aligned} \tag{4.8}$$

Proof. By Lemma 4.3, $\mathcal{F}$ is χ^*-condensing on $\mathcal{PC}_0$. By the same argument as in the proof of Theorem 3.4, we have $\mathcal{F}$ is closed and condition (4.8) ensures the existence of a number $R > 0$ such that $\mathcal{F}(B_R) \subset B_R$, where B_R is the closed ball in $\mathcal{PC}_0$ centered at 0 with radius R.

Now we are able to state that $\mathcal{F}\colon B_R \to Kv(B_R)$ has a nonempty compact fixed point set according to Theorem 2.11. The proof complete. □

5. Special case

In this section, we consider the problem without impulsive condition, as a special case of (1.1)–(1.3).

$$ {}^C D_0^\alpha u(t) - Au(t) \in F(t, u(t), u_t), \quad t > 0, \tag{5.1}$$
$$ u(s) + g(u)(s) = \varphi(s), \quad s \in [-h, 0]. \tag{5.2}$$

Definition 5.1. A function $u \in C([-h, +\infty]; X)$ is said to be an integral solution of problem (5.1)–(5.2) if and only if $u(t) + g(u)(t) = \varphi(t)$ for $t \in [-h, 0]$, and
$$u(t) = S_\alpha(t)[\varphi(0) - g(u)(0)] + \int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)f(s)\,ds$$
for any $t > 0$, where $f \in \mathcal{P}_F^p(u)$.

We will prove the existence of decay integral solutions with polynomial decay rate for problem (5.1)–(5.2). To this end, we will consider the solution operator $\mathcal{F}$ on the following space

$$\mathcal{BC}^\gamma = \{u \in C([-h,+\infty);X) : t^\gamma\|u(t)\| = O(1) \text{ as } 0 < t \to \infty\},$$

where γ is a positive number chosen later. This space is endowed with the supremum norm

$$\|u\|_{\mathcal{BC}} = \sup_{t\ge -h} \|u(t)\|,$$

and it becomes a closed subspace of the Banach space

$$BC = \{u \in C([-h,+\infty);X) : \|u(t)\|_{\mathcal{BC}} < \infty\}.$$

On $\mathcal{BC}^\gamma$ we use the following MNCs

$$d_T(D) = \sup_{t\ge T}\sup_{x\in D}\|x(t)\|, \qquad d_\infty(D) = \lim_{T\to\infty} d_T(D),$$
$$\chi_\infty(D) = \sup_{T>0}\chi_T(\pi_T(D)), \qquad \chi^*_{\mathcal{BC}}(D) = \chi_\infty(D) + d_\infty(D).$$

By the same proof as in Lemma 4.1, we have $\chi^*_{\mathcal{BC}}$ is a regular MNC on $\mathcal{BC}^\gamma$.

Now we prove the existence of decay integral solutions of problem (5.1)–(5.2) with a polynomial decay rate. As in Section 4, we will show that $\mathcal{F}$ keeps $\mathcal{BC}^\gamma$ invariant and $\mathcal{F}$ is $\chi^*_{\mathcal{BC}}$-condensing on $\mathcal{BC}^\gamma$. For problem (5.1)–(5.2), we assume that (F), (G) is satisfied for all $T > 0$ and

(A*) The operator A satisfies (A) such that

$$\|S_\alpha(t)\| \le C_S t^{-\delta} \quad \text{and} \quad \|P_\alpha(t)\| \le C_P t^{-\delta},$$

where $0 < \delta < \alpha$.

The following proposition gives a sufficient condition for (A*).

Proposition 5.2. [7] *If the semigroup $W(\cdot)$ is exponentially stable, i.e.,*

$$\|W(t)\| \le Me^{-at} \quad \textit{for some } a, M > 0,$$

then there exist two positive numbers C_S and C_P such that

$$\|S_\alpha(t)\| \le C_S t^{-\alpha}, \quad \|P_\alpha(t)\| \le C_P t^{-\alpha}, \quad \forall\, t > 0.$$

Lemma 5.3. *Let* (A*) *hold and* (F) *and* (G) *hold for all $T > 0$. Then $\mathcal{F}(\mathcal{BC}^\gamma) \subset \mathcal{BC}^\gamma$ for all $\gamma \le \delta$.*

Proof. Now, let $u \in \mathcal{BC}^\gamma$ and $z \in \mathcal{F}(u)$. Then one can find $f \in \mathcal{P}_F^p(u)$ such that

$$z(t) = S_\alpha(t)[\varphi(0) - g(u)(0)] + \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s)\, ds, \quad \forall\, t > 0.$$

We have

$$\begin{aligned}\|z(t)\| &\le \|S_\alpha(t)\|(\|\varphi\|_{\mathcal{C}_h} + \|g(u)\|_{\mathcal{C}_h}) + \int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\| m(s)(\|u(s)\| + \|u_s\|_{\mathcal{C}_h})\, ds \\ &= I_1(t) + I_2(t) + I_3(t) + I_4(t),\end{aligned}$$

where

$$\begin{aligned}I_1(t) &= \|S_\alpha(t)\|(\|\varphi\|_{\mathcal{C}_h} + \|g(u)\|_{\mathcal{C}_h}),\\ I_2(t) &= \int_0^{t/2} (t-s)^{\alpha-1}\|P_\alpha(t-s)\| m(s)(\|u(s)\| + \|u_s\|_{\mathcal{C}_h})\, ds,\\ I_3(t) &= \int_{t/2}^{t-1} (t-s)^{\alpha-1}\|P_\alpha(t-s)\| m(s)(\|u(s)\| + \|u_s\|_{\mathcal{C}_h})\, ds,\\ I_4(t) &= \int_{t-1}^{t} (t-s)^{\alpha-1}\|P_\alpha(t-s)\| m(s)(\|u(s)\| + \|u_s\|_{\mathcal{C}_h})\, ds.\end{aligned}$$

Since $\|S_\alpha(t)\| = O(t^{-\delta})$ as $t \to \infty$ and $\gamma < \delta$, we have $t^\gamma I_1(t) = o(1)$ as $t \to \infty$.

In what follows, we denote by C a generic constant, which may be change from line to line. Considering I_2, one has

$$\begin{aligned}I_2(t) &= \int_0^{t/2} (t-s)^{\alpha-1}\|P_\alpha(t-s)\| m(s)(\|u(s)\| + \|u_s\|_{\mathcal{C}_h})\, ds\\ &\le C \int_0^{t/2} (t-s)^{-1} m(s)\, ds,\end{aligned}$$

thanks to the fact that $\|P_\alpha(t)\| \le Ct^{-\delta}$ for $t > 0$ and the boundedness of $(\|u(s)\| + \|u_s\|_{\mathcal{C}_h})$. Using the Hölder inequality we have

$$\begin{aligned}I_2(t) &\le C \left[\int_0^{t/2} (t-s)^{-p/(p-1)}\, ds\right]^{(p-1)/p} \|m\|_{L^p(\mathbb{R}^+)}\\ &= C(p-1)^{-(p-1)/p}(2^{1/(p-1)} - 1)\|m\|_{L^p(\mathbb{R}^+)} t^{-1/p}.\end{aligned}$$

Therefore, $t^\gamma I_2(t) = O(1)$ as $t \to \infty$. Now, we observes that $t^\gamma\|u(t)\| = O(1)$ as $t \to \infty$. Then for all $t > h$ we get

$$\begin{aligned}t^\gamma \|u_t\|_{\mathcal{C}_h} &\le t^\gamma \sup_{\rho\in[-h,0]} \|u(t+\rho)\|\\ &= t^\gamma \sup_{\rho\in[-h,0]} (t+\rho)^{-\gamma}(t+\rho)^\gamma\|u(t+\rho)\|\\ &\le t^\gamma (t-h)^{-\gamma} \sup_{\rho\in[-h,0]} (t+\rho)^\gamma\|u(t+\rho)\|\\ &= O(1) \quad \text{as } t \to \infty.\end{aligned}$$

Regarding I_3, for $t > \max\{2h, 2\}$ we have

$$\begin{aligned}
t^\gamma I_3(t) &= t^\gamma \int_{t/2}^{t-1} s^{-\gamma}(t-s)^{\alpha-1}\|P_\alpha(t-s)\| m(s)(s^\gamma\|u(s)\| + s^\gamma\|u_s\|_{\mathcal{C}_h})\, ds \\
&\le Ct^\gamma \int_{t/2}^{t-1} \left(\frac{t}{2}\right)^{-\gamma} (t-s)^{-1} m(s)\, ds \\
&\le C2^\gamma \int_{t/2}^{t-1} (t-s)^{-1} m(s)\, ds,
\end{aligned}$$

thanks to the fact that $s^\gamma\|u(s)\| + s^\gamma\|u_s\|_{\mathcal{C}_h}$ is bounded for $s \ge t/2 > h$. Applying the Hölder inequality again, we obtain

$$\begin{aligned}
t^\gamma I_3(t) &\le C \left[\int_{t/2}^{t-1} (t-s)^{-p/(p-1)}\, ds\right]^{(p-1)/p} \|m\|_{L^p(\mathbb{R}^+)} \\
&\le C(p-1)^{(p-1)/p} \left[1 - \left(\frac{t}{2}\right)^{-1/(p-1)}\right]^{(p-1)/p} \|m\|_{L^p(\mathbb{R}^+)} \\
&= O(1) \quad \text{as } t \to \infty.
\end{aligned}$$

Similar to I_4, we have

$$\begin{aligned}
t^\gamma I_4(t) &= t^\gamma \int_{t-1}^{t} s^{-\gamma}(t-s)^{\alpha-1}\|P_\alpha(t-s)\| m(s)(s^\gamma\|u(s)\| + s^\gamma\|u_s\|_{\mathcal{C}_h})\, ds \\
&\le C\left(\frac{t}{t-1}\right)^\gamma \int_{t-1}^{t} (t-s)^{\alpha-1} m(s)\, ds,
\end{aligned}$$

here we use the boundedness of $\|P_\alpha(t-s)\|$ and $(s^\gamma\|u(s)\| + s^\gamma\|u_s\|_{\mathcal{C}_h})$ for $s \ge t-1$. Then

$$\begin{aligned}
t^\gamma I_4(t) &\le C\left(\frac{t}{t-1}\right)^\gamma \left(\int_{t-1}^{t} (t-s)^{p(\alpha-1)/(p-1)}\, ds\right)^{(p-1)/p} \|m\|_{L^p(\mathbb{R}^+)} \\
&= C\left(\frac{t}{t-1}\right)^\gamma \left(\frac{p-1}{p\alpha-1}\right)^{(p-1)/p} \|m\|_{L^p(\mathbb{R}^+)} \\
&= O(1) \quad \text{as } t \to \infty.
\end{aligned}$$

Summing up, we get $t^\gamma\|z(t)\| = O(1)$ as $t \to \infty$. Equivalently, $z \in \mathcal{BC}^\gamma$. The proof is complete. □

The χ^*-condensing property of $\mathcal{F}$ will be proved in the following lemma.

Lemma 5.4. *Let* (A*) *hold and* (F), (G) *hold for all* $T > 0$. *Then* $\mathcal{F}$ *is* $\chi^*_{\mathcal{BC}}$*-condensing on* $\mathcal{BC}^\gamma$ *with* $0 < \gamma < \delta$ *provided that*

$$\ell^* = \eta S_\alpha^\infty + 8\sup_{t\ge 0}\int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|_\chi k(s)\, ds < 1. \tag{5.3}$$

Proof. Let $D \subset \mathcal{BC}^\gamma$ be a bounded set. We first show that $d_\infty(\mathcal{F}(D)) = 0$. Indeed, taking $r > 0$ such that $\|x\|_{\mathcal{BC}} \leq r$ for all $x \in D$, we have

$$t^\gamma \|\mathcal{F}(x)(t)\| = O(1) \quad \text{as } t \to \infty$$

according to the proof of Lemma 5.3. This means that

$$\|\mathcal{F}(x)(t)\| \leq Ct^{-\gamma}, \quad \forall\, x \in D$$

for all t large enough. Equivalently, for a large T, one has $d_T(\mathcal{F}(D)) \leq CT^{-\gamma}$. Then

$$d_\infty(\mathcal{F}(D)) = \lim_{T\to\infty} d_T(\mathcal{F}(D)) = 0. \tag{5.4}$$

On the other hand, by the same argument as in the proof of Lemma 3.3, we have

$$\chi_\infty(\mathcal{F}(D)) \leq \ell^* \chi_\infty(D). \tag{5.5}$$

Now, it follows from (5.4) and (5.5) that

$$\chi^*_{\mathcal{BC}}(\mathcal{F}(D)) \leq \ell^* \chi^*_{\mathcal{BC}}(D),$$

and condition (5.3) give us the condensing property of $\mathcal{F}$ on $\mathcal{BC}^\gamma$. The proof is complete. □

And we have the stability result for problem (5.1)–(5.2) stated in the following theorem.

Theorem 5.5. *Assume that the hypotheses of Lemma* 5.4 *hold. Then problem* (5.1)–(5.2) *has at least one integral solution on* $[-h, +\infty)$ *satisfying* $\|u(t)\| = O(t^{-\gamma})$ *provided that*

$$\liminf_{r\to\infty} \left[\frac{1}{r} S^\infty_\alpha \Psi_g(r) + 2\sup_{t>0} \int_0^t (t-s)^{\alpha-1} \|P_\alpha(t-s)\| m(s)\, ds\right] < 1. \tag{5.6}$$

Proof. By Lemma 5.4, $\mathcal{F}$ is $\chi^*_{\mathcal{BC}}$-condensing on $\mathcal{BC}^\gamma$. By the same argument as in the proof of Theorem 3.4, we have $\mathcal{F}$ is closed and condition (5.6) ensures the existence of a stable closed ball B_R in $\mathcal{BC}^\gamma$.

Now we are able to state that $\mathcal{F}\colon B_R \to Kv(B_R)$ has a nonempty compact fixed point set according to Theorem 2.11. The proof complete. □

6. Application

We consider the following fractional partial differential system:

$$\partial_t^\gamma u(x,t) = (-i\Delta+\sigma)^{1/2}u(x,t) + f(t,x,u(t,x)) + \sum_{i=1}^{m} b_i(x)v_i(t), \quad x\in\mathbb{R}^2,\ t>0, \tag{6.1}$$

$$\begin{gathered} v_i(t) \in \left[\int_{\mathcal{O}} k_{1,i}(y)u(t-h,y)\,dy, \int_{\mathcal{O}} k_{2,i}(y)u(t-h,y)\,dy\right], \\ 1\le i\le m, \quad x\in\mathbb{R}^2, \quad t\in\mathbb{R}^+\setminus\{t_k : k\in\mathbb{N}\}, \end{gathered} \tag{6.2}$$

$$u(x,s) + \sum_{i=1}^{M} c_i u(x,\tau_i+s) = \varphi(x,s), \quad s\in[-h,0], \tag{6.3}$$

where ∂_t^γ stands for the Caputo derivative of order $\gamma\in(0,1)$ with respect to t, $\mathcal{O}$ is a bounded domain in $\mathbb{R}^2$.

Let $X=L^3(\mathbb{R}^2)$ with the norm $\|\cdot\|$, $\sigma>0$ is a suitable constant, $-i\Delta$ is Schrödinger operator. Let

$$\widehat{A} = (-i\Delta+\sigma)^{1/2}, \quad D(\widehat{A}) = W^{1,3}(\mathbb{R}^2).$$

Definition 6.1. Let $-1<\gamma<0$ and $0<\omega<\pi/2$. By $\Theta_\omega^\gamma(X)$ we denote the family of all linear closed operator $A\colon D(A)\subset X\to X$ which satisfy

(1) $\sigma(A)\subset S_\omega=\{z\in\mathbb{C}\setminus\{0\} : |\arg z|\le\omega\}\cup\{0\}$ and

(2) for every $\omega<\mu<\pi$ there exists a constant C_μ such that

$$\|R(z;A)\|\le C_\mu|z|^\gamma \quad \text{for all } z\in\mathbb{C}\setminus S_\mu.$$

A linear operator A will be called an almost sectorial operator on X if $A\in\Theta_\omega^\gamma(X)$.

Theorem 6.2. *For each fixed $t\in S^0_{\pi/2}$, $S_\alpha(t)$ and $P_\alpha(t)$ are linear and bounded operators on X. Moreover, there exist constant $C_S=C(\alpha,\gamma)>0$, $C_P=C(\alpha,\gamma)>0$ such that for all $t>0$,*

$$\|S_\alpha(t)\|\le C_S t^{-\alpha(1+\gamma)}, \quad \|P_\alpha(t)\|\le C_P t^{-\alpha(1+\gamma)}.$$

It follows from [15] that $\widehat{A}$ is an almost sectorial operator, $\widehat{A}\in\Theta_\omega^{-\frac{1}{6}\alpha}$ for some $0<\omega<\pi/2$. Thus, we have

$$\|S_\alpha(t)\|\le C_S t^{-\frac{5}{6}\alpha}, \quad \|P_\alpha(t)\|\le C_P t^{-\frac{5}{6}\alpha}. \tag{6.4}$$

Then (A*) is satisfied.

Now let $g(u)(s)(x)=\sum_{i=1}^{M} c_i u(x,\tau_i+s)$, then

$$\|g(u_1)-g(u_2)\|_{\mathcal{C}_h}\le M^{2/3}\max\{c_i\}_{i=1}^m\|u_1-u_2\|_{\mathcal{PC}}.$$

Therefore, (G) satisfies with $\eta=M^{2/3}\max\{c_i\}_{i=1}^m$ and $\Psi_g=\eta\,\mathrm{Id}$.

For the multivalued part, we assume that

(1) $b_i \in L^3(\mathbb{R}^2)$, $k_{j,i} \in L^{3/2}(\mathcal{O})$, $j = 1, 2$, $1 \le i \le m$,

(2) $f : \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ such that $f(\,\cdot\,, x, z)$ is measurable, $f(t, \cdot\,, z)$ is measurable, $f(t, x, 0) = 0$ and there exists $\kappa \in L^p_{\mathrm{loc}}(\mathbb{R}^+; L^\infty(\mathbb{R}^2))$ verifying

$$|f(t,x,z_1) - f(t,x,z_2)| \le \kappa(t,x)|z_1 - z_2|, \quad \forall\, t > 0,\ x \in \mathbb{R}^2,\ z_1, z_2 \in \mathbb{R}. \tag{6.5}$$

Now let $F_1 : \mathbb{R}^+ \times X \to X$, $F_2 : C([-h,0]; X) \to \mathcal{P}(X)$ be such that

$$F_1(t,v)(x) = f(t,x,v(x)),$$
$$F_2(w)(x) = \sum_{i=1}^m b_i(x) \left[\int_{\mathcal{O}} k_{1,i}(y) w(-h,y)\,dy, \int_{\mathcal{O}} k_{2,i}(y) w(-h,y)\,dy\right].$$

Then problem (6.1)–(6.2) is exactly a prototype of system (1.1)–(1.3) with $F(t,v,w) = F_1(t,v) + F_2(w)$.

It follows from (6.5) that

$$\|F_1(t,v_1) - F_1(t,v_2)\| \le \|\kappa(t,\cdot\,)\|_{L^\infty(\mathbb{R}^2)} \|v_1 - v_2\|. \tag{6.6}$$

It is easily seen that F_1 is a continuous (single-valued) map. In addition, F_2 is a multimap with closed convex values and the range lying in a finite dimensional space $\operatorname{span}\{b_1, \dots, b_n\} \subset X$. Then one sees that F_2 maps any bounded set into a relatively compact set. The facts that F_2 has a closed graph can be testified by a simple argument. Thus, F_2 is u.s.c. multimap (thanks to 2.8) with convex closed and compact values, and so is F.

The inequality (6.6) ensures that

$$\chi(\mathcal{F}_1(t,B)) \le \|\kappa(t,\cdot\,)\|_{L^\infty(\mathbb{R}^2)} \chi(B).$$

On the other hand, for a bounded set $C \subset C([-h,0]; X)$, we see that $F_2(t,C)$ is bounded subset of the finite dimensional space formed by $\{b_i\}_{i=1}^m$. So

$$\chi(\mathcal{F}_2(t,C)) = 0.$$

Let $F(t,v,w) = F_1(t,v) + F_2(w)$. Then

$$\chi(F(t,B,C)) \le \chi(F_1(t,B)) + \chi(F_2(C)) \le \|\kappa(t,\cdot\,)\|_{L^\infty(\mathbb{R}^2)} \chi(B)$$

for all $B \in \mathcal{B}(X)$, $C \in \mathcal{B}(C([-h,0]; X))$. Thus F satisfies (F4) for $k(t) = \|\kappa(t,\cdot\,)\|_{L^\infty(\mathbb{R}^2)}$.

Now, we check (F3). It is easy to see that

$$\|F_1(t,v)\| \le \|\kappa(t,\cdot\,)\|_{L^\infty(\mathbb{R}^2)} \|v\|,$$
$$\|F_2(w)\| \le \sum_{i=1}^m \|b_i\| \max\{\|k_{1,i}\|_{L^2(\mathcal{O})}, \|k_{2,i}\|_{L^2(\mathcal{O})}\} \cdot \|w\|_{C([-h,0];X)}.$$

Then, (F3) takes place with

$$m(t) = \max\left\{ \|\kappa(t,\cdot)\|_{L^\infty(\mathbb{R}^2)}, \sum_{i=1}^{m} \|b_i\| \max\{\|k_{1,i}\|_{L^2(\mathcal{O})}, \|k_{2,i}\|_{L^2(\mathcal{O})}\} \right\}.$$

By the above description for (6.1)–(6.3), we can apply Theorem 4.4 to get the existence of decay integral solutions. Moreover, inequality (6.4) ensures that, system (6.1)–(6.3) has a integral solutions with decay rate described by $t^\gamma \|u(t,\cdot)\| = O(1)$ as $t \to \infty$, for $0 < \gamma < -\frac{5}{6}\alpha$.

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