

## A Method-of-lines Approach for Solving American Option Problems

Min-Sun Horng, Tzyy-Leng Horng\* and Chih-Yuan Tien

**Abstract.** The early exercise property of American option changes the original Black-Scholes equation to an inequality that cannot be solved via traditional finite difference method. Therefore, finding the early exercise boundary prior to spatial discretization is a must in each time step. This overhead slows down the computation and the accuracy of solution relies on if the early exercise boundary can be accurately located. A simple numerical method based on finite difference and method of lines is proposed here to overcome this difficulty in American option valuation. Our method averts the otherwise necessary procedure of locating the optimal exercise boundary before applying finite difference discretization. The method is efficient and flexible to all kinds of pay-off. Computations of American put, American call with dividend, American strangle, two-factor American basket put option, and two-factor convertible bond with embedded call and put options are demonstrated to show the efficiency of the current method.

### 1. Introduction

In last two decades, the problem of pricing American options has been investigated extensively both in numerical methods and analytical approximations. These two approaches both encounter the arduous challenge which is known as a free boundary value problem arising from the early exercise feature of American options. The difficulty associated with the valuation of American options stems from the fact that the optimal exercise boundary must be determined as a part of the solution. Unfortunately, the early exercise boundary cannot be solved in a close form, and, consequently, nor can the option's value. Up to now, reviewing all relevant literatures, most efforts have been exerted mainly on locating the free boundary that sets the domain for the Black-Scholes equation.

For analytic approximations, Johnson [19] adopted an interpolation scheme to price the American put option for a non-dividend paying stock. MacMillan [22] used a quadratic approximation, which involved solving an approximate partial differential equation (PDE) for the early exercise premium, the amount by which the value of an American option

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exceeds a European one, to evaluate an American put option. Geske and Johnson [15] obtained a valuation formula for American put option expressed in terms of a series of compound-option functions. Following [22], Ju and Zhong [20] presented an efficient and accurate approximate formula for pricing American options on a dividend paying stock.

For numerical approximations, the most popular numerical methods for pricing American options can be classified to lattice method, Monte Carlo simulation and finite difference method. Sure, besides finite difference methods, there are other popular numerical method based on discretization for solving PDEs like finite element method, boundary element method, spectral and pseudo-spectral methods and etc. Here we just use finite difference to stand for methods of this kind. In fact, finite difference method ranks as the most popular one among its kind in financial engineering. The lattice method is simple and still widely used for evaluating American options. It was first introduced by Cox et al. [9], and the convergence of the lattice method for American options is proved by Amin and Khanna [1]. The Monte Carlo method is also popular among financial practitioners. It is appealing, simple to implement for pricing European options, and especially has advantage of pricing multi-asset options. For pricing American options, Monte Carlo method requests some further modification. Fu [13, 14] priced American-style options by using Monte Carlo method in conjunction with gradient-based optimization techniques. Duck et al. [11] proposed a technique which generates monotonically varying data to enhance the accuracy and reliability of Monte Carlo-based method in handling early exercise features.

The application of finite difference method to price American options can be first found in [4, 5, 26]. Jaillet et al. [18] showed the convergence of the finite difference method. A comparison of different numerical methods for American options pricing was discussed in [6, 16].

Generally, there still exist some difficulties in using these numerical methods. For finite difference method, the difficulty arises from the early exercise property, which changes the original Black-Scholes equation to an inequality that cannot be solved via traditional finite difference process. Therefore, finding the early exercise boundary prior to spatial discretization (discretization on underlying asset) is a must in each time step. This overhead slows down the computation and the accuracy of solution relies on if the early exercise boundary can be accurately located. For lattice method and Monte Carlo simulation, although they do not have the free boundary value problem, they would need some extra efforts to compute the Greeks and the optimal exercise boundary, which are mostly desired in practice besides option valuation.

In this paper, we propose a simple numerical method based on finite difference method and method of lines (MOL) to overcome the difficulty mentioned above in American

option valuation. Particularly, our method averts the otherwise necessary procedure of locating the optimal exercise boundary before applying finite difference discretization. This method is efficient, flexible to all kinds of pay-off, and easy to implement when compared with many other methods. The results also show that our method possesses the optimal accuracy intrinsic to finite difference discretization, and thereby make it a powerful tool for practitioners when evaluating American option and financial derivatives having American option feature.

This paper is organized as follows. Section 2 is devoted to the description of method of lines. Section 3 describes the core idea of our scheme of evaluating American option through the spirit of Black-Scholes inequality with demonstrations through successful computation of American put, American call with dividend, and American strangle options. Optimal accuracy is also shown in the example of American put option compared with an exhaustive result from binomial tree. Section 4 depicts the scheme of locating the optimal exercise boundary, and shows the optimal exercise boundaries together with the option values for cases computed in Section 3. The dual optimal exercise boundaries in American strangle option are particularly compared with Chiarella and Ziogas [8], and satisfactory agreement is observed. Section 5 extends the current method to compute multi-asset American options with a demonstration of evaluating a two-factor American basket put option. The option value with the associated optimal exercise boundary is shown in the context. Section 6 further demonstrates the power of current method by applying to evaluate the price of a two-factor callable and puttable convertible bond which is a popular financial product in American style nowadays. Besides the price evaluated, its early exercise regime is also computed.

## 2. Method of lines

Under the usual assumptions, Black and Scholes [2] and Merton [23] have shown that the price  $V$  of any contingent claim written on a stock, whether it is American or European, satisfies the famous Black-Scholes equation:

$$(2.1) \quad \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S \frac{\partial V}{\partial S} - rV = 0,$$

where volatility  $\sigma$ , the risk-free rate  $r$ , and dividend yield  $D$  are all assumed to be constants. The value of any particular contingent claim is determined by the terminal and boundary conditions. For an American option, notice that the PDE only holds in the not-yet-exercised region. At the place where the option should be exercised immediately, the equality sign in equation (2.1) would turn into an inequality one. That means the option value  $V(S, t)$  at each time follows either  $V(S, t) = f(S, t)$  for the early exercised region or  $\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S \frac{\partial V}{\partial S} - rV = 0$  for the not-yet-exercised region, where

$f(S, t)$  is the payoff of an American option at time  $t$ . There is a moving boundary that separates these two regions and makes the whole problem a free boundary problem for Black-Scholes equation. Generally, this early exercised boundary is difficult to locate and there is no simple closed-form expression for it. Most finite difference methods nowadays exert their efforts on locating the early exercise boundary that is a must before finite difference discretization can be applied to the not-yet-exercised region.

The method of lines (MOL), a popular method for solving PDEs in engineering, was first promoted by Liskovets [21]. The idea is first reducing a time-dependent PDE to a system of ordinary differential equations (ODEs) in time via semi-discretization in space. Then this system of ODEs in time can further be solved efficiently by many well-developed ODE solvers. This methodology has been successfully applied to the valuation of American options on common stock, and is found to be accurate and efficient [7, 17, 24]. To explain how MOL is employed, here we simply take valuation of a European call as an example. Incorporating with 2<sup>nd</sup> order finite difference scheme, we can discretize equation (2.1) on a uniform mesh of underlying asset price  $\{S_i\}_{i=0}^N$  into a semi-discrete system of ODEs in time:

$$(2.2) \quad \frac{dV_i}{dt} = -\frac{1}{2}\sigma^2 S_i^2 \frac{V_{i+1} - 2V_i + V_{i-1}}{(\Delta S)^2} - (r - D)S_i \frac{V_{i+1} - V_{i-1}}{2\Delta S} + rV_i, \quad i = 1, \dots, N-1.$$

The boundary conditions of equation (2.1) are

$$\frac{\partial V(S, t)}{\partial S} \rightarrow 1 \quad \text{as } S \rightarrow \infty, \quad 0 \leq t \leq T, \quad \text{and} \quad V(0, t) = 0, \quad 0 \leq t \leq T.$$

Both boundary conditions above exhibit linear behavior close to boundaries, and hence they can be incorporated into this system of ODEs by compensating

$$(2.3) \quad \frac{dV_0}{dt} = -(r - D)S_0 \frac{-3V_0 + 4V_1 - V_2}{2\Delta S} + rV_0,$$

$$(2.4) \quad \frac{dV_N}{dt} = -(r - D)S_N \frac{3V_N - 4V_{N-1} + V_{N-2}}{2\Delta S} + rV_N,$$

for equation (2.2) with  $S \rightarrow \infty$  being truncated to  $S_{\max}$  which is specified by  $S_{\max} = 3K$  or  $S_{\max} = 4K$  considering both computational efficiency and acceptable boundary error. Equations (2.2)–(2.4) pose as an ordinary differential initial value problem in time and can be solved by many efficient ODE solvers which have been developed in hundreds of years. Actually, traditional finite difference method incorporating with explicit Euler scheme in time integration is equivalent to choose forward Euler scheme as the ODE solver in MOL. Likewise, popular Crank-Nicolson finite difference scheme is equivalent to select the Adams-Moulton one-step method (trapezoidal rule) as the ODE solver in MOL. As noted in [24], the free boundary initially moves with infinite speed but slows down very quickly. Therefore, this stiff problem would be hard to implement efficiently by

plain finite difference methods. To evaluate American option efficiently, it would request a self-adjusting-in-time-step solver for time integration, and this can be done easily by novel ODE solvers nowadays featuring self-adjusting variable step size and order (VSVO). Many VSVO type ODE solvers have been collected in popular MATLAB<sup>®</sup> ODE suite. In the current study, we selected ode23 from the suite to be our demonstrating ODE solver. Ode23 implements an explicit Runge-Kutta (2, 3) pair developed by Bogacki and Shampine [3]. It features by an adaptive step size controlled by specified error tolerance and is an efficient on-step solver for moderately stiff ODE's. More details of this solver can be found in [3].

### 3. Valuation of American options

American options differ from European ones by that the holder can select to exercise at any time before the expiry date. This early exercise feature of American options causes the free boundary problem of Black-Scholes equation. So far all finite difference methods for pricing American options, including those employing MOL mentioned above, request to find out the optimal exercise boundary first in each time step. This procedure is a must, and accurate location of this optimal exercise boundary is crucial to the overall accuracy. That is why many analytical mathematicians study this subject and compete on deriving a better analytic approximation of optimal exercise boundary that is crucial to finite difference methods.

However, our current method does not request this optimal exercise boundary at all in each time step, and this optimal exercise boundary can be extracted from the numerical results afterwards if wanted. Due to the saving of overhead on computing this optimal exercise boundary, our method is much more efficient compared with other numerical methods. The key idea of our method is to modify equation (2.2) to

$$(3.1) \quad \frac{dV_i}{dt} = \min \left( -\frac{1}{2}\sigma^2 S_i^2 \frac{V_{i+1} - 2V_i + V_{i-1}}{(\Delta S)^2} - (r - D)S_i \frac{V_{i+1} - V_{i-1}}{\Delta S} + rV_i, 0 \right),$$

where  $i = 1, \dots, N - 1$ . This idea is based on the fact that the value of an American option, when evolving backward in time, can not allow smaller than its final pay-off at any time before expiration. If it is smaller than its final pay-off, the option should then be early exercised at that spot. This spirit can also be observed in evaluating American options by binomial tree. To elaborate our point, for those nodes  $i$  that do not need early exercise, their option values would be governed by the backward time evolution of Black-Scholes equation and

$$\frac{dV_i}{dt} = \min \left( -\frac{1}{2}\sigma^2 S_i^2 \frac{V_{i+1} - 2V_i + V_{i-1}}{(\Delta S)^2} - (r - D)S_i \frac{V_{i+1} - V_{i-1}}{\Delta S} + rV_i, 0 \right)$$

$$= -\frac{1}{2}\sigma^2 S_i^2 \frac{V_{i+1} - 2V_i + V_{i-1}}{(\Delta S)^2} - (r - D)S_i \frac{V_{i+1} - V_{i-1}}{\Delta S} + rV_i.$$

For option values at other nodes  $i$  that would fall below their final pay-off if following the backward evolution of Black-Scholes equation, it would request an early exercise and then

$$\frac{dV_i}{dt} = \min \left( -\frac{1}{2}\sigma^2 S_i^2 \frac{V_{i+1} - 2V_i + V_{i-1}}{(\Delta S)^2} - (r - D)S_i \frac{V_{i+1} - V_{i-1}}{\Delta S} + rV_i, 0 \right) = 0.$$

Equation (3.1) actually can deduce the Black-Scholes inequality for American options:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S \frac{\partial V}{\partial S} - rV \leq 0.$$

Solving equation (3.1) is straight forward and easy to implement in the frame work of MOL. Extracting the optimal exercise boundary, if wanted, can be done afterwards from the result by re-computing  $dV_i/dt$ ,  $\forall i$ , at each time step, and locating the single zero of  $dV_i/dt$  through interpolation. This trajectory of single zero would be the optimal exercise boundary. The details of computing this optimal exercise boundary will be discussed in a later section below. Actually, this idea can be implemented by conventional finite difference techniques such as forward Euler and Crank-Nicolson methods too. We can simply integrate the Black-Scholes equation backward in time at each time step, and then replace the option value falling below the final pay-off by it at those nodes that need an early exercise. This can be justified rigorously by thinking of Black-Scholes inequality above as Black-Scholes equation with an additional a priori unknown forcing function. The only purpose of this forcing function is to make sure that, once the prices at those nodes fall below the pay-off when evolving backward in time, the forcing function will compensate those prices to become the pay-off value. One may still argue about the adoption of this unknown a priori forcing function, and this can be justified by the governing equation (3.2) for American put option shown below. We can see the option price retains its continuity in both the function itself and its spatial derivative all the time at the interface (the early exercise boundary) and then for the whole domain  $[0, S_{\max}]$ . This implies the existence of a forcing function that is non-zero only on those early-exercised nodes. Though forward Euler or Crank-Nicolson finite difference scheme may then look like a simpler way to evaluate American options by adopting the idea above, however, as mentioned before, the free boundary moves extremely fast near the expiry date, which would request much finer mesh in time near the expiry date. This makes these two conventional finite difference techniques with fixed time step very inefficient to reach small time error near the expiry date. If the mesh of time is not appropriately resolved near the expiry date, it may cause large errors in both the option value and optimal exercise boundary. Under this situation, MOL employing VSVO-type error-control ODE solvers would be a much more superior method. Here, we use American put option, American call option with dividend and American strangle option to demonstrate how our simple approach would work.

### 3.1. American put option

In Figure 3.1, the optimal exercise boundary  $S_{f,P}(t)$  separates the underlying asset domain in two regions, continuation and stopping regions. On the stopping region  $[0, S_{f,P}(t)) \times [0, T]$ ,  $V(S, t) = K_P - S$ , while on the continuation region  $[S_{f,P}(t), \infty) \times [0, T]$ ,  $V(S, t)$  needs to satisfy the following free-boundary problem:

$$\begin{aligned}
 (3.2) \quad & \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S \frac{\partial V}{\partial S} - rV = 0, & S_{f,P}(t) < S, 0 \leq t < T, \\
 & V(S, T) = \max(K_P - S, 0), & S \geq 0, \\
 & V_{S_{f,P}}(S_{f,P}(t), t) = K_P - S_{f,P}(t), & 0 \leq t \leq T, \\
 & \lim_{S \rightarrow S_{f,P}(t)} \frac{\partial V}{\partial S} = -1, & 0 \leq t \leq T, \\
 & S_{f,P}(T) = \min\left(K_P, \frac{rK_P}{D}\right).
 \end{aligned}$$

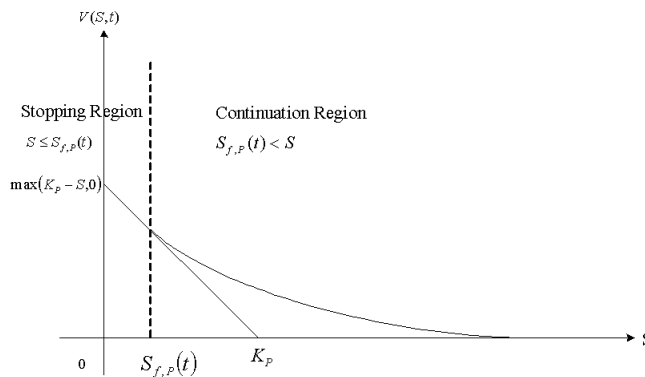


Figure 3.1: Illustration of continuation region, stopping region and optimal exercise boundary for American put option.

Naturally, finding out the optimal exercise boundary would be a must before integrating equation (3.2). Instead, our simple method solves equation (3.1) with linear boundary conditions (2.3)–(2.4) and the following terminal condition

$$V(S, T) = \max(K_P - S, 0).$$

The specific parameter values adopted here in our computation are  $r = 10\%$ ,  $\sigma = 40\%$ ,  $D = 0$ ,  $T = 1$ ,  $K_P = 1/5$ , and stock price ranges from  $S = 0$  to  $S = 1$  with the infinite domain of  $S$  being truncated by five times of the exercise price  $K_P$ . The error analysis of our result is reported in Table 3.1. The first part in this table lists the calculated option price at the exercise price. For calculating absolute error, we calculated a binomial

tree with exhaustive  $N_b = 10,000$  time steps and used the result as the exact solution  $V_{\text{exa}}(S_i, 0)$ . Here, we again limit the maximum time step to be  $10^{-4}$  so that the total error would be dominated only by the spatial error. The column of absolute error again shows perfect 2<sup>nd</sup> order convergence. In the second part, the maximum absolute error (MAE),  $\max_{0 \leq i \leq N} |V(S_i, 0) - V_{\text{exa}}(S_i, 0)|$ , shows that the order of accuracy deteriorates slightly from 2<sup>nd</sup> order, and the location of MAE is mostly close to the free boundary except at  $N = 400$ , in which the location of MAE is close to exercise price. This deterioration may be due to the fact that  $V$  fails to be twice differentiable at the optimal exercise boundary. As we know, error analysis of finite difference is based on Taylor expansion. 2<sup>nd</sup> order of accuracy on the whole range of  $S$  would request the justification of twice differentiability on all  $S$ . As to the free boundary, its location computing is illustrated in Section 4. From Table 3.1, the free boundary seems to converge in first-order sense without rigorous analysis provided here.

At the Money			
$N$	Binomial Price	MOL Price	Absolute Error
50		$2.36815 \times 10^{-2}$	$2.35270 \times 10^{-4}$
100	$2.39167 \times 10^{-2}$	$2.38493 \times 10^{-2}$	$6.74212 \times 10^{-5}$
200		$2.38999 \times 10^{-2}$	$1.68044 \times 10^{-5}$
400		$2.39125 \times 10^{-2}$	$4.24050 \times 10^{-6}$
Whole Underlying Price Range			
$N$	Maximum Abs. Error	Location	Free Boundary
50	$3.44314 \times 10^{-4}$	$1.400 \times 10^{-1}$	$1.453 \times 10^{-1}$
100	$9.58743 \times 10^{-5}$	$1.500 \times 10^{-1}$	$1.337 \times 10^{-1}$
200	$3.08001 \times 10^{-5}$	$1.350 \times 10^{-1}$	$1.333 \times 10^{-1}$
400	$5.18810 \times 10^{-6}$	$1.825 \times 10^{-1}$	$1.330 \times 10^{-1}$

Table 3.1: Error analysis for American put option. The parameter values are  $r = 10\%$ ,  $\sigma = 40\%$ ,  $T = 1$ ,  $K_P = 1/5$  and price of underlying asset ranges from  $S = 0$  to  $S = 1$ . The maximum time step is set to  $10^{-4}$  in ode23.

### 3.2. American call option with dividend

In Figure 3.2, the optimal exercise boundary  $S_{f,C}(t)$  separates the underlying asset domain to continuation and stopping regions. On the stopping region  $[S_{f,C}(t), \infty) \times [0, T]$ ,



$V(S, t) = S - K_C$ , while on the continuation region  $[0, S_{f,C}(t)] \times [0, T]$ ,  $V(S, t)$  needs to be solved from the following free-boundary problem for American call options:

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S \frac{\partial V}{\partial S} - rV &= 0, & 0 < S < S_{f,C}(t), 0 \leq t < T, \\ V(S, T) &= \max(S - K_C, 0), & S \geq 0, \\ V_{S_{f,C}}(S_{f,C}(t), t) &= S_{f,C}(t) - K_C, & 0 \leq t \leq T, \\ \lim_{S \rightarrow S_{f,C}(t)} \frac{\partial V}{\partial S} &= 1, & 0 \leq t \leq T, \\ S_{f,C}(T) &= \max\left(K_C, \frac{rK_C}{D}\right). \end{aligned}$$

The above opting pricing problem can be solved exactly in the same way as pricing American put option before only with the terminal condition changed to

$$V(S, T) = \max(S - K_C, 0).$$

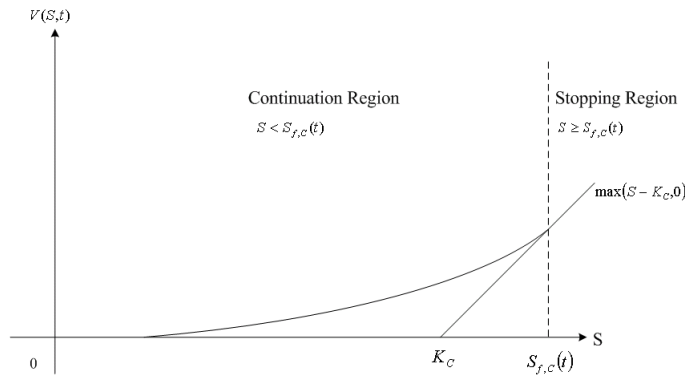


Figure 3.2: Illustration of continuation region, stopping region and optimal exercise boundary for American call option with dividend.

Table 3.2 reports the error analysis of American call option pricing. The specific parameter values are  $r = 9\%$ ,  $\sigma = 40\%$ ,  $D = 10\%$ ,  $T = 1$ ,  $K_C = 1/5$  and stock price ranges again from  $S = 0$  to  $S = 1$ . Same as before, the infinite domain of  $S$  is truncated by five times of the exercise price  $K_C$ . The first part in this table lists the calculated option price at the exercise price. Again, for calculating absolute error, we calculated a binomial tree with exhaustive  $N_b = 10,000$  time steps and used the result as the exact solution. Here, both the local error at the exercise price and MAE show perfect 2<sup>nd</sup> order of convergence, with MAE happening near exercise price instead of optimal exercise boundary as happening above in the case of American put.

At the Money			
$N$	Binomial Price	MOL Price	Absolute Error
50		$2.86049 \times 10^{-2}$	$2.28149 \times 10^{-4}$
100	$2.88331 \times 10^{-2}$	$2.87764 \times 10^{-2}$	$5.68926 \times 10^{-5}$
200		$2.88191 \times 10^{-2}$	$1.39543 \times 10^{-5}$
400		$2.88299 \times 10^{-2}$	$3.13296 \times 10^{-6}$
Whole Underlying Price Range			
$N$	Maximum Abs. Error	Location	Free Boundary
50	$2.28149 \times 10^{-4}$	$2.000 \times 10^{-1}$	$3.525 \times 10^{-1}$
100	$5.81994 \times 10^{-5}$	$1.900 \times 10^{-1}$	$3.531 \times 10^{-1}$
200	$1.51350 \times 10^{-5}$	$1.900 \times 10^{-1}$	$3.530 \times 10^{-1}$
400	$4.32402 \times 10^{-6}$	$1.975 \times 10^{-1}$	$3.530 \times 10^{-1}$

Table 3.2: Error analysis for American call option with dividend. The parameter values were  $r = 9\%$ ,  $D = 10\%$ ,  $\sigma = 40\%$ ,  $T = 1$ ,  $K_C = 1/5$  and price of underlying asset ranges from  $S = 0$  to  $S = 1$ . The maximum time step is set to  $10^{-4}$  in ode23.

### 3.3. American strangle option

To demonstrate the flexibility of our method that can be applied to any kind of pay-off function, here we apply our method to the option pricing of an American strangle position, unusually seen on the market, studied first by Chiarella and Ziogas [8]. American strangle pay-off can be comprehended as a combination of American put and call, and there would be two free boundaries when option evolves backward in time. In Figure 3.3, the optimal exercise boundaries  $S_{f,P}(t)$  and  $S_{f,C}(t)$  separates the underlying asset domain into continuation and stopping regions. On the stopping region  $[0, S_{f,P}(t)) \times [0, T]$ ,  $V(S, t) = K_P - S$ , and  $[S_{f,C}(t), \infty) \times [0, T]$ ,  $V(S, t) = S - K_C$ , while on the continuation region  $[S_{f,P}(t), S_{f,C}(t)] \times [0, T]$ ,  $V(S, t)$  is governed by the following free-boundary problem for American strangle:

$$\begin{aligned}
 \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S \frac{\partial V}{\partial S} - rV &= 0, & S_{f,P}(t) < S < S_{f,C}(t), 0 \leq t < T, \\
 V(S, T) &= \max(K_P - S, 0) + \max(S - K_C, 0), & S \geq 0, \\
 V(S_{f,P}(t), t) &= K_P - S_{f,P}(t), & 0 \leq t \leq T, \\
 V(S_{f,C}(t), t) &= S_{f,C}(t) - K_C, & 0 \leq t \leq T, \\
 \lim_{S \rightarrow S_{f,P}(t)} \frac{\partial V}{\partial S} &= -1, & \lim_{S \rightarrow S_{f,C}(t)} \frac{\partial V}{\partial S} &= 1, & 0 \leq t \leq T,
 \end{aligned}$$

$$S_{f,P}(T) = \min \left( K_P, \frac{rK_P}{D} \right), \quad S_{f,C}(T) = \max \left( K_C, \frac{rK_C}{D} \right).$$

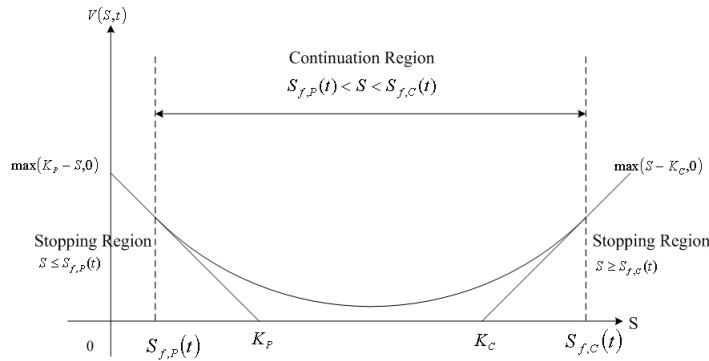


Figure 3.3: Illustration of continuation region, stopping region and dual optimal exercise boundaries for American strangle option from Chiarella and Ziogas [8].

Though the pay-off of American strangle is a combination of put and call positions, its current value would not just be the sum of the associated American put and call. This is chiefly because the dual optimal exercise boundaries,  $S_{f,P}$  and  $S_{f,C}$ , are not independent. Chiarella and Ziogas [8] first derived a set of complicated coupled integral equations for these dual early exercise boundaries. Then they solved the Black-Scholes equation for the option value in the continuation region by Crank-Nicolson finite difference method. Using our simple method, it needs only to change the terminal condition to be the following pay-off function

$$V(S, T) = \max(K_P - S, 0) + \max(S - K_C, 0),$$

and follow the procedure same as above.

Table 3.3 reports our result of American strangle pricing. The parameter values  $r = 5\%$ ,  $\sigma = 20\%$ ,  $D = 10\%$ ,  $T = 1$ ,  $K_P = 1$ ,  $K_C = 1.5$  are adopted from Table 1 of [8]. The infinite domain in  $S$  is truncated by five times of  $K_C$  in the current computation. In this table, we quote the Crank-Nicolson finite difference result from Table 1 of [8], where they employed 1460 time steps and 120,000 space-nodes in their finite difference calculation. We might as well treat this exhaustive solution as exact. The American strangle prices calculated by our simple approach with various number of space nodes and the associated absolute errors are listed in the table. Here we did not limit maximum time step to a very small number as before (just chose ordinary  $10^{-2}$  for the maximum time step) in ode23. The errors show fast convergence to the exact solution. The computational efficiency of our method can be especially noted by that our  $N = 800$  results almost match with every digit of the exact solutions that used  $N = 120,000$  space nodes in [8]. We conclude that

our simple method can be easily and robustly applied to all kinds of pay-off functions and have a satisfactory accuracy with an economic spatial resolution.

Method, $N$	$S$				
	0.75	1.00	1.25	1.50	1.75
CN, 120000	0.275648	0.100319	0.038560	0.092314	0.255619
MOL, 100	0.275677 (0.000029)	0.100094 (0.000225)	0.038632 (0.000072)	0.091799 (0.000515)	0.254837 (0.000782)
MOL, 200	0.275659 (0.000011)	0.100283 (0.000036)	0.038566 (0.000006)	0.092196 (0.000118)	0.255507 (0.000112)
MOL, 400	0.275650 (0.000002)	0.100319 (0.000000)	0.038563 (0.000003)	0.092307 (0.000007)	0.255597 (0.000022)
MOL, 800	0.275648 (0.000000)	0.100329 (0.000010)	0.038563 (0.000003)	0.092336 (0.000022)	0.255632 (0.000013)

Table 3.3: Error analysis for American strangle option. The parameter values are  $r = 5\%$ ,  $\sigma = 20\%$ ,  $D = 10\%$ ,  $T = 1$ ,  $K_P = 1$ ,  $K_C = 1.5$  and price of underlying asset ranges from  $S = 0$  to  $S = 7.5$ . The maximum time step is set to  $10^{-2}$  in ode23. The numbers in the parentheses are the absolute error.

#### 4. Estimating the optimal exercise boundary

Different from other methods, the optimal exercise boundary is not requested at each time step for our method. However, it can be extracted from the numerical result afterwards if wanted. We first re-compute  $(\partial V/\partial t)_i$ ,  $\forall i$ , at each time step by equation (2.2). Then the optimal exercise boundary  $S_{\text{opt}}$  would be the single zero of  $\frac{\partial V}{\partial t}(S, t) = 0$  as mentioned before. Instead of doing time-consuming root finding,  $S_{\text{opt}}$  can be simply interpolated by  $(\partial V/\partial t)_i$ ,  $\forall i$ . The way is to see  $S$  as function of  $\partial V/\partial t$  instead, since  $(\partial V/\partial t)_i$  vs.  $S_i$  is monotonic across  $S_{\text{opt}}$ . We can then interpolate to find  $S_{\text{opt}}$  through  $S_i$  vs.  $(\partial V/\partial t)_i$ . There are other more accurate ways of locating  $S_{\text{opt}}$  like utilizing Delta value for example. Taking American put as an example, we can extrapolate to find  $S_{\text{opt}}$  that approximates  $\frac{\partial V}{\partial S}(S_{\text{opt}}, t) = -1$  through  $S_i$  vs.  $(\partial V/\partial S)_i$  in the continuation region near optimal exercise boundary. Figure 4.1 shows respectively the optimal exercise boundary  $S_{\text{opt}}(t) = S_{f,P}(t)$  of the American put option in Table 3.1,  $S_{\text{opt}}(t) = S_{f,C}(t)$  of the American call option with dividend in Table 3.2, and the dual optimal exercise boundaries  $S_{f,P}(t)$  and  $S_{f,C}(t)$  of the American strangle option in Table 3.3. Here, we also computed American strangle

option with  $r = 10\%$ ,  $\sigma = 20\%$ ,  $D = 5\%$ ,  $T = 1$ ,  $K_P = 1$ ,  $K_C = 1.1$  and compare our dual free boundaries with their counterparts in Figures 2 and 3 of [8] in Figure 4.2. Obviously, our result agrees very well with [8].

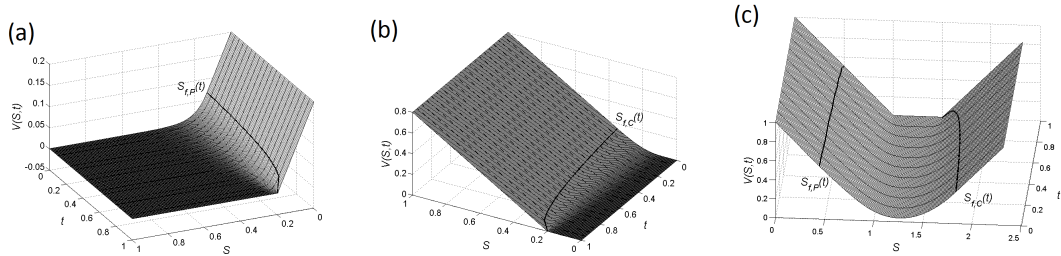


Figure 4.1: Optimal exercise boundaries together with option value are shown for (a) American put option, (b) American call option with dividend, and (c) American strangle option.

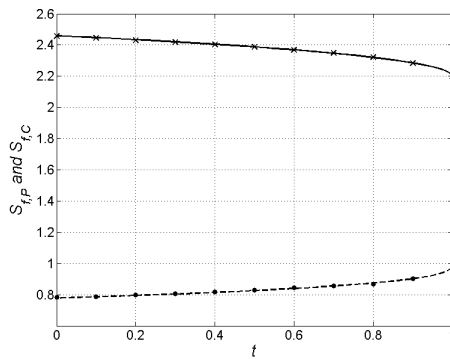


Figure 4.2: Comparison of dual optimal exercise boundaries computed by the current method with their counterparts in Chiarella and Ziogas [8]. Solid and dash lines are  $S_{f,C}(t)$  and  $S_{f,P}(t)$  computed by current method respectively;  $\times$  and  $\bullet$  are their counterparts from Chiarella and Ziogas [8].

### 5. Valuation of multi-asset American options

Our current method can be extended directly to evaluate multi-asset American options. Taking multi-asset American basket put option as an example. The equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \rho_{ij} \sigma_i \sigma_j S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i=1}^n (r - D_i) S_i \frac{\partial V}{\partial S_i} - rV = 0$$

subject to the pay-off function

$$V(S_1, S_2, \dots, S_N, T) = \max \left( K - \sum_{i=1}^n \alpha_i S_i, 0 \right)$$

with  $\sum_{i=1}^n \alpha_i = 1$ . Literatures discussing numerical methods for multi-asset American options are rare. Nielsen et al. [25] used penalty method to compute multi-asset American options. Their method is only first order of accuracy in both time and space (underlying asset price). Fasshauer et al. [12] employed novel meshless method to do the same thing. Though a computational result for two-factor American basket put option was shown, neither article above pointed out the optimal exercise boundary. Besides, these two methods are far too complicated for most financial practitioners to implement. Here we also demonstrate our simple method with a result for two-factor American basket put option as shown in Figure 5.1. In addition to the option value, we also computed its optimal exercise boundary afterwards in terms of the contour  $\partial V/\partial t = 0$  and show that in Figure 5.1(b).

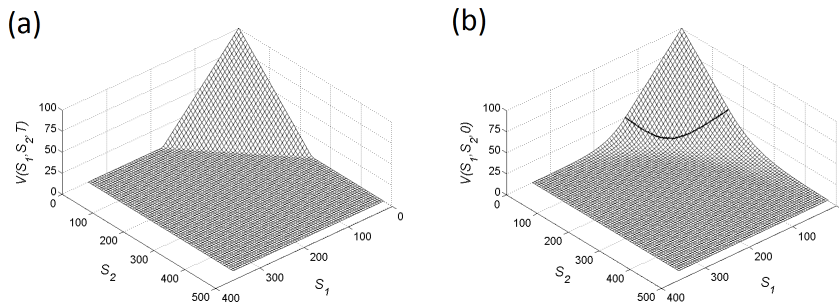


Figure 5.1: Two-factor American basket put option valued at (a)  $t = T$  and (b)  $t = 0$  and the associated optimal exercise boundary in (b). The parameters used here are  $r = 4.5\%$ ,  $\sigma_1 = 25\%$ ,  $\sigma_2 = 35\%$ ,  $D_1 = D_2 = 0$ ,  $\rho_{12} = -0.65$ ,  $\alpha_1 = 0.58$ ,  $\alpha_2 = 0.42$ ,  $T = 1$ ,  $K = 100$ .

## 6. Valuation of callable and putable convertible bond

Here we further demonstrate the advantage of current method by applying it to evaluate a two-factor convertible bond with embedded call and put options that was computed by de Frutos [10]. The pricing of such callable and putable convertible bond is much more difficult than pricing a straight forward two-factor American option in Section 5 due to its complicated callable and putable options. A convertible bond is a contract that the holder will be paid by a known fixed amount, the principal, or the holder can exchange for the issuer's other securities (mostly stock), called conversion, at a given date in the future, the maturity. A callable bond gives the issuer the right to buy back the bond for a fixed price, whereas a putable bond gives holder the right to return the bond to issuer for a fixed price. Usually a convertible bond with American feature allows the conversion (by holder), calling (by issuer) and putting (by holder) to be exercised at any time during the life of bond. Usually the life span of a convertible bond is longer than ordinary stock

option, and therefore the fluctuation of interest rate has to be considered. Following [10], we used Cox-Ingersoll-Ross (CIR) model to describe the dynamics of interest rate. Under this consideration, pricing such a convertible bond is governed by the following two-factor (stock price and interest rate) modified Black-Scholes equation (in non-early-exercised regime),

$$(6.1) \quad \frac{\partial V}{\partial t} + \frac{1}{2}\omega_S^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\omega_S\omega_r S r^{1/2} \frac{\partial^2 V}{\partial S \partial r} + \frac{1}{2}\omega_r^2 r \frac{\partial^2 V}{\partial r^2} + (r - D)S \frac{\partial V}{\partial S} + (a - br) \frac{\partial V}{\partial r} - rV + \kappa = 0.$$

The detailed derivation of equation (6.1) and those callable/puttable early-exercised constraints can be found in [10]. A case of price evaluation in [10] is repeated here by the current method. The result is shown in Figure 6.1, and it agrees well with [10]. However, the early-exercised region was not computed in [10], but it is easily tracked here by current method and shown in Figure 6.1(b). Note that the range of interest rate  $r$  is truncated up to an unrealistic large value  $r = 1$  on purpose to avoid boundary error.

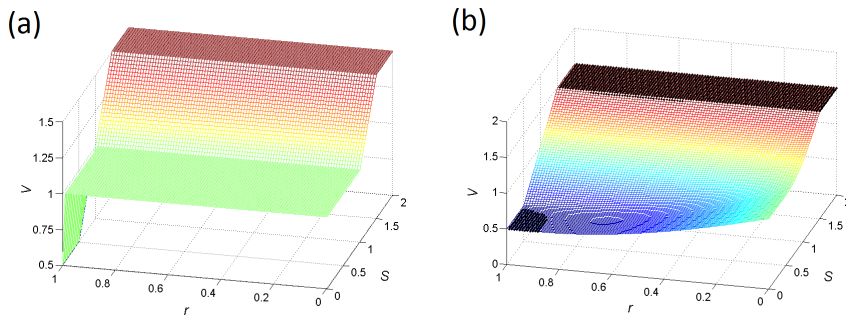


Figure 6.1: Two-factor callable/puttable convertible bond valued at (a)  $t = T = 1$  and (b)  $t = 0$ . The parameters used here are  $\omega_S = 0.35$ ,  $\omega_r = 0.75$ ,  $\rho = -0.3$ ,  $a = 0.04$ ,  $b = 0.08$ ,  $CP = 1.5$ ,  $PP = 0.5$ ,  $D = 0$ ,  $\kappa = 0$ , which are adopted from [10]. The symbol notations used above are also referred to [10]. The dark-colored areas in (b) are early-exercised regions, in which obviously the one happening at large  $S$  is for early call by issuer, and the other happening at large  $r$  is for early put by holder.

### 7. Conclusion

We have introduced an efficient numerical method to evaluate American options in this article. This simple method is much easier to implement compared with those numerical methods requesting the early exercise boundary calculated in advance at each time step. Not requesting the early exercise boundary makes it flexible to suit all kinds of pay-off function. The optimal exercise boundary can be easily extracted afterwards from the

computed option values at each time step if wanted. Various ways like interpolating on the Theta or Delta values can do the purpose efficiently. Most of all, this efficient method can be directly extended to evaluate multi-factor general American option problems such as two-factor American option and convertible bond with embedded call and put options as demonstrated above. Also, tracing their optimal exercise boundaries/regimes afterwards is easy by current method no matter how complicated those early-exercised constraints would be.

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