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Eigenvalue Problem for a System of Singular ODEs with a Perturbed q-Laplace operator

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Abstract. Our purpose is to characterize the eigenvalue interval for a system of boundary value problems with a one-dimensional perturbed q-Laplace operator. We consider both sublinear and superlinear nonlinearities with a possible singularity at zero. The tools applied here are based on variational methods and properties of the Fenchel transform.

1. Introduction

Consider the eigenvalue problem for the following system of ODEs

$$\begin{cases} -\left((a_i(t)|u_i'(t)|^{q_i-2}u_i')' + \frac{ka_i(t)}{t}|u_i'(t)|^{q_i-2}u_i'\right) & \\ \quad = F_i(t,u(t)) + \lambda G_i(t,u(t)) & \text{a.e. in } (0,T), \\ u_i'(0) = 0 \quad \text{and} \quad u_i(T) = 0 & \text{for all } i \in \{1,2,\ldots,n\} \end{cases} \tag{1.1}$$

with $q_i \geq 2$, $k > 1$, $T > 0$, $a_i \in C^1([0,T])$, $u := (u_1,\ldots,u_n)$, where F_i and G_i denote the partial derivatives of F and G respectively, with respect to the ith variable: $F_i := \partial F/\partial u_i$, $G_i := \partial G/\partial u_i$. We discuss an interval of values of the parameter λ where one can establish the existence of positive solutions of (1.1). We also characterize the monotonicity of the solution. Throughout this paper we assume

(A1) $F, G\colon (0,T) \times I \to \mathbb{R}$, where I is a subset of $\mathbb{R}^n$, are Gateaux differentiable with respect to the second variable u for a.a. $t \in (0,1)$ and measurable with respect to t for all $u \in I$, λ is a real number such that $F(t,\cdot) + \lambda G(t,\cdot)$ is convex, $t \mapsto t^k[F(t,\mathbf{0}) + \lambda G(t,\mathbf{0})]$ belongs to $L(0,T)$.

(A2) For each $i \in \{1,2,\ldots,n\}$ and for almost all $t \in (0,T)$ and all $u \in I$,

$$F_i(t,u) + \lambda G_i(t,u) \geq 0$$

and $t \mapsto F_i(t,\mathbf{0}) + \lambda G_i(t,\mathbf{0})$ is not identically zero in a certain subset of $(0,T)$ with positive measure.

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(A3) For each $i \in \{1, 2, \ldots, n\}$, $a_i \in C^1([0, T])$ and

$$a_{\min} := \min_{i \in \{1,\ldots,n\}} \min_{t \in [0,T]} a_i(t) > 0.$$

(A4) There exists a positive $d \in \mathbb{R}$, such that $[0, d]^n := \underbrace{[0, d] \times \cdots \times [0, d]}_{n} \subset I$, and $\varphi_\lambda^i \in L^{q_i}(0, T)$, with $q_i' = q_i/(q_i - 1)$, such that for all $i \in \{1, 2, \ldots, n\}$, the following assertions hold

$$F_i(t, u) + \lambda G_i(t, u) \leq \varphi_\lambda^i(t), \quad t \in (0, T)$$

for all $u \in [0, d]^n$ and

$$\int_0^T \varphi_\lambda^i(r)\, dr \leq \left(\frac{d}{T}\right)^{q_i - 1} a_{\min}.$$

Definition 1.1. By a solution of our problem we mean a function $u \in C^1([0, T])$, $u := (u_1, \ldots, u_n)$, such that $t^k a_i(t)|u_i'|^{q_i-2}u_i' \in A([0, T])$ for all $i \in \{1, 2, \ldots, n\}$, and satisfies (1.1) a.e. in $(0, T)$. Here $A([0, T])$ denotes the space of absolutely continuous functions $v := (v_1, \ldots, v_n)$ such that $v_i'/t^k \in L^{q_i'}(0, T)$.

In the literature some authors have discussed positive solutions for systems of nonlinear ODE's containing the perturbed q-Laplace operator (see [1, 3–10] and the references therein). In this paper we consider nonlinearities F and G which may have a singularity (with respect to the first variable) at zero. Motivated by [8–10] in this paper we present methods based on the calculus of variation. Our main tool is the Fenchel conjugate and the subdifferential of convex functions. Therefore we start with the definition of the functional J and consider (1.1) as the Euler-Lagrange equation of J,

$$J_\lambda(u) = \int_0^T t^k \left(-H_\lambda(t, u(t)) + \sum_{i=1}^n \frac{1}{q_i} a_i(t)|u_i'(t)|^{q_i} \right) dt,$$

where

$$H_\lambda(t, u) := \begin{cases} F(t, u) + \lambda G(t, u) & \text{if } u \in [0, d]^n,\ t \in [0, T], \\ +\infty & \text{if } u \in \mathbb{R}^n \setminus [0, d]^n,\ t \in [0, T]. \end{cases}$$

Our assumptions do not guarantee that J is necessarily bounded in its natural domain. Thus, we consider J on the following set

$$\begin{aligned} U := \{u = (u_1, \ldots, u_n) \in C^1([0, T]) \mid t^k a_i(t)|u_i'|^{q_i-2}u_i' \in A([0, T]), \\ u_i(T) = 0, u_i'(0) = 0 \text{ and } 0 < u_i(t) \leq d \text{ for all } t \in [0, T] \text{ and } i = 1, 2, \ldots, n\} \end{aligned}$$

which is associated with the definition of our solution.

Lemma 1.2. *Under assumptions* (A1)–(A4) *the set* U *has the following property: for each* $u := (u_1, \dots, u_n) \in U$ *there exists* $\widetilde{u} := (\widetilde{u}_1, \dots, \widetilde{u}_n) \in U$ *such that for each* $i \in \{1, 2, \dots, n\}$ *and a.e. in* $(0, T)$

$$-\left(a_i(t)t^k |\widetilde{u}_i'(t)|^{q_i-2}\widetilde{u}_i'(t)\right)' = t^k (F_i(t, u(t)) + \lambda G_i(t, u(t))). \tag{1.2}$$

Proof. Consider an arbitrary $u := (u_1, \dots, u_n) \in U$ and fix $i \in \{1, 2, \dots, n\}$. It is easy to check that $\widetilde{u} := (\widetilde{u}_1, \dots, \widetilde{u}_n)$ with

$$\widetilde{u}_i(t) = \int_t^T \left(\frac{1}{a_i(s)s^k} \int_0^s r^k (F_i(r, u(r)) + \lambda G_i(r, u(r)))\, dr\right)^{1/(q_i-1)} ds$$

satisfies (1.2) a.e. in $(0, T)$. Indeed, it is clear that $\widetilde{u}_i(T) = 0$, $\widetilde{u}_i \in C([0, T]) \cap C^1((0, T])$. Moreover, since

$$\widetilde{u}_i'(t) = -\left(\frac{1}{a_i(t)t^k} \int_0^t r^k (F_i(r, u(r)) + \lambda G_i(r, u(r)))\, dr\right)^{1/(q_i-1)}$$

we have

$$\begin{aligned} &a_i(t)t^k |\widetilde{u}_i'(t)|^{q_i-2}\widetilde{u}_i'(t) \\ &= -a_i(t)t^k \left[\left(\frac{1}{a_i(t)t^k} \int_0^t r^k (F_i(r, u(r)) + \lambda G_i(r, u(r)))\, dr\right)^{1/(q_i-1)}\right]^{q_i-1} \\ &= -\int_0^t r^k (F_i(r, u(r)) + \lambda G_i(r, u(r)))\, dr \end{aligned}$$

which gives (1.2). Now it suffices to prove that $\widetilde{u} \in U$. We start with the observation that, by the above assertion, we have that $t \mapsto t^k a_i(t)|u_i'|^{q_i-2}u_i'(t)$ belongs to $A([0, T])$. In the next step we check the smoothness of $\widetilde{u}$. From Hölder's inequality, we have for all $t \in [0, T]$,

$$\begin{aligned} |\widetilde{u}_i'(t)|^{q_i-1} &= \frac{1}{a_i(t)t^k} \int_0^t r^k (F_i(r, u(r)) + \lambda G_i(r, u(r)))\, dr \\ &\le \frac{1}{a_i(t)t^k} \left(\int_0^t l^{q_i k}\, dl\right)^{1/q_i} \left(\int_0^t (F_i(l, u(l)) + \lambda G_i(t, u(t)))^{q_i'}\, dl\right)^{1/q_i'} \\ &\le \frac{1}{a_i(t)t^k} \left(\frac{1}{q_i k + 1}\right)^{1/q_i} t^{k+1/q_i} \left(\int_0^T (\varphi_\lambda^i(l))^{q_i'}\, dl\right)^{1/q_i'} \\ &\le \frac{1}{a_{\min}} \left(\frac{1}{q_i k + 1}\right)^{1/q_i} \left(\int_0^T (\varphi_\lambda^i(l))^{q_i'}\, dl\right)^{1/q_i'} t^{1/q_i}. \end{aligned}$$

Therefore, passing to the limit, one has $\lim_{t\to 0^+} \widetilde{u}_i'(t) = 0$. Finally, one can see that $\widetilde{u} \in C^1([0, T])$ with $\widetilde{u}_i'(0) = 0$. In the last part we show that $\widetilde{u}_i(t) \le d$ on $[0, T]$. To this end it suffices to note that the definition of $\widetilde{u}$ and assumption (A4) give

$$\begin{aligned}\widetilde{u}_i(t) &= \int_t^T \left(\frac{1}{a_i(s)s^k}\int_0^s r^k(F_i(r,u(r))+\lambda G_i(r,u(r)))\,dr\right)^{1/(q_i-1)} ds\\ &\le \int_0^T \left(\frac{1}{a_{\min}s^k}\int_0^s r^k\varphi_\lambda^i(r)\,dr\right)^{1/(q_i-1)} ds\\ &\le \left(\frac{1}{a_{\min}}\right)^{1/(q_i-1)} T\left(\int_0^T \varphi_\lambda^i(r)\,dr\right)^{1/(q_i-1)} \le d.\end{aligned}$$

To sum up, we see that $\widetilde{u}\in U$ and satisfies (1.2). Thus U has the required property. □

Theorem 1.3. *If assumptions* (A1)–(A4) *hold and* $\{u^m\}_{m\in N}\subset U$ *is a minimizing sequence of the functional* $J\colon U\to\mathbb{R}$ *then there exists a sequence* $\{v^m\}_{m\in N}\subset W^{1,q_1'}(0,T)\times\cdots\times W^{1,q_n'}(0,T)$, *with* $v^m:=(v_1^m,\dots,v_n^m)$, *such that for each* $i\in\{1,\dots,n\}$ *we have*

$$-(v_i^m)'(t)=t^k\left(\frac{\partial}{\partial u_i}H_\lambda(t,u^m(t))\right)\quad \text{a.e. in } (0,1) \tag{1.3}$$

and

$$\lim_{m\to\infty}\int_0^T \frac{1}{q_i'(t^k a_i(t))^{q_i'/q_i}}|v_i^m(t)|^{q_i'}+\frac{1}{q_i}a_i(t)t^k|(u_i^m)'(t)|^{q_i}-(u_i^m)'(t)v_i^m(t)\,dt=0. \tag{1.4}$$

Proof. We start the proof with the boundedness from below of J on U. To show this fact we note that, taking into account the convexity of $F(t,\cdot)+\lambda G(t,\cdot)$, one can derive for each $u\in U$ the following chain of inequalities

$$\begin{aligned}J_\lambda(u) &= \int_0^T t^k\left(-H_\lambda(t,u(t))+\sum_{i=1}^n\frac{1}{q_i}a_i(t)|u_i'(t)|^{q_i}\right)dt\\ &\ge -\int_0^T t^k(H_\lambda(t,u(t)))\,dt\\ &\ge -\int_0^T t^kH_\lambda(t,\mathbf{0})\,dt-\sum_{i=1}^n\int_0^T t^ku_i(t)\frac{\partial}{\partial u_i}H_\lambda(t,u(t))\,dt\\ &\ge -\int_0^T t^kH_\lambda(t,\mathbf{0})\,dt-dT^k\sum_{i=1}^n\int_0^T\varphi_\lambda^i(t)\,dt,\end{aligned} \tag{1.5}$$

which leads to the estimate $-\infty<\min:=\inf_{u\in U}J(u)<+\infty$. Thus, for each $\varepsilon>0$ there exists $m_0\in\mathbb{N}$ such that $J(u^m)<\varepsilon+\min$ for all $m\ge m_0$. Lemma 1.2 guarantees for each m the existence of $\widetilde{u}^m:=(\widetilde{u}_1,\dots,\widetilde{u}_n)\subset U$ such that

$$-(a_i(t)t^k|\widetilde{u}_i'(t)|^{q_i-2}\widetilde{u}_i'(t))'=t^k\left(\frac{\partial}{\partial u_i}H_\lambda(t,u(t))\right)\quad\text{a.e. in } (0,T). \tag{1.6}$$

Now define $\{v^m\}_{m\in\mathbb{N}}\subset W^{1,q_1'}(0,T)\times\cdots\times W^{1,q_n'}(0,T)$ as follows:

$$v_i^m(t):=t^ka_i(t)|(\widetilde{u}_i^m)'(t)|^{q_i-2}(\widetilde{u}_i^m)'(t)\quad\text{for } t\in(0,T). \tag{1.7}$$

For the reader's convenience we recall the Fenchel equality (see, e.g., [2, Proposition **I**.5.1]): let $K\colon W \to \overline{\mathbb{R}}$ be convex and lower semicontinuous, where W is a Banach space. Then for all $w^* \in W^*$ and $w \in W$ we have

$$w^* \in \partial K(w) \quad \Longleftrightarrow \quad K(w) + K^*(w^*) = \langle w^*, w\rangle, \tag{1.8}$$

where K^* is the Fenchel transform of K, $\langle\,\cdot\,,\cdot\,\rangle$ is the dual pair for spaces W and W^* and ∂ denotes the subdifferential in the sense of convex analysis. Precisely,

$$K^*(w^*) := \sup_{w\in W}\{\langle w, w^*\rangle - F(w)\}$$

and

$$\partial K(u) := \{w^* \in W^* \mid \text{ for all } w \in W, K(w) \geq K(u) + \langle w^*, w-u\rangle\}.$$

We apply (1.8) twice in this proof. First consider the functional $L^{q_i}(0,T) \ni w \mapsto K(w) := \int_0^T \frac{1}{q_i} a_i(t)|w(t)|^{q_i}\,dt$, $i \in \{1,\dots,n\}$. It is easy to show $K^*(w^*) = \int_0^T \frac{1}{q_i'(t^k a_i(t))^{q_i'/q_i}}|w^*(t)|^{q_i'}\,dt$. Thus, by (1.7), we get

$$K^*(v_i^m) = \int_0^T (\widetilde{u}_i^m)'(t) v_i^m(t)\,dt - K((\widetilde{u}_i^m)')$$

for each $i \in \{1,\dots,n\}$, which can be rewritten as

$$\begin{aligned}
&\sum_{i=1}^n \int_0^T \frac{1}{q_i'(t^k a_i(t))^{q_i'/q_i}} |v_i^m(t)|^{q_i'}\,dt \\
&= \sum_{i=1}^n \left[\int_0^T (\widetilde{u}_i^m)'(t) v_i^m(t)\,dt - \int_0^T \frac{1}{q_i} a_i(t)|(\widetilde{u}_i^m)'(t)|^{q_i}\,dt\right].
\end{aligned}$$

Finally, one can obtain

$$\begin{aligned}
&\sum_{i=1}^n \int_0^T \frac{1}{q_i'(t^k a_i(t))^{q_i'/q_i}} |v_i^m(t)|^{q_i'}\,dt \\
&\leq \sup_{u\in U} \sum_{i=1}^n \left[\int_0^T u_i'(t) v_i^m(t)\,dt - \int_0^T \frac{1}{q_i} a_i(t)|u_i'(t)|^{q_i}\,dt\right] \\
&\leq \sup_{\substack{z\in L^2(0,T)\\ z=(z_1,\dots,z_n)}} \sum_{i=1}^n \left[\int_0^T z_i(t) v_i^m(t)\,dt - \int_0^T \frac{t^k}{q_i} a_i(t)|z_i'(t)|^{q_i}\,dt\right] \\
&= \sum_{i=1}^n \int_0^T \frac{1}{q_i'(t^k a_i(t))^{q_i'/q_i}} |v_i^m(t)|^{q_i'}\,dt.
\end{aligned}$$

Thus, we have equality in the above chain of inequalities for all $m \in \mathbb{N}$, namely

$$\begin{aligned}
&\sup_{u\in U} \sum_{i=1}^n \left[\int_0^T u_i'(t) v_i^m(t)\,dt - \int_0^T \frac{t^k}{q_i} a_i(t)|u_i'(t)|^{q_i}\,dt\right] \\
&= \sum_{i=1}^n \int_0^T \frac{1}{q_i'(t^k a_i(t))^{q_i'/q_i}} |v_i^m(t)|^{q_i'}\,dt.
\end{aligned} \tag{1.9}$$

From (1.6), we derive for all $m \in \mathbb{N}$,

$$-(v_i^m)'(t) \in t^k \partial_u \{H_\lambda(t, u^m(t))\} = t^k \left\{ \frac{\partial}{\partial u_i} H_\lambda(t, u(t)) \right\} \quad \text{a.e. in } (0,T),$$

where ∂_u denotes the subdifferential of $u \mapsto H_\lambda(t,u)$ for t fixed, which gives (1.3). Let H_λ^* denote the Fenchel conjugate of the function $H_\lambda(t,\cdot)$, namely for given $t \in [0,T]$ and $\lambda > 0$ fixed, $H_\lambda^*(t,v) := \sup_{u \in \mathbb{R}^n} \{\langle u, v\rangle - H_\lambda(t,u)\}$, where $\langle u, v \rangle := \sum_{i=1}^n u_i v_i$. Then applying (1.3) and again the Fenchel equality (1.8), for the functional $L^{q_1}(0,T) \times \cdots \times L^{q_n}(0,T) \ni u \mapsto \int_0^T t^k H_\lambda(t, u(t))\, dt$, we infer that for each $m \geq m_0$,

$$\begin{aligned} \min + \varepsilon > J(u^m) &= \int_0^T t^k \left(-(H_\lambda(t, u^m(t))) + \sum_{i=1}^n \frac{1}{q_i} a_i(t) |(u_i^m)'(t)|^{q_i} \right) dt \\ &= \int_0^T t^k H_\lambda^* \left(t, -\frac{(v^m)'(t)}{t^k} \right) dt + \sum_{i=1}^n \int_0^T u_i^m(t)(v_i^m)'(t)\, dt \qquad (1.10) \\ &\quad + \sum_{i=1}^n \int_0^T \frac{t^k}{q_i} a_i(t) |(u_i^m)'(t)|^{q_i}\, dt, \end{aligned}$$

where for all $(t, v, \lambda) \in (0,T) \times \mathbb{R}^n \times \mathbb{R}$. On the other hand, from the definition of the Fenchel conjugate we see for all $u \in U$,

$$\begin{aligned} \min = \inf_{u \in U} J(u) &\leq \sum_{i=1}^n \int_0^T \frac{t^k}{q_i} a_i(t) |(u_i)'(t)|^{q_i}\, dt - \int_0^T t^k H_\lambda(t, u(t))\, dt \\ &\leq \sum_{i=1}^n \int_0^T \frac{t^k}{q_i} a_i(t) |(u_i)'(t)|^{q_i}\, dt + \int_0^T t^k H_\lambda^* \left(t, -\frac{(v^m)'(t)}{t^k} \right) dt \\ &\quad + \sum_{i=1}^n \int_0^T u_i(t)(v_i^m)'(t)\, dt, \end{aligned}$$

which implies for all $m \in \mathbb{N}$,

(1.11)
$$\begin{aligned} \min &\leq \inf_{u \in U} \left[\sum_{i=1}^n \int_0^T \frac{t^k}{q_i} a_i(t) |(u_i)'(t)|^{q_i}\, dt + \int_0^T t^k H_\lambda^* \left(t, -\frac{(v^m)'(t)}{t^k} \right) dt - \sum_{i=1}^n \int_0^T u_i'(t) v_i^m(t)\, dt \right] \\ &= \int_0^T t^k H_\lambda^* \left(t, -\frac{(v^m)'(t)}{t^k} \right) dt - \sup_{u \in U} \sum_{i=1}^n \left[\int_0^T u_i'(t) v_i^m(t)\, dt - \int_0^T \frac{t^k}{q_i} a_i(t) |(u_i)'(t)|^{q_i}\, dt \right]. \end{aligned}$$

Combining (1.9) and (1.11) one has

$$\min \leq \int_0^T t^k H_\lambda^* \left(t, -\frac{(v^m)'(t)}{t^k} \right) dt - \sum_{i=1}^n \int_0^T \frac{1}{q_i'(t^k a_i(t))^{q_i'/q_i}} |v_i^m(t)|^{q_i'}\, dt \qquad (1.12)$$

for all $m \in \mathbb{N}$. Finally, (1.10) and (1.12) imply

$$\begin{aligned}
0 &\le \sum_{i=1}^{n} \left(\int_0^T \frac{t^k}{q_i} a_i(t) |(u_i^m)'(t)|^{q_i} \, dt - \int_0^T (u_i^m)'(t) v_i^m(t) \, dt + \int_0^T \frac{1}{q_i'(t^k a_i(t))^{q_i'/q_i}} |v_i^m(t)|^{q_i'} \, dt \right) \\
&= \sum_{i=1}^{n} \int_0^T \frac{1}{q_i'(t^k a_i(t))^{q_i'/q_i}} |v_i^m(t)|^{q_i'} \, dt - \int_0^T t^k H_\lambda^* \left(t, -\frac{(v^m)'(t)}{t^k} \right) dt \\
&\quad + \sum_{i=1}^{n} \int_0^T \frac{t^k}{q_i} a_i(t) |(u_i^m)'(t)|^{q_i} \, dt + \sum_{i=1}^{n} \int_0^T u_i^m(t) (v_i^m)'(t) \, dt + \int_0^T t^k H_\lambda^* \left(t, -\frac{(v^m)'(t)}{t^k} \right) dt \\
&\le -\min + \min + \varepsilon = \varepsilon
\end{aligned}$$

for all $m \ge m_0$. Taking into account the fact that $\varepsilon > 0$ was arbitrary, we get

$$\lim_{m \to \infty} \sum_{i=1}^{n} \left(\int_0^T \frac{1}{q_i} a_i(t) |(u_i^m)'(t)|^{q_i} \, dt - \int_0^T (u_i^m)'(t) v_i^m(t) \, dt + \int_0^T \frac{1}{q_i'(t^k a_i(t))^{q_i'/q_i}} |v_i^m(t)|^{q_i'} \, dt \right) = 0.$$

Since each component of the above sum is nonnegative, one has (1.4). $\square$

Theorem 1.4. *Assume that* (A1)–(A4) *are satisfied. Then there exists at least one solution* $\overline{u} \in U$ *of* (1.1) *which is a decreasing function in* $(0,T)$*. Moreover* $\overline{u}$ *is a minimizer of* $J \colon U \to \mathbb{R}$*.*

Proof. Let $\{u^m\}_{m\in\mathbb{N}}$ be a minimizing sequence of $J \colon U \to \mathbb{R}$. It is clear that for $a \in \mathbb{R}$ sufficiently large, we have $J(u^m) \le a$ for all $m \in \mathbb{N}$. We start with the observation that $\{u_i^m\}_{m\in\mathbb{N}}$ is bounded in $L^{q_i}(0,T)$ so up to a subsequence, $\{u_i^m\}_{m\in\mathbb{N}}$ tends weakly to a certain $\overline{u}_i \in L^{q_i}(0,T)$. Moreover (1.5) implies the boundedness of each of the sequences $\{t^{k/q_i}(u_i^m)'\}_{m\in\mathbb{N}}$ and $\{(t^k u_i^m)'\}_{m\in\mathbb{N}}$ in the space $L^{q_i}(0,T)$. Consequently, $\{t^k u_i^m\}_{m\in\mathbb{N}}$ (up to a subsequence) is weakly convergent in $W_0^{1,q_i}(0,T)$ to a certain $\overline{z}_i \in W_0^{1,q_i}(0,T)$. Thus the Rellich-Kondrashov theorem (see, e.g., [4]) guarantees the uniform convergence in $[0,T]$. Therefore we derive that $\overline{z}_i(t) = t^k \overline{u}_i(t)$ and further $\overline{u}_i$ is continuous on $(0,T]$ and $0 \le \overline{u}_i \le d$ on $(0,T]$. We must now prove that $\overline{u}_i \in C^1([0,T])$. Note Theorem 1.3 guarantees the existence of a sequence $\{v_i^m\}_{m\in\mathbb{N}} \subset W^{1,q_i'}(0,T)$ such that

$$-(v_i^m)'(t) = t^k \left(\frac{\partial}{\partial u_i} H_\lambda(t, u^m(t)) \right) \quad \text{for a.a. } t \in (0,T) \tag{1.13}$$

and

$$\lim_{m \to \infty} \int_0^T \frac{1}{q_i'(t^k a_i(t))^{q_i'/q_i}} |v_i^m(t)|^{q_i'} + \frac{1}{q_i} a_i(t) t^k |(u_i^m)'(t)|^{q_i} - (u_i^m)'(t) v_i^m(t) \, dt = 0. \tag{1.14}$$

Taking into account (1.13) one has the boundedness of $\{(v_i^m)'/t^k\}_{m\in\mathbb{N}}$ and $\{(v_i^m)'\}_{m\in\mathbb{N}}$ in the space $L^{q_i'}(0,T)$ and further, going if necessary to a subsequence, one has the weak convergence of $\{(v_i^m)'\}_{m\in\mathbb{N}}$ and $\{(v_i^m)'/t^k\}_{m\in\mathbb{N}}$ in $L^{q_i'}(0,T)$. On the other hand, (1.14) gives the boundedness of $\{(v_i^m)\}_{m\in\mathbb{N}}$ in $L^{q_i'}(0,T)$, which implies that (up to a subsequence) $\{(v_i^m)'\}_{m\in\mathbb{N}}$ in $W^{1,q_i'}(0,T)$ tends weakly to $\overline{v}_i \in W^{1,q_i'}(0,T)$. Therefore we obtain the uniform convergence of $\{(v_i^m)'\}_{m\in\mathbb{N}}$ to $\overline{v}_i$ in $[0,T]$. Now the continuity and nonpositivity of v_i^m for all $m \in \mathbb{N}$, give the continuity and nonpositivity of $\overline{v}_i$.

Now we claim that for a.a. $t \in (0,T)$,

$$\overline{v}_i'(t) = -t^k \left(\frac{\partial}{\partial u_i} H_\lambda(t, \overline{u}(t)) \right) \quad \text{and} \quad \overline{v}_i(t) = t^k a_i(t) |\overline{u}_i'(t)|^{q_i-2} \overline{u}_i'(t), \tag{1.15}$$

where $\overline{u} := (\overline{u}_1, \dots, \overline{u}_n)$. We start with the observation that (1.13), the properties of the Fenchel conjugate and the properties of both sequences $\{u_i^m\}_{m\in\mathbb{N}}$ and $\{(v_i^m)'\}_{m\in\mathbb{N}}$ imply the following chain of inequalities

$$\begin{aligned} 0 &\geq \liminf_{m\to\infty} \int_0^T \langle v^m(t), u^m(t)\rangle + t^k H_\lambda^* \left(t, -\frac{(v^m)'(t)}{t^k} \right) + t^k H_\lambda(t, u^m(t))\, dt \\ &\geq \int_0^T \langle \overline{v}'(t), \overline{u}(t)\rangle + t^k H_\lambda^* \left(t, -\frac{\overline{v}'(t)}{t^k} \right) + t^k H_\lambda(t, \overline{u}(t))\, dt \geq 0. \end{aligned}$$

We derive the first assertion of (1.15). To obtain the other one we note that (1.14) gives

$$\begin{aligned} 0 &\geq \liminf_{m\to\infty} \int_0^T \frac{1}{q_i'(t^k a_i(t))^{q_i'/q_i}} |v_i^m(t)|^{q_i'} + \frac{1}{q_i} a_i(t) t^k |(u_i^m)'(t)|^{q_i} - (u_i^m)'(t) v_i^m(t)\, dt \\ &\geq \int_0^T \frac{1}{q_i'(t^k a_i(t))^{q_i'/q_i}} |\overline{v}_i(t)|^{q_i'} + \frac{1}{q_i} a_i(t) t^k |(\overline{u}_i)'(t)|^{q_i} - (\overline{u}_i)'(t) \overline{v}_i(t)\, dt \geq 0. \end{aligned}$$

Applying again the properties of the Fenchel transform, we obtain

$$\overline{v}_i(t) = t^k a_i(t) |\overline{u}_i'(t)|^{q_i-2} \overline{u}_i'(t) \quad \text{a.e. in } (0,T)$$

what is our claim. To sum up, (1.15) and the definition of H_λ allow us to see that $\overline{u}$ is a solution of our problem, namely for each $i \in \{1, \dots, n\}$,

$$(t^k a_i(t) |\overline{u}_i'(t)|^{q_i-2} \overline{u}_i'(t))' = -t^k (F_i(t, \overline{u}(t)) + \lambda G_i(t, \overline{u}(t))) \quad \text{for a.a. } t \in (0,T).$$

Now the reasoning in the proof of Lemma 1.2 leads to the conclusion that $\overline{u} \in C^1([0,T])$ with $\overline{u}'(0) = 0$, $\overline{u}(T) = 0$ and the last assertion of (1.15) guarantees $t^k a_i(t) |\overline{u}_i'(t)|^{q_i-2} \overline{u}_i'(t) \in L^{q_i'}(0,T)$. All these facts imply that $\overline{u} \in U$.

Now we investigate the monotonicity of each coordinate of $\overline{u}$. Fix $i \in \{1, 2, \dots, n\}$. We show that $u_i'(t) < 0$ for all $t \in (0,T)$. Let $h_i(t) = t^k a_i(t) |u_i'(t)|^{q_i-2} u_i'(t)$ for all $t \in [0,T]$. Therefore $h_i'(t) < 0$ for all $t \in (0,T)$ and further h_i is decreasing. Taking into account the boundary condition we have $h_i(t) < h(0) = 0$ for $t \in (0,T)$, and further, from the positivity of $a_i(t)$ in $[0,T]$, we obtain $u'(t) < 0$ for $t \in (0,T)$.

To prove the last assertion of our theorem it suffices to note that the uniform convergence of $\{u_i^m\}_{m\in\mathbb{N}}$ to $\overline{u}_i$ and the weak convergence of $\{t^{k/q_i}(u_i^m)'\}_{m\in\mathbb{N}}$ in $L^{q_i}(0,T)$ to $t^{k/q_i}\overline{u}_i'$, give the following assertion

$$\begin{aligned}\inf_{u\in U} J(u) &= \liminf_{m\to\infty} \int_0^T t^k(-H_\lambda(t,u^m(t))) + \sum_{i=1}^n \frac{1}{q_i} a_i(t)t^k|(u_i^m)'(t)|^{q_i}\,dt \\ &\geq \int_0^T t^k(-H_\lambda(t,\overline{u}(t))) + \sum_{i=1}^n \frac{1}{q_i} a_i(t)t^k|(\overline{u}_i)'(t)|^{q_i}\,dt = J(\overline{u}). \qquad \square\end{aligned}$$

Example 1.5. Consider the following boundary value problem

$$\begin{aligned} &-\left((a_1(t)|u_1'(t)|^4u_1'(t))' + \frac{k}{t}a_1(t)|u_1'(t)|^4u_1'(t)\right) \\ &\qquad = \frac{1}{\sqrt{t}}(u_1^3(t)+u_1^7(t)+u_1(t)) + \lambda(u_2(t)+1), \\ &-\left((a_2(t)|u_2'(t)|^2u_2'(t))' + \frac{k}{t}a_2(t)|u'(t)|^2u'(t)\right) \\ &\qquad = \frac{1}{\sqrt[3]{t}}\frac{1}{3-u_2(t)} + \lambda u_1(t) \quad \text{a.e. in } (0,2), \\ &\qquad\qquad u_i'(0) = 0 \quad \text{and} \quad u_i(2) = 0, \end{aligned} \tag{1.16}$$

where a_1, a_2 satisfy (A3) and such that $a_{\min} \geq 230$ (e.g., for $i = 1,2$, $a_i(t) = b_i/(1+t^{m_i})$, where $m_i \geq 1$ and $b_i \geq 230$). We find an interval of eigenvalues λ such that (1.16) possesses at least one positive solution in the set

$$\begin{aligned} U := \{&u = (u_1,u_2) \in C^1([0,T]) \mid u_i(2) = 0 \text{ and } u_i'(0) = 0 \text{ and } u_i(t) \leq 1 \\ &\text{for } t\in[0,2] \text{ and } a_i|u_i'|^2u_i' \in A([0,T]), i = 1,2\}. \end{aligned}$$

Moreover for each $i = 1,2$, u_i is decreasing in $(0,2)$.

We consider (1.1) with $T = 2$, $q_1 = 6$, $q_2 = 4$, $d = 1$ and

$$\begin{aligned} F(t,(u_1,u_2)) &= \frac{1}{\sqrt{t}}\left(\frac{1}{4}u_1^4 + \frac{1}{8}u_1^8 + \frac{1}{2}u_1^2\right) - \frac{1}{\sqrt[3]{t}}\ln(3-u_2), \\ G(t,(u_1,u_2)) &= (u_2u_1 + u_1). \end{aligned}$$

We show that all the assumptions of Theorem 1.4 are satisfied in this case. It is easy to show that for all $\lambda \in [-1/3, 1/3]$, $H_\lambda(t,\cdot\,)$ is convex in $[0,1]\times[0,1]$. Moreover,

$$\begin{aligned} \frac{\partial}{\partial u_1}H_\lambda(t,(u_1,u_2)) &= \frac{1}{\sqrt{t}}(u_1^3+u_1^7+u_1) + \lambda(u_2+1), \\ \frac{\partial}{\partial u_2}H_\lambda(t,(u_1,u_2)) &= \frac{1}{\sqrt[3]{t}}\frac{1}{3-u_2} + \lambda u_1. \end{aligned}$$

Therefore both derivatives are positive in $(0,2)\times[0,1]^2$ and $\frac{\partial}{\partial u_i}H(t,(0,0))\neq 0$. In the next step we show that (A4) is satisfied. For $\varphi^1_\lambda(t)=2/\sqrt{t}+2\lambda$ and $\varphi^2_\lambda(t)=1/(2\sqrt[3]{t})+\lambda$, $t\in(0,2)$, we have

$$\int_0^2 \frac{\partial}{\partial u_1}H_\lambda(t,(u_1(t),u_2(t)))\,dt \le \int_0^2 \varphi^1_\lambda(t)\,dt = 6\sqrt{2}+4\lambda$$

and

$$\int_0^2 \frac{\partial}{\partial u_2}H_\lambda(t,(u_1(t),u_2(t)))\,dt \le \int_0^T \varphi^2_\lambda(t)\,dt = \frac{3}{4}2^{2/3}+2\lambda.$$

In order to guarantee that $\int_0^T \varphi^i_\lambda(r)\,dr \le (d/T)^{q_i-1}a_{\min}$, for $i=1,2$, we have to find $\lambda\in[-1/3,1/3]$ such that

$$6\sqrt{2}+4\lambda \le \left(\frac{1}{2}\right)^5 a_{\min} \quad \text{and} \quad \frac{3}{4}2^{2/3}+2\lambda \le \left(\frac{1}{2}\right)^3 a_{\min}.$$

Taking into account that inequality $a_{\min}\ge 230$ one sees that

$$\frac{1}{16}a_{\min}-\frac{3}{8}2^{2/3} > \frac{1}{128}a_{\min}-\frac{3}{2}\sqrt{2} \ge \frac{230}{128}-\frac{3}{2}\sqrt{2} > -\frac{1}{3}.$$

Finally, λ has to satisfy two conditions: $\lambda\in[-1/3,1/3]$ and $\lambda\le\frac{1}{128}a_{\min}-\frac{3}{2}\sqrt{2}$. So for $\lambda\in I_{\text{eigen}}:=\left[-\frac{1}{3},\frac{230}{128}-\frac{3}{2}\sqrt{2}\right]$, Theorem 1.4 guarantees the existence of positive solutions with the above properties.

Remark 1.6. We wish to emphasize that our approach can be applied for both sub and superquadratic cases. Thus we can consider for example F and G defined as $c(t)(e^{u_1}+e^{u_2})$ or $c(t)\sqrt{5+u_1^2+u_2^2}$ for an appropriate constant d and a function c sufficiently smooth and small (in the sense of the maximum norm). We do not need to control the growth of the nonlinearity $F+\lambda G$. Rather we have to control only its value in $(0,2)\times[0,d]^2$.

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