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A Fourth Order Singular Elliptic Problem Involving p-biharmonic Operator

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Abstract. In this paper, a fourth order singular elliptic problem involving p-biharmonic operator with Dirichlet boundary condition is considered. The existence of at least one weak solution is proved in two different cases of the nonlinear term at the origin. The results are obtained by applying the critical points principle of Ricceri, variational methods and Rellich's inequality. Also an example is presented to verify the results.

1. Introduction

Nonlinear singular elliptic problems have been studied intensively in recent years and arise in some parts of science such that boundary layer phenomena for viscous fluids, chemical heterogeneous catalysts, chemical catalyst kinetics, and theory of heat conduction in electrically conducting materials.

These kinds of problems also appear in glacial advance, in transport of coal slurries down conveyor belts and in some other geophysical and industrial contents, see [8].

This paper studies the existence of at least one weak solution for the following singular problem.

$$\begin{cases} \Delta_p^2 u = \mu \frac{|u|^{p-2}u}{|x|^{2p}} + \lambda f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p^2 u := \Delta(|\Delta u|^{p-2}\Delta u)$ denotes the p-biharmonic operator, $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain containing the origin and with smooth boundary $\partial\Omega$, $1 < p < N/2$ and $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function on which we put some conditions in order to obtain our results.

Singular elliptic problems involving p-Laplace and p-biharmonic operators have been studied by many authors, for instance Ghoussoub et al. in [5] study the following singular problem

$$\begin{cases} -\Delta_p u = \mu \frac{|u|^{q-2}u}{|x|^{s}} + \lambda |u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

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where $q \leq (p-s)/((N-p)p) := p^*(s)$, $0 \leq s \leq p < N$ and $p \leq r \leq p^*(0)$, moreover, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator.

In [12] the following problem is studied

$$\begin{cases} M\left(\int_\Omega |\Delta u|^p \,\mathrm{d}x\right) \Delta_p^2 u - \frac{a}{|x|^{2p}}|u|^{p-2}u = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 < p < N/2$ and $M\colon [0,+\infty[\to \mathbb{R}$ is a continuous function. As a special case, by putting $M \equiv 1$ and $\mu = 0$ the problem in [12] converts to our problem. But, we study the existence of at least one weak solution under different conditions and in two cases of the nonlinear term at the origin which varies our results from those in [12]. Also, Makvand et al. [7] study the following p-biharmonic singular problem

$$\begin{cases} \Delta_p^2 u + a(x)\frac{|u|^{q-2}}{|x|^{2q}} = \lambda f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a \in L^\infty(\Omega)$ with $\operatorname{ess}_\Omega \inf a(x) > 0$, $1 < q < N/2 < p$ and $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function. We can refer to [3,4,6,9,11,13] as some references on singular elliptic problems.

2. Preliminaries

Here, we recall some theorems and definitions as follows.

Let $X := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\| = \left(\int_\Omega |\Delta u(x)|^p \,\mathrm{d}x\right)^{1/p},$$

where $1 < p < N/2$ and $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain containing the origin and with smooth boundary $\partial\Omega$. Also let $\|\cdot\|_q$ denote the norm of $L^q(\Omega)$, i.e.,

$$\|u\|_q := \left(\int_\Omega |u(x)|^q \,\mathrm{d}x\right)^{1/q}.$$

We remind Rellich's inequality [2], which is

$$\int_\Omega \frac{|u(x)|^p}{|x|^{2p}} \,\mathrm{d}x \leq \frac{1}{H}\int_\Omega |\Delta u(x)|^p \,\mathrm{d}x \quad \text{for all } u \in X,$$

where $H := \big(N(p-1)(N-2p)/p^2\big)^p$. Let

$$F(x,t) := \int_0^t f(x,\xi)\,\mathrm{d}\xi \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}.$$

Set $p^* := pN/(N-p)$. Therefore, by Sobolev embedding, there exists a positive constant c, such that

$$\|u\|_{p^*} \le c\|u\| \quad \text{for all } u \in X,$$

where

$$c := \pi^{-1/2} N^{-1/p} \left(\frac{p-1}{N-p}\right)^{1-1/p} \left[\frac{\Gamma(1+N/2)\Gamma(N)}{\Gamma(N/p)\Gamma(N+1-N/p)}\right]^{1/N}$$

and

$$\Gamma(t) := \int_0^{+\infty} z^{t-1} e^{-z}\,\mathrm{d}z \quad \text{for all } t > 0$$

denotes the Gamma function.

For fixed $q \in [1, p^*)$, the compact embedding $X \hookrightarrow L^q(\Omega)$ says that there exists $c_q > 0$ such that

$$\|u\|_q \le c_q\|u\| \quad \text{for all } u \in X. \tag{2.1}$$

Now we consider the energy functional $I_{\mu,\lambda}\colon X \to \mathbb{R}$ as

$$I_{\mu,\lambda}(u) := \Phi_\mu(u) - \lambda\Psi(u) \quad \text{for all } u \in X, \tag{2.2}$$

where

$$\Phi_\mu(u) := \frac{1}{p}\int_\Omega |\Delta u(x)|^p\,\mathrm{d}x - \frac{\mu}{p}\int_\Omega \frac{|u(x)|^p}{|x|^{2p}}\,\mathrm{d}x$$

and

$$\Psi(u) := \int_\Omega F(x, u(x))\,\mathrm{d}x.$$

By Rellich's inequality, one has

$$\left(\frac{H-\mu}{pH}\right)\|u\|^p \le \Phi_\mu(u) \le \frac{\|u\|^p}{p} \quad \text{for all } u \in X, \tag{2.3}$$

which implies that Φ_μ is well defined and coercive. Also Φ_μ is Gâteaux differentiable and weakly lower semicontinuous functional in X whose derivative is the functional $\Phi' \in X^*$ give by [10]

$$\Phi'_\mu(u)(v) = \int_\Omega |\Delta u(x)|^{p-2}\Delta u(x)\Delta v(x)\,\mathrm{d}x - \mu\int_\Omega \frac{|u(x)|^{p-2}u(x)v(x)}{|x|^{2p}}\,\mathrm{d}x$$

for every $v \in X$.

Moreover Ψ is well defined and continuously Gâteaux differentiable functional (by standard arguments) whose Gateâux derivative is a compact operator Ψ' from X to X given by

$$\Psi'(u)(v) = \int_\Omega f(x, u(x))v(x)\,\mathrm{d}x$$

for every $v \in X$.

Definition 2.1. For fixed real parameters μ and λ, a function $u\colon \Omega \to \mathbb{R}$ is said to be a weak solution of (1.1) if $u \in X$ and

$$\int_\Omega |\Delta u(x)|^{p-2}\Delta u(x)\Delta v(x)\,\mathrm{d}x - \mu\int_\Omega \frac{|u(x)|^{p-2}}{|x|^{2p}}u(x)v(x)\,\mathrm{d}x - \lambda\int_\Omega f(x,u(x))v(x)\,\mathrm{d}x = 0$$

for every $v \in X$.

Therefore, the critical points of $I_{\mu,\lambda}$ are exactly the weak solutions of (1.1).

We recall the following theorem [1] (a version of the Ricceri's principle [10]) to study the weak solution of problem (1.1).

Theorem 2.2. *Let X be a reflexive real Banach space, and let $\Phi, \Psi\colon X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is strongly continuous, sequentially weakly lower semicontinuous and coercive. Further, assume that Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Psi$, put*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\left(\sup_{v\in\Phi^{-1}(]-\infty,r[)}\Psi(v)\right) - \Psi(u)}{r - \Phi(u)},$$

then, for every $r > \inf_X \Phi$ and each $\lambda \in (0, 1/\varphi(r))$, the restriction of $I_\lambda := \Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty,r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.

3. A weak solution

Here, we prove the existence of at least one weak solution for the problem (1.1). The main tool used here is the Ricceri's theorem verified by G. Bonanno.

Theorem 3.1. *Let $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying*

$$|f(x,t)| \le a_1 + a_2|t|^{q-1} \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}, \tag{3.1}$$

where $q \in]1, p^*[$ and a_1, a_2 are two non-negative constants, also $f(x,0) \neq 0$ in Ω. Then for every $\mu \in [0, H[$ there exists a positive parameter

$$\widetilde{\lambda}_\mu := q\sup_{\gamma>0}\left(\frac{\gamma^{p-1}}{qa_1c_1\left(\frac{pH}{H-\mu}\right)^{1/p} + a_2c_q^q\left(\frac{pH}{H-\mu}\right)^{q/p}\gamma^{q-1}}\right)$$

such that for every $\lambda \in]0, \widetilde{\lambda}_\mu[$, the problem (1.1) admits at least one non-trivial weak solution $\widetilde{u}_\lambda \in X$ with $\lim_{\lambda\to 0^+}\|\widetilde{u}_\lambda\| = 0$ and the function $g(\lambda) := I_{\mu,\lambda}(\widetilde{u}_\lambda)$ ($I_{\mu,\lambda}$ as in (2.2)) is strictly decreasing and negative in $]0, \widetilde{\lambda}_\mu[$.

Proof. In order to apply Theorem 2.2, fix $\mu \in [0, H[$ and $\lambda \in]0, \widetilde{\lambda}_\mu[$, also let $X := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $\Phi := \Phi_\mu$ and Ψ be the functional as in Section 2. We prove the theorem in the following steps.

Step 1. As we know $\Phi\colon X \to \mathbb{R}$ is continuously Gâteaux differentiable, sequentially weakly lower semicontinuous and coercive functional as $\mu \in [0, H[$, also $\inf_{u \in X} \Phi(u) = 0$. Moreover $\Psi\colon X \to \mathbb{R}$ is continuously Gâteaux differentiable.

By (3.1), one has

$$F(x,t) \le a_1|t| + a_2\frac{|t|^q}{q} \tag{3.2}$$

for all $(x,t) \in \Omega \times \mathbb{R}$ and since $\lambda \in]0, \widetilde{\lambda}_\mu[$, there exists $\widetilde{\gamma} > 0$ such that

$$\lambda < \widetilde{\lambda}_\mu(\widetilde{\gamma}) := \frac{q\widetilde{\gamma}^{p-1}}{qa_1c_1\left(\frac{pH}{H-\mu}\right)^{1/p} + a_2c_q^q\left(\frac{pH}{H-\mu}\right)^{q/p}\widetilde{\gamma}^{q-1}}. \tag{3.3}$$

From (3.2), one has

$$\Psi(u) = \int_\Omega F(x, u(x))\,\mathrm{d}x \le a_1\|u\|_1 + \frac{a_2}{q}\|u\|_q^q,$$

therefore, due to (2.3),

$$\|u\| < \left(\frac{rpH}{H-\mu}\right)^{1/p} \tag{3.4}$$

for $u \in X$ and $\Phi(u) < r$.

Now from (2.1) and (3.4) for every $u \in X$ with $\Phi(u) < r$ we get

$$\Psi(u) < c_1a_1\left(\frac{pH}{H-\mu}\right)^{1/p} r^{1/p} + a_2\frac{c_q^q}{q}\left(\frac{pH}{H-\mu}\right)^{q/p} r^{q/p},$$

then

$$\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) \le c_1a_1\left(\frac{pH}{H-\mu}\right)^{1/p} r^{1/p} + a_2\frac{c_q^q}{q}\left(\frac{pH}{H-\mu}\right)^{q/p} r^{q/p}.$$

Hence

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} \le c_1a_1\left(\frac{pH}{H-\mu}\right)^{1/p} r^{1/p-1} + a_2\frac{c_q^q}{q}\left(\frac{pH}{H-\mu}\right)^{q/p} r^{q/p-1}$$

for every $r \in (0, \infty)$. In particular, for $r := \widetilde{\gamma}^p$ we get

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,\widetilde{\gamma}^p[)} \Psi(u)}{\widetilde{\gamma}^p} \le c_1a_1\left(\frac{pH}{H-\mu}\right)^{1/p} \widetilde{\gamma}^{1-p} + a_2\frac{c_q^q}{q}\left(\frac{pH}{H-\mu}\right)^{q/p} \widetilde{\gamma}^{q-p}. \tag{3.5}$$

Therefore, since $0 \in \Phi^{-1}(]-\infty, \widetilde{\gamma}^p[)$ and $\Phi(0) = \Psi(0) = 0$,

$$\varphi(\widetilde{\gamma}^p) := \inf_{u\in\Phi^{-1}(]-\infty,\widetilde{\gamma}^p[)} \frac{\left(\sup_{v\in\Phi^{-1}(]-\infty,\widetilde{\gamma}^p[)} \Psi(u)\right) - \Psi(u)}{\widetilde{\gamma}^p - \Phi(u)} \leq \frac{\sup_{v\in\Phi^{-1}(]-\infty,\widetilde{\gamma}^p[)} \Psi(v)}{\widetilde{\gamma}^p}.$$

Now, from (3.3), (3.5) and above inequality, one has

$$\begin{aligned}\varphi(\widetilde{\gamma}^p) \leq \frac{\sup_{v\in\Phi^{-1}(]-\infty,\widetilde{\gamma}^p[)} \Psi(v)}{\widetilde{\gamma}^p} &\leq c_1 a_1 \left(\frac{pH}{H-\mu}\right)^{1/p} \widetilde{\gamma}^{1-p} + a_2 \frac{c_q^q}{q} \left(\frac{pH}{H-\mu}\right)^{q/p} \widetilde{\gamma}^{q-p} \\ &< \frac{1}{\lambda},\end{aligned}$$

hence

$$\lambda \in \left]0, \frac{q\widetilde{\gamma}^{p-1}}{q_1 a_1 c_1 \left(\frac{pH}{H-\mu}\right)^{1/p} + a_2 c_q^q \left(\frac{pH}{H-\mu}\right)^{q/p} \widetilde{\gamma}^{q-1}}\right[\subseteq \left]0, \frac{1}{\varphi(\widetilde{\gamma}^p)}\right[.$$

In conclusion, based on Theorem 2.2, there exists a critical point $\widetilde{u}_\lambda \in \Phi^{-1}(]-\infty, \widetilde{\gamma}^p[)$ for $I_{\mu,\lambda}$ which is a global minimum of the $I_{\mu,\lambda}$ in $\Phi^{-1}(]-\infty, \widetilde{\gamma}^p[)$, moreover, $\widetilde{u}_\lambda \neq 0$ as $f(x, 0) \neq 0$ in Ω.

Step 2. Now, we show that $\lim_{\lambda\to 0^+} \|\widetilde{u}_\lambda\| = 0$ and the function $g(\lambda) := I_{\mu,\lambda}(\widetilde{u}_\lambda)$ is negative and strictly decreasing in $]0, \widetilde{\lambda}_\mu(\widetilde{\gamma})[$.

Since Φ is coercive, there exists a positive constant M such that

$$\|\widetilde{u}_\lambda\| \leq M$$

as $\widetilde{u}_\lambda \in \Phi^{-1}(]-\infty, \widetilde{\gamma}^p[)$, for every $\lambda \in]0, \widetilde{\lambda}_\mu(\widetilde{\gamma})[$.

Hence due to the compactness of Ψ', there exists a positive number L such that

$$|\Psi'(\widetilde{u}_\lambda)(\widetilde{u}_\lambda)| \leq \|\Psi'(\widetilde{u}_\lambda)\|\|\widetilde{u}_\lambda\| < LM^2 \tag{3.6}$$

for every $\lambda \in]0, \widetilde{\lambda}_\mu(\widetilde{\gamma})[$.

On the other hand as $\widetilde{u}_\lambda$ is a critical point of $I_{\mu,\lambda}$, we get

$$I'_{\mu,\lambda}(\widetilde{u}_\lambda)(\widetilde{u}_\lambda) = 0,$$

which implies that

$$\Phi(\widetilde{u}_\lambda) - \lambda \int_\Omega f(x, \widetilde{u}_\lambda(x))\widetilde{u}_\lambda(x)\, dx = 0 \quad \text{for all } \lambda \in]0, \widetilde{\lambda}_\mu(\widetilde{\gamma})[. \tag{3.7}$$

Therefore, by (3.6) and (3.7), one has

$$\lim_{\lambda\to 0^+} \Phi(\widetilde{u}_\lambda) = 0.$$

Further, by (2.3),

$$\|\widetilde{u}_\lambda\|^p \leq \frac{pH}{H-\mu}\Phi(\widetilde{u}_\lambda) \quad \text{for all } \lambda \in]0, \widetilde{\lambda}_\mu(\widetilde{\gamma})[\,,$$

then we can conclude that

$$\lim_{\lambda \to 0^+} \|\widetilde{u}_\lambda\| = 0.$$

Also, since restriction of $I_{\mu,\lambda}$ to $\Phi^{-1}(]-\infty, \widetilde{\gamma}^p[)$ admits a global minimum that is a local minimum of $I_{\mu,\lambda}$ in X, the map $g(\lambda) := I_{\mu,\lambda}(\widetilde{u}_\lambda)$ is negative in $]0, \widetilde{\lambda}_\mu(\widetilde{\gamma})[$, because $\widetilde{u}_\lambda \neq 0$ and $I_{\mu,\lambda}(0) = 0$.

Now assume that $\widetilde{u}_{\lambda_1}$ and $\widetilde{u}_{\lambda_2}$ are critical points of $I_{\mu,\lambda}$ (local minimum) for λ_1 and λ_2 respectively in $]0, \widetilde{\lambda}_\mu(\widetilde{\gamma})[$ with $\lambda_1 < \lambda_2$. Set

$$E_i := \inf_{u \in \Phi^{-1}(]-\infty, \widetilde{\gamma}^p[)} \left(\frac{\Phi(u)}{\lambda_i} - \Psi(u) \right) = \frac{1}{\lambda_i} I_{\mu,\lambda_i}(\widetilde{u}_{\lambda_i}), \quad i = 1, 2.$$

Obviously $E_i < 0$ for $i = 1, 2$, and since $\lambda_1 < \lambda_2$, we get $E_2 \leq E_1$, therefore,

$$I_{\mu,\lambda_2}(\widetilde{u}_{\lambda_2}) = \lambda_2 E_2 \leq \lambda_2 E_1 < \lambda_1 E_1 = I_{\mu,\lambda_1}(\widetilde{u}_{\lambda_1}).$$

Hence the proof is complete as $\widetilde{\lambda} \in]0, \widetilde{\lambda}_\mu[$ is arbitrary. □

Remark 3.2. Recalling $\widetilde{\lambda}_\mu$,

$$\widetilde{\lambda}_\mu := q \sup_{\gamma > 0} \left(\frac{\gamma^{p-1}}{q a_1 c_1 \left(\frac{pH}{H-\mu}\right)^{1/p} + a_2 c_q^q \left(\frac{pH}{H-\mu}\right)^{q/p} \gamma^{q-1}} \right)$$

and by direct computations, we get

$$\widetilde{\lambda}_\mu = \begin{cases} +\infty & \text{if } 1 < q < p, \\ \frac{H-\mu}{a_2 c_p^p H} & \text{if } q = p, \\ \dfrac{q \widetilde{\gamma}_{\max}^{p-1}}{q a_1 c_1 \left(\frac{pH}{H-\mu}\right)^{1/p} + a_2 c_q^q \left(\frac{pH}{H-\mu}\right)^{q/p} \widetilde{\gamma}_{\max}^{q-1}} & \text{if } q \in]p, p^*[\,, \end{cases}$$

in which

$$\widetilde{\gamma}_{\max} := \left(\frac{H-\mu}{pH}\right)^{1/p} \left(q \frac{a_1 c_1}{a_2 c_q^q} \left(\frac{p-1}{q-p} \right) \right)^{1/(q-1)}.$$

The following theorem shows the existence of at least one non-trivial weak solution for the problem (1.1) in the case that the function f vanishes at the origin.

Theorem 3.3. *Let $f \colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function with $f(x, 0) = 0$ satisfying*

$$|f(x,t)| \leq a_1 + a_2 |t|^{q-1} \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}, \tag{3.8}$$

where $q \in]p, p^*[$ and a_1, a_2 are non-negative constants.

Also, assume that there exists a non-empty open set $D \subseteq \Omega$ and a set $B \subseteq D$ with $\operatorname{meas}(B) > 0$ *such that*

$$\limsup_{t\to 0^+} \frac{\inf_{x\in B} F(x,t)}{t^p} = +\infty \quad \textit{and} \quad \liminf_{t\to 0^+} \frac{\inf_{x\in D} F(x,t)}{t^p} > -\infty, \tag{3.9}$$

then for arbitrary fixed $\mu \in [0, H[$ there exists a number $\widetilde{\lambda}_\mu > 0$ as follows:

$$\widetilde{\lambda}_\mu := \frac{q\widetilde{\gamma}_{\max}^{p-1}}{qa_1c_1\left(\frac{pH}{H-\mu}\right)^{1/p} + a_2c_q^q\left(\frac{pH}{H-\mu}\right)^{q/p}\widetilde{\gamma}_{\max}^{q-1}},$$

where

$$\widetilde{\gamma}_{\max} := \left(\frac{H-\mu}{pH}\right)^{1/p}\left(q\frac{a_1c_1}{a_2c_q^q}\left(\frac{p-1}{q-p}\right)\right)^{1/(q-1)},$$

then for every $\lambda \in]0, \widetilde{\lambda}_\mu[$, there exists at least one non-trivial weak solution $\widetilde{u}_\lambda \in X := W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ for the problem (1.1).

Proof. Our aim is to show that $\widetilde{u}_\lambda \neq 0$. To this end it is sufficient to show that

$$\limsup_{\|u\|\to 0^+} \frac{\Psi(u)}{\Phi(u)} = +\infty \tag{3.10}$$

for Ψ and $\Phi := \Phi_\mu$ introduced in Theorem 3.1.

In fact from (3.10), there exists a sequence $\{w_k\} \subset X$ with $\lim_{k\to+\infty} \|w_k\| = 0$, such that $w_k \in \Phi^{-1}(]-\infty, \widetilde{\gamma}^p[)$, and

$$\Phi(w_k) - \lambda\Psi(w_k) < 0$$

for k large enough. Finally, since $\widetilde{u}_\lambda$ is a global minimum for restriction of $I_{\mu,\lambda}$ to $\Phi^{-1}(]-\infty, \widetilde{\gamma}^p[)$, one has

$$\Phi(\widetilde{u}_\lambda) - \lambda\Psi(\widetilde{u}_\lambda) < 0 = \Phi(0) - \lambda\Psi(0).$$

In order to prove (3.10), by (3.9), there exist a sequence of positive numbers $\{t_k\}$ with $\lim_{k\to+\infty} t_k = 0$ and two constants $\delta > 0$ and Γ such that

$$\lim_{k\to+\infty} \frac{\inf_{x\in B} \int_0^{t_k} f(x,\xi)\,\mathrm{d}\xi}{|t_k|^p} = +\infty \tag{3.11}$$

and we get

$$\inf_{x\in D} \int_0^t f(x,\xi)\,\mathrm{d}\xi \le \Gamma|t|^p$$

for every $t \in [0, \delta]$.

Now, by Urysohn's theorem, there exist a set $C \subseteq B$ with positive measure and a function $v \in X$ such that $v(x) \in [0,1]$, $v(x) = 1$ for every $x \in C$ and $v(x) = 0$ for every $x \in \Omega \setminus D$.

Assume that $Q > 0$, then there exists a constant $T > 0$ such that

$$Q < \frac{T(\operatorname{meas}(C)) + \Gamma \int_{D\setminus C} |v(x)|^p \,\mathrm{d}x|}{\Phi(v)}.$$

From (3.11), there exists $N_0 \in N$ such that $t_k < \delta$ and

$$\inf_{x\in B} \int_0^{t_k} f(x,\xi)\,\mathrm{d}\xi \geq T|t_k|^p$$

for all $k > N_0$. Therefore, by above inequalities, we get

$$\begin{aligned}\frac{\Psi(t_k v)}{\Phi(t_k v)} &= \frac{\int_C \left(\int_0^{t_k} f(x,\xi)\,\mathrm{d}\xi\right)\,\mathrm{d}x + \int_{D\setminus C}\left(\int_0^{t_k v} f(x,\xi)\,\mathrm{d}\xi\right)\,\mathrm{d}x}{\Phi(t_k^p v)} \\ &\geq \frac{(T\operatorname{meas}(C)) + \Gamma\int_{D\setminus C}|v(x)|^p\,\mathrm{d}x}{\Phi(v)} > Q.\end{aligned}$$

Hence (3.10) is proved as Q is arbitrary. □

By an idea in [3], here is an example of our result for a particular function f which vanishes at zero.

Example 3.4. Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a bounded domain and $1 < p < N/2$. Consider the following singular problem

$$\begin{cases} \Delta_p^2 u = \mu \frac{|u|^{p-2}u}{|x|^{2p}} + \lambda(\alpha(x)|u|^{r-2}u + \beta(x)|u|^{s-2}u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.12}$$

where $\alpha, \beta\colon \Omega \to \mathbb{R}$ are two continuous positive and bounded functions and $1 < r < p < s < p^*$. Then for every $\mu \in [0, H[$ and $\lambda \in \Lambda$, the problem (3.12) has at least one non-trivial weak solution $\widetilde{u}_\lambda \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that

$$\lim_{\lambda\to 0^+} \|\widetilde{u}_\lambda\| = 0,$$

where

$$\Lambda := \left]0, \frac{s\widetilde{\gamma}_{\max}^{p-1}}{2M_0\left\{sc_1\left(\frac{pH}{H-\mu}\right)^{1/p} + c_s^s\left(\frac{pH}{H-\mu}\right)^{s/p}\widetilde{\gamma}_{\max}^{s-1}\right\}}\right[, \quad M_0 := \max\{\|\alpha\|_\infty, \|\beta\|_\infty\}$$

and

$$\widetilde{\gamma}_{\max} := \left(\frac{H-\mu}{pH}\right)^{1/p}\left(s\frac{c_1}{c_s^s}\left(\frac{p-1}{s-p}\right)\right)^{1/(s-1)}.$$

For $f(x,t) := \alpha(x)|t|^{r-2}t + \beta(x)|t|^{s-2}t$, we see that $f(x,0) = 0$ and

$$|f(x,t)| \leq 2\max\{\|\alpha\|_\infty, \|\beta\|_\infty\}(1 + |t|^{s-1}) \quad \text{for all } (x,t) \in \Omega\times\mathbb{R}.$$

Therefore condition (3.8) is satisfied. Moreover, by standard arguments, we get

$$\lim_{\xi \to 0^+} \frac{\int_\Omega F(x,\xi)\,\mathrm{d}x}{\xi^p} = +\infty.$$

Hence, thanks to Theorem 3.3, the result is achieved.

4. Conclusion

In this paper we study a singular p-biharmonic problem with Dirichlet boundary condition. The existence of at least one weak solution is proved in two cases. First we get the results where $f(x,0) \neq 0$ and then, adding different conditions and using the Urysohn's theorem we prove the existence of at least one weak solution in the case that $f(x,0) = 0$. Our main tools are Ricceri's theorem which is verified by G. Bonanno, Rellich's inequality and variational method.

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