

## Distributions of Branch Points of Some Algebroid Functions

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**Abstract.** In this paper, we investigate the distribution of branch points of some algebroid functions with infinitely many branch points. A sufficient condition that algebroid function has infinitely many branch points is given. By the Ahlfors' theory of covering surfaces, we also obtain the relationship between the Borel cluster directions of branch points and the Borel directions.

### 1. Introduction

The value distribution of meromorphic functions due to R. Nevanlinna (see [16] for standard references) was extended to the corresponding theory of algebroid functions by H. Selberg [8], E. Ullrich [13], and G. Valiron [14] in the 1930s. In the theory of algebroid functions, one main difference is processing branch points, compared with the meromorphic functions. The branch points theorem established by G. Valiron [14] is one of the important theorems in this field. It shows that the valence function of branch points for algebroid function is less than its characteristic function. Many results on the existence of singular directions for algebroid functions are based on the branch points theorem, see [4, 6, 12, 15]. Up to now, lots of important theorems for meromorphic functions have not been extended to algebroid functions for their multivaluedness and the complexity of their branch points. For some algebroid functions, their branch points are easy to find. One of the examples is the square root  $\sqrt{z}$ . And there exist some algebroid functions with infinitely many branch points, see Example 2.1 and Example 2.3. But usually, one does not know the number of branch points and their distributions by their definitions. The purpose of this paper is to study the distribution of branch points of those algebroid functions with infinitely many branch points. We give a sufficient condition that an algebroid function has infinitely many branch points. The relationship between the Borel cluster direction of branch points and the Borel direction is obtained.

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Received March 28, 2018; Accepted September 9, 2018.

Communicated by Sai-Kee Yeung.

2010 *Mathematics Subject Classification.* 30D30.

*Key words and phrases.* general algebroid functions, branch points, Borel direction.

The research is supported by National Science Foundation of China Grant No. #11501127 and Grant No. #11771153, Natural Science Foundation of Guangdong Province Grant No. #2016A030313686, Guangdong Universities (Basic and Applied Research) Major Project Grant No. #2017KZDXM038 and Guangzhou City Social Science Federation “Yangcheng Young Scholars” Grant No. #18QNXR35.

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Suppose that  $A_0(z), A_1(z), \dots, A_k(z)$  are analytic functions in a simply connected region  $D$  without any common zeros and  $A_k(z) \not\equiv 0$ , then the bivariate complex function

$$(1.1) \quad \Psi(z, W) = A_k(z)W^k + A_{k-1}(z)W^{k-1} + \dots + A_0(z) = 0$$

defines a  $k$ -valued algebroid function  $W(z)$  in the region  $D$  (see [3, 10]). If  $k = 1$ , we get a meromorphic function by (1.1). If the above equation is irreducible,  $W(z)$  is said to be an irreducible algebroid function. Most results on algebroid functions (see [3, 7, 10]), for example the first and second fundamental theorem, are based on the hypothesis that they are irreducible. However, it is difficult to judge whether an algebraic equation is irreducible. Even an irreducible algebroid function in a region  $D$  may become reducible in a smaller region. For convenience, we represent (1.1) as

$$(1.2) \quad \Psi(z, W) = \Psi_1(z, W) \cdot \Psi_2(z, W) \cdots \Psi_t(z, W) = 0, \quad t \geq 1,$$

where  $\Psi_j(z, W)$  ( $j = 1, \dots, t$ ) are irreducible polynomials in the region  $D$ . If  $t = 1$ , we get an irreducible algebroid function in  $D$ . The algebroid function defined by (1.2) is said to be a general algebroid function [9].

From now on we will confine our consideration to the general algebroid functions. For the general algebroid function  $W(z)$  defined by (1.2), the resultant of each irreducible factor  $\Psi_j(z, W)$  ( $j = 1, 2, \dots, t$ ) and its partial derivative  $\Psi_{jW}(z, W)$  verifies  $R(\Psi_j, \Psi_{jW})(z) \not\equiv 0$ . Obviously, points in the set  $Q_W = Q_W(D) := \bigcup_{j=1}^t \{z \in D \mid R(\Psi_j, \Psi_{jW})(z) = 0\}$  are isolated. We say that the points in the sets  $Q_W(D)$ ,  $P_W(D) := \{z \mid A_k(z) = 0\}$ ,  $S_W(D) := P_W(D) \cup Q_W(D)$  and  $T_W(D) := D - S_W(D)$  are multiple points, poles, critical points and regular points, respectively. For a regular point  $z_0 \in T_W(D)$ , there are  $k$  regular function elements of  $W(z)$ , which are  $\{(w_j(z), B_r(z_0))\}_{j=1}^k$ . For  $a \in S_W(D)$ , there is a critical function element  $(q(z), B_r(a))$ , where  $q(z)$  can be expanded to Puiseux series

$$q(z) = \sum_{n=u}^{\infty} B_n(z-a)^{n/s}, \quad B_u \neq 0$$

in  $B_r(a) := \{|z-a| < r\} \subset D$ . If  $u < 0$ , the pair  $(q(z), B_r(a))$  is called a pole element of  $W(z)$  (in particular when  $s = 1$ , it is a meromorphic function element). When  $s > 1$ , the pair  $(q(z), B_r(a))$  is an algebraic function element and  $a$  is a branch point of order  $s - 1$ . When  $s = 1$  and  $u \geq 0$ , the pair  $(q(z), B_r(a))$  is a regular function element. Therefore, the domain of an irreducible algebroid function composed by these function elements is a connected Riemann surface, see [1]. And different sheets are connected by the algebraic function elements. The domain of a general algebroid function can be decomposed into finitely many connected Riemann surfaces. The first fundamental theorem, the second fundamental theorem and the branch points theorem are also true for the case of general algebroid functions, see [10].

For an angular domain  $E = \Delta(\theta, \varepsilon) =: \{z \mid \theta - \varepsilon < \arg z < \theta + \varepsilon\}$ , we denote by  $n(r, E, W)$  and  $\bar{n}(r, E, W)$  the number of, respectively, poles repeated according to their multiplicities and the distinct poles in  $\{|z| < r\} \cap E$ . Similarly,  $n(r, E, W = a)$  is the number of roots of the equation  $W(z) = a$  in  $\{|z| < r\} \cap E$ . We shall breviate notation  $n(r, W)$  for  $n(r, C, W)$ . Let

$$\begin{aligned}
 m(r, W) &= \frac{1}{2k\pi} \int_0^{2\pi} \ln^+ |W(re^{i\theta})| \, d\theta, \\
 N(r, W) &= \frac{1}{k} \left[ \int_0^r \frac{n(t, W) - n(0, W)}{t} \, dt + n(0, W) \ln r \right], \\
 N_\chi(r, W) &= \frac{1}{k} \left[ \int_0^r \frac{n_\chi(t, W) - n_\chi(0, W)}{t} \, dt + n_\chi(0, W) \ln r \right], \\
 S(r, W) &= \frac{1}{\pi} \sum_{n=1}^k \iint_{|z| \leq r} \left( \frac{|w'_j(re^{i\theta})|}{1 + |w_j(re^{i\theta})|^2} \right)^2 r \, dr d\theta, \\
 T_0(r, W) &= \frac{1}{k} \int_0^r \frac{S(t, W)}{t} \, dt \quad \text{and} \quad T(r, W) = N(r, W) + m(r, W),
 \end{aligned}$$

where  $n_\chi(r, W)$  is the number of branch points in  $|z| \leq r$ , counting with their orders. Moreover,  $S(r, W)$  is a conformal invariant and is called the mean covering number of  $|z| \leq r$  into  $W$ -sphere. We call  $T_0(r, W)$  the Ahlfors-Shimzu characteristic function of  $W(z)$ . It is known from [3, 9] that

$$T_0(r, W) = N(r, W) + m(r, W) + O(1).$$

**Definition 1.1.** (1) Let  $n(r)$  be a non-negative continuous function on interval  $(0, 1)$ , the order of  $n(r)$  is defined as

$$p(n(r)) := \limsup_{r \rightarrow 1^-} \frac{\ln n(r)}{-\ln(1 - r)}.$$

(2) Let  $n(r)$  be a non-negative continuous function on interval  $(0, \infty)$ , the order of  $n(r)$  is defined as

$$p(n(r)) := \limsup_{r \rightarrow \infty} \frac{\ln n(r)}{\ln r}.$$

*Remark 1.2.* (1) Let  $W(z)$  be a general algebroid function in the complex plane or the unit disc. The order of  $T(r, W)$  is called the order of function  $W(z)$ , denoted by  $\rho(W) =: p(T(r, W)) = \rho$ . Especially, if  $\rho \in (0, +\infty)$ ,  $W(z)$  is said to be of finite order. If  $\rho = \infty$ ,  $W(z)$  is said to be of infinite order.

(2) For a  $k$ -valued general algebroid function in the complex plane of order  $\rho \in (0, +\infty]$ , if there exists a ray  $L(\theta_0) = \{z \mid \arg z = \theta_0\}$  such that for any  $\varepsilon > 0$  and  $a \in \mathbb{C} \cup \{\infty\}$  with at most  $2k$  exceptional values, the order of  $\bar{n}(r, \Delta(\theta_0, \varepsilon), W(z) = a)$  is  $\rho$ , then  $L(\theta_0)$

is said to be the Borel direction of  $W(z)$ . Correspondingly, the exceptional value is said to be the Borel exceptional value.

(3) For a  $k$ -valued general algebroid function in the whole plane of order  $\rho \in (0, +\infty]$ , if there exists a half line  $L(\theta_0) = \{z \mid \arg z = \theta_0\}$  such that for any  $\varepsilon > 0$ , the order of  $n_x(r, \Delta(\theta_0, \varepsilon), W(z))$  is  $\rho$ , then  $L(\theta_0)$  is said to be the Borel cluster direction of branch points.

In this paper, we shall obtain the following results:

**Theorem 1.3.** *Suppose that  $W(z)$  is a  $k$ -valued ( $k > 1$ ) algebroid function of order  $\rho \in (0, +\infty]$  defined by (1.2) in the complex plane. If there are 3 distinct Borel exceptional values of  $W(z)$ , that is,*

$$(1.3) \quad \limsup_{r \rightarrow +\infty} \frac{\ln \bar{n}(r, W = a_j)}{\ln r} < \rho, \quad j = 1, 2, 3,$$

*then there exists at least one Borel cluster direction of branch points.*

*Remark 1.4.* This theorem gives a sufficient condition that an algebroid function has infinitely many branch points. Examples 2.1 and 2.2 show that Theorem 1.3 is sharp.

**Theorem 1.5.** *Suppose that  $W(z)$  is a  $k$ -valued algebroid function of order  $\rho \in (0, +\infty]$  defined by (1.2) in the complex plane. If the half line  $L(\theta_0)$  is a Borel direction of  $W(z)$ , then there are only 2 possibilities:*

- (1)  $L(\theta_0)$  is a Borel direction with 2 exceptional values at most,
- (2)  $L(\theta_0)$  is a Borel cluster direction of branch points.

**Theorem 1.6.** *Suppose  $W(z)$  is a  $k$ -valued algebroid function of order  $\rho \in (1/2, +\infty)$  defined by (1.2) in the complex plane. For  $\eta \in (\pi/(2\rho), \pi)$ , if there is some  $\eta' \in (0, \eta)$ , such that*

$$\limsup_{r \rightarrow \infty} \frac{\ln n_\chi(r, \Delta(\theta_0, \eta'), W)}{\ln r} = \rho,$$

*then for any  $a \in \mathbb{C} \cup \{\infty\}$  with  $2k$  exceptional values at most,*

$$(1.4) \quad \limsup_{r \rightarrow +\infty} \frac{\ln \bar{n}(r, \Delta(\theta_0, \eta), W = a)}{\ln r} = \rho.$$

*Remark 1.7.* (1) Example 2.3 shows the angular domain  $\Delta(\theta_0, \eta)$  is precise, where  $\eta \in (\pi/(2\rho), \pi)$ .

(2) The authors do not know whether there is a Borel direction of  $W(z)$  in the domain  $\Delta(\theta_0, \eta)$ .

**Theorem 1.8.** *Suppose  $W(z)$  is a  $k$ -valued algebroid function of infinite order defined by (1.2) in the complex plane. Then the Borel cluster direction of branch points is also a Borel direction of  $W(z)$ .*

## 2. Examples

**Example 2.1.** Consider the following equation

$$\Psi(z, W_1) = (e^z - 1)W_1^2 - 1 = 0.$$

Then  $W_1(z)$  satisfies the hypothesis of Theorem 1.3.

*Proof.* Under the given assumption, we have

$$W_1(z) = \frac{1}{\sqrt{e^z - 1}}.$$

And  $W_1(z)$  is a 2-valued algebroid function in the complex plane. By [11], the order of  $W_1(z)$  is

$$\rho(W_1) = \rho(e^z - 1) = 1.$$

For  $e^z \neq 0, \infty$ , we can obtain that  $0, \pm i$  are 3 Borel exceptional values of  $W_1(z)$ . By [4], the function  $e^z - 1$  has at least one Borel direction  $L(\theta_0)$  of order 1. This implies that for all  $\varepsilon > 0$  and  $a \in \mathbb{C} \cup \{\infty\}$  (with 2 exceptional values at most), we have

$$\limsup_{r \rightarrow +\infty} \frac{\ln \bar{n}(r, \Delta(\theta_0, \varepsilon), e^z - 1 = a)}{\ln r} = 1.$$

Since  $e^z - 1 \neq -1, \infty$  and

$$n_\chi(r, \Delta(\theta_0, \varepsilon), W_1(z)) = \bar{n}(r, \Delta(\theta_0, \varepsilon), e^z - 1 = 0),$$

we have the order of  $n_\chi(r, \Delta(\theta_0, \varepsilon), W)$  is  $p(n_\chi(r, \Delta(\theta_0, \varepsilon), W)) = 1$ . Therefore,  $L(\theta_0)$  is a Borel cluster direction of  $W_1(z)$ . □

**Example 2.2.** Consider the equation

$$\Psi(z, W_2) = e^z W_2^2 - 1 = 0,$$

which defines a general algebroid function  $W_2(z)$  of order 1. It has 2 Borel exceptional values, that is,  $0, \infty$ . And  $W_2(z)$  has no branch points in the complex plane. Note that  $W_2(z)$  is a reducible algebroid function.

**Example 2.3.** Let  $W_3(z) = \sqrt{A(z)}$  be a 2-valued algebroid function defined by the equation

$$(2.1) \quad \Psi(z, W_3) = W_3^2 - A(z) = 0,$$

where

$$A(z) = \frac{\prod_{n=2}^{\infty} \left(1 - \frac{z}{n \ln^2 n}\right)}{\prod_{n=2}^{\infty} \left(1 + \frac{z}{n \ln^2 n}\right)} = \prod_{n=2}^{\infty} \frac{n \ln^2 - z}{n \ln^2 + z}.$$

Then  $\rho(W_3) = 1$  and  $L(0)$  is the Borel cluster direction of branch points of order 1. Further, for all  $\eta \in (\pi/2, \pi)$  and  $a \in \mathbb{C} \cup \{\infty\}$  (with at most 4 exceptional values), we deduce that the equality (1.4) holds, that is,

$$\limsup_{r \rightarrow +\infty} \frac{\ln \bar{n}(r, \Delta(0, \eta), W_3(z) = a)}{\ln r} = 1.$$

But for any  $\eta \in (0, \pi/2]$ , there are infinitely many complex numbers such that (1.4) does not hold in the angular domain  $\Delta(0, \eta)$ .

*Proof.* We first estimate  $\rho(W_3)$ . It follows from an example in [2, p. 29] that the order of holomorphic function

$$\Pi(z) = \prod_{n=2}^{\infty} \left( 1 + \frac{z}{n \ln^2 n} \right)$$

is 1. According to the Jensen formula, we have

$$T(r, \Pi(z)) = T\left(r, \frac{1}{\Pi(z)}\right) + O(1).$$

Further,

$$\rho\left(\frac{1}{\Pi(z)}\right) = 1.$$

Noticing that  $\rho(\Pi(z)) = \rho(\Pi(-z)) = 1$ , we obtain that  $\rho(A(z)) \leq 1$ .

On the other hand, let  $\Pi(-z) = 0$ , we have  $z_n = n \ln^2 n$  and  $\arg z_n = 0$ . Further, the multiplicity of  $z_n$  is 1. Choosing  $r_n = n \ln^2 n + 1/2$ , then for all  $\varepsilon > 0$ , we have

$$\bar{n}(r_n, \Delta(0, \varepsilon), A(z) = 0) = n.$$

Hence, we have

$$\begin{aligned} 1 \geq \rho(A(z)) &\geq \limsup_{r \rightarrow +\infty} \frac{\ln \bar{n}(r, \Delta(0, \varepsilon), A(z) = 0)}{\ln r} \\ &\geq \lim_{r \rightarrow +\infty} \frac{\ln \bar{n}(r_n, \Delta(0, \varepsilon), A(z) = 0)}{\ln r_n} = 1. \end{aligned}$$

Therefore,  $\rho(A(z)) = \rho(W_3) = 1$ .

Let us turn to prove the second part. For all  $\varepsilon > 0$ , according to (2.1), given that  $W_3(z) = \sqrt{A(z)}$ , we have

$$\limsup_{r \rightarrow +\infty} \frac{\ln n_\chi(r, \Delta(0, \varepsilon), W_3)}{\ln r} = \limsup_{r \rightarrow +\infty} \frac{\ln \bar{n}(r, \Delta(0, \varepsilon), A(z) = 0)}{\ln r} = 1.$$

This asserts that  $L(0)$  is a Borel cluster direction of branch points.

By  $\rho(A(z)) = 1$  and a theorem in [4], we can obtain there is at least one Borel direction of  $A(z)$  on the whole plane, denoted by  $L(\theta_0)$ . For  $\text{Re}(z) > 0$ , we have  $|A(z)| < 1$ .

Similarly,  $|A(z)| > 1$  when  $\operatorname{Re}(z) < 0$ . Then there are no Borel directions in the domain  $\Delta(0, \pi/2)$  nor in  $\Delta(\pi, \pi/2)$ . Hence  $\theta_0 = \pi/2$  or  $\theta_0 = -\pi/2$ . This means that for any  $\varepsilon > 0$  and  $b \in \mathbb{C} \cup \{\infty\}$  (with 2 exceptional values at most), we have

$$\limsup_{r \rightarrow +\infty} \frac{\ln \bar{n}(r, \Delta(\theta_0, \varepsilon), A(z) = b)}{\ln r} = 1.$$

Hence for all  $\eta \in (\pi/2, \pi)$ ,  $a = b^2 \in \mathbb{C} \cup \{\infty\}$  (with 4 exceptional at most), choosing  $\varepsilon = \eta/2 - \pi/4 > 0$ , we have

$$\Delta(0, \eta) \supset \Delta(\theta_0, \varepsilon).$$

Therefore,

$$\begin{aligned} 1 &\geq p(2\bar{n}(r, \Delta(0, \eta), W_3 = b)) \\ &= p(\bar{n}(r, \Delta(0, \eta), W_3 = b) + \bar{n}(r, \Delta(0, \eta), W_3 = -b)) \\ &\geq p(\bar{n}(r, \Delta(\theta_0, \varepsilon), A = b^2)) \geq 1. \end{aligned}$$

But in the angular domain  $\Delta(0, \pi/2)$ , it is easy to see that  $|W(z)| < 1$ . This implies that for any  $\eta \in (0, \pi/2]$  and any complex number  $a$  verifying  $|a| \geq 1$ , we have  $p(\bar{n}(r, \Delta(0, \eta), W = a)) = 0 < 1$ . □

### 3. Some lemmas

The next two lemmas are proved by G. Valiron [14], when  $W(z)$  is an irreducible algebraic function.

**Lemma 3.1** (The branch points theorem). *Let  $W(z)$  be a  $k$ -valued algebraic function in  $\{|z| < R\}$ , then for  $r \in (0, R)$ ,*

$$N_\chi(r, W) \leq 2(k - 1)T(r, W).$$

**Lemma 3.2** (The second fundamental theorem). *Suppose  $W(z)$  is an irreducible  $k$ -valued algebraic function in the unit disc  $B := \{|z| < 1\}$  defined by (1.1). Let  $a_j$  ( $j = 1, 2, \dots, q$ ) be  $q$  distinct finite or infinite complex numbers. Then for  $r \in (0, 1)$ , we have*

$$(q - 2)T(r, W) < \sum_{j=1}^q \bar{N}(r, W = a_j) + N_x(r, W) + E(r, W).$$

Here the error term  $E(r, W)$  satisfies

$$E(r, W) = O \left\{ \ln T(r, W) + \ln \frac{1}{1 - r} \right\},$$

when  $r \rightarrow 1^-$ , outside a set  $E_0$  of  $r$  such that  $\int_{E_0} dr/(1 - r) \leq 2$ . In particular the above equation holds for some  $r$  in the interval  $(\tau, \tau')$  provided that  $\tau \in (0, 1)$ ,  $\tau' \in (1 - (1 - \tau)/e^2, 1)$ .

*Remark 3.3.* It has been proved that two lemmas above are also true for general algbroid functions [10].

**Lemma 3.4.** [5] *Suppose that  $W(z)$  is an irreducible  $k$ -valued algebroid function in the complex plane. Let  $a_1, a_2, \dots, a_q$  ( $q \geq 3$ ) be  $q$  distinct complex numbers on the sphere with radius  $\delta \in (0, 1/2)$ . For  $\varepsilon > \varepsilon' > 0$  and  $r > r' > 2$ , we have*

$$(q - 2)S(r', \Delta(\theta_0, \varepsilon'), W) \leq n_\chi(r, \Delta(\theta_0, \varepsilon), W) + \sum_{j=1}^q \bar{n}(r, \Delta(\theta_0, \varepsilon), W = a_j) + \frac{2^{56}k\pi^{24} \ln r}{\delta^{38}(\varepsilon - \varepsilon')(\ln r - \ln r')} + (q - 2)S(1/r', \Delta(\theta_0, \varepsilon'), W).$$

*Remark 3.5.* By [10], we can also prove that Lemma 3.4 is true for general algebroid function.

Let  $\theta_0 \in [0, 2\pi)$ ,  $\eta \in (0, \pi]$ . The function

$$h(z) = \frac{(ze^{-i\theta_0})^{\pi/(2\eta)} - 1}{(ze^{-i\theta_0})^{\pi/(2\eta)} + 1}$$

maps the angular domain  $\Delta(\theta_0, \eta)$  onto the unit disc  $|h| < 1$ . Let

$$z(h) = e^{i\theta_0} \left( \frac{1+h}{1-h} \right)^{2\eta/\pi}$$

be its inverse mapping, which maps the unit disc  $|h| < 1$  onto the angular domain  $\Delta(\theta_0, \eta)$ .

**Lemma 3.6.** *For the mapping  $h(z)$ , we have*

(a) *for all  $\eta' \in (0, \eta)$  and  $r > 1$ ,*

$$\left\{ h(pe^{i\theta}) \mid p \in [1, r], \theta \in (\theta_0 - \eta', \theta_0 + \eta') \right\} \subset \left\{ |h| < \sqrt{1 - r^{-\pi/(2\eta)} \cos \frac{\eta'\pi}{2\eta}} \right\};$$

(b)  $\left\{ |h| < \sqrt{1 - 4r^{-\pi/(2\eta)}} \right\} \subset \left\{ h(pe^{i\theta}) \mid p \in (0, r), \theta \in (\theta_0 - \eta, \theta_0 + \eta) \right\}$ ,  $r > 1$ .

*Proof.* Without loss of generality, we assume  $\theta_0 = 0$ . Since

$$|h(pe^{i\theta})| = \left| \frac{p^{\pi/(2\eta)} \cos \frac{\theta\pi}{2\eta} - 1 + ip^{\pi/(2\eta)} \sin \frac{\theta\pi}{2\eta}}{p^{\pi/(2\eta)} \cos \frac{\theta\pi}{2\eta} + 1 + ip^{\pi/(2\eta)} \sin \frac{\theta\pi}{2\eta}} \right| = \sqrt{\frac{p^{\pi/\eta} + 1 - 2p^{\pi/(2\eta)} \cos \frac{\theta\pi}{2\eta}}{p^{\pi/\eta} + 1 + 2p^{\pi/(2\eta)} \cos \frac{\theta\pi}{2\eta}}}$$

then

$$|h(pe^{i\theta})|^2 = \frac{2p^{\pi/\eta} + 2}{p^{\pi/\eta} + 1 + 2p^{\pi/(2\eta)} \cos \frac{\theta\pi}{2\eta}} - 1 = 1 - \frac{4p^{\pi/(2\eta)} \cos \frac{\theta\pi}{2\eta}}{p^{\pi/(2\eta)} + 1 + 2p^{\pi/(2\eta)} \cos \frac{\theta\pi}{2\eta}}.$$



(a) We first estimate the maximum of  $\{|h(pe^{i\theta})|^2 \mid p \in [-\eta, \eta'], p \in [1, r]\}$ :

$$\begin{aligned} & \max\{|h(pe^{i\theta})|^2 \mid \theta \in [-\eta', \eta], p \in [0, r]\} \\ &= \max \left\{ 1 - \frac{4 \cos \frac{\theta\pi}{2\eta}}{p^{\pi/(2\eta)} + p^{-\pi/(2\eta)} + 2 \cos \frac{\theta\pi}{2\eta}} \mid \theta \in [-\eta', \eta'], p \in [1, r] \right\} \\ &\leq \max \left\{ 1 - \frac{4 \cos \frac{\eta'\pi}{2\eta}}{p^{\pi/(2\eta)} + p^{-\pi/(2\eta)} + 2} \mid p \in [1, r] \right\} \\ &\leq 1 - \frac{4 \cos \frac{\eta'\pi}{\eta}}{r^{\pi/(2\eta)} + 3} \leq 1 - r^{-\pi/(2\eta)} \cos \frac{\eta'\pi}{2\eta}. \end{aligned}$$

(b) We then turn to estimate the minimum of  $\{|h(pe^{i\theta})|^2 \mid \theta \in [-\eta, \eta], p \in [0, r]\}$ :

$$\begin{aligned} & \min\{|h(pe^{i\theta})| \mid \theta \in [-\eta, \eta], p \in [0, r]\} \\ &= \min \left\{ \frac{2r^{\pi/\eta} + 2}{r^{\pi/\eta} + 1 + 2r^{\pi/(2\eta)} \cos \frac{\theta\pi}{2\eta}} - 1 \mid \theta \in [-\eta, \eta] \right\} \\ &= \frac{2r^{\pi/\eta} + 2}{r^{\pi/\eta} + 1 + 2r^{\pi/(2\eta)}} - 1 = 1 - \frac{4}{r^{\pi/(2\eta)} + r^{-\pi/(2\eta)} + 2} \geq 1 - 4r^{-\pi/(2\eta)}. \quad \square \end{aligned}$$

**Lemma 3.7.** *Suppose that  $n(r)$  is a non-negative continuous function defined on interval  $(0, A)$ . Let*

$$N(r) = \int_0^r \frac{n(t) - n(0)}{t} dt + n(0) \ln r.$$

Then the order of  $n(r)$  and  $N(r)$  satisfy

- (1) when  $A = +\infty$ ,  $p(n(r)) = p(N(r))$ ;
- (2) when  $A = 1$ ,  $p(n(r)) - 1 = p(N(r))$ .

*Proof.* We suppose that  $B$  and  $\delta$  are some constants. They may be different when they appear in different places.

(1) When  $A = +\infty$ , we can obtain the desired result by the two inequalities:

$$n(r) \ln 2 \leq \int_r^{2r} \frac{n(t)}{t} dt \leq N(2r)$$

and

$$N(r) - B = \int_\delta^r \frac{n(t)}{t} dt \leq n(r) \ln r.$$

(2) When  $A = 1$ , by

$$n(r) \frac{1-r}{2} \leq \int_r^{(1+r)/2} \frac{n(t)}{t} dt \leq N\left(\frac{1+r}{2}\right),$$

we have  $p(n(r)) - 1 \leq p(N(r))$ .

Now we prove  $p(n(r)) - 1 \geq p(N(r))$ . Otherwise, there exists some  $\varepsilon_0 > 0$  such that when  $r$  is sufficiently close to  $1^-$ , we have

$$\begin{aligned} N(r) - B &= \int_{\delta}^r \frac{n(t)}{t} dt < \int_{\delta}^r \frac{(1-t)^{-p(N(r))-1+\varepsilon_0}}{t} dt \\ &\leq \frac{1}{\delta}(1-r)^{-p(N(r))-1} \int_{\delta}^r (1-t)^{\varepsilon_0} dt \\ &< \frac{1}{\delta(1+\varepsilon_0)} \left(\frac{1}{1-r}\right)^{p(N(r))-\varepsilon_0} + B, \end{aligned}$$

or

$$p(N(r)) < p(N(r)) - \varepsilon_0,$$

which is a contradiction. □

#### 4. Proofs of theorems

*Proof of Theorem 1.3. Case 1.* We first prove that Theorem 1.3 is true when  $W(z)$  is of finite order  $\rho \in (0, +\infty)$ .

Step 1. Now we prove for  $m$  angular domains  $\Delta\theta_m^i = \{z \mid |\arg z - (2i\pi)/m| < 2\pi/m\}$  ( $i = 1, 2, \dots, m-1, m \in \mathbb{Z}_+$ ), there is at least a domain  $\Delta\theta_m^i$  such that

$$\limsup_{r \rightarrow +\infty} \frac{\ln n_{\chi}(r, \Delta\theta_m^i, W)}{\ln r} \geq \rho.$$

Otherwise, for every  $\Delta\theta_m^i$  ( $i = 0, 1, \dots, m-1$ ), we have

$$\limsup_{r \rightarrow +\infty} \frac{\ln n_{\chi}(r, \Delta\theta_m^i, W)}{\ln r} < \rho.$$

Then there is a  $\sigma_0 \in (0, \rho)$ , such that when  $r$  is sufficiently large, we have

$$n_{\chi}(r, \Delta_m^i, W) < r^{\rho-\sigma_0}.$$

It follows from (1.3) that for the  $\sigma_0$  and  $r$  above,

$$\bar{n}(r, W = a_j) < r^{\rho-\sigma_0}, \quad j = 1, 2, 3.$$

Set  $\Delta'(\theta_m^i) = \{z \mid |\arg z - (2i\pi)/m| \leq \pi/m\}$ . According to Lemma 3.4, we have

$$\begin{aligned} S(r, \Delta'\theta_m^i, W) &\leq \sum_{j=1}^3 \bar{n}(2r, \Delta\theta_m^i, W = a_j) + n_{\chi}(2r, \Delta\theta_m^i, W) + O(\ln r) \\ &\leq \sum_{j=1}^3 \bar{n}(2r, \Delta\theta_m^i, W = a_j) + r^{\rho-\sigma_0} + O(\ln r). \end{aligned}$$

Adding the above inequality from  $i = 0$  to  $m - 1$ , it follows that

$$\begin{aligned} S(r, W) &\leq 2 \sum_{j=1}^3 \bar{n}(2r, W = a_j) + mr^{\rho-\sigma_0} + O(\ln r) \\ &< (6 + m)(2r)^{\rho-\sigma_0} + O(\ln r). \end{aligned}$$

Dividing both sides by  $r$  and letting  $r \rightarrow +\infty$ , we have

$$\limsup_{r \rightarrow \infty} \frac{\ln T(r, W)}{\ln r} = \limsup_{r \rightarrow \infty} \frac{\ln S(r, W)}{\ln r} \leq \rho - \sigma_0 < \rho,$$

which is a contradiction.

Step 2. Choose a subsequence of  $\{\theta_m^{i_m}\}$ , denoted by  $\{\theta_m\}$  such that

$$\lim_{m \rightarrow +\infty} \theta_m = \theta_0.$$

Then for all  $\varepsilon > 0$ , we have

$$\limsup_{r \rightarrow \infty} \frac{\ln n_\chi(r, \Delta(\theta_0, \varepsilon), W)}{\ln r} \geq \rho.$$

It follows from Lemmas 3.1 and 3.7 that

$$\rho = p(T(r, W)) \geq p(N_\chi(r, W)) = p(n_\chi(r, W)) \geq p(n_\chi(r, \Delta(\theta_0, \varepsilon), W)) \geq \rho.$$

Case 2. When  $\rho(W) = +\infty$ , we can obtain the desired result by replacing  $\rho$  in Case 1 with any sufficiently large positive number  $M$ . □

*Proof of Theorem 1.5.* We just need to prove (2) holds when  $\rho \in (0, \infty)$ . First, we prove that for any  $\varepsilon' \in (0, \varepsilon_0)$ ,

$$\limsup_{r \rightarrow \infty} \frac{\ln S(r, \Delta(\theta_0, \varepsilon'), W(z))}{\ln r} \geq \rho.$$

For  $\varepsilon'$  we mentioned above and all  $\sigma \in (0, \rho)$ , there exists some sequence  $\{r_n\}$  such that for any  $a \in \mathbb{C} \cup \{\infty\}$  with  $2k$  exceptional values at most,

$$\bar{n}(r_n, \Delta(\theta_0, \varepsilon'), W(z) = a) > r_n^{\rho-\sigma}.$$

Let

$$E_n = \{a \in S \mid \bar{n}(r_n, \Delta(\theta_0, \varepsilon')) > r_n^{\rho-\sigma}\},$$

where  $S$  is the complex sphere. Then

$$\pi S(r_n, \Delta(\theta_0, \varepsilon'), W) > r_n^{\rho-\sigma} m(E_n),$$

where  $m(E_n)$  is the Lebesgue measure of  $E_n$ . Since  $\#\{S - E_n\} < 2k$ , further  $m(E_n) = \pi$ , then we have

$$\limsup_{r \rightarrow +\infty} \frac{\ln S(r, \Delta(\theta_0, \varepsilon'), W(z))}{\ln r} \geq \rho.$$

Replacing  $r'$  and  $r$  in Lemma 3.4 with  $r$  and  $2r$ , we can obtain the desired result. □

*Proof of Theorem 1.6.* Without loss of generality we suppose  $\theta_0 = 0$ . The mapping

$$h(z) = \frac{(ze^{-i\theta_0})^{\pi/(2\eta)} - 1}{(ze^{-i\theta_0})^{\pi/(2\eta)} + 1}$$

maps the angular domain  $\Delta(0, \eta)$  onto the unit disc  $\{|h| < 1\}$ . Set

$$R' = \sqrt{1 - \cos \frac{\eta'\pi}{2\eta} r^{-\pi/(2\eta)}}.$$

By Lemma 3.6(a), we have

$$\rho = \limsup_{r \rightarrow +\infty} \frac{\ln n_\chi(r, \Delta(0, \eta'), W)}{\ln r} \leq \limsup_{R \rightarrow 1^-} \frac{\ln n_\chi(R', W(z(h)))}{-\ln(1 - R')} \cdot \frac{\pi}{2\eta}.$$

Applying Lemmas 3.1 and 3.7(2), we have the order of  $T(R, W(z(h)))$  in  $|h| < 1$  is

$$p(T(R, W(z(h)))) \geq p(N_\chi(R, W(z(h)))) = p(n_\chi(R, W(z(h)))) - 1 \geq \frac{2\eta\rho}{\pi} - 1 > 0.$$

According to Lemmas 3.2 and 3.7, we have for any  $a \in \mathbb{C} \cup \{\infty\}$  with  $2k$  exceptional values at most,

$$p(\bar{n}(R, W(z(h)) = a)) = p(\bar{N}(R, W(z(h)) = a)) + 1 \geq \frac{2\eta\rho}{\pi}.$$

Let

$$r' = \left(\frac{4}{1 - R^2}\right)^{\pi/(2\eta)}.$$

By Lemma 3.6(b), we have

$$\frac{2\eta\rho}{\pi} \leq \limsup_{R \rightarrow 1^-} \frac{\ln \bar{n}(R, W(z(h)) = a)}{-\ln(1 - R)} = \limsup_{r' \rightarrow +\infty} \frac{\ln \bar{n}(r', \Delta(0, \eta), W(z) = a)}{\frac{\pi}{2\eta} \ln r'}.$$

Hence for any  $a \in \mathbb{C} \cup \{\infty\}$  with  $2k$  exceptional values at most, we have

$$p(n(r, \Delta(0, \eta), W(z) = a)) = \rho. \quad \square$$

*Proof of Theorem 1.8.* Suppose that  $L(0)$  is a Borel cluster direction of  $W(z)$ . Let  $\varepsilon$  be a sufficiently small positive number. Then for any  $\varepsilon > \varepsilon' > 0$  and  $M > \pi/(2\varepsilon)$ , we have

$$p(n_\chi(r, \Delta(0, \varepsilon'), W(z))) > M.$$

Replacing  $\eta, \eta'$  and  $\rho$  in Theorem 1.6 with  $\varepsilon, \varepsilon/2$  and  $M$  respectively yields

$$p(\bar{n}(r, \Delta(0, \varepsilon), W(z) = a)) \geq M,$$

where  $a$  is any finite or infinite complex numbers with  $2k$  exceptional values at most. This attains our purpose. □

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