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Spectral Approximations for Nonlinear Fractional Delay Diffusion Equations with Smooth and Nonsmooth Solutions

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Abstract. A fully discrete scheme is proposed for the nonlinear fractional delay diffusion equations with smooth solutions, where the fractional derivative is described in Caputo sense with the order α $(0 < \alpha < 1)$. The scheme is constructed by combining finite difference method in time and Legendre spectral approximation in space. Stability and convergence are proved rigorously. Moreover, a modified scheme is proposed for the equation with nonsmooth solutions by adding correction terms to the approximations of fractional derivative operator and nonlinear term. Numerical examples are carried out to support the theoretical analysis.

1. Introduction

Fractional differential equations have received considerable attention because they are more accurate for describing some certain phenomenons than classical integer-order differential models. They have been applied in various fields of science and engineering, such as physics, chemistry, biology, viscoelasticity and finance [6, 19, 21–23]. Some analytical methods have been used to solve the fractional differential equations, for example, Fourier transform method, Laplace transform method, Mellin transform method and Green function method. The analytical methods do not work well for the majority of fractional differential equations, especially for nonlinear problems. Thus some numerical methods are considered, such as finite difference methods [2, 5, 11, 29, 30], finite element methods [9, 12, 13, 34, 36], spectral methods [4, 16, 17, 37] and other numerical methods [7, 20, 26, 32].

Time delay occurs frequently in realistic world and it has numerous applications in mathematical modeling, such as population dynamics [14] and automatics control systems with feedback [28]. The numerical solutions for fractional differential equations with delay have been investigated by some authors. Zayernouri et al. [33] applied the fractional

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basic functions called Jacobi polyfractonomials as the basis functions in spectral and discontinuous spectral element methods of Petrov-Galerkin type to solve fractional delay differential equations. Saeed et al. [27] proposed the Chebyshev wavelet method for the fractional delay differential equations. Yaghoobi et al. [31] developed a scheme based on a cubic spline interpolation to solve a class of nonlinear variable-order fractional differential equation with delay. Rahimkhani et al. [24] introduced a new operational matrix based on Bernoulli wavelets to solve fractional delay differential equations. The numerical approximations for fractional partial differential equations with delay is limited. Rihan [25] provided an unconditionally stable implicit difference approximation for time fractional partial differential equations with and without time delay. Hao et al. [10] constructed a linearized quasi-compact finite difference scheme for semilinear space fractional diffusion equations with a fixed time delay.

In this paper, we consider the following nonlinear fractional delay diffusion equations

$$\begin{cases} {}_0^CD_t^\alpha u(x,t) - \nu\frac{\partial^2 u(x,t)}{\partial x^2} = f(u(x,t),u(x,t-s)) + g(x,t) & (x,t)\in(-1,1)\times(0,T], \\ u(-1,t)=0, \quad u(1,t)=0 & t\in[0,T], \\ u(x,t)=\varphi(x,t) & (x,t)\in(-1,1)\times[-s,0], \end{cases} \tag{1.1}$$

where $\nu > 0$ is the diffusion coefficient, $s > 0$ is the time delay and $\varphi(x,t)$ is a given function. The Caputo derivative ${}_0^CD_t^\alpha u(x,t)$ is defined as

$$ {}_0^CD_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u'(x,s)}{(t-s)^\alpha}\,\mathrm{d}s, \quad 0<\alpha<1. $$

We assume that the function f satisfies the following Lipschitz condition

$$ |f(u_1,v_1) - f(u_2,v_2)| \le L(|u_1-v_1| + |u_2-v_2|), \tag{1.2}$$

where L is a positive constant.

The purpose of this paper is to study numerical solution of equation (1.1). We construct fully discrete schemes based on finite difference method in time and Legendre spectral approximation in space for the equation with smooth and nonsmooth solutions. More precisely, we apply $L1$ formulation to discretize Caputo derivative and use $f(2u(x,t_{k-1}) - u(x,t_{k-2}), u(x,t_{k-n}))$ to approximate nonlinear term $f(u(x,t_k),u(x,t_{k-n}))$. Stability and convergence are proved for the problem with smooth assumption. For the equation with nonsmooth solutions, we follow Lubich's correction approach [18] by adding correction terms to $L1$ formulation and the approximation of nonlinear term, which not only makes the new approximation exact for low regularity terms of the solutions but also maintains accuracy for high regularity terms. Numerical results have verified the theoretical analysis. To the best of our knowledge, there is no work on studying the delay problem with nonsmooth solutions.

The rest of the paper is organized as follows. Section 2 gives some preliminaries and notations. In Section 3, we construct a fully discrete Legendre spectral scheme and analyze its stability and convergence. In the next section, a modified scheme is proposed by adding correction terms for the problem with nonsmooth solutions. Numerical examples are presented in Section 5. Final section is for some conclusions.

2. Preliminaries and notations

For abbreviation, we denote $\partial_x^l u(x,t) = \partial^l u(x,t)/\partial x^l$ and $\Lambda = (-1,1)$. For any real $m \geq 0$, let $H^m(\Lambda)$ be the Sobolev space endowed with the norm $\|\cdot\|_m$ and the seminorm $|\cdot|_m$ in the usual sense. In particular, $L^2(\Lambda) = H^0(\Lambda)$. We also define the space-time Sobolev space $L^\infty(-s,T;H^m(\Lambda))\colon (-s,T) \to H^m(\Lambda)$ with the norm

$$\|\phi\|_{L^\infty(-s,T;H^m(\Lambda))} = \sup_{-s\leq t\leq T} \|\phi\|_{H^m(\Lambda)} < \infty.$$

Let N be a positive integer, $\mathbb{P}_N(\Lambda)$ stands for the set of all polynomials of degree at most N. We define the spaces $H_0^1(\Lambda) = \{v : v \in H^1(\Lambda), v(\pm 1) = 0\}$ and $\mathbb{P}_N^0 = \{v \in \mathbb{P}_N : v(\pm 1) = 0\}$. Throughout the paper, c denotes a generic positive constant.

We introduce the orthogonal projection $\pi_N^{1,0}\colon H_0^1(\Lambda) \to \mathbb{P}_N^0$, such that for any $v \in H_0^1(\Lambda)$,

$$(\partial_x \pi_N^{1,0} u, \partial_x v) = (\partial_x u, \partial_x v), \quad \forall\, v \in \mathbb{P}_N^0.$$

For the orthogonal projection $\pi_N^{1,0}$, we have

Lemma 2.1. [3, p. 288] *For any $u \in H_0^1(\Lambda) \cap H^m(\Lambda)$, it holds*

$$\|\pi_N^{1,0} u - u\|_k \leq c N^{k-m} \|u\|_m, \quad k = 0,1.$$

We define time step $\tau = s/n$ and $M = [T/\tau]$. Denote $t_k = k\tau$ and $u^k = u(\,\cdot\,, t_k)$, $-n \leq k \leq M$. To discretize the Caputo derivative, we introduce the $L1$ formulation as follows:

$$D_\tau^\alpha u^k := \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left(a_0^{(\alpha)} u^k - \sum_{j=1}^{k-1} (a_{k-1-j}^{(\alpha)} - a_{k-j}^{(\alpha)}) u^j - a_{k-1}^{(\alpha)} u^0 \right) \tag{2.1}$$

with

$$a_l^{(\alpha)} = (l+1)^{(1-\alpha)} - l^{(1-\alpha)}, \quad l \geq 0.$$

Then the truncation error satisfies

Lemma 2.2. [30, Lemma 4.1] *For $0 < \alpha < 1$ and $u(t) \in C^2[0,T]$, then*

$$|{}_0^C D_t^\alpha u^k - D_\tau^\alpha u^k| \leq c_u \tau^{2-\alpha},$$

where c_u is related to $u''(t)$.

According to [1], we can deduce that

$$(D_\tau^\alpha u^k, u^k) \geq \frac{1}{2} D_\tau^\alpha \|u^k\|^2. \tag{2.2}$$

Finally, we present the results of the Gronwall type inequality.

Lemma 2.3. [15, Lemma 3.1] *Suppose that the nonnegative sequences* $\{\omega^n, g^n \mid n = 0, 1, 2, \ldots\}$ *satisfy*

$$D_\tau^\alpha \omega^n \leq \lambda_1 \omega^n + \lambda_2 \omega^{n-1} + g^n, \quad n \geq 1,$$

where λ_1 *and* λ_2 *are positive constants. Then there exists a positive constant* $\tau^* = \sqrt[\alpha]{1/(2\Gamma(2-\alpha)\lambda_1)}$ *such that, when* $\tau \leq \tau^*$,

$$\omega^n \leq 2\left(\omega^0 + \frac{t_n^\alpha}{\Gamma(1+\alpha)} \max_{0 \leq j \leq n} g^j\right) E_\alpha(2\lambda t_n^\alpha), \quad 1 \leq n \leq N,$$

where $E_\alpha(z) = \sum_{k=0}^{\infty} z^k/\Gamma(1+k\alpha)$ *is the Mittag-Leffler function and* $\lambda = \lambda_1 + \lambda_2/(2 - 2^{1-\alpha})$.

Using the same arguments as in the proof of Lemma 2.3, one can obtain more general result as follows:

Corollary 2.4. *If the nonnegative sequences* $\{\omega^n, g^n \mid n = 0, 1, 2, \ldots\}$ *satisfy*

$$D_\tau^\alpha \omega^n \leq \lambda_1 \omega^n + \lambda_2 \omega^{n-1} + \lambda_3 \omega^{n-2} + \lambda_4 \omega^{n-k} + g^n, \quad n \geq k,$$

where λ_1, λ_2, λ_3 *and* λ_4 *are positive constants. Then there exists a positive constant* $\tau^* = \sqrt[\alpha]{1/(2\Gamma(2-\alpha)\lambda_1)}$ *such that, when* $\tau \leq \tau^*$,

$$\omega^n \leq 2\left((1 + \Gamma(2-\alpha)\tau^\alpha(\lambda_2 + \lambda_3 + \lambda_4))\omega^0 + \frac{t_n^\alpha}{\Gamma(1+\alpha)} \max_{0 \leq j \leq n} g^j\right) E_\alpha(2\lambda t_n^\alpha)$$

with $\lambda = \lambda_1 + \lambda_2/(2 - 2^{1-\alpha}) + \lambda_3/(2 - 2^{1-\alpha})^2 + \lambda_4/(2 - 2^{1-\alpha})^k$.

3. Stability and convergence of fully discrete scheme for smooth solutions

3.1. Fully discrete scheme and its stability

Applying $L1$ formulation to discrete Caputo derivative and $f(2u^{k-1} - u^{k-2}, u^{k-n})$ to approach $f(u^k, u^{k-n})$, we construct a linearized Legendre spectral scheme for (1.1). The scheme in weak formulation is as follows: find $\{u_N^k\}_{k=1}^M \in \mathbb{P}_N^0$, such that

$$(D_\tau^\alpha u_N^k, v_N) + \nu(\partial_x u_N^k, \partial_x v_N) = (f(2u_N^{k-1} - u_N^{k-2}, u_N^{k-n}), v_N) + (g^k, v_N), \quad \forall\, v_N \in \mathbb{P}_N^0 \tag{3.1}$$

with $u_N^k = \pi_N^{1,0} \varphi^k$, $-n \leq k \leq 0$.

It is a linear iteration scheme and its well-posedness is guaranteed by the well-known Lax-Milgram lemma. At each time level, one only needs to solve a system of linear equations.

Assume that $\{\widetilde{u}_N^k\}_{k=1}^M$ is the solution of

$$(D_\tau^\alpha \widetilde{u}_N^k, v_N)+\nu(\partial_x \widetilde{u}_N^k, \partial_x v_N) = (f(2\widetilde{u}_N^{k-1}-\widetilde{u}_N^{k-2}, \widetilde{u}_N^{k-n}), v_N)+(\widetilde{g}^k, v_N), \quad \forall\, v_N \in \mathbb{P}_N^0 \tag{3.2}$$

with initial conditions $\widetilde{u}_N^k = \pi_N^{1,0}\varphi^k$, $-n \le k \le 0$.

Next, we present the stability result in the following.

Theorem 3.1. *The fully discrete scheme* (3.1) *is unconditionally stable in the sense that for all $\tau > 0$, it holds*

$$\|u_N^k - \widetilde{u}_N^k\|^2 \le C \max_{1\le k\le M} \|g^k - \widetilde{g}^k\|^2.$$

Proof. Denote $\eta_N^k = u_N^k - \widetilde{u}_N^k$. Subtracting (3.2) from (3.1), it holds

$$\begin{aligned}&(D_\tau^\alpha \eta_N^k, v_N) + \nu(\partial_x \eta_N^k, \partial_x v_N)\\ &= (f(2u_N^{k-1} - u_N^{k-2}, u_N^{k-n}) - f(2\widetilde{u}_N^{k-1} - \widetilde{u}_N^{k-2}, \widetilde{u}_N^{k-n}), v_N) + (g^k - \widetilde{g}^k, v_N).\end{aligned} \tag{3.3}$$

According to (1.2) and using Hölder inequality and Young's inequality, we derive that

$$\begin{aligned}&(f(2u_N^{k-1} - u_N^{k-2}, u_N^{k-n}) - f(2\widetilde{u}_N^{k-1} - \widetilde{u}_N^{k-2}, \widetilde{u}_N^{k-n}), v_N)\\ &\le L(\|2\eta_N^{k-1} - \eta_N^{k-2}\| + \|\eta_N^{k-n}\|)\|v_N\|\\ &\le 3L^2\|2\eta_N^{k-1} - \eta_N^{k-2}\|^2 + 3L^2\|\eta_N^{k-n}\|^2 + \frac{1}{6}\|v_N\|^2\\ &\le 24L^2\|\eta_N^{k-1}\| + 6L^2\|\eta_N^{k-2}\|^2 + 3L^2\|\eta_N^{k-n}\|^2 + \frac{1}{6}\|v_N\|^2,\end{aligned}$$

and

$$(g^k - \widetilde{g}^k, v_N) \le 3\|g^k - \widetilde{g}^k\|^2 + \frac{1}{12}\|v_N\|^2.$$

Then (3.3) becomes

$$\begin{aligned}&(D_\tau^\alpha \eta_N^k, v_N) + \nu(\partial_x \eta_N^k, \partial_x v_N)\\ &\le \frac{1}{4}\|v_N\|^2 + 24L^2\|\eta_N^{k-1}\| + 6L^2\|\eta_N^{k-2}\|^2 + 3L^2\|\eta_N^{k-n}\|^2 + 3\|g^k - \widetilde{g}^k\|^2.\end{aligned}$$

Taking $v_N = \eta_N^k$ and using (2.2), we can deduce that

$$\begin{aligned}&\frac{1}{2}D_\tau^\alpha\|\eta_N^k\|^2 + \nu\|\partial_x \eta_N^k\|^2\\ &\le \frac{1}{4}\|\eta_N^k\|^2 + 24L^2\|\eta_N^{k-1}\|^2 + 6L^2\|\eta_N^{k-2}\|^2 + 3L^2\|\eta_N^{k-n}\|^2 + 3\|g^k - \widetilde{g}^k\|^2,\end{aligned}$$

namely,

$$D_\tau^\alpha\|\eta_N^k\|^2 \le \frac{1}{2}\|\eta_N^k\|^2 + 48L^2\|\eta_N^{k-1}\|^2 + 12L^2\|\eta_N^{k-2}\|^2 + 6L^2\|\eta_N^{k-n}\|^2 + 6\|g - \widetilde{g}\|^2.$$

By means of Corollary 2.4, there exists a positive constant $\tau^* = \sqrt[\alpha]{1/\Gamma(2-\alpha)}$, when $\tau < \tau^*$, we have

$$\|\eta_N^k\|^2 \le \frac{12t_k^\alpha}{\Gamma(1+\alpha)} E_\alpha(2\lambda t_k^\alpha) \max_{1\le k\le M} \|g^k - \widetilde{g}^k\|^2$$

with $\lambda = 1/2 + 48L^2/(2-2^{1-\alpha}) + 12L^2/(2-2^{1-\alpha})^2 + 6L^2/(2-2^{1-\alpha})^n$. By simple calculation, we know that $\tau^* \ge 1$ for all $0 < \alpha < 1$. Thus the scheme is unconditionally stable. □

3.2. Convergence analysis

In this subsection, we investigate the convergence of fully discrete scheme (3.1) using error estimation.

Theorem 3.2. *Let $\{u^k\}_{k=-n}^M$ be the exact solution of equation* (1.1) *and $\{u_N^k\}_{k=-n}^M$ the solution of* (3.1). *Suppose that ${}_0^CD_t^\alpha u \in L^\infty(0,T;H^m(\Lambda))$, $u \in L^\infty(-s,T;H^m(\Lambda))$, we have*

$$\|u^k - u_N^k\| \le C(N^{-m} + \tau^{2-\alpha}), \quad 1 \le k \le M, \tag{3.4}$$

where C is independent of N and τ.

Proof. Denote $u^k - u_N^k = (u^k - \pi_N^{1,0}u^k) + (\pi_N^{1,0}u^k - u_N^k) \triangleq \widetilde{e}_N^k + \widehat{e}_N^k$. The weak formulation of equation (1.1) is

$$({}_0^CD_t^\alpha u^k, v_N) + \nu(\partial_x u^k, \partial_x v_N) = (f(u^k, u^{k-n}), v_N) + (g, v_N). \tag{3.5}$$

Subtracting (3.1) from (3.5) and owing to the definition of orthogonal projection, the error equation satisfies

$$(D_\tau^\alpha \widehat{e}_N^k, v_N) + \nu(\partial_x \widehat{e}_N, \partial_x, v_N) \triangleq R_1^k + R_2^k, \tag{3.6}$$

where

$$\begin{aligned} R_1^k &= (f(u^k, u^{k-n}) - f(2u_N^{k-1} - u_N^{k-2}, u_N^{k-n}), v_N), \\ R_2^k &= (D_\tau^\alpha \pi_N^{1,0} u^k - {}_0^CD_t^\alpha u^k, v_N). \end{aligned}$$

We next estimate the right-hand terms R_1^k and R_2^k. For the first term R_1^k,

$$\begin{aligned} R_1^k &= (f(u^k, u^{k-n}) - f(2u^{k-1} - u^{k-2}, u^{k-n}), v_N) \\ &\quad + (f(2u^{k-1} - u^{k-2}, u^{k-n}) - f(2u_N^{k-1} - u_N^{k-2}, u_N^{k-n}), v_N) \\ &\triangleq R_{11}^k + R_{12}^k. \end{aligned} \tag{3.7}$$

Applying Taylor expansion, it holds

$$\begin{aligned} f(u^k, u^{k-n}) &= f(2u^{k-1} - u^{k-2}, u^{k-n}) + (u^k - 2u^{k-1} + u^{k-2}) f_1'(\xi, u^{k-n}) \\ &= f(2u^{k-1} - u^{k-2}, u^{k-n}) + \widetilde{c}_u \tau^2, \end{aligned}$$

furthermore, by means of Hölder inequality and Young's inequality, we have

$$\begin{aligned} R_{11}^k &\le \|f(u^k, u^{k-n}) - f(2u^{k-1} - u^{k-2}, u^{k-n})\| \|v_N\| \\ &\le \widetilde{c}_u \tau^4 + \frac{1}{36}\|v_N\|^2. \end{aligned} \tag{3.8}$$

According to (1.2), we can deduce that

$$\begin{aligned} R_{12}^k &\le L(\|2e_N^{k-1} - e_N^{k-2}\| + \|e_N^{k-n}\|)\|v_N\| \\ &\le L(\|2\widehat{e}_N^{k-1} - \widehat{e}_N^{k-2}\| + \|\widehat{e}_N^{k-n}\| + \|2\widetilde{e}_N^{k-1} - \widetilde{e}_N^{k-2}\| + \|\widetilde{e}_N^{k-n}\|)\|v_N\| \\ &\le 48L^2\|\widehat{e}_N^{k-1}\|^2 + 12L^2\|\widehat{e}_N^{k-2}\|^2 + 6L^2\|\widehat{e}_N^{k-n}\|^2 + 48L^2\|\widetilde{e}_N^{k-1}\|^2 \\ &\quad + 12L^2\|\widetilde{e}_N^{k-2}\|^2 + 6L^2\|\widetilde{e}_N^{k-n}\|^2 + \frac{1}{6}\|v_N\|^2, \end{aligned} \tag{3.9}$$

moreover, owing to Lemma 2.1, it holds

$$\|\widetilde{e}_N^{k-1}\|^2 \le cN^{-2m}\|u^{k-1}\|_m^2, \quad \|\widetilde{e}_N^{k-2}\|^2 \le cN^{-2m}\|u^{k-2}\|_m^2, \quad \|\widetilde{e}_N^{k-n}\|^2 \le cN^{-2m}\|u^{k-n}\|_m^2,$$

then (3.9) becomes

$$\begin{aligned} R_{12}^k \le\ & 48L^2\|\widehat{e}_N^{k-1}\|^2 + 12L^2\|\widehat{e}_N^{k-2}\|^2 + 6L^2\|\widehat{e}_N^{k-n}\|^2 \\ & + cN^{-2m}\|u\|_{L^\infty(-s,T;H^m(\Lambda))}^2 + \frac{1}{6}\|v_N\|^2. \end{aligned} \tag{3.10}$$

Substituting (3.8) and (3.10) into (3.7), we can derive that

$$\begin{aligned} R_1^k \le\ & \frac{7}{36}\|v_N\|^2 + 48L^2\|\widehat{e}_N^{k-1}\|^2 + 12L^2\|\widehat{e}_N^{k-2}\|^2 + 6L^2\|\widehat{e}_N^{k-n}\|^2 \\ & + cN^{-2m}\|u\|_{L^\infty(-s,T;H^m(\Lambda))}^2 + \widetilde{c}_u\tau^4. \end{aligned} \tag{3.11}$$

For the second term R_2^k, it holds

$$\begin{aligned} R_2^k &= (D_\tau^\alpha \pi_N^{1,0}u^k - {}_0^C D_t^\alpha \pi_N^{1,0}u^k, v_N) + ({}_0^C D_t^\alpha \pi_N^{1,0}u^k - {}_0^C D_t^\alpha u^k, v_N) \\ &= (\pi_N^{1,0}(D_\tau^\alpha u^k - {}_0^C D_t^\alpha u^k), v_N) - ({}_0^C D_t^\alpha \widetilde{e}_N^k, v_N) \\ &\triangleq R_{21}^k + R_{22}^k, \end{aligned} \tag{3.12}$$

using (2.1) and Poincaré inequality, it holds

$$\begin{aligned} R_{21}^k &\le \|\pi_N^{1,0}(D_\tau^\alpha u^k - {}_0^C D_t^\alpha u^k)\|^2 + \frac{1}{36}\|v_N\|^2 \\ &\le \|D_\tau^\alpha \partial_x u^k - {}_0^C D_t^\alpha \partial_x u^k\|^2 + \frac{1}{36}\|v_N\|^2 \\ &\le c_{1,u}\tau^{4-2\alpha} + \frac{1}{36}\|v_N\|^2, \end{aligned}$$

furthermore, according to Lemma 2.1, we have

$$\begin{aligned} R_{22}^k &\le cN^{-2m}\|{}_0^CD_t^\alpha u^k\|_m^2 + \frac{1}{36}\|v_N\|^2 \\ &\le cN^{-2m}\|{}_0^CD_t^\alpha u\|_{L^\infty(0,T;H^m(\Lambda))}^2 + \frac{1}{36}\|v_N\|^2. \end{aligned}$$

Thus (3.12) becomes

$$R_2^k \le cN^{-2m}\|{}_0^CD_t^\alpha u\|_{L^\infty(0,T;H^m(\Lambda))}^2 + c_{3,u}\tau^{4-2\alpha} + \frac{1}{18}\|v_N\|^2. \tag{3.13}$$

Substituting (3.11) and (3.13) into (3.6), we can infer that

$$\begin{aligned} &(D_\tau^\alpha \widehat{e}_N^k, v_N) + \nu(\partial_x \widehat{e}_N^k, \partial_x v_N) \\ &\le \frac{1}{4}\|v_N\|^2 + 48L^2\|\widehat{e}_N^{k-1}\|^2 + 12L^2\|\widehat{e}_N^{k-2}\|^2 + 6L^2\|\widehat{e}_N^{k-n}\|^2 + \widetilde{G}, \end{aligned} \tag{3.14}$$

where

$$\widetilde{G} = cN^{-2m}\left(\|{}_0^CD_t^\alpha u\|_{L^\infty(0,T;H^m(\Lambda))}^2 + \|u\|_{L^\infty(-s,T;H^m(\Lambda))}^2\right) + c_u\tau^{4-2\alpha}.$$

Taking $v_N = \widehat{e}_N^k$ in (3.14) and applying (2.2), we can conclude that

$$\frac{1}{2}D_\tau^\alpha\|\widehat{e}_N^k\|^2 + \nu\|\partial_x\widehat{e}_N^k\|^2 \le \frac{1}{4}\|\widehat{e}_N^k\|^2 + 48L^2\|\widehat{e}_N^{k-1}\|^2 + 12L^2\|\widehat{e}_N^{k-2}\|^2 + 6L^2\|\widehat{e}_N^{k-n}\|^2 + \widetilde{G},$$

namely,

$$D_\tau^\alpha\|\widehat{e}_N^k\|^2 \le \frac{1}{2}\|\widehat{e}_N^k\|^2 + 96L^2\|\widehat{e}_N^{k-1}\|^2 + 24L^2\|\widehat{e}_N^{k-2}\|^2 + 12L^2\|\widehat{e}_N^{k-n}\|^2 + G$$

with $G = 2\widetilde{G}$. For $-n \le k \le 0$, $u_N^k = \pi_N^{1,0}u^k$, then we have $\widehat{e}_N^{k-n} = 0$, $k-n \le 0$. According to Corollary 2.4, there exists a positive constant $\tau^* = \sqrt[\alpha]{1/\Gamma(2-\alpha)} \ge 1$ such that, when $\tau \le \tau^*$,

$$\|\widehat{e}_N^k\|^2 \le 2G\frac{t_k^\alpha}{\Gamma(1+\alpha)}E_\alpha(2\lambda t_k^\alpha), \quad 1 \le k \le M$$

with $\lambda = 1/2 + 96L^2/(2-2^{1-\alpha}) + 24L^2/(2-2^{1-\alpha})^2 + 12L^2/(2-2^{1-\alpha})^n$. Finally, by means of triangle inequality and Lemma 2.1, we complete the proof of (3.4). □

4. Modified fully discrete spectral scheme for nonsmooth solutions

Although the initial conditions and source terms are smooth, the solutions of fractional differential equation can be nonsmooth and even have strong singularity at $t = 0$. The rate of convergence maybe deteriorated significantly. To make up for the lost accuracy near $t = 0$, we follow Lubich's approach [18] by adding correction terms.

First, we introduce the modified $L1$ formulation which has been applied in [35]

$${}_0^CD_t^\alpha u^k = D_\tau^\alpha u^k + \tau^{-\alpha}\sum_{j=1}^{m_1}\omega_{k,j}(u^j - u^0), \tag{4.1}$$

where the starting weights $\omega_{k,j}$ are chosen such that (4.1) is exact for $u = t^{\sigma_r}$, $0 < \sigma_r < \sigma_{r+1}$, $1 \le r \le m_1$. Specifically, the starting weights $\omega_{k,j}$ can be computed from the following linear system

$$\sum_{j=1}^{m_1} \omega_{k,j} j^{\sigma_r} = \frac{\Gamma(1+\sigma_r)}{\Gamma(1+\sigma_r-\alpha)} k^{\sigma_r-\alpha} - \frac{1}{\Gamma(2-\alpha)} \left(a_0 k^{\sigma_r} - \sum_{j=1}^{k-1} (a_{k-1-j}^{(\alpha)} - a_{k-j}^{(\alpha)}) j^{\sigma_r} \right).$$

According to [8], it is known that the above linear system is ill-conditioned. However, the accuracy can significantly increase for the equations with low regularity solutions by adding a few correction terms, which does not make that the condition number of the exponential Vandermonde matrix too large. Thus the starting weights $\{\omega_{k,j}\}$ can be solved accurately.

Next, we add correction terms for the approximation of nonlinear term in the following. Choosing the starting weights $\widetilde{\omega}_{j,k}$ such that the following equality is exact for $u = t^{\sigma_r}$, $0 < \sigma_r < \sigma_{r+1}$, $1 \le r \le m_2$,

$$f(u^k, u^{k-n}) = f(2u^{k-1} - u^{k-2}, u^{k-n}) + \sum_{j=1}^{m_2} \widetilde{\omega}_{k,j} (f(u^j, u^{k-n}) - f(u^0, u^{k-n})).$$

Then the modified fully discrete scheme for equation (1.1) in weak formulation is as follows: find $\{u_N^k\}_{k=1}^M \in \mathbb{P}_N^0$, such that

$$\begin{aligned} &(D_\tau^\alpha u_N^k, v_N) + \nu(\partial_x u_N^k, \partial_x v_N) \\ &= (f(2u^{k-1} - u^{k-2}, u^{k-n}), v_N) + (g, v_N) - \tau^{-\alpha} \sum_{j=1}^{m_1} \omega_{k,j} (u^j - u^0, v_N) \\ &\quad + \sum_{j=1}^{m_2} \widetilde{\omega}_{k,j} (f(u^j, u^{k-n}) - f(u^0, u^{k-n}), v_N), \quad \forall\, v_N \in \mathbb{P}_N^0 \end{aligned} \tag{4.2}$$

with $u_N^k = \pi_N^{1,0} \varphi^k$, $-n \le k \le 0$.

The modified scheme (4.2) is exact for low regularity terms t^{σ_r}, $1 \le r \le m$ by adding correction terms, therefore, it has higher accuracy in temporal discretization than scheme (3.1) for the equation with nonsmooth solutions.

5. Numerical experiment

In this section, we present some numerical examples to support the theoretical analysis.

5.1. The implementation of the schemes

As with [17], we also evaluate integrals using numerical quadratures. More precisely, we choose Legendre-Gauss-Lobatto integration formulas. Let $\{x_j, \omega_j\}_{j=0}^N$ be Legendre-Gauss-Lobatto quadrature nodes and weights, namely, $\{x_j\}_{j=0}^N$ are the zeros of $(1-x^2)L_N'(x)$

and weights are expressed as

$$\omega_j = \frac{2}{N(N+1)} \frac{1}{L_N^2(x_j)}, \quad 0 \le j \le N,$$

where $L_N(x)$ stands for Legendre polynomial with degree N. Then the discrete inner product relative to Legendre-Gauss-Lobatto quadrature is defined as follows:

$$(u, v)_N = \sum_{j=0}^{N} u(x_j) v(x_j) \omega_j. \tag{5.1}$$

With the above quadrature nodes and weights, it holds

$$\int_{-1}^{1} p(x)\,\mathrm{d}x = \sum_{j=0}^{N} p(x_j)\omega_j, \quad \forall\, p \in \mathbb{P}_{2N-1}.$$

Denoting by $\{h_j\}_{j=0}^N$ Lagrange basis polynomials associated with Legendre-Gauss-Lobatto points $\{x_j\}_{j=0}^N$, we have

$$h_j(x_i) = \delta_{ij}, \quad \forall\, i, j \in \{0, 1, \ldots, N\},$$

where δ_{ij} denotes the Kronecker-delta function. Then expanding u_N^k in terms of the Lagrange basis polynomials

$$u_N^k(x) = \sum_{j=0}^{N} \widehat{u}_j^k h_j(x)$$

with $\widehat{u}_j^k = u_N^k(x_j)$, unknowns of discrete solution.

As $u_N^k(\pm 1) = 0$, we choose $v_N = h_i(x)$, $i = 1, 2, \ldots, N-1$. We rewrite scheme (3.1) as

$$\sum_{j=1}^{N-1} (h_j, h_i)_N \widehat{u}_j^k + \gamma\nu \sum_{j=1}^{N-1} (\partial_x h_j, \partial_x h_i)_N \widehat{u}_j^k = F_N^k(h_i), \quad 1 \le i \le N-1, \tag{5.2}$$

where

$$\begin{aligned} F_N^k(h_i) = {} & \gamma(f(2u_N^{k-1} - u_N^{k-2}, u_N^{k-n}), h_i)_N + \gamma(g^k, h_i)_N \\ & + \sum_{j=0}^{k-2} (a_j^{(\alpha)} - a_{j+1}^{(\alpha)})(u_N^{k-j-1}, h_i)_N + a_{k-1}^{(\alpha)}(u_N^0, h_i)_N \end{aligned}$$

with $\gamma = \tau^\alpha \Gamma(2-\alpha)$.

Similarly, the modified scheme (4.2) can be written as

$$\sum_{j=1}^{N-1} (h_j, h_i)_N \widehat{u}_j^k + \gamma\nu \sum_{j=1}^{N-1} (\partial_x h_j, \partial_x h_i)_N \widehat{u}_j^k = \widetilde{F}_N^k(h_i), \quad 1 \le i \le N-1, \tag{5.3}$$

where

$$\begin{aligned}\widetilde{F}_N^k(h_i) &= \gamma(f(2u_N^{k-1}-u_N^{k-2},u_N^{k-n}),h_i)_N+\gamma(g^k,h_i)_N\\&\quad+\sum_{j=0}^{k-2}(a_j^{(\alpha)}-a_{j+1}^{(\alpha)})(u_N^{k-j-1},h_i)_N+a_{k-1}^{(\alpha)}(u_N^0,h_i)_N\\&\quad+\gamma\sum_{j=1}^{m_2}\widetilde{\omega}_{k,j}(f(u^j,u^{k-n})-f(u^0,u^{k-n}),h_i)_N-\Gamma(2-\alpha)\sum_{j=1}^{m_1}\omega_{k,j}(u^j-u^0,h_i)_N.\end{aligned}$$

Moreover, according to (5.1), it holds

$$(h_j,h_i)_N=\sum_{l=0}^{N}h_j(x_l)h_i(x_l)\omega_l=\omega_i\delta_{ij},\quad (\partial_x h_j,\partial_x h_i)_N=\sum_{l=0}^{N}\partial_x h_j(x_l)\partial_x h_i(x_l)\omega_l.$$

Then one can obtain system of linear equations for (5.2) and (5.3) in the following,

$$\widehat{u}_i^k\omega_i+\gamma\nu\sum_{j=1}^{N-1}\sum_{l=0}^{N}\partial_x h_j(x_l)\partial_x h_i(x_l)\omega_l\widehat{u}_j^k=F_N^k(h_i),\quad 1\le i\le N-1, \tag{5.4}$$

$$\widehat{u}_i^k\omega_i+\gamma\nu\sum_{j=1}^{N-1}\sum_{l=0}^{N}\partial_x h_j(x_l)\partial_x h_i(x_l)\omega_l\widehat{u}_j^k=\widetilde{F}_N^k(h_i),\quad 1\le i\le N-1, \tag{5.5}$$

where the nonlinear terms $F_N^k(h_i)$ and $\widetilde{F}_N^k(h_i)$ on the right-hand side can be computed by

$$\begin{aligned}(f(2u_N^{k-1}-u_N^{k-2},u_N^{k-n}),h_i)_N&=\sum_{l=0}^{N}f(2u_N^{k-1}(x_l)-u_N^{k-2}(x_l),u_N^{k-n}(x_l))h_i(x_l)\omega_l\\&=\sum_{l=0}^{N}f(2\widehat{u}_l^{k-1}-\widehat{u}_l^{k-2},\widehat{u}_l^{k-n})h_i(x_l)\omega_l\\&=f(2\widehat{u}_i^{k-1}-\widehat{u}_i^{k-2},\widehat{u}_i^{k-n})\omega_i,\end{aligned}$$

and $\sum_{j=1}^{m_2}\widetilde{\omega}_{k,j}(f(u^j,u^{k-n})-f(u^0,u^{k-n}),h_i)_N$ can be obtained analogously.

5.2. Numerical results

We now present some numerical examples to support theoretical analysis. First, we consider the fractional Hutchinson's equation with smooth solution and describe convergence rates in temporal and spatial directions.

Example 5.1. We consider the following fractional Hutchinson's equation

$$\begin{cases}{}_0^C D_t^\alpha u(x,t)-\partial_x^2 u(x,t)=u(x,t)(1-u(x,t-0.1))+g(x,t) & (x,t)\in(-1,1)\times(0,T],\\ u(-1,t)=0,\quad u(1,t)=0 & t\in[0,T],\\ u(x,t)=t^{2+\alpha}\sin(\pi x) & (x,t)\in(-1,1)\times[-0.1,0],\end{cases}$$

where

$$g(x,t)=\frac{\Gamma(3+\alpha)}{2}t^2\sin(\pi x)+\pi^2t^{2+\alpha}\sin(\pi x)-t^{2+\alpha}\sin(\pi x)(1-(t-0.1)^{2+\alpha}\sin(\pi x)).$$

The exact solution is $u(x,t)=t^{2+\alpha}\sin(\pi x)$.

We first investigate the temporal accuracy by choosing N big enough to eliminate spatial error. Taking $T=1$ and $N=15$, Table 5.1 shows the discrete L^2 errors $\|u^k-u_N^k\|_N$ and L^∞ errors $\|u^k-u_N^k\|_{\infty,N}$ and the associated temporal convergence rates for different α, where the convergence rate is computed by $\log_{\tau_1/\tau_2}(e_1/e_2)$.

τ	$\alpha=0.01$		$\alpha=0.99$		$\alpha=0.01$		$\alpha=0.99$	
	L^2 Error	Rate	L^2 Error	Rate	L^∞ Error	Rate	L^∞ Error	Rate
$0.1/2^2$	3.9936e-04	1.9983	6.7679e-03	0.9600	4.5884e-04	1.9984	7.4645e-03	0.9429
$0.1/2^3$	9.9958e-05	1.9997	3.4792e-03	0.9851	1.1484e-04	1.9998	3.8829e-03	0.9773
$0.1/2^4$	2.4994e-05	2.0002	1.7577e-03	0.9975	2.8713e-05	2.0002	1.9722e-03	0.9938
$0.1/2^5$	6.2478e-06	2.0005	8.8037e-04	1.0037	7.1771e-06	2.0005	9.9033e-04	1.0019
$0.1/2^6$	1.5675e-06	2.0012	4.3905e-04	1.0068	1.7936e-06	2.0012	4.9450e-04	1.0059
$0.1/2^7$	3.9003e-07	*	2.1848e-04	*	4.4804e-07	*	2.4623e-04	*

Table 5.1: Errors and temporal convergence rates for Example 5.1.

From Table 5.1, $(2-\alpha)$-order temporal accuracy has been obtained for L^2 error, which is consistent with theoretical analysis. In addition, L^∞ error can also attain $(2-\alpha)$-order temporal accuracy.

Next, we present spatial convergence rate by choosing τ sufficiently small to avoid the contamination of the temporal error. Taking $T=1$ and $\tau=0.001$, Figure 5.1 plots the L^2 errors and L^∞ errors in semi-log scale with respect to the polynomial degree N for $\alpha=0.2$. Analogously, Figure 5.2 shows the errors for $\alpha=0.8$ with $\tau=0.0001$.

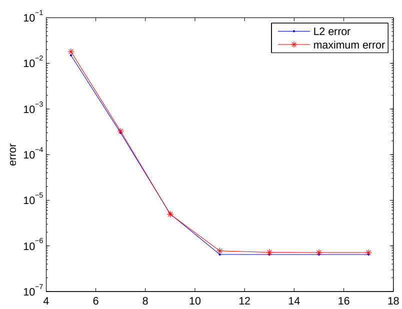

Figure 5.1: Errors as a function of the polynomial degree N for $\alpha=0.2$.

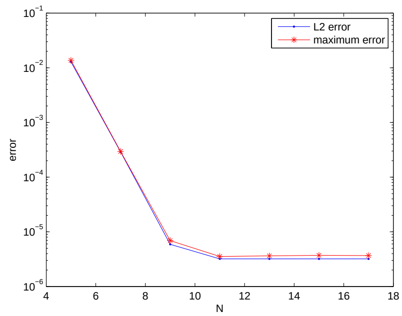

Figure 5.2: Errors as a function of the polynomial degree N for $\alpha = 0.8$.

In Figures 5.1 and 5.2, the L^2 errors and L^∞ errors decay exponentially, that is to say, we obtain spectral accuracy in spatial direction for smooth solution.

Since the spectral method depends on the regularity of the solution, we investigate the equation with limit regularity solution and present the spatial accuracy in following example.

Example 5.2. We consider the following equation

$$\begin{cases} {}_0^C D_t^\alpha u(x,t) - \nu \partial_x^2 u(x,t) = -u^2(x,t) + u(x,t-0.2) + g(x,t) & (x,t) \in (-1,1) \times (0,T], \\ u(-1,t) = 0, \quad u(1,t) = 0 & t \in [0,T], \\ u(x,t) = t^2(1-x^2)x^{13/3} & (x,t) \in (-1,1) \times [-0.2,0], \end{cases}$$

where

$$\begin{aligned} g(x,t) = & \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} - \nu t^2 \left(\frac{130}{9} x^{7/3} - \frac{304}{9} x^{13/3} \right) + t^4 (1-x^2)^2 x^{26/3} \\ & - (t-0.2)^2 (1-x^2) x^{13/3}. \end{aligned}$$

The exact solution is $u(x,t) = t^2(1-x^2)x^{13/3}$.

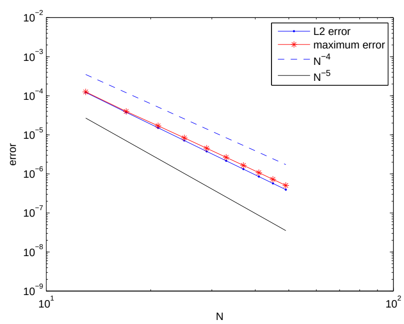

Figure 5.3: Errors as a function of the polynomial degree N for $\alpha = 0.1$.

It is easily seen that $u \in H^4(\Lambda)$, but $u \notin H^5(\Lambda)$. Taking $T = 1$ and $\tau = 0.001$, Figures 5.3 and 5.4 present the L^2 errors and L^∞ errors with respect to polynomial degree N for different α. To make a close comparison, we also plot the decay rates with N^{-4} and N^{-5}.

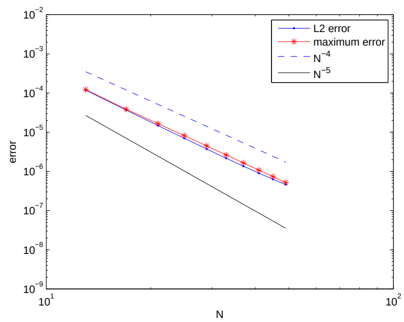

Figure 5.4: Errors as a function of the polynomial degree N for $\alpha = 0.5$.

From Figures 5.3 and 5.4, it is obviously observed that both of L^2 errors and L^∞ errors decay with a rate between N^{-4} and N^{-5}, which is accordance with theoretical analysis.

Finally, we investigate the equation with nonsmooth solution and present temporal convergence rates in following example.

Example 5.3. We consider the following equation

$$\begin{cases} {}_0^C D_t^\alpha u(x,t) - \nu \partial_x^2 u(x,t) = u(x,t) - (u(x,t-0.1))^2 + g(x,t) & (x,t) \in (-1,1) \times (0,T], \\ u(-1,t) = 0, \quad u(1,t) = 0 & t \in [0,T], \\ u(x,t) = (t^\alpha + t^3)\sin(\pi x) & (x,t) \in (-1,1) \times [-0.1, 0], \end{cases}$$

where

$$\begin{aligned} g(x,t) = & \left(\Gamma(1+\alpha) + \frac{6}{\Gamma(4-\alpha)} t^{3-\alpha} \right) \sin(\pi x) + \nu \pi^2 (t^\alpha + t^3) \sin(\pi x) \\ & - (t^\alpha + t^3)\sin(\pi x) + ((t-0.1)^\alpha + (t-0.1)^3)^2 (\sin(\pi x))^2. \end{aligned}$$

The exact solution is $u(x,t) = (t^\alpha + t^3)\sin(\pi x)$.

Taking $T = 1$ and $N = 25$, Tables 5.2 and 5.3 show the L^2 errors and L^∞ errors and associated temporal convergence rates for $\alpha = 0.2$, which are obtained by solving systems of linear equations (5.4) and (5.5). More precisely, errors and temporal convergence rates are obtained by adding correction terms for three cases: $m_1 = 0$, $m_2 = 0$; $m_1 = 1$, $m_2 = 0$ and $m_1 = 1$, $m_2 = 1$. In addition, we also present errors and temporal convergence rates for $\alpha = 0.8$ in Table 5.4.

τ	$m_1 = 0,\ m_2 = 0$		$m_1 = 1,\ m_2 = 0$		$m_1 = 1,\ m_2 = 1$	
	Error	Rate	Error	Rate	Error	Rate
$0.1/2^5$	1.7830e-05	1.3045	2.4169e-06	0.7092	5.6767e-06	2.1348
$0.1/2^6$	7.2186e-06	1.1963	1.4783e-06	0.8721	1.2926e-06	2.1751
$0.1/2^7$	3.1502e-06	1.1310	8.0771e-07	1.0617	2.8622e-07	2.2332
$0.1/2^8$	1.4384e-06	1.0954	3.8693e-07	1.1512	6.0876e-08	2.3255
$0.1/2^9$	6.7319e-07	1.0765	1.7422e-07	1.1933	1.2145e-08	2.4959
$0.1/2^{10}$	3.1921e-07	*	7.6189e-08	*	2.1531e-09	*

Table 5.2: L^2 errors and temporal convergence rates for Example 5.3.

τ	$m_1 = 0,\ m_2 = 0$		$m_1 = 1,\ m_2 = 0$		$m_1 = 1,\ m_2 = 1$	
	Error	Rate	Error	Rate	Error	Rate
$0.1/2^5$	2.2823e-05	1.3098	2.7844e-06	0.8112	7.2587e-06	2.1347
$0.1/2^6$	9.2066e-06	1.2023	1.5868e-06	0.7522	1.6529e-06	2.1751
$0.1/2^7$	4.0011e-06	1.1363	9.4209e-07	1.0159	3.6600e-07	2.2331
$0.1/2^8$	1.8203e-06	1.0994	4.6590e-07	1.1308	7.7847e-08	2.3254
$0.1/2^9$	8.4952e-07	1.0796	2.1275e-07	1.1827	1.5532e-08	2.4957
$0.1/2^{10}$	4.0196e-07	*	9.3719e-08	*	2.7539e-09	*

Table 5.3: L^∞ errors and temporal convergence rates for Example 5.3.

τ	$m_1 = 0,\ m_2 = 0$		$m_1 = 1,\ m_2 = 0$		$m_1 = 0,\ m_2 = 0$		$m_1 = 1,\ m_2 = 0$	
	L^2 Error	Rate	L^2 Error	Rate	L^∞ Error	Rate	L^∞ Error	Rate
$0.1/2^5$	2.3242e-04	1.1898	2.5256e-04	1.1811	2.8366e-04	1.1911	3.1183e-04	1.1810
$0.1/2^6$	1.0189e-04	1.1997	1.1138e-04	1.1891	1.2423e-04	1.2014	1.3753e-04	1.1890
$0.1/2^7$	4.4359e-05	1.2062	4.8848e-05	1.1937	5.4022e-05	1.2084	6.0371e-05	1.1937
$0.1/2^8$	1.9226e-05	1.2110	2.1355e-05	1.1964	2.3378e-05	1.2137	2.6371e-05	1.1963
$0.1/2^9$	8.3501e-06	1.2148	9.3185e-06	1.1979	1.0080e-05	1.2182	1.1508e-05	1.1979
$0.1/2^{10}$	3.5780e-06	*	4.0620e-06	*	4.3323e-06	*	5.0165e-06	*

Table 5.4: Errors and temporal convergence rates for Example 5.3.

From Table 5.2, it is obviously seen that L^2 errors do not attain $(2-\alpha)$-order temporal accuracy for smaller fractional order α by solving systems of linear equations (5.4), which is caused by low the regularity of solution. After adding one correction term, namely, we correct $L1$ formulation by (4.1), the temporal convergence rates do not improve, but the errors become smaller. Then continuing to add one correction term for nonlinear term, the temporal convergence rates are really improved. It also demonstrates that we can obtain better accuracy for low regularity solution only by adding a few correction terms. In addition, similar results have been obtained for L^∞ errors in Table 5.3.

It is observed that the accuracy almost does not improve for $\alpha = 0.8$ in Table 5.4. It is because that the solution has higher regularity and the temporal accuracy is lower for big α. According to above numerical results, it can be seen that adding suitable correction terms not only makes the new approximations exact for low regularity terms of solutions but also maintains accuracy for high regularity terms.

The numerical results are in accordance with theoretical analysis. We obtain $(2-\alpha)$-order temporal accuracy and spectral accuracy in space for smooth solution. For the solution with limited regularity, algebra accuracy in space has been obtained. By adding a few correction terms to the approximations of fractional derivative and nonlinear term for the problem with nonsmooth solutions, the temporal accuracy can be improved for small α, and it can also maintains accuracy for high regularity terms. It should be pointed that only modifying the $L1$ formulation can not attain the desired accuracy for small α, the nonlinear term also need to be modified.

6. Conclusion

We have constructed finite difference/spectral scheme for nonlinear fractional delay diffusion equation with smooth solutions. It has been proved that the scheme is unconditionally stable and convergent with order $O(\tau^{2-\alpha}+N^{-m})$. For the problems with nonsmooth solutions, we add a few correction terms to the approach of fractional derivative and nonlinear term, which not only makes the new approximations exact for low regularity terms of solutions but also maintains accuracy for high regularity terms. Numerical examples have been presented to confirm our theoretical results.

It should be pointed that the schemes constructed in this paper for smooth and nonsmooth solutions can be extended to solve the fractional multidelay equations, where nonlinear term $f(u(x,t),u(x,t-s))$ is replaced by $f(u(x,t),u(x,t-s_1),\ldots,u(x,t-s_l))$. Similarly, the unconditional stability and convergence will be obtained.

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