TAIWANESE JOURNAL OF MATHEMATICS
Vol. 23, No. 1, pp. 63–85, February 2019
DOI: 10.11650/tjm/180805

Skew Generalized Power Series Rings and the McCoy Property

Masoome Zahiri, Rasul Mohammadi, Abdollah Alhevaz* and Ebrahim Hashemi

Abstract. Given a ring R, a strictly totally ordered monoid $(S, \preceq)$ and a monoid homomorphism $\omega\colon S \to \mathrm{End}(R)$, one can construct the skew generalized power series ring $R[[S, \omega, \preceq]]$, consisting all of the functions from a monoid S to a coefficient ring R whose support is artinian and narrow, where the addition is pointwise, and the multiplication is given by convolution twisted by an action ω of the monoid S on the ring R. In this paper, we consider the problem of determining some annihilator and zero-divisor properties of the skew generalized power series ring $R[[S, \omega, \preceq]]$ over an associative non-commutative ring R. Providing many examples, we investigate relations between McCoy property of skew generalized power series ring, namely (S, ω)-McCoy property, and other standard ring-theoretic properties. We show that if R is a local ring such that its Jacobson radical $J(R)$ is nilpotent, then R is (S, ω)-McCoy. Also if R is a semicommutative semiregular ring such that $J(R)$ is nilpotent, then R is (S, ω)-McCoy ring.

1. Introduction and definitions

Throughout this paper, all monoids and rings are with identity element that is inherited by submonoids and subrings and preserved under homomorphisms, but neither monoids nor rings are assumed to be commutative. Moreover, $\mathbb{N}$ will denote the set of natural numbers and $\mathbb{Z}$ will denote the ring of integers. We denote the identity matrix and unit matrices in the full matrix ring $M_n(R)$, by I_n and E_{ij}, respectively. We also adopt the notation $R[S]$ to represent the monoid ring of a monoid S over a ring R, and by $J(R)$, $\mathrm{Nil}(R)$ and $\mathrm{Nil}^*(R)$, we mean the Jacobson radical of a ring R, the set of all nilpotent elements in R and the upper nilradical (i.e., the sum of all nil ideals) of R, respectively. All other terminology is standard, and definitions can be found in [14].

In their pioneering work [22], Nielsen introduced McCoy property of associative rings which have since become one of the widely used tools for studying the zero-divisors and annihilators of a ring extensions. Recall that a ring R is said to be *right McCoy* if the equation $f(x)g(x) = 0$ implies $f(x)c = 0$ for some nonzero $c \in R$, where $f(x)$, $g(x)$

Received January 7, 2018; Accepted August 6, 2018.
Communicated by Kunio Yamagata.
2010 *Mathematics Subject Classification*. Primary: 16S35; Secondary: 16N40, 06F05.
Key words and phrases. skew generalized power series ring, (S, ω)-McCoy ring, strictly ordered monoid, unique product monoid, reversible ring, semi-regular ring.
*Corresponding author.

are nonzero polynomials in $R[x]$. *Left McCoy rings* are defined dually and they satisfy dual properties. A ring R is called *McCoy* if it is both left and right McCoy. This nomenclature was used by them since it was McCoy [20, Lemma 1] who initially showed that any commutative ring satisfies this condition. These rings, though may look a bit specific, were studied by many authors so far and are related to important problems. Systematic studies of one-sided McCoy rings were started in [22] and next continued in a number of papers. There are many ways to generalize the McCoy condition, and we direct the reader to the papers [2, 3, 5, 11, 19, 21, 28, 29] for a nice introduction to these topics.

McCoy's theorem fails in general for the case of formal power series ring $R[[x]]$ over a commutative ring R, by [9]. In fact, Fields [9, Theorem 5] proved that if R is a commutative Noetherian ring with identity in which $(0) = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ is a shortest primary representation of (0), then $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$ is a zero-divisor in $R[[x]]$ if and only if there is a nonzero element $r \in R$ which satisfies $rf(x) = 0$. He also provided an example showing that the condition "R is Noetherian" is not redundant [9, Example 3]. Moreover, Camillo and Nielsen [5, Section 3], constructed an example showing that formal power series rings over an associative non-commutative McCoy ring R, need not be McCoy in general.

As a continuation of the above results on McCoy rings, Yang et al. [28], introduced the concept of power-serieswise McCoy ring. A ring R is called *right power series-wise McCoy* if there exists a nonzero annihilator $c \in R$, with $f(x)c = 0$, for any $f(x), g(x) \in R[[x]]$ satisfying $f(x)g(x) = 0$. This condition was introduced by Yang et al. to develop an annihilator theory for power series rings. *Left power series-wise McCoy rings* are defined in the dual way, or, formally, by saying that a ring R is left power series-wise McCoy if and only if the ring R^{op} opposite to R is right power series-wise McCoy. Throughout this paper, we call these rings *McCoy of power series type.* Following Başer et al. [3], we say that a ring R with an endomorphism σ is *σ-skew McCoy* if whenever polynomials $f(x)$, $g(x)$ in skew polynomial ring $R[x;\sigma]$ satisfy $f(x)g(x) = 0$ then $f(x)c = 0$ for some nonzero $c \in R$. A stronger condition than σ-skew McCoy was introduced and studied by Alhevaz and Kiani in [2]. A ring R with an endomorphism σ is *σ-skew power serieswise McCoy* if whenever power series $f(x)$, $g(x)$ in skew power series ring $R[[x;\sigma]]$ satisfy $f(x)g(x) = 0$ then $f(x)c = 0$ for some nonzero $c \in R$. In [11], Hashemi extended the McCoy notion to monoid rings. If R is a ring and S a monoid, then R is called a *right McCoy ring relative to monoid S* (*right S-McCoy ring*) if whenever elements $\alpha = \sum_{i=1}^{m} a_i s_i$ and $\beta = \sum_{j=1}^{n} b_j t_j$ of the monoid ring $R[S]$ satisfy $\alpha\beta = 0$, then $\alpha c = 0$ for some nonzero $c \in R$. *Left S-McCoy rings* are defined similarly. If R is both left and right S-McCoy, then we say R is *S-McCoy.*

Recently in [1], Alhevaz and Hashemi unified the above versions of McCoy rings by introducing the notion of (S, ω)-McCoy ring, where $(S, \leq)$ is a strictly ordered monoid and $\omega\colon S \to \mathrm{End}(R)$, $s \mapsto \omega_s$, is a monoid homomorphism. A ring R is called *right* (S, ω)-*McCoy* if whenever $fg = 0$ for nonzero elements $f, g \in R[[S, \omega, \leq]]$, then there exists a nonzero element $c \in R$ such that $fc = 0$ or equivalently $f(s).\omega_s(c) = 0$ for all $s \in S$ [1, Definition 2.18]. *Left* (S, ω)-*McCoy* rings are defined similarly, and a ring R is called (S, ω)-McCoy if whenever it is left and right (S, ω)-McCoy.

We continue by recalling the structure of the skew generalized power series ring construction, introduced in [18]. In order to recall the structure of this ring construction, we need some definitions. In this paper "an order" on a set will always mean "a partial order". For a ring R, the monoid of endomorphisms of R (with composition of endomorphisms as the operation) is denoted by $\mathrm{End}(R)$. Let $(S, \leq)$ be an ordered set. Then $(S, \leq)$ is called *artinian* if every strictly decreasing sequence of elements of S is finite and $(S, \leq)$ is called *narrow* if every subset of pairwise order-incomparable elements of S is finite. Thus, $(S, \leq)$ is artinian and narrow if and only if every nonempty subset of S has at least one but only a finite number of minimal elements. Clearly, the union of a finite family of artinian and narrow subsets of an ordered set as well as any subset of an artinian and narrow set are again artinian and narrow.

A monoid S (written multiplicatively) equipped with an order $\leq$ is called an *ordered monoid* if for any $s_1, s_2, t \in S$, $s_1 \leq s_2$ implies $s_1 t \leq s_2 t$ and $ts_1 \leq ts_2$. Moreover, if $s_1 < s_2$ implies $s_1 t < s_2 t$ and $ts_1 < ts_2$, then $(S, \leq)$ is said to be *strictly ordered.* A monoid S is said to be *totally orderable* if $(S, \leq)$ is an ordered monoid for some total order $\leq$. Also by [16, Definition 4.11], a monoid $(S, \leq)$ is called *quasitotally ordered* (and that $\leq$ is a *quasitotal order* on S) if $\leq$ can be refined to an order $\preceq$ with respect to which S is a strictly totally ordered monoid. For a strictly ordered monoid S and a ring R, in the 1990s, Elliott and Ribenboim [7] defined the ring of generalized power series $R[[S]]$ consisting of all maps from S to R whose support is artinian and narrow with the the pointwise addition and the convolution multiplication. This construction provided interesting examples of rings (see e.g., Elliott and Ribenboim [7], Ribenboim [25, 26]) and it was extensively studied by many authors.

In [18], Mazurek and Ziembowski, introduced a "twisted" version of the Ribenboim construction and study when it produces a von Neumann regular ring. Now we recall the construction of the skew generalized power series ring introduced in [18]. Let R be a ring, $(S, \leq)$ a strictly ordered monoid, and $\omega\colon S \to \mathrm{End}(R)$ a monoid homomorphism. For $s \in S$, let ω_s denote the image of s under ω, that is $\omega_s = \omega(s)$. Let A be the set of all functions $f\colon S \to R$ such that the $\mathrm{Supp}(f) = \{s \in S : f(s) \neq 0\}$ is artinian and narrow. Then, for any $s \in S$ and $f, g \in A$, the set

$$X_s(f, g) = \{(x, y) \in \mathrm{Supp}(f) \times \mathrm{Supp}(g) : s = xy\}$$

is finite. Thus, one can define the product $fg\colon S \to R$ of $f, g \in A$ as follows:

$$fg(s) = \sum_{(u,v)\in X_s(f,g)} f(u)\omega_u(g(v))$$

(by convention, a sum over the empty set is 0). With pointwise addition and multiplication as defined above, A becomes a ring, called *the ring of skew generalized power series* with coefficients in R and exponents in S (one can think of a map $f\colon S \to R$ as a formal series $\sum_{s\in S} r_s s$, where $r_s = f(s) \in R$) and denoted either by $R[[S, \omega, \leq]]$ (or by $R[[S, \omega]]$, when there is no ambiguity concerning the order $\leq$) (see [16, 18]). The skew generalized power series construction embraces a wide range of classical ring-theoretic extensions, including skew polynomial rings, skew power series rings, skew Laurent polynomial rings, skew group rings, Malcev Neumann Laurent series rings, and of course the untwisted versions of all of these. We will use the symbol 1 to denote the identity elements of the monoid S, the ring R, and the ring $R[[S, \omega, \leq]]$, as well as the trivial monoid homomorphism $1\colon S \to \mathrm{End}(R)$ that sends every element of S to the identity endomorphism.

For each $r \in R$ and $s \in S$, let $c_r, e_s \in R[[S, \omega, \leq]]$ defined by

$$c_r(x) = \begin{cases} r & \text{if } x = 1, \\ 0 & \text{if } x \in S \setminus \{1\}, \end{cases} \qquad e_s(x) = \begin{cases} 1 & \text{if } x = s, \\ 0 & \text{if } x \in S \setminus \{s\}. \end{cases}$$

It is clear that $r \to c_r$ is a ring embedding of R into $R[[S, \omega, \leq]]$ and $s \to e_s$ is a monoid embedding of S into the multiplicative monoid of the ring $R[[S, \omega, \leq]]$, and $e_s c_r = c_{\omega_s(r)} e_s$.

The construction of the skew generalized power series rings generalizes some classical ring constructions such as polynomial rings ($S = \mathbb{N} \cup \{0\}$ under usual addition, with the trivial order, and ω is trivial), monoid rings (trivial order, and ω is trivial), skew polynomial ring $R[x; \sigma]$ for some $\sigma \in \mathrm{End}(R)$ ($S = \mathbb{N} \cup \{0\}$ under usual addition, with the trivial order, and $\omega_1 = \sigma$), skew Laurent polynomial ring $R[x, x^{-1}; \sigma]$ for some $\sigma \in \mathrm{Aut}(R)$ ($S = \mathbb{Z}$ under usual addition, with the trivial order, and $\omega_1 = \sigma$), skew monoid rings (with trivial order), skew power series ring $R[[x; \sigma]]$ for some $\sigma \in \mathrm{End}(R)$ ($S = \mathbb{N} \cup \{0\}$ under usual addition, with the usual order, and $\omega_1 = \sigma$), skew Laurent power series ring $R[[x, x^{-1}; \sigma]]$ for some $\sigma \in \mathrm{Aut}(R)$ ($S = \mathbb{Z}$ with usual addition, with the usual order, and $\omega_1 = \sigma$), the Mal'cev-Neumann construction ($(S, ., \leq)$ a totally ordered group and trivial ω) the Mal'cev-Neumann construction of twisted Laurent series rings ($(S, ., \leq)$ a totally ordered group; see [14, p. 230]), and generalized power series rings (ω is trivial; see [26, Section 4]).

In this paper, we study the McCoy property of skew generalized power series rings. Providing many examples, we investigate the relationship between McCoy property of skew generalized power series ring, namely (S, ω)-McCoy property, and other standard ring-theoretic properties. Let $(S, \preceq)$ be a strictly totally ordered monoid, $\omega\colon S \to \mathrm{End}(R)$

is a monoid homomorphism such that ω_s is injective for all $s \in S \setminus \{1\}$. We will show that, if R is a local ring such that $J(R)$ is nilpotent, then R is (S, ω)-McCoy. Moreover, we show that if R is abelian, semiregular and S-compatible ring with $(J(R))^2 = 0$, then R is (S, ω)-McCoy. Also, if R is a semiregular, semicommutative and S-compatible ring such that $J(R)$ is a nilpotent ideal, then R is (S, ω)-McCoy. Our results in this paper, extend to skew generalized power series rings many results known earlier for special ring extensions such as power series and monoid rings. Nevertheless, we wish to emphasize that many of our results over skew generalized power series ring are new even in the case of power series rings and monoid rings.

2. (S, ω)-McCoy rings

Recall by [16, Definition 2.1] that, a ring R is called (S, ω)-*Armendariz* if whenever $fg = 0$ for $f, g \in R[[S, \omega, \leq]]$, then $f(s) \cdot \omega_s(g(t)) = 0$ for all $s, t \in S$. We start this section by definition of the McCoy rings in the case of skew generalized power series ring over general non-commutative rings, as appears in [1, Definition 2.18].

Definition 2.1. A ring R is called *right* (S, ω)-*McCoy* if whenever $fg = 0$ for nonzero elements $f, g \in R[[S, \omega, \leq]]$, then there exists a nonzero element $c \in R$ such that $fc = 0$, or equivalently $f(s).\ \omega_s(c) = 0$ for all $s \in S$. *Left* (S, ω)-*McCoy* rings are defined similarly, and a ring R is called (S, ω)-McCoy if whenever it is left and right (S, ω)-McCoy.

One might notice that if $S = \{1\}$ then every ring is (S, ω)-McCoy. One can see easily that each (S, ω)-Armendariz ring is (S, ω)-McCoy, but the converse is not true in general. In the following, we will see that with some limitations on S, the order $\leq$ and the monoid homomorphism ω, the notion of (S, ω)-McCoy ring collapse to their special cases in the literature.

Example 2.2. (a) Suppose R is McCoy ring, as in [22]. This is the special case of (S, ω)-McCoy ring, where $S = \mathbb{N} \cup \{0\}$ with usual addition, with the trivial order $\leq$ and trivial ω.

(b) Suppose R is σ-skew McCoy ring for some $\sigma \in \mathrm{End}(R)$, as in [3]. This is the special case of (S, ω)-McCoy ring, where $S = \mathbb{N} \cup \{0\}$ with usual addition, with the trivial order $\leq$ and ω is determined by $\omega(1) = \sigma$.

(c) Suppose R is McCoy of power series type, as in [28]. This is the special case of (S, ω)-McCoy ring, where $S = \mathbb{N} \cup \{0\}$ with usual addition, with its natural linear order $\leq$ and trivial ω.

(d) Suppose R is McCoy ring relative to a monoid S, as in [11]. This is the special case of (S, ω)-McCoy ring, where S is given, with the trivial order $\leq$ and trivial ω.

(e) Suppose R is σ-skew McCoy ring of Laurent polynomial type for some $\sigma \in \mathrm{Aut}(R)$, as in [2]. This is the special case of (S, ω)-McCoy ring, where $S = \mathbb{Z}$ with usual addition, with the trivial order $\leq$ and ω is determined by $\omega(1) = \sigma$.

(f) Suppose R is σ-skew McCoy ring of Laurent series type for some $\sigma \in \mathrm{Aut}(R)$, as in [2]. This is the special case of (S, ω)-McCoy ring, where $S = \mathbb{Z}$ with usual addition, with the usual order $\leq$ and ω is determined by $\omega(1) = \sigma$.

In the rest of the paper we will need some well-known zero-divisor conditions. We introduce them now. A ring R is called *reduced* if it has no nonzero nilpotent element. According to Cohn [6], a ring R is called *reversible* if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Prior to Cohn's work, reversible rings were studied under the name *completely reflexive* by Mason in [17] and under the name *zero commutative*, or *zc*, by Habeb in [10]. In his monograph [27] on distributive lattices arising in ring theory, Tuganbaev investigates a property called *commutative at zero*, which is equivalent to the reversible condition on rings. Note that for the class of reversible rings the set of all left annihilators of any element $a \in R$ coincide with the set of its all right annihilators and we denote it by $\mathrm{ann}_R(a)$. Due to Bell [4], a ring R is called to satisfy the *Insertion-of-Factors-Property* if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. In the literature, such a ring also goes by the name *semicommutative*, *SI* or *zero insertive*. In this note, we choose "a semicommutative ring" among them, so as to cohere with other related references. Moreover, due to Feller [8], a ring is *right* (resp., *left*) *duo* if every right (resp., left) ideal is an ideal. A ring R is called *abelian* if all idempotents in R are central. Simple computations show that reversible as well (one-sided) duo rings are semicommutative and abelian. The study of these rings is one of the central topics in noncommutative ring theory because of the famous Köthe's problem which ask whether every one-sided nil ideal of any associative ring is contained in a two-sided nil ideal of the ring. As observed by Bell [4], these rings fulfill the requirements of the Köthe's Conjecture.

It is known that reduced rings are McCoy of power series type, and McCoy rings of power series type are McCoy; but the converses are not true (see [28]). Notice also that reversible as well as right duo rings are McCoy (see, [22, Theorem 2] and [5, Theorem 8.2]). Because of these results, it is natural to ask whether every reversible or right duo ring is McCoy of power series type, or generally (S, ω)-McCoy.

Example 2.3. There exists a commutative (and hence reversible and right duo) local ring that is not McCoy of power series type.

Proof. Let $\mathbb{Z}_2$ be the ring of integers modulo 2 and let $R_0 = \mathbb{Z}_2[x_1, x_2, \ldots]$ be the polynomial ring with commuting indeterminates x_i over $\mathbb{Z}_2$, and I be the ideal of R_0 generated by x_i^{i+1} for each i. Set $R = R_0/I$. Assume that $S = \mathbb{N} \cup \{0\}$ under addition, with the usual linear order, and $\omega\colon S \to \mathrm{End}(R)$ is determined by $\omega_n = \omega(n) = \mathrm{Id}_R$ for each $n \in \mathbb{N} \cup \{0\}$. Then clearly R is commutative and local ring, but one can see that R is not McCoy of power series type. For, let $f(y) = \sum_{i=1}^{\infty} x_i^i y^{i-1}$ in $R[[y]] \cong R[[S, \omega, \leq]]$. Then $f^2(y) = 0$, whereas there does not exist a nonzero element c of R which annihilates all the coefficients of $f(y)$ simultaneously, as desired. $\square$

Following the above example, it is natural to ask the following question.

Question 2.4. Under what conditions, reversible rings are (S, ω)-McCoy?

In the following, we study this question by providing some conditions for a ring R and a monoid S such that the ring R is (S, ω)-McCoy.

Let P and Q be non-empty subsets of a monoid M. An element s is called a *u.p.-element* (*unique product element*) of $PQ = \{pq : p \in P, q \in Q\}$ if it is uniquely presented in the form $s = pq$ where $p \in P$ and $q \in Q$. Recall that a monoid M is called a *u.p.-monoid* (*unique product monoid*) if for any two non-empty finite subsets $P, Q \subseteq M$ there exist a u.p.-element in PQ. Unique product monoids and groups play an important role in ring theory, for example providing a positive case in the zero-divisor problem for group rings (see also [23]), and their structural properties have been extensively studied (see [15, 23, 24]). The class of u.p.-monoids includes the right and the left totally ordered monoids, sub-monoids of a free group, and torsion-free nilpotent groups. Every u.p.-monoid M has no non-unity element of finite order. Thus if M is a u.p.-monoid and $1 \neq g \in M$, then the set $\{1, g, g^2, \ldots\}$ is infinity.

Marks et al. [15], introduced new classes of u.p.-monoids as follows: An ordered monoid $(M, \leq)$ is called an *artinian narrow unique product monoid* (or an *a.n.u.p.-monoid*, or simply *a.n.u.p.*) if for every two artinian and narrow subsets A and B of M there exists a u.p.-element in the product AB. Also an ordered monoid $(M, \leq)$ is called a *minimal artinian narrow unique product monoid* (or a *m.a.n.u.p.-monoid*, or simply *m.a.n.u.p.*) if for every two artinian and narrow subsets A and B of M there exist $a \in \min A$ and $b \in \min B$ such that ab is a u.p.-element of AB.

For an ordered monoid $(M, \leq)$, Marks et al. [15] proved the following implications:

$$\begin{gathered} M \text{ is a commutative, torsion-free and cancellative monoid} \\ \Downarrow \\ (M, \leq) \text{ is quasitotally ordered} \\ \Downarrow \\ (M, \leq) \text{ is a m.a.n.u.p.-monoid} \\ \Downarrow \\ (M, \leq) \text{ is an a.n.u.p.-monoid} \end{gathered}$$

$$\Downarrow$$

$$M \text{ is a u.p.-monoid.}$$

Also it is shown that all of the implications in diagram above are irreversible.

Recall that a monoid S is *cancellative* if for all $s, t, z \in S$, $s \neq t$ implies $sz \neq tz$ and $zs \neq zt$. Also, a monoid S is called *aperiodic* if for any $s \in S \setminus \{1\}$ and $m, n \in \mathbb{N}$, we have $s^m \neq s^n$. By [16, Lemma 4.2], for strictly ordered monoid $(S, \leq)$, if R is (S, ω)-Armendariz then the monoid S is cancellative and aperiodic. We know that each strictly totally ordered monoid is cancellative in general. A detailed analysis of the proof of [16, Lemma 4.2] shows that the hypothesis of being strictly ordered for a monoid S, is not necessary and not used in the proof. In the following, by a similar way we can show that the result is also true for a larger class of (S, ω)-McCoy rings.

Lemma 2.5. *Let R be any ring, S be any monoid and $\omega\colon S \to \mathrm{End}(R)$ a monoid homomorphism such that ω_s is injective for all $s \in S \setminus \{1\}$. If R is (S, ω)-McCoy ring, then the monoid S is cancellative and aperiodic.*

Proof. We first show that S is cancellative. Let $s, t \in S$ be such that $s \neq t$. Suppose, towards a contradiction, that there exists $z \in S$ such that $sz = tz$. Then in $R[[S, \omega, \leq]]$ we have $(e_s - e_t)e_z = 0$, and since R is (S, ω)-McCoy, it follows that $(e_s - e_t)c = 0$ for some nonzero $c \in R$, or equivalently $(e_s - e_t)(s).\omega_s(c) = 0$ for all $s \in S$. Then $1.\omega_s(c) = 0$, and hence $c = 0$, since ω_s is injective, which is a contradiction. Similarly, one can prove that $s \neq t$ implies $zs \neq zt$. Hence S is cancellative.

Now, we show that S is aperiodic. Suppose, towards a contradiction, that S is not aperiodic. Then from above, and using the cancellative property of S, we deduce that there exists $s \in S \setminus \{1\}$ such that $s^n = 1$ for some $n \in \mathbb{N}$. We can assume that $s^i \neq 1$ for each $i \in \{1, 2, \ldots, n-1\}$. Since $(1 - e_s)(1 + e_s + e_{s^2} + \cdots + e_{s^{n-1}}) = 0$, then from (S, ω)-McCoy property of R we deduce that $(1 - e_s)c = 0$ for some nonzero $c \in R$, or equivalently $(1 - e_s)(1).c = 0$. Then $c = 0$, which is a contradiction. Hence S is aperiodic. $\square$

Recall that an endomorphism σ of a ring R is called *rigid* if for every $a \in R$, we have $a\sigma(a) = 0$ if and only if $a = 0$. Also by [12], an endomorphism σ of a ring R is called *compatible* if for every $a, b \in R$, we have $ab = 0$ if and only if $a\sigma(b) = 0$. Let R be any ring, $(S, \leq)$ a strictly ordered monoid and also $\omega\colon S \to \mathrm{End}(R)$ a monoid homomorphism. Following [16], a ring R is called S-compatible (or (S, ω)-compatible) if ω_s is compatible for each $s \in S$. A ring R is called S-rigid (or (S, ω)-rigid) if ω_s is rigid for each $s \in S$. By [12, Lemma 2.2], one can see that a ring R is (S, ω)-rigid if and only if R is (S, ω)-compatible and reduced. Also each compatible endomorphism is injective.

Note that each (S, ω)-rigid ring is (S, ω)-McCoy, but the converse is not true in general. In what follows, we can find many examples of non-reduced (S, ω)-McCoy rings. Recall that an element a of a ring R is called *regular* if there exists $b \in R$ such that $a = aba$. A ring R is said to be (*von Neumann*) *regular* if every element of R is regular. Next, we prove that when R is regular and (S,ω)-compatible, then (S,ω)-McCoy rings coincide with (S,ω)-rigid rings.

Theorem 2.6. *Let R be a regular ring, $(S,\leq)$ a u.p.-monoid and $\omega\colon S \to \mathrm{End}(R)$ a monoid homomorphism. If R is S-compatible and (S,ω)-McCoy, then R is (S,ω)-rigid.*

Proof. Since R is S-compatible, hence it suffices to show that R is reduced. First of all notice that, since R is S-compatible, for every $s \in S$ the endomorphism ω_s is idempotent-stabilizing, that is, $\omega_s(e) = e$ for every $e \in R$. Since abelian regular rings are reduced, it suffices to show that R is abelian. Assume on the contrary that there exist $e^2 = e \in R$ and $r \in R$ with $er - re \neq 0$. Then $er(1-e) \neq 0$ or $(1-e)re \neq 0$. Let $a = er(1-e) \neq 0$. Since R is regular, there exists $b \in R$ with $aba = a$, where we may assume $b = (1-e)be$. Then, it is easy to see that $a^2 = 0 = b^2$, $(ab)^2 = ab$ and $(ba)^2 = ba$. Since ω_s is idempotent-stabilizing endomorphism, we have $\omega_s(ab) = ab$ and $\omega_s(ba) = ba$ for each $s \in S$. Also, since $aba = a$ we have $(1-ab)a = 0$ and hence $(1-ab)\omega_s(a) = 0$. Now, consider two nonzero elements $f = c_b + c_{(1-ab)}e_s$ and $g = c_{(ba)} - c_a e_s \in R[[S,\omega,\leq]]$. Using the above computations, we have $fg = 0$. So there exists nonzero $c \in R$ such that $fc = 0$, since R is (S,ω)-McCoy ring. This yields $(1-ab)\omega_s(c) = 0$ and $bc = 0$. From $bc = 0$ we have $b\omega_s(c) = 0$, since R is S-compatible. Also from $(1-ab)\omega_s(c) = 0$ we have $\omega_s(c) - ab\omega_s(c) = 0$. Hence $\omega_s(c) = \omega_s(c) - ab\omega_s(c) = 0$. Since ω_s is compatible and hence is injective, we get $c = 0$, a contradiction. The case $(1-e)re \neq 0$ can be handled similarly. Therefore R is abelian. $\square$

Now we are in a good position to state our main results in this paper. Before proving our first main theorem, we need to introduce some definitions and notations. A ring R is called *semiregular* if $R/J(R)$ is regular and idempotents lift modulo $J(R)$ (i.e., if, whenever $a^2 - a \in J(R)$, there exists $e^2 = e \in R$ such that $e - a \in J(R)$).

Let I be an ideal of R. Then we denote

$$I[[S,\omega,\preceq]] = \{f \in R[[S,\omega,\preceq]] \mid f(s) \in I \text{ for every } s \in S\}.$$

Our next result provides an important and large class of (S,ω)-McCoy rings.

Theorem 2.7. *Let R be a semicommutative ring, $(S,\preceq)$ a strictly totally ordered monoid and $\omega\colon S \to \mathrm{End}(R)$ a monoid homomorphism. If R is S-compatible and semiregular ring such that $J(R)$ is nilpotent, then R is (S,ω)-McCoy.*

Proof. We prove only the right case and the proof of the left case is similar. If $J(R) = 0$, then R will be regular, and since every semicommutative regular ring is reduced, then R is (S,ω)-McCoy.

If $J(R) \neq 0$, then since $J(R)$ is nilpotent, there exists an integer $k \geq 2$ such that $J(R)^{k-1} \neq 0 = J(R)^k$. If $f \in J(R)[[S, \omega, \preceq]]$, then $f(s) \in J(R)$ and so $f(s)J(R)^{k-1} = 0$. Taking $0 \neq r \in J(R)^{k-1}$, we get $f(s)r = 0$ for every $s \in S$, i.e., R is right (S, ω)-McCoy.

Now, assume that $f \notin J(R)[[S, \omega, \preceq]]$. Let $A = R[[S, \omega, \preceq]]$ and $f, g \in A$ with $fg = 0$ where $g \neq 0$. We consider the following two cases:

Case 1. If $J(R)g(s) = 0$ for every $s \in S$, then it suffices to show that the set

$$\mathrm{Supp}'(f) = \{s \in S \mid c_{f(s)}g \neq 0\} \subseteq \mathrm{Supp}(f)$$

is empty, because in this case by using S-compatibility and semicommutativity of R, we get $f(s)g(t) = 0$ for all $s, t \in S$, while $g \neq 0$.

Hence, assume towards a contradiction, that $\mathrm{Supp}'(f)$ is a non-empty set. Since S is strictly totally ordered monoid, there exists an element $x_0 \in S$ such that x_0 is minimal element in $\mathrm{Supp}'(f)$ with respect to the order $\preceq$. Again, since S is strictly totally ordered monoid, we can write $\mathrm{Supp}(g) = \{y_0, y_1, y_2, \ldots\}$, where $y_0 \prec y_1 \prec y_2 \prec \cdots$, so y_0 is minimal in $\mathrm{Supp}(g)$. Hence $f(x_0')g(u) = 0$ and $f(v)g(y_0') = 0$ for any $u, v \in S$ and $x_0' \prec x_0$ and $y_0' \prec y_0$.

Since $(s,t) \in X_{(x_0,y_0)}(f,g)\backslash\{(x_0,y_0)\}$, hence we get $s \prec x_0$ and $y_0 \prec t$, this is because y_0 is the unique minimal element in $\mathrm{Supp}(g)$. Hence, for every $(s,t) \in X_{(x_0,y_0)}(f,g)\backslash\{(x_0,y_0)\}$ we get $f(s)g(t) = 0$ and then by S-compatibility of R, we have $f(s)\omega_s(g(t)) = 0$. From this and $\sum_{(s,t)\in X_{(x_0,y_0)}(f,g)} f(s)\omega_s(g(t)) = 0$, we must have $f(x_0)\omega_{x_0}(g(y_0)) = 0$ and then $f(x_0)g(y_0) = 0$.

As R is semiregular, there exist $r_0 \in R$ and $j_0 \in J(R)$ such that $f(x_0)r_0f(x_0) + j_0 = f(x_0)$. Hence, $f(x_0)r_0f(x_0)r_0 + J(R) = f(x_0)r_0 + J(R)$, i.e., $f(x_0)r_0 + J(R)$ is an idempotent element of $R/J(R)$. If $f(x_0)r_0 \in J(R)$, then we get $f(x_0) = (f(x_0)r_0f(x_0) + j_0) \in J(R)$, and so $c_f(x_0)g = 0$, a contradiction. Hence, we assume that $f(x_0)r_0 + J(R)$ is a nonzero idempotent of $R/J(R)$. It follows that $f_1 = fc_{r_0f(x_0)} \neq 0$. Next, we show that $f_1(s)g(t) = 0$ for every $s, t \in S$. Notice that y_1 is the unique minimal element in the set $\mathrm{Supp}(g) \setminus \{y_0\}$. We consider the set

$$X_{x_0y_1}(f_1, g) = \{(s,t) \in \mathrm{Supp}(f_1) \times \mathrm{Supp}(g) \mid st = x_0y_1\}.$$

Notice that, if $(s,t) \in X_{x_0y_1}(f_1,g) \setminus \{(x_0,y_1)\}$, then either $x_0 \prec s$, $t \prec y_1$ or $s \prec x_0$, $y_1 \prec t$. If $t \prec y_1$, then we must have $t = y_0$, because y_0 is the minimal element of $\mathrm{Supp}(g)$, and so in this case we get $f_1(s)g(t) = 0$. If $s \prec x_0$, then from definition of $\mathrm{Supp}'(f)$ and the fact that x_0 is the minimal element of $\mathrm{Supp}'(f)$ we get $f(s)g(t) = 0$. Hence for every $(s,t) \in X_{x_0y_1}(f_1,g) \setminus \{(x_0,y_1)\}$ we have $f(s)g(t) = 0$. Considering $f_1g(x_0y_1)$, by applying S-compatibility of R we have $f_1(x_0)\omega_{f_1(x_0)}g(y_1) = 0$. Then $f_1(x_0)g(y_1) = 0$ and from $j_0g(y_1) = 0$ we obtain that $f(x_0)g(y_1) = 0$. This yields that $f_1(s_1)g(y_1) = 0$ for all $s_1 \in S$.

Continuing this process, we deduce that $f(x_0)g(y_i) = 0$ for all $y_i \in \mathrm{Supp}(g)$, contradicting the fact that x_0 is a minimal element in $\mathrm{Supp}'(f)$. Hence $\mathrm{Supp}'(f)$ is empty, as wanted.

Case 2. If $J(R)g(s) \neq 0$ for some $s \in S$, then from $J(R)^k = 0$, there exists an integer $m \leq k$ such that $J(R)^m g(s_1) \neq 0$ for some $s_1 \in S$ while $J(R)^{m+1}g(y) = 0$ for every $y \in S$. Hence, there exist elements $j_1, \dots, j_m \in J(R)$ such that $c_{j_1 \cdots j_m} g \neq 0$ while $J(R)c_{j_1 \cdots j_m} g(y') = 0$ for every $y' \in S$. If we replace g by $g_1 = c_{j_1 \cdots j_m} g$, then $fg_1 = 0$ and we reduce to the previous case. Now, the proof is complete. □

We have the following immediate corollaries.

Corollary 2.8. *Let R be a ring, $(S, \preceq)$ a strictly totally ordered monoid and $\omega \colon S \to \mathrm{End}(R)$ a monoid homomorphism. If R is reversible semiregular and S-compatible ring such that $J(R)$ is nilpotent ideal, then R is (S, ω)-McCoy.*

Corollary 2.9. *If R is a reversible semiregular ring such that $J(R)$ is nilpotent ideal, then R is McCoy of power series type.*

Corollary 2.10. *If R is a semiregular ring with nilpotent $J(R)$ such that $A = R[[x]]$ is semicommutative, then R is McCoy of power series type.*

We continue by proving another main result of this paper, which shows that a local ring R with nilpotent $J(R)$ is (S, ω)-McCoy.

Theorem 2.11. *Let R be a ring, $(S, \preceq)$ a strictly totally ordered monoid, and $\omega \colon S \to \mathrm{End}(R)$ a monoid homomorphism. If R is S-compatible and local ring such that $J(R)$ is nilpotent, then R is (S, ω)-McCoy.*

Proof. We prove only the right case. Let $A = R[[S, \omega, \preceq]]$ and also $f, g \in A$ with $fg = 0$ where $g \neq 0$. We claim that both f and g are in $J(R)[[S, \omega, \preceq]]$. Assume on the contrary that $f \notin J(R)[[S, \omega, \preceq]]$ or $g \notin J(R)[[S, \omega, \preceq]]$. We may assume that $f \notin J(R)[[S, \omega, \preceq]]$. Let

$$\mathrm{Supp}'(f) = \{s \in S \mid f(s) \notin J(R)\}.$$

As $(S, \preceq)$ is a strictly totally ordered monoid, we can write

$$\mathrm{Supp}'(f) = \{x_0, x_1, x_2, \dots\} \quad \text{and} \quad \mathrm{Supp}(g) = \{y_0, y_1, y_2, \dots\},$$

where $x_0 \prec x_1 \prec x_2 \prec \cdots$ and $y_0 \prec y_1 \prec y_2 \prec \cdots$. Note that by the definition of $\mathrm{Supp}'(f)$, we have $f(s) \in J(R)$, for any $s \prec x_0$. Since R is a local ring, $f(x_0)$ is left invertible in R and then $Rf(x_0) + J(R) = R$, since $J(R)$ is a maximal left ideal of R. This means that there exist elements $u \in R$ and $j \in J(R)$ such that $uf(x_0) + j = 1$. As j is a nilpotent element, hence there exists an element $v \in R$ such that $vuf(x_0) = 1$.

From $fg = 0$, we get $c_{vu}fg = 0$. On the other hand, there exists an integer m such that $j_1 \cdots j_{m+1}g(s) = 0$, for every $s \in S$ and $j_1, \ldots, j_{m+1} \in J(R)$ while $j'_1 \cdots j'_m g(h) \neq 0$ for some $h \in \mathrm{Supp}(g)$ and for some elements $j'_1, \ldots, j'_m$ of $J(R)$. Hence

$$\mathrm{Supp}'(g) = \{s \in S \mid J(R)^m g(s) \neq 0\}$$

is a non-empty subset of S. As $f(s') \in J(R)$ for any $s' \prec x_0$, it then follows that

$$jf(s')g(t') = 0 \quad \text{for every } s' \prec x_0 \text{ and } t' \in S \text{ and } j \in J(R)^m. \tag{2.1}$$

As $(S, \preceq)$ a strictly totally ordered monoid, hence $\mathrm{Supp}'(g)$ has a unique minimal element, say $h \in \mathrm{Supp}'(g)$. So,

$$jRg(t_1) = 0 \quad \text{for every } t_1 \prec h \text{ and } j \in J^m(R). \tag{2.2}$$

Now, consider the set $X_{x_0h}(c_{vu}f, g) = \{(s,t) \in \mathrm{Supp}(c_{vu}f) \times \mathrm{Supp}(g) \mid st = x_0h\}$. If $(s,t) \in X_{x_0h}(c_{vu}f, g)$, $s \prec x_0$, then we must have $t \succ h$ otherwise $st \prec x_0t \preceq x_0h = st$, a contradiction. Similarly, if $(s,t) \in X_{x_0h}(c_{vu}f, g)$, $s \succ x_0$, then $t \prec h$.

Now, applying (2.1) and (2.2), we get $jvuf(s)g(t) = 0$ for every $j \in J(R)^m$ and $(s,t) \in X_{x_0h}(c_{vu}f, g)$ with either $t \prec h$ or $s \prec x_0$.

As R is S-compatible, then for every $j \in J(R)^m$, we have

$$\sum_{\substack{(s,t)\in X_{x_0h} \\ t \prec h}} c_j c_{vu} f(s)\omega_s(g(t)) = \sum_{\substack{(s',t')\in X_{x_0h} \\ s' \prec x_0}} c_j c_{vu} f(s')\omega_{s'}(g(t')) = 0.$$

Since, for every $j \in J(R)^m$, we have

$$\begin{aligned} 0 = c_j c_{vu} fg(x_0h) = jvuf(x_0)\omega_{x_0}(g(h)) &+ \sum_{\substack{(s,t)\in X_{x_0h} \\ t \prec h}} c_{vu} f(s)\omega_s(g(t)) \\ &+ \sum_{\substack{(s',t')\in X_{x_0h} \\ s' \prec x_0}} c_{vu} f(s')\omega_{s'} g(t'), \end{aligned}$$

then S-compatibility of R implies that

$$0 = jvuf(x_0)g(h) = jg(h),$$

a contradiction. Hence $f, g \in J(R)[[S, \omega, \preceq]]$. As $J(R)$ is nilpotent, there exists an integer k such that $J(R)^k \neq 0 = J(R)^{k+1}$. Taking $0 \neq r \in J(R)^k$, we get $f(s)r = 0$ for every $s \in S$, i.e., R is right (S, ω)-McCoy. □

Remark 2.12. Note that the hypothesis "$J(R)$ is nilpotent" in Theorem 2.11, is not superfluous. By Example 2.3, R is local and $J(R)$ is nil, while R is not McCoy of power series type.

Corollary 2.13. *If R is a Noetherian local ring, then R is McCoy of power series type.*

Proof. Since for a Noetherian ring R, the Jacobson radical $J(R)$ is nilpotent, hence the result follows by Theorem 2.11. $\square$

The following observation can be found in [27, Theorem 6.39(1)].

Lemma 2.14. *An abelian ring R is semiperfect if and only if it is a finite direct sum of local rings.*

Corollary 2.15. *Let R be an abelian semiperfect ring such that $J(R)$ is nilpotent. Then R is McCoy of power series type.*

Corollary 2.16. *Let R be an abelian left continuous ring with a.c.c. on essential left ideals. Then R is left McCoy ring of power series type.*

Since semiperfect rings are Morita invariant, then for every semiperfect ring R with nilpotent $J(R)$, the ring $M_n(R)$ is also semiperfect and $J(M_n(R))$ is nilpotent, while $M_n(R)$ is not in general McCoy of power series type. This shows that the hypothesis "R is abelian" in Corollary 2.15 is not superfluous.

We have the following implications:

$$\begin{array}{ccccc} \text{local} & \longrightarrow & \text{semiperfect} & \longrightarrow & \text{semiregular} \\ & & & & \uparrow \\ & & & & \text{regular} \end{array}$$

Note that if R is a regular ring, then we have $J(R) = 0$, but regular rings are not in general McCoy of power series type. For example, $M_n(F)$ is a regular ring but not McCoy of power series type.

By the above implications, one may ask the following natural question:

Question 2.17. If R is an abelian semiregular ring such that $J(R)$ is nilpotent, is R McCoy of power series type?

In the following result, we give a partial answer to the above question.

Theorem 2.18. *Let R be a ring, $(S, \preceq)$ a strictly totally ordered monoid and $\omega\colon S \to \mathrm{End}(R)$ a monoid homomorphism. If R is abelian, semiregular and S-compatible ring with $J(R)^2 = 0$, then R is (S, ω)-McCoy.*

Proof. We prove only the right case. In this case, without loss of generality, we can assume that $J(R) \neq 0$, because if $J(R) = 0$, then R will be regular, and since every abelian regular ring is reduced, then there is nothing to prove. Let $A = R[[S, \omega, \preceq]]$ and also $f, g \in A$ with $fg = 0$ where $g \neq 0$. We can also assume that $g \in J(R)[[S, \omega, \preceq]]$, because if $g \notin J(R)[[S, \omega, \preceq]]$ then there exists an element $t \in S$ such that $g(t) \notin J(R)$. From semiregularity of R, there exists a nonzero idempotent $e \in R$ such that $(g(t)R + J(R))/J(R) = (eR + J(R))/J(R)$, so that $g(t)J(R) \neq 0$. Hence there exists an element $j \in J(R)$ such that $g(t)j \neq 0$. As R is a S-compatible ring, $gc_j \neq 0$. Since $gc_j \in J(R)[[S, \omega, \preceq]]$ and $fgc_j = 0$, hence we can replace g by gc_j, if needed. We consider the following two cases:

Case 1. If $f \in J(R)[[S, \omega, \preceq]]$, then since $g \in J(R)[[S, \omega, \preceq]]$ and $J(R)^2 = 0$, we get $f(s)\omega_s(g(t)) = 0$ for every $s, t \in S$, and the result follows.

Case 2. If $f \notin J(R)[[S, \omega, \preceq]]$, then there exists $s \in S$ such that $f(s) \notin J(R)$. Assume that

$$\mathrm{Supp}'(f) = \{s \in S \mid f(s) \notin J(R) \text{ and } c_{f(s)}g \neq 0\} \subseteq \mathrm{Supp}(f).$$

If $\mathrm{Supp}'(f)$ is empty we are done, so assume by way of contradiction, that $\mathrm{Supp}'(f)$ is non-empty and let

$$X'_s(f, g) = \{(x, y) \in \mathrm{Supp}'(f) \times \mathrm{Supp}(g) \mid s = xy\}.$$

Using S-compatibility of R, we get $f(s)g(t) = 0$, for all $t \in S$ and $s \in S \setminus \mathrm{Supp}'(f)$. As S is a strictly totally ordered monoid, we can write

$$\begin{aligned} \mathrm{Supp}'(f) &= \{x_0, x_1, x_2, \ldots\}, \quad \text{where } x_i \prec x_{i+1} \text{ for } i = 0, 1, \ldots, \\ \mathrm{Supp}(g) &= \{y_0, y_1, y_2, \ldots\}, \quad \text{where } y_i \prec y_{i+1} \text{ for } i = 0, 1, \ldots. \end{aligned}$$

Note that $f(s)g(t) = 0$ for every $s, t \in S$ with $s < x_0$. Consider the set $X_{x_0y_0}(f, g)$. If $(x_0, y_0) \neq (s, t) \in X_{x_0y_0}(f, g)$, then we can see that $s \neq x_0$ and $t \neq y_0$. Hence we get either $s \prec x_0$, $t \succ y_0$ or $x_0 \prec s$, $y_0 \succ t$.

Note that for every $s \prec x_0$, we get $s \in S \setminus \mathrm{Supp}'(f)$, so by above $f(s)g(t) = 0$ for all $t \in S$. Similarly, if $t \prec y_0$, then $g(t) = 0$ and hence $f(s)g(t) = 0$ for all $s \in S$. So that

$$\sum_{\substack{(s,t) \in X_{x_0y_0} \\ s \prec x_0}} f(s)\omega_s(g(t)) = \sum_{\substack{(s,t) \in X_{x_0y_0} \\ t \prec y_0}} f(s)\omega_s(g(t)) = 0.$$

Therefore, $0 = fg(x_0y_0) = f(x_0)\omega_{x_0}(g(y_0))$, and then S-compatibility of R implies that

$$f(x_0)g(y_0) = 0. \tag{2.3}$$

Since R is a semiregular ring, there exists an element $r_0 \in R$ such that

$$f(x_0)r_0f(x_0) + J(R) = f(x_0) + J(R).$$

As idempotents lift module $J(R)$ and $r_0f(x_0)+J(R)$ is an idempotent of $R/J(R)$, there exists a central idempotent $e_0=e_0^2$ of R such that $r_0f(x_0)+J(R)=e_0+J(R)$. Hence,

$$f(x_0)r_0f(x_0)+J(R)=f(x_0)e_0+J(R)=e_0f(x_0)+J(R).$$

It is easy to see that $c_{r_0f(x_0)}f$ is nonzero and $c_{r_0f(x_0)}fg=0$.

Next, we will prove by induction that $f(x_0)r_0f(x_0)g(y_i)=0$ for every $y_i\in \mathrm{Supp}(g)$. From $J(R)^2=0$ and $g\in J(R)[[S,\omega,\preceq]]$, we get $J(R)g(y_0)=0$. It then follows that

$$\begin{aligned} r_0f(x_0)f(s)g(y_0)&=(r_0f(x_0)+J(R))f(s)g(y_0)\\ &=(r_0f(x_0)+J(R))(f(s)+J(R))g(y_0)\\ &=(e_0+J(R))(f(s)+J(R))g(y_0)\\ &=(f(s)+J(R))(e_0+J(R))g(y_0)\\ &=(f(s)+J(R))(r_0f(x_0)+J(R))g(y_0)\\ &=f(s)r_0f(x_0)g(y_0)\quad \text{for every } s\in S. \end{aligned}\tag{2.4}$$

Comparing (2.3) and (2.4), we get

$$f(s)r_0f(x_0)g(y_0)=f(x_0)r_0f(s)g(y_0)=0$$

for every $s\in S$. So the base case of our induction is established. Now, assume by induction that k is an integer such that $r_0f(x_0)f(s)g(y_i)=0$ for every $s\in S$ and $i=0,1,\dots,k-1$, and consider the set $X_{x_0y_k}(c_{r_0f(x_0)}f,g)$. If $(s,t)\in X_{x_0y_k}(c_{r_0f(x_0)}f,g)\setminus\{(x_0,y_k)\}$, then we can see that $s\neq s_0$ and $t\neq y_k$. Then, it is not hard to see that we have either $s\succ x_0$, $t\prec y_k$ or $s\prec x_0$, $t\succ y_k$. If $t\prec y_k$, then by induction hypothesis, we get $c_{r_0f(x_0)}fg(st)=0$, for every $s\in S$. If $s\prec x_0$, then since x_0 is a minimal element in $\mathrm{Supp}'(f)$, we get $f(s)g(t)=0$ for every $t\in S$. So,

$$\begin{aligned} 0&=c_{r_0f(x_0)}fg(x_0y_k)\\ &=r_0f(x_0)f(x_0)\omega_{f(x_0)}(g(y_k))\\ &\quad+\sum_{\substack{(s,t)\in X_{x_0y_k}(c_{r_0f(x_0)}f,g)\\ s<x_0}} r_0f(x_0)f(s)\omega_{f(s)}(g(t))\\ &\quad+\sum_{\substack{(s',t')\in X_{x_0y_k}(c_{r_0f(x_0)}f,g)\\ s'>x_0}} r_0f(x_0)f(s')\omega_{f(s')}(g(t'))\\ &=r_0f(x_0)f(x_0)\omega_{f(x_0)}(g(y_k)). \end{aligned}$$

As R is S-compatible, we get $r_0f(x_0)f(x_0)g(y_k)=0$ and from $J(R)g(y_k)=0$ we deduce that for every integer $k = 0, 1, \ldots,$

$$\begin{aligned} 0 &= (r_0 f(x_0) f(x_0) + J(R)) g(y_k) = (r_0 f(x_0) + J(R))(f(x_0) + J(R)) g(y_k) \\ &= (e_0 + J(R))(f(x_0) + J(R)) g(y_k) = (f(x_0) + J(R))(e_0 + J(R)) g(y_k) \\ &= (f(x_0) r_0 f(x_0) + J(R)) g(y_k) = f(x_0) r_0 f(x_0) g(y_k) = 0, \end{aligned}$$

as needed. Hence, $f(x_0) r_0 f(x_0) g(y_i) = 0$ for every $y_i \in \mathrm{Supp}(g)$. Since $J(R) g(y_i) = 0$ for every $y_i \in \mathrm{Supp}(g)$, and $f(x_0) r_0 f(x_0) g(y_i) = 0$ for every $y_i \in \mathrm{Supp}(g)$, consequently

$$f(x_0) g(y_i) = (f(x_0) + J(R)) g(y_i) = (f(x_0) r_0 f(x_0) + J(R)) g(y_i) = 0$$

for every $y_i \in \mathrm{Supp}(g)$, contradicting the fact $x_0 \in \mathrm{Supp}'(f)$. Hence $\mathrm{Supp}'(f)$ is empty, and therefore R is right (S, ω)-McCoy. $\square$

Example 2.19. [14, Exercise 4.26] Let R be a commutative Q-algebra generated by $\{x_1, x_2, \ldots\}$ with the relations $x_i x_j = 0$ for all i, j. Then R is a semiprimary ring (i.e., $R/J(R)$ is semisimple and $J(R)$ is nilpotent). This is an example of a commutative, semiprimary (and hence semiregular) ring such that $J(R) = 0$. Hence, it satisfies the conditions of Theorem 2.7, but not Theorem 2.25, because $\mathrm{Nil}(R)$ is not Noetherian as a right R-module.

Example 2.20. [13, Example 2.1] Let $\mathbb{Z}_2$ be the field of integers modulo 2 and $A = \mathbb{Z}_2[a_0, a_1, a_2, b_0, b_1, b_2, c]$ be the free algebra of polynomials with zero constant terms in noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ over $\mathbb{Z}_2$. Note that A is a ring without identity and consider an ideal of the ring $\mathbb{Z}_2 + A$, say I, generated by $a_0 b_0, a_0 b_1 + a_1 b_0, a_0 b_2 + a_1 b_1 + a_2 b_0, a_1 b_2 + a_2 b_1, a_2 b_2, a_0 r b_0, a_2 r b_2, b_0 a_0, b_0 a_1 + b_1 a_0, b_0 a_2 + b_1 a_1 + b_2 a_0, b_1 a_2 + b_2 a_1, b_2 a_2, b_0 r a_0, b_2 r a_2, (a_0 + a_1 + a_2) r (b_0 + b_1 + b_2), (b_0 + b_1 + b_2) r (a_0 + a_1 + a_2)$, and $r_1 r_2 r_3 r_4$, where $r, r_1, r_2, r_3, r_4 \in A$. Then clearly $A^4 \subseteq I$. Next, let $R = (\mathbb{Z}_2 + A)/I$ and consider $R[x] \cong (\mathbb{Z}_2 + A)[x]/I[x]$. Then, it already proved that R is reversible while $R[x]$ is not semicommutative. On the other hand, it is clear that R is a semiregular ring and $J(R)$ is nilpotent, because $R/J(R) \simeq \mathbb{Z}_2$ and $J(R)^4 = 0$. Hence, this is an example that satisfies the conditions of Theorem 2.7 and Corollary 2.8.

Example 2.21. Let F be a field and consider the ring

$$R_3 = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in F \right\}.$$

Then R_3 is a semicommutative ring, while R_3 is not reversible. Also, $R_3[[x]]$ is semicommutative and clearly $J(R_3)$ is nilpotent. So, this is an example that satisfies the conditions of Theorem 2.7, but not the conditions of Corollary 2.8 and Theorem 2.25.

Example 2.22. Consider the ring of integers $\mathbb{Z}$. It is clear that $\mathbb{Z}$ is a zip ring and $J(\mathbb{Z}) = 0$. Hence, it satisfies the conditions of Theorem 2.25, but not the conditions of Theorem 2.7 and Corollary 2.8, since $\mathbb{Z}$ is not a semiregular ring.

Lemma 2.23. *Let R be a ring, $(S, \preceq)$ a strictly totally ordered monoid and $\omega\colon S \to \mathrm{End}(R)$ a monoid homomorphism. Suppose that R is S-compatible and $\mathrm{Nil}(R)$ is a nilpotent ideal. Then $f \in \mathrm{Nil}(R[[S, \omega, \preceq]])$ if and only if $f(s) \in \mathrm{Nil}(R)$ for every $s \in S$.*

Proof. For the forward direction, assume by the contrary that, there exists an integer k such that $f^k = 0 \neq f^{k-1}$, while $f(s) \notin \mathrm{Nil}(R)$ for some $s \in S$. Set

$$T = \{s \in S \mid f(s) \notin \mathrm{Nil}(R)\}.$$

To prove the result, it suffices to show that T is an empty set. As S is a strictly totally ordered monoid, T has a unique minimal element, say s_0. Let $W = \{s \in S \mid s \prec s_0\}$. Then, clearly $f(W) = \{f(s) \mid s \prec s_0\} \subseteq \mathrm{Nil}(R)$. Since S is a strictly totally ordered monoid, $\mathrm{Supp}(f) \cap W$ is a finite chain, namely $\{w_0, \ldots, w_n\}$, where $w_0 \prec w_1 \prec \cdots \prec w_n \prec s_0$. We consider the following two cases:

Case 1. If $k = 2$, then consider the set

$$X_{s_0 s_0}(f, f) = \{(x, y) \in \mathrm{Supp}(f) \times \mathrm{Supp}(f) \mid xy = s_0 s_0\}.$$

If $(x, y) \in X_{s_0 s_0}(f, f)$ and $(x, y) \neq (s_0, s_0)$, then we get $x \notin T$ or $y \notin T$, because if $x, y \in T$, then we get $s_0 \prec x$, $s_0 \prec y$, so $s_0 s_0 \prec x s_0 \prec xy = s_0 s_0$, a contradiction. Thus for every $(s_0, s_0) \neq (x, y) \in X_{s_0 s_0}(f, f)$, we have $f(x)f(y) \in \mathrm{Nil}(R)$, because $f(x) \in \mathrm{Nil}(R)$ or $f(y) \in \mathrm{Nil}(R)$ and $\mathrm{Nil}(R)$ is an ideal of R. As R is S-compatible, we get

$$f(x)\omega_x(f(y)) \in \mathrm{Nil}(R) \quad \text{for every } (s_0, s_0) \neq (x, y) \in X_{s_0 s_0}(f, f). \tag{2.5}$$

Since

$$\sum_{(t,s) \in X_{s_0 s_0}(f,f)} f(t)\omega_t(f(s)) = 0,$$

we get

$$f(s_0)\omega_{s_0}(f(s_0)) + \sum_{\substack{(s_0,s_0) \neq (x,y) \\ (x,y) \in X_{s_0 s_0}(f,f)}} f(x)\omega_x(f(y)) = 0. \tag{2.6}$$

As $\mathrm{Nil}(R)$ is an ideal of R, comparing (2.5) and (2.6), we get $f(s_0)\omega_{s_0}(f(s_0)) \in \mathrm{Nil}(R)$ and the S-compatiblity of R implies that $f(s_0)f(s_0) \in \mathrm{Nil}(R)$. So, $f(s_0) \in \mathrm{Nil}(R)$. But this contradicts the fact that s_0 is an element of T. Hence T is an empty set and $f(s) \in \mathrm{Nil}(R)$ for every $s \in S$. Therefore, for every $s \in \mathrm{Supp}(f)$, we get $f(s) \in \mathrm{Nil}(R)$ with $f^2 = 0$.

Case 2. If $k \geq 3$, since $(f^{k-1})^2 = 0$, then by Case 1, we get $f^{k-1}(s) \in \mathrm{Nil}(R)$, for every $s \in S$. Hence, we must have $f^{k-1}(s_0^{k-1}) \in \mathrm{Nil}(R)$. It then follows that

$$\begin{aligned} f^{k-1}(s) &= f(s_0)\omega_{s_0}(f(s_0))\cdots\omega_{s_0^{k-2}}(f(s_0)) \\ &\quad + \sum_{(s_0,\ldots,s_0)\neq(x_1,\ldots,x_{k-1})\in X_{s_0^{k-1}}(f,\ldots,f)} f(x_1)\omega_{x_1}(f(x_2))\cdots\omega_{x_1\cdots x_{k-1}}(f(x_{k-1})) \\ &\quad \in \mathrm{Nil}(R). \end{aligned}$$

It is not hard to see that for every $(s_0,\ldots,s_0) \neq (x_1,\ldots,x_{k-1}) \in X_{s_0^{k-1}}$, there exists x_j, $1 \leq j \leq k-1$, such that $x_j \prec s_0$. As R is S-compatible and $\mathrm{Nil}(R)$ is an ideal of R, we have

$$\sum_{(s_0,\ldots,s_0)\neq(x_1,\ldots,x_{k-1})\in X_{s_0^{k-1}}(f,\ldots,f)} f(x_1)\omega_{x_1}(f(x_2))\cdots\omega_{x_1\cdots x_{k-1}}(f(x_{k-1})) \in \mathrm{Nil}(R).$$

Consequently, we obtain that

$$f(s_0)\omega_{s_0}\big(f(s_0)\cdots\omega_{s_0^{k-2}}(f(s_0))\big) \in \mathrm{Nil}(R).$$

Applying the S-compatiblity of R, we get $f(s_0)^{k-1} \in \mathrm{Nil}(R)$ and hence $f(s_0) \in \mathrm{Nil}(R)$, a contradiction. Therefore, in any case, T is an empty set, as desired.

The backward direction is straightforward. □

Proposition 2.24. *Let R be a ring, $(S, \preceq)$ a strictly totally ordered monoid and $\omega\colon S \to \mathrm{End}(R)$ a monoid homomorphism. Suppose that R is S-compatible, reversible and* $\mathrm{Nil}(R)$ *is Noetherian as a right R-module. If $f, g \in R[[S,\omega,\preceq]]$ such that $fg = 0$, then for every finite subset $T \subseteq \mathrm{Supp}(f)$, we have $r_R(f(T)) \neq 0$.*

Proof. Suppose that $A = R[[S,\omega,\preceq]]$. We break the proof into three cases:

Case 1. If $f \in \mathrm{Nil}(A)$, then for any finite subset $T \subseteq \mathrm{Supp}(f)$, there exists an integer m which is minimal with respect to $(\{f(t) \mid t \in T\})^m = 0$, so $f(T)r = 0$, where r is a nonzero element of $(\{f(t) \mid t \in T\})^{m-1}$.

Case 2. If $f \notin \mathrm{Nil}(A)$ but $g \in \mathrm{Nil}(A)$, then since $\mathrm{Nil}(R)$ is right Noetherian, the right ideal of R generated by $g(S) = \{g(s) \mid s \in S\}$ is finitely generated. Hence there exists a finite subset $D = \{s'_0, \ldots, s'_m\}$ of S such that $g(S)R = \sum_{i=0}^m g(s'_i)R$. It then follows that $l_R(g(S)) = l_R\big(\{g(s'_0), \ldots, g(s'_m)\}\big)$.

As S is strictly totally ordered monoid, we can assume that $s'_0 \prec s'_1 \prec \cdots \prec s'_m$. Let

$$D_1 = \{s \in \mathrm{Supp}(g) \mid s \preceq s'_m\}.$$

Then D_1 is a finite subset of S. We may assume that $D_1 = \{s_0, s_1, \ldots, s'_m\}$ with $s_0 \prec s_1 \prec \cdots \prec s'_m$.

Now consider the finite subset T of $\mathrm{Supp}(f)$. Let t_n be the maximal element of T with respect to $\preceq$. Then $T_1 = \{s \in \mathrm{Supp}(f) \mid s \preceq t_n\}$ is a finite set, namely $T_1 = \{t_0, \ldots, t_n\}$, where $t_0 \prec t_1 \prec \cdots \prec t_n$.

Assume that

$$Q = \{t_i \in T_1 \mid c_{f(t_i)}g = 0, 0 \le i \le n\},$$

and define

$$f_1(s) = \begin{cases} 0 & \text{if } s \in Q, \\ f(s) & \text{otherwise.} \end{cases}$$

Since for every $s \in S$, we have

$$X_s(f,g) = X_s(f_1,g) \cup \{(x,y) \mid x \in Q, xy = s\},$$

then we get $f_1 g = 0$. If $T_1 = Q$, then there is nothing to prove. Hence, without loss of generality, we can assume that $\{t_0, \ldots, t_{v-1}\} \subseteq Q$ and $\{t_v, \ldots, t_n\} \cap Q = \emptyset$.

Next, we show that $f_1(t_v)g(s_0) = 0$. For, let $(x,y) \in X_{t_v s_0}(f_1,g)$ with $x \neq t_v$. Then, we get either $x \prec t_v$ or $t_v \prec x$. If $x \prec t_v$, then $x \in Q$, so $x \notin \mathrm{Supp}(f_1)$. If $t_v \prec x$, then we must have $y \prec s_0$ and so $g(y) = 0$, because s_0 is the minimal element of $\mathrm{Supp}(g)$. Hence

$$f_1(x)g(y) = 0, \quad \text{where } x \neq t_v,\ (x,y) \in X_{t_v s_0}(f_1,g). \tag{2.7}$$

Using $f_1 g(t_v s_0) = 0$ and (2.7), we get $f_1(t_v)\omega_{t_v}(g(s_0)) = 0$ and then S-compatibility of R implies that $f_1(t_v)g(s_0) = 0$. As $c_{f(t_v)}g \neq 0$, there exists an element $s \in \mathrm{Supp}(g)$ such that $f_{t_v}g(s) \neq 0$. Since R is reversible and S-compatible, we get $g(s)f(t_v) \neq 0$ and hence $gc_{f(t_v)} \neq 0$.

Let $0 \neq g_1 = gc_{f(t_v)}$. Then $g_1(s_0) = 0$. From $f_1 g = 0$ we obtain $f_1 g_1 = 0$. It is easy to see that $f_1(t_v)\omega_{t_v}(g_1(s_1)) = 0$ and so $f_1(t_v)g_1(s_1) = 0$. As R is reversible and $f_1(t_v) = f(t_v)$, we get $g_1(s_1)f_1(t_v) = g(s_1)f(t_v)f(t_v) = 0$. Continuing in this way, there exists an integer z_1 minimal with respect to the property that $g(s_0)(f(t_v))^{z_1} = \cdots = g(s'_m)(f(t_v))^{z_1} = 0$. Since $g(S) \subseteq \sum_{i=0}^{m} g(s'_i)R$, then using reversibility and S-compatibility of R, we get $gc_{(f(t_v))^{z_1}} = 0$, while $gc_{(f(t_v))^{z_1-1}} \neq 0$.

Take $g_2 = gc_{f(t_v)^{z_1-1}}$. Then $fg_2 = 0$ and clearly $f(t_v)g_2(s) = 0$ for every $s \in S$. Assume that

$$Q_1 = \{t_i \in T_1 \mid c_{f(t_i)}g_2 = 0, 0 \le i \le n\},$$

and define

$$f_2(s) = \begin{cases} 0 & \text{if } s \in Q_1, \\ f(s) & \text{otherwise.} \end{cases}$$

Since for every $s \in S$, we have

$$X_s(f,g_2) = X_s(f_2,g_2) \cup \{(x,y) \mid x \in Q_1, xy = s\},$$

then we conclude that $f_2g_2 = 0$. Clearly $Q \cup \{t_v\} \subseteq Q_1$. If $T_1 = Q_1$, then there is nothing to prove. Hence, without loss of generality, we can assume that $\{t_0, \ldots, t_{v-1}, t_v, \ldots, t_{l_1}\} \subseteq Q_1$ and $\{t_{l_1+1}, \ldots, t_n\} \cap Q_1 = \emptyset$.

By a similar method as used above, there exists a minimal integer z_2 such that $c_{f_2(t_{l_1+1})^{z_2}} g_2 = g_2 c_{f_2(t_{l_1+1})^{z_2}} g_2 = 0$, whereas $c_{f_2(t_{l_1+1})^{z_2-1}} g_2 \neq 0$ and $g_2 c_{f_2(t_{l_1+1})^{z_2-1}} \neq 0$. Take $g_3 = g_2 c_{f_2(t_{l_1+1})^{z_2-1}}$. Then, there exists an integer $l_1 < l_2$ such that $f(t_i)g_3(s) = 0$ for every $i = 0, 1, \ldots, l_1, \ldots, l_2$ and $s \in \{s_0, s_1, \ldots, s'_m\}$. Continuing this process finite number of times, we can find a nonzero function $g' = gc_r$, where $r \in R$, such that $f(t_i)g'(s) = 0$ for all $i = 0, 1, \ldots, n$ and $s \in \{s_0, s_1, \ldots, s'_m\}$. As $g(S) \subseteq \sum_{i=0}^{m} g(s'_i)R$, we get $g(s)\sigma_s(r) \in \sum_{i=0}^{m} g(s'_i)R$ for every $s \in S$, i.e., $g'(S) \subseteq \sum_{i=0}^{m} g(s'_i)R$. Therefore, $f(t_i)g(S) = 0$ for all $i = 0, 1, \ldots, n$. This means that in this case $r_R(T_1) \neq 0$.

Case 3. Both $f, g \notin \mathrm{Nil}(A)$ and $T = \{t_0, t_1, \ldots, t_n\}$ is a finite subset of $\mathrm{Supp}(f)$ with $f(t_i) \notin \mathrm{Nil}(R)$ for $0 \leq i \leq n$. If there exists $s \in S$ such that $g(s)\,\mathrm{Nil}(R) \neq 0$, then for some suitable element $n \in \mathrm{Nil}(R)$ we get $0 \neq g_1 = gc_n$. As $\mathrm{Nil}(R)$ is a nilpotent ideal of R, we get $g_1 \in \mathrm{Nil}(A)$. Since $fg_1 = 0$, replacing g_1 by g, then g falls under Case 2.

Now, let $gc_n = 0$ for every $n \in \mathrm{Nil}(R)$. Then it is not hard to see that, $f(t)g(s) \in \mathrm{Nil}(R)$ for every $t \in \mathrm{Supp}(f)$, $s \in \mathrm{Supp}(g)$. Let $s' \in \mathrm{Supp}(g)$ be such that $g(s') \notin \mathrm{Nil}(R)$. Then from $f(t_i)g(s') \in \mathrm{Nil}(R)$ and $\mathrm{Nil}(R)g(s') = 0$, we get $f(t_i)g(s')g(s') = 0$ for $0 \leq i \leq n$. As $g(s')g(s') \neq 0$, then we get $r_R(T) \neq 0$, and the proof is complete. □

Using the above proposition, we can provide another class of (S, ω)-McCoy rings. We conclude by showing that reversible right zip ring R such that $\mathrm{Nil}(R)$ is Noetherian as a right R-module, is a right (S, ω)-McCoy ring.

Theorem 2.25. *Let R be a ring, $(S, \preceq)$ a strictly totally ordered monoid and $\omega \colon S \to \mathrm{End}(R)$ a monoid homomorphism. If R is S-compatible, reversible right zip ring and $\mathrm{Nil}(R)$ is Noetherian as a right R-module, then R is right (S, ω)-McCoy.*

Proof. Let $f, g \in R[[S, \omega, \preceq]]$ be such that $fg = 0$ with $g \neq 0$. Take $\mathrm{Im}(f) = \{f(s) \mid s \in S\}$. If we show that $r_R(\mathrm{Im}(f)) \neq 0$, then by S-compatiblity of R we get $fc_d = 0$ for some nonzero element $d \in r_R(\mathrm{Im}(f))$, and the proof is complete. Assume by contrary that $r_R(\mathrm{Im}(f)) = 0$. As R is a right zip ring, there exists a finite subset T of $\mathrm{Im}(f)$ such that $r_R(T) = 0$, a contradiction to Proposition 2.24. Hence R is right (S, ω)-McCoy. □

Acknowledgments

We thank the anonymous referee for constructive comments and suggestions, especially for generalizing the statement of Theorem 2.7. The first author was partially supported by Eqlid University and the research of the third and fourth named authors was in part supported by a grant from Shahrood University of Technology.

References

[1] A. Alhevaz and E. Hashemi, *An alternative perspective on skew generalized power series rings*, Mediterr. J. Math. **13** (2016), no. 6, 4723–4744.

[2] A. Alhevaz and D. Kiani, *McCoy property of skew Laurent polynomials and power series rings*, J. Algebra Appl. **13** (2014), no. 2, 1350083, 23 pp.

[3] M. Başer, T. K. Kwak and Y. Lee, *The McCoy condition on skew polynomial rings*, Comm. Algebra **37** (2009), no. 11, 4026–4037.

[4] H. E. Bell, *Near-rings in which each element is a power of itself*, Bull. Austral. Math. Soc. **2** (1970), 363–368.

[5] V. Camillo and P. P. Nielsen, *McCoy rings and zero-divisors*, J. Pure Appl. Algebra **212** (2008), no. 3, 599–615.

[6] P. M. Cohn, *Reversible rings*, Bull. London Math. Soc. **31** (1999), no. 6, 641–648.

[7] G. A. Elliott and P. Ribenboim, *Fields of generalized power series*, Arch. Math. (Basel) **54** (1990), no. 4, 365–371.

[8] E. H. Feller, *Properties of primary noncommutative rings*, Trans. Amer. Math. Soc. **89** (1958), 79–91.

[9] D. E. Fields, *Zero divisors and nilpotent elements in power series rings*, Proc. Amer. Math. Soc. **27** (1971), 427–433.

[10] J. M. Habeb, *A note on zero commutative and duo rings*, Math. J. Okayama Univ. **32** (1990), 73–76.

[11] E. Hashemi, *McCoy rings relative to a monoid*, Comm. Algebra **38** (2010), no. 3, 1075–1083.

[12] E. Hashemi and A. Moussavi, *Polynomial extensions of quasi-Baer rings*, Acta Math. Hungar. **107** (2005), no. 3, 207–224.

[13] N. K. Kim and Y. Lee, *Extensions of reversible rings*, J. Pure Appl. Algebra **185** (2003), no. 1-3, 207–223.

[14] T. Y. Lam, *A First Course in Noncommutative Rings*, Second edition, Graduate Texts in Mathematics **131**, Springer-Verlag, New York, 2001.

[15] G. Marks, R. Mazurek and M. Ziembowski, *A new class of unique product monoids with applications to ring theory*, Semigroup Forum **78** (2009), no. 2, 210–225.

[16] ______, *A unified approach to various generalizations of Armendariz rings*, Bull. Aust. Math. Soc. **81** (2010), no. 3, 361–397.

[17] G. Mason, *Reflexive ideals*, Comm. Algebra **9** (1981), no. 17, 1709–1724.

[18] R. Mazurek and M. Ziembowski, *On von Neumann regular rings of skew generalized power series*, Comm. Algebra **36** (2008), no. 5, 1855–1868.

[19] ______, *On right McCoy rings and right McCoy rings relative to u.p.-monoids*, Commun. Contemp. Math. **17** (2015), no. 4, 1550049, 10 pp.

[20] N. H. McCoy, *Annihilators in polynomial rings*, Amer. Math. Monthly **64** (1957), 28–29.

[21] R. Mohammadi, A. Moussavi and M. Zahiri, *On annihilations of ideals in skew monoid rings*, J. Korean Math. Soc. **53** (2016), no. 2, 381–401.

[22] P. P. Nielsen, *Semi-commutativity and the McCoy condition*, J. Algebra **298** (2006), no. 1, 134–141.

[23] J. Okniński, *Semigroup Algebras*, Monographs and Textbooks in Pure and Applied Mathematics **138**, Marcel Dekker, New York, 1991.

[24] D. S. Passman, *The Algebraic Structure of Group Rings*, Pure and Applied Mathematics, Wiley-Interscience, New York, 1977.

[25] P. Ribenboim, *Special properties of generalized power series*, J. Algebra **173** (1995), no. 3, 566–586.

[26] ______, *Semisimple rings and von Neumann regular rings of generalized power series*, J. Algebra **198** (1997), no. 2, 327–338.

[27] A. A. Tuganbaev, *Semidistributive Modules and Rings*, Mathematics and its Applications **449**, Kluwer Academic Publishers, Dordrecht, 1998.

[28] S. Yang, X. Song and Z. Liu, *Power-serieswise McCoy rings*, Algebra Colloq. **18** (2011), no. 2, 301–310.

[29] M. Zahiri, A. Moussavi and R. Mohammadi, *On annihilator ideals in skew polynomial rings*, Bull. Iranian Math. Soc. **43** (2017), no. 5, 1017–1036.

Masoome Zahiri
Department of Mathematics, Faculty of Sciences, Higher Education Center of Eghlid, Eghlid, Iran
E-mail address: m.zahiri86@gmail.com

Rasul Mohammadi
Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
E-mail address: mohamadi.rasul@yahoo.com

Abdollah Alhevaz and Ebrahim Hashemi
Faculty of Mathematical Sciences, Shahrood University of Technology, P.O. Box: 316-3619995161, Shahrood, Iran
E-mail address: a.alhevaz@shahroodut.ac.ir, eb_hashemi@yahoo.com