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# Gramian Schauder Basic Measures and Gramian Uniformly Bounded Linearly Stationary Processes

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Abstract. For a normal Hilbert $B(H)$-module $X$ gramian Schauder basic $X$-valued measures are considered. Some equivalence conditions are given for an $X$-valued measure to be gramian Schauder basic. As an application gramian uniformly bounded linearly stationary $X$-valued processes are characterized.

## 1. Introduction

For a Hilbert space $H$ a normal Hilbert $B(H)$-module is defined as a model for the space $S(K, H)$ of Hilbert-Schmidt class operators from another Hilbert space $K$ to $H$, where $B(H)$ is the algebra of all bounded linear operators on $H$ (cf. Kakihara [6] and Ozawa [11]).

If $Y$ is a Banach space and $\xi$ is a $Y$-valued countably additive measure, Dunford-Schwartz integrability of a complex valued function is defined (cf. Dunford and Schwartz [4]) and the set $L^1_{DS}(\xi)$ of all complex valued Dunford-Schwartz integrable functions is shown to be a Banach space with a suitable norm (cf. Abreu and Salehi [2]). Using these objects Abreu and Salehi defined and characterized $Y$-valued Schauder basic measures. Moreover, they applied it to characterize uniformly bounded linearly stationary Hilbert space valued processes.

Our objective is to define normal Hilbert $B(H)$-module valued gramian Schauder basic measures together with their characterization. In order to do this let $X$ be a normal Hilbert $B(H)$-module and $(\Theta, \mathfrak{A})$ be a measurable space. In addition to the usual scalar valued inner product $X$ has the trace class operator valued inner product, called a gramian. For an $X$-valued measure $\xi$ we have to construct an $L^1$-space consisting not only of complex valued functions but also of operator valued functions. Recently a suitable such space $\mathfrak{L}^1_*(\xi)$ is defined and shown to be a Banach space (cf. Kakihara [7]), where the pseudo (weak) Radon-Nikodým derivative $\xi'$ with respect to a dominating measure is used. Then $X$-valued gramian Schauder basic measures are obtained as an analogy of [2], where their characterization is also discussed.

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Finally, $X$-valued gramian uniformly bounded linearly stationary processes on a locally compact abelian group are studied in terms of gramian Schauder basic measures. Originally this kind of Hilbert space valued processes are considered by Tjøstheim and Thomas [12] and Niemi [9] in connection with shift operator groups. Then Abreu and Salehi [2] gave a measure theoretic view on them.

In Section 2, some necessary definitions and preliminary results will be stated. In Section 3, the structure of the space $\mathfrak{L}^1_*(\xi)$ is studied for a normal Hilbert $B(H)$-module valued measure $\xi$ and in Section 4 gramian Schauder basic measures are considered in detail. Finally in Section 5 we shall develop several types of uniformly bounded linearly stationary processes in the normal Hilbert $B(H)$-module setting.

## 2. Preliminaries

Let $H$ be a Hilbert space, $B(H)$ be the algebra of all bounded linear operators on $H$ and $T(H)$ be the Banach space of all trace class operators in $B(H)$. Then a **normal pre-Hilbert $B(H)$-module** is defined as a left $B(H)$-module $X$ with a $T(H)$-valued **gramian** $[\cdot,\cdot]\colon X \times X \to T(H)$ such that

(1) $[x, x] \geq 0$ for $x \in X$, 0 being the null operator;

(2) $[x, x] = 0$ if and only if $x = 0$;

(3) $[a \cdot x + b \cdot y, z] = a[x, z] + b[y, z]$ for $x, y, z \in X$ and $a, b \in B(H)$;

(4) $[x, y]^* = [y, x]$ for $x, y \in X$,

where the action of $B(H)$ on $X$ is denoted by

$$B(H) \times X \ni (a, x) \mapsto a \cdot x \in X.$$

We can define a norm and an inner product respectively by

$$\|x\|_X = \|[x, x]\|_\tau^{1/2}, \quad (x, y)_X = \mathrm{tr}[x, y], \quad x, y \in X,$$

where $\|\cdot\|_\tau$ is the trace norm and $\mathrm{tr}(\cdot)$ is the trace. If $X$ is complete with respect to the norm, then it is called a **normal Hilbert $B(H)$-module**.

A typical example of a normal Hilbert $B(H)$-module is the space $S(K, H)$ of all Hilbert-Schmidt class operators from $K$ to $H$, where $K$ is another Hilbert space and the module action of $B(H)$ and the gramian are respectively given by

$$a \cdot x = ax, \quad [x, y] = xy^*, \quad a \in B(H),\ x, y \in S(K, H).$$

Another such example is the tensor product Hilbert space $K \otimes H$, where the module action and the gramian are respectively defined by

$$\begin{aligned} a \cdot (f \otimes \phi) &= f \otimes (a\phi), \\ [f \otimes \phi, g \otimes \psi] &= (f, g)_K \phi \otimes \overline{\psi} \end{aligned} \tag{2.1}$$

for $a \in B(H)$, $\phi, \psi \in H$ and $f, g \in K$. Here, $(\,\cdot\,,\cdot\,)_K$ is the inner product in $K$ and $\phi \otimes \overline{\psi}$ is the one-dimensional operator on $H$ given by $(\phi \otimes \overline{\psi})\psi' = (\psi', \psi)_H \phi$ for $\psi' \in H$, $(\,\cdot\,,\cdot\,)_H$ being the inner product in $H$.

Since for any normal Hilbert $B(H)$-module $X$, there exists a Hilbert space $K$ such that $X \cong K \otimes H$, **we shall assume that $X = K \otimes H$ from now on** (cf. Kakihara [6, p. 30]). **Also we shall assume that $H$ is separable.**

An operator $S$ on a normal Hilbert $B(H)$-module $X$ is called a **module map** if

$$S(a \cdot x) = a \cdot (Sx), \quad a \in B(H),\ x \in X.$$

Let $A(X)$ denote the set of all bounded module maps in $B(X)$, the set of all bounded linear operators on $X$. Since $X = K \otimes H$, we can identify $A(X) = B(K) \otimes \mathbf{1} = B(K)$, $\mathbf{1}$ being the identity operator on $H$.

For a subset $X_0 \subset X$ we denote by $\mathfrak{S}(X_0)$ the closed submodule generated by $X_0$, i.e., the closure of

$$\left\{ \sum_{k=1}^{n} a_k \cdot x_k : a_k \in B(H), x_k \in X_0, 1 \le k \le n, n \in \mathbb{N} \right\},$$

$\mathbb{N}$ being the set of all positive integers.

Let $(\Theta, \mathfrak{A})$ be a measurable space, $\mathfrak{A}$ being a $\sigma$-algebra of subsets of $\Theta$, and consider $X$-valued countably additive measures on $\mathfrak{A}$. The set of all such measures is denoted by $\mathrm{ca}(\mathfrak{A}, X)$. An $X$-valued measure $\xi \in \mathrm{ca}(\mathfrak{A}, X)$ is said to be **orthogonally scattered** if

$$A, B \in \mathfrak{A},\ A \cap B = \emptyset \quad \Longrightarrow \quad (\xi(A), \xi(B))_X = 0.$$

Let $\mathrm{caos}(\mathfrak{A}, X)$ denote the set of all orthogonally scattered measures in $\mathrm{ca}(\mathfrak{A}, X)$. $\xi$ is said to be **gramian orthogonally scattered**, denoted $\xi \in \mathrm{cagos}(\mathfrak{A}, X)$, if

$$A, B \in \mathfrak{A},\ A \cap B = \emptyset \quad \Longrightarrow \quad [\xi(A), \xi(B)] = 0.$$

For an $X$-valued measure $\xi \in \mathrm{ca}(\mathfrak{A}, X)$, in addition to semivariation and variation, **operator semivariation** $\|\xi\|_o(A)$ on $A \in \mathfrak{A}$ is defined by

$$\|\xi\|_o(A) = \sup \left\{ \left\| \sum_{\Delta \in \pi} a_\Delta \cdot \xi(\Delta) \right\|_X : a_\Delta \in B(H), \|a_\Delta\| \le 1, \Delta \in \pi \in \Pi(A) \right\}, \tag{2.2}$$

where $\Pi(A)$ is the set of all finite measurable partitions of $A$. If $a_\Delta \in B(H)$ is replaced by a complex number $\alpha_\Delta$ with $|\alpha_\Delta| \le 1$ in (2.2), then the **semivariation** $\|\xi\|(A)$ is obtained. Let $\mathrm{bca}(\mathfrak{A}, X)$ denote the set of all $X$-valued measures in $\mathrm{ca}(\mathfrak{A}, X)$ of bounded operator semivariation.

For $x = f \otimes \phi \in X = K \otimes H$ and $\psi \in H$ define $\langle x, \psi \rangle_H \in K$ by

$$\langle x, \psi \rangle_H = \langle f \otimes \phi, \psi \rangle_H = (\phi, \psi)_H f. \tag{2.3}$$

In general, for $x = \sum_{n=1}^\infty f_n \otimes \phi_n \in X$, we can define

$$\langle x, \psi \rangle_H = \left\langle \sum_{n=1}^\infty f_n \otimes \phi_n, \psi \right\rangle_H = \sum_{n=1}^\infty (\phi_n, \psi)_H f_n \in K,$$

where the series converges in the norm $\|\cdot\|_K$. Let $\{\phi_n\}_{n=1}^\infty$ be an orthonormal basis of $H$. Then, note that each $x \in X$ and its norm can be written as

$$x = \sum_{n=1}^\infty \langle x, \phi_n \rangle_H \otimes \phi_n, \quad \|x\|_X^2 = \sum_{n=1}^\infty \|\langle x, \phi_n \rangle_H\|_K^2, \tag{2.4}$$

where the first series converges in the norm $\|\cdot\|_X$. Then the following characterization of a gramian orthogonally scattered measure is obtained (cf. [6, p. 66, Proposition 18]).

**Proposition 2.1.** *For an $X$-valued measure $\xi \in \mathrm{ca}(\mathfrak{A}, X)$ the following conditions are equivalent.*

(1) $\xi \in \mathrm{cagos}(\mathfrak{A}, X)$.

(2) $\xi_\phi \in \mathrm{caos}(\mathfrak{A}, K)$ *for every $\phi \in H$, where $\xi_\phi(\cdot) = \langle \xi(\cdot), \phi \rangle_H$.*

(3) $\xi_\phi$ *and* $\xi_\psi$ *are mutually orthogonally scattered for every $\phi, \psi \in H$. That is, if $A, B \in \mathfrak{A}$ are disjoint, then $(\xi_\phi(A), \xi_\psi(B))_K = 0$.*

*Proof.* Let $\{\phi_n\}_{n=1}^\infty$ be an orthonormal basis of $H$. Let $\phi, \psi \in H$, $A, B \in \mathfrak{A}$ and write $\xi(A)$ and $\xi(B)$ using (2.4) by

$$\xi(A) = \sum_{n=1}^\infty g_n \otimes \phi_n, \quad \xi(B) = \sum_{n=1}^\infty h_n \otimes \phi_n,$$

where $g_n = \langle \xi(A), \phi_n \rangle_H$, $h_n = \langle \xi(B), \phi_n \rangle_H \in K$, $n \ge 1$. Then we see that

$$\begin{aligned}(\xi_\phi(A), \xi_\psi(B))_K &= (\langle \xi(A), \phi \rangle_H, \langle \xi(B), \psi \rangle_H)_K \\ &= \left( \left\langle \sum_{n=1}^\infty g_n \otimes \phi_n, \phi \right\rangle_H, \left\langle \sum_{m=1}^\infty h_m \otimes \phi_m, \psi \right\rangle_H \right)_K\end{aligned}$$

$$
\begin{aligned}
&= \left( \sum_{n=1}^{\infty} g_n (\phi_n, \phi)_H, \sum_{m=1}^{\infty} h_m (\phi_m, \psi)_H \right)_K \\
&= \sum_{n,m=1}^{\infty} (g_n, h_m)_K (\phi_n, \phi)_H \overline{(\phi_m, \psi)}_H \\
&= \sum_{n,m=1}^{\infty} (g_n, h_m)_K ((\psi, \phi_m)_H \phi_n, \phi)_H \\
&= \sum_{n,m=1}^{\infty} (g_n, h_m)_K ((\phi_n \otimes \overline{\phi}_m)\psi, \phi)_H \\
&= \left( \sum_{n,m=1}^{\infty} (g_n, h_m)_K (\phi_n \otimes \overline{\phi}_m)\psi, \phi \right)_H \\
&= ([\xi(A), \xi(B)]\psi, \phi)_H, \qquad \text{by (2.1)},
\end{aligned}
$$

and hence

$$
(\xi_\phi(A), \xi_\psi(B))_K = ([\xi(A), \xi(B)]\psi, \phi)_H.
$$

It follows from the above that (1) and (3) are equivalent. (3) $\Rightarrow$ (2) is obvious. (2) $\Rightarrow$ (3) follows from

$$
([\xi(A), \xi(B)]\psi, \phi)_H = 0 \text{ for } \phi, \psi \in H \quad \iff \quad ([\xi(A), \xi(B)]\phi, \phi)_H = 0 \text{ for } \phi \in H.
$$

This completes the proof. □

## 3. Structure of $\mathfrak{L}^1_*(\xi)$

First we define the Dunford-Schwartz integral for a $\mathbb{C}$-valued function with respect to a Banach space valued measure and then define Schauder basic measures, where $\mathbb{C}$ is the set of all complex numbers. So let $(\Theta, \mathfrak{A})$ be a measurable space as before and $Y$ be a Banach space with the norm $\|\cdot\|_Y$. Consider a $Y$-valued countably additive measure $\xi$ on $\mathfrak{A}$, denoted $\xi \in \mathrm{ca}(\mathfrak{A}, Y)$. Then the Dunford-Schwartz integrability is defined as follows.

**Definition 3.1.** Let $\xi \in \mathrm{ca}(\mathfrak{A}, Y)$. A measurable complex valued function $f$ on $\Theta$ is said to be **Dunford-Schwartz integrable** (**DS-integrable**) with respect to $\xi$ if there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset L^0(\Theta)$ of $\mathbb{C}$-valued $\mathfrak{A}$-simple functions such that

(i) $f_n \to f$ $\xi$-a.e.;

(ii) $\left\{ \int_A f_n \, d\xi \right\}_{n=1}^{\infty}$ is a Cauchy sequence in $Y$ for every $A \in \mathfrak{A}$.

The **integral** of $f$ with respect to $\xi$ over $A \in \mathfrak{A}$ is defined by

$$
\int_A f \, d\xi = \lim_{n \to \infty} \int_A f_n \, d\xi,
$$

where the integral in the right side is defined in an obvious way.

Let $L^1_{\mathrm{DS}}(\xi)$ denote the set of all $\mathbb{C}$-valued functions on $\Theta$ that are DS-integrable with respect to $\xi$. For $f \in L^1_{\mathrm{DS}}(\xi)$ define $\xi_f(\cdot)$ and $\|f\|_{s,\xi}$ by

$$\xi_f(A) = \int_A f\,d\xi, \quad A \in \mathfrak{A}, \tag{3.1}$$

$$\|f\|_{s,\xi} = \|\xi_f\|(\Theta). \tag{3.2}$$

Then, clearly $\xi_f \in \mathrm{ca}(\mathfrak{A}, Y)$ for $f \in L^1_{\mathrm{DS}}(\xi)$ and $\|\cdot\|_{s,\xi}$ is a norm on $L^1_{\mathrm{DS}}(\xi)$. Abreu and Salehi [2] proved that the space $(L^1_{\mathrm{DS}}(\xi), \|\cdot\|_{s,\xi})$ is a Banach space. Let $T_\xi \colon L^1_{\mathrm{DS}}(\xi) \to Y$ be given by

$$T_\xi f = \xi_f(\Theta) = \int_\Theta f\,d\xi, \quad f \in L^1_{\mathrm{DS}}(\xi). \tag{3.3}$$

**Definition 3.2.** Let $\xi \in \mathrm{ca}(\mathfrak{A}, Y)$ be a $Y$-valued countably additive measure and the operator $T_\xi \colon L^1_{\mathrm{DS}}(\xi) \to Y$ be defined by (3.3). Then, $\xi$ is said to be **Schauder basic** if $T_\xi$ is one-to-one and its range is closed.

Let $\xi \in \mathrm{ca}(\mathfrak{A}, Y)$. It follows from (3.2) and (3.3) that $\|T_\xi\| \le 1$, so that $\xi$ is Schauder basic if and only if $T_\xi$ is bounded below, i.e., there exists a constant $\alpha > 0$ such that

$$\|T_\xi f\|_Y \ge \alpha \|f\|_{s,\xi}, \quad f \in L^1_{\mathrm{DS}}(\xi)$$

(cf. [1, p. 70]). Here are some equivalence conditions for a Hilbert space valued measure to be Schauder basic obtained by Abreu and Salehi [2] (see also Niemi [9, 10]), which is stated as follows.

**Proposition 3.3.** *Let $\mathcal{H}$ be a Hilbert space and $\xi \in \mathrm{ca}(\mathfrak{A}, \mathcal{H})$. Then the following conditions are equivalent.*

(1) *$\xi$ is Schauder basic.*

(2) *There exists a bounded operator $Q \in B(\mathcal{H})$ with a bounded inverse such that $Q\xi \in \mathrm{caos}(\mathfrak{A}, \mathcal{H})$.*

(3) *There exist an $m \in \mathrm{ca}(\mathfrak{A}, \mathbb{R}^+)$ and a constant $M > 0$ such that for any $f \in L^0(\Theta)$,*

$$\frac{1}{M}\int_\Theta |f|^2\,dm \le \left\|\int_\Theta f\,d\xi\right\|_{\mathcal{H}}^2 \le M\int_\Theta |f|^2\,dm.$$

(4) *There exists a constant $\alpha > 0$ such that for any $\{A_1, A_2, \ldots, A_n\} \in \Pi(\Theta)$ and $\alpha_j \in \mathbb{C}$ with $|\alpha_j| \le 1$, $1 \le j \le n$,*

$$\frac{1}{\alpha}\sum_{j=1}^n |\alpha_j|^2 \|\xi(A_j)\|_{\mathcal{H}}^2 \le \left\|\sum_{j=1}^n \alpha_j \xi(A_j)\right\|_{\mathcal{H}}^2 \le \alpha \sum_{j=1}^n |\alpha_j|^2 \|\xi(A_j)\|_{\mathcal{H}}^2.$$

In our normal Hilbert $B(H)$-module setting, to obtain similar equivalence conditions we need to consider operator valued functions and create a space corresponding to $L^1_{\mathrm{DS}}(\xi)$ for a Banach space valued measure $\xi$. First we state some measurability concepts for operator valued functions. Then, we define an integrability concept for such functions using a pseudo Radon-Nikodým derivative and construct an $L^1$ space for a normal Hilbert $B(H)$-module valued measure.

**Definition 3.4.** A $B(H)$-valued function $\Phi$ on $\Theta$ is said to be **$\mathfrak{A}$-measurable** if, for every $\phi \in H$, the $H$-valued function $\Phi(\cdot)\phi$ is strongly measurable, i.e., if there exists a sequence $\{\phi_n\}_{n=1}^{\infty} \subset L^0(\Theta; H)$ of $H$-valued $\mathfrak{A}$-simple functions such that $\|\Phi(\theta)\phi - \phi_n(\theta)\|_H \to 0$ for every $\theta \in \Theta$.

Let $O(H)$ denote the set of all linear operators $a$ with domain $\mathfrak{D}(a) \subseteq H$. An $O(H)$-valued function $\Phi$ on $\Theta$ is said to be **$\mathfrak{A}$-measurable** if there exists a sequence $\{\Phi_n\}_{n=1}^{\infty}$ of $B(H)$-valued $\mathfrak{A}$-measurable functions on $\Theta$ such that $\|\Phi_n(\theta)\phi - \Phi(\theta)\phi\|_H \to 0$ for every $\theta \in \Theta$ and $\phi \in \mathfrak{D}(\Phi(\theta))$.

We shall denote the action of $a \in B(H)$ on $x \in X$ simply by $ax$ instead of $a \cdot x$. Let $\xi \in \mathrm{bca}(\mathfrak{A}, X)$ be an $X$-valued measure of bounded operator semivariation. There always exists a positive finite measure $\nu \in \mathrm{ca}(\mathfrak{A}, \mathbb{R}^+)$ such that $\xi \ll \nu$. If $\xi$ is of bounded variation, denoted $\xi \in \mathrm{vca}(\mathfrak{A}, X)$, then it has an ordinary Radon-Nikodým derivative $\xi' = d\xi/d\nu$ with respect to $\nu$. If this is not the case we shall consider a pseudo Radon-Nikodým derivative as follows, where we called it a weak Radon-Nikodým derivative in [7].

**Definition 3.5.** Let $\xi \in \mathrm{bca}(\mathfrak{A}, X)$ and $\xi \ll \nu$ with a finite $\nu \in \mathrm{ca}(\mathfrak{A}, \mathbb{R}^+)$. Then, $\xi$ is said to have a **pseudo Radon-Nikodým derivative** $\xi'$ with respect to $\nu$, denoted $\xi' = d\xi/d\nu$, if there exists a sequence $\{\Phi_n\}_{n=1}^{\infty} \subset L^0(\Theta; X)$ of $X$-valued $\mathfrak{A}$-simple functions on $\Theta$ such that

(1) $\|\Phi_n(\theta) - \xi'(\theta)\|_X \to 0$ for $\nu$-a.e. $\theta$;

(2) $\left\| \int_A \Phi_n \, d\nu - \xi(A) \right\|_X \to 0$ for $A \in \mathfrak{A}$.

The sequence $\{\Phi_n\}_{n=1}^{\infty}$ is called a **determining sequence** for the $X$-valued function $\xi'$. We may write

$$\int_A \xi' \, d\nu = \lim_{n\to\infty} \int_A \Phi_n \, d\nu = \xi(A), \quad A \in \mathfrak{A}.$$

It is shown that a pseudo Radon-Nikodým derivative does not depend on a determining sequence if the derivative exists (cf. [7]). Using a pseudo Radon-Nikodým derivative we can define a Dunford-Schwartz type integral of operator valued functions.

**Definition 3.6.** Let $\xi \in \mathrm{bca}(\mathfrak{A}, X)$ have a pseudo Radon-Nikodým derivative $\xi' = d\xi/d\nu$ with respect to a dominating measure $\nu$.

(1) An $O(H)$-valued function $\Phi$ on $\Theta$ is said to be **$\xi$-integrable** if $\Phi(\theta)\xi'(\theta) \in X$ $\nu$-a.e. $\theta$ and if there exists a sequence $\{\Phi_n\}_{n=1}^{\infty} \subset L^0(\Theta; B(H))$ of $B(H)$-valued $\mathfrak{A}$-simple functions on $\Theta$ such that

(a) $\|\Phi_n(\theta)\xi'(\theta) - \Phi(\theta)\xi'(\theta)\|_X \to 0$ for $\nu$-a.e. $\theta$;

(b) The sequence $\left\{\int_A \Phi_n \, d\xi\right\}_{n=1}^{\infty}$ is a Cauchy sequence in $X$ in the norm $\|\cdot\|_X$ for every $A \in \mathfrak{A}$, where for $\Psi = \sum_{k=1}^{m} a_k 1_{A_k} \in L^0(\Theta; B(H))$ we define

$$\int_A \Psi \, d\xi = \sum_{k=1}^{m} a_k \xi(A \cap A_k),$$

$1_A$ being the indicator function of $A \in \mathfrak{A}$. In this case, the **integral** of $\Phi$ with respect to $\xi$ over $A \in \mathfrak{A}$ is defined by

$$\int_A \Phi \, d\xi = \int_A \Phi\xi' \, d\nu = \lim_{n\to\infty} \int_A \Phi_n \, d\xi.$$

Let $\mathfrak{L}^1(\xi)$ denote the set of all $O(H)$-valued $\mathfrak{A}$-measurable functions on $\Theta$ that are $\xi$-integrable.

(2) For a function $\Phi \in \mathfrak{L}^1(\xi)$ we define an $X$-valued measure $\xi_\Phi$ by

$$\xi_\Phi(A) = \int_A \Phi \, d\xi, \quad A \in \mathfrak{A}.$$

Finally, a special subset $\mathfrak{L}^1_*(\xi)$ of $\mathfrak{L}^1(\xi)$ is defined as

$$\mathfrak{L}^1_*(\xi) = \{\Phi \in \mathfrak{L}^1(\xi) : \xi_\Phi \in \mathrm{bca}(\mathfrak{A}, X)\},$$

where the norm of $\Phi \in \mathfrak{L}^1_*(\xi)$ is given by the total operator semivariation

$$\|\Phi\|_{o,\xi} = \|\xi_\Phi\|_o(\Theta).$$

For $\xi \in \mathrm{bca}(\mathfrak{A}, X)$ it is shown that $(\mathfrak{L}^1_*(\xi), \|\cdot\|_{o,\xi})$ is a Banach space and the set $L^0(\Theta; B(H))$ of all $B(H)$-valued $\mathfrak{A}$-simple functions is dense in it (cf. [7]).

It may be interesting to consider scalar valued functions $f$ on $\Theta$ instead of operator valued functions. So let $L^1_{\mathrm{DS}}(\xi)$ be the set of all $\mathbb{C}$-valued functions that are DS-integrable with respect to $\xi$ as given in Definition 3.1, while $L^1(\xi)$ be the set of $\mathbb{C}$-valued measurable functions that are $\xi$-integrable using the pseudo Radon-Nikodým derivative as given in Definition 3.6. Then, it was proved in [7] that $L^1(\xi) = L^1_{\mathrm{DS}}(\xi)$. Let us define

$$L^1_*(\xi) = \{f \in L^1(\xi) : \xi_f \in \mathrm{bca}(\mathfrak{A}, X)\}, \tag{3.4}$$

$$\|f\|_{o,\xi} = \|\xi_f\|_o(\Theta), \quad f \in L^1_*(\xi), \tag{3.5}$$

where $\xi_f$ is defined by (3.1). Then we have the following.

**Proposition 3.7.** *Let $\xi \in \mathrm{bca}(\mathfrak{A}, X)$ have a pseudo Radon-Nikodým derivative with respect to $\nu$. Then, the space $(L^1_*(\xi), \|\cdot\|_{o,\xi})$ given by* (3.4) *and* (3.5) *is a Banach space. Moreover, the set $L^0(\Theta)$ of $\mathbb{C}$-valued $\mathfrak{A}$-simple functions is dense in it.*

*Proof.* Let $\{f_n\}_{n=1}^\infty \subset L^1_*(\xi)$ be a Cauchy sequence. Then it is a Cauchy sequence in $(L^1(\xi), \|\cdot\|_{s,\xi})$ since $\|f_n - f_m\|_{s,\xi} \le \|f_n - f_m\|_{o,\xi} \to 0$ as $n, m \to \infty$. Since the space $(L^1(\xi), \|\cdot\|_{s,\xi})$ is a Banach space there exists a unique $f \in L^1(\xi)$ such that $\|f_n - f\|_{s,\xi} \to 0$ as $n \to \infty$. Let $\pi = \{A_1, \dots, A_k\} \in \Pi(\Theta)$, $a_j \in B(H)$ with $\|a_j\| \le 1$ for $1 \le j \le k$. For any $\varepsilon > 0$ choose an $n_0 = n_0(\varepsilon, \pi) \in \mathbb{N}$ such that

$$\|\xi_{f_n}(A_j) - \xi_f(A_j)\|_X < \frac{\varepsilon}{k}, \quad 1 \le j \le k,\ n \ge n_0.$$

Then it follows that for $n \ge n_0$,

$$\begin{aligned}\left\|\sum_{j=1}^k a_j \xi_f(A_j)\right\|_X &\le \left\|\sum_{j=1}^k a_j(\xi_f - \xi_{f_n})(A_j)\right\|_X + \left\|\sum_{j=1}^k a_j \xi_{f_n}(A_j)\right\|_X \\ &\le \sum_{j=1}^k \|a_j\| \|(\xi_f - \xi_{f_n})(A_j)\|_X + \|\xi_{f_n}\|_o(\Theta) \\ &< \varepsilon + \|f_n\|_{o,\xi} < \infty.\end{aligned}$$

This implies that

$$\|f\|_{o,\xi} \le \lim_{n\to\infty} \|f_n\|_{o,\xi} < \infty$$

since $\{\|f_n\|_{o,\xi}\}_{n=1}^\infty$ is convergent. Thus we have $f \in L^1_*(\xi)$ and

$$0 = \lim_{n\to\infty} \lim_{m\to\infty} \|f_n - f_m\|_{o,\xi} = \lim_{n\to\infty} \|f_n - f\|_{o,\xi},$$

so that $L^1_*(\xi)$ is closed.

To see that $L^0(\Theta)$ is dense in $L^1_*(\xi)$ let $f \in L^1_*(\xi)$ and let $\{f_n\}_{n=1}^\infty \subset L^0(\Theta)$ be its determining sequence of $\mathbb{C}$-valued functions. Then the rest of the proof is similarly done as in the proof of Theorem 3.4.5 in [7], showing $\|f_n - f\|_{o,\xi} \to 0$. □

For $\xi \in \mathrm{bca}(\mathfrak{A}, X)$ consider two sets given by

$$L^1(\xi) \otimes B(H) = \left\{\sum_{j=1}^n f_j \otimes a_j : f_j \in L^1(\xi), a_j \in B(H), 1 \le j \le n, n \in \mathbb{N}\right\},$$

$$L^1_*(\xi) \otimes B(H) = \left\{\sum_{j=1}^n f_j \otimes a_j : f_j \in L^1_*(\xi), a_j \in B(H), 1 \le j \le n, n \in \mathbb{N}\right\},$$

where $f \otimes a$ is a function on $\Theta$ defined by $(f \otimes a)(\theta) = f(\theta)a$ for $f \in L^1(\xi)$ and $a \in B(H)$. It is clear that

$$L^1(\xi) \otimes B(H) \subseteq \mathfrak{L}^1(\xi), \quad L^1_*(\xi) \otimes B(H) \subseteq \mathfrak{L}^1_*(\xi).$$

Let us denote the closure of each space with respect to $\|\cdot\|_{s,\xi}$ and $\|\cdot\|_{o,\xi}$ respectively by

$$L^1(\xi) \otimes_{s,\xi} B(H) \subseteq \mathfrak{L}^1(\xi), \quad L^1_*(\xi) \otimes_{o,\xi} B(H) \subseteq \mathfrak{L}^1_*(\xi).$$

Since the space $L^0(\Theta; B(H))$ is dense in all four spaces mentioned above it is easy to see the following proposition.

**Proposition 3.8.** *For $\xi \in \mathrm{bca}(\mathfrak{A}, X)$ it holds that*

$$L^1(\xi) \otimes_{s,\xi} B(H) = \mathfrak{L}^1(\xi), \quad L^1_*(\xi) \otimes_{o,\xi} B(H) = \mathfrak{L}^1_*(\xi).$$

## 4. Gramian Schauder basic measures

Banach space valued Schauder basic measures were mentioned in Definition 3.2. In this section we introduce gramian Schauder basic measures in our setting. We assume that $X = K \otimes H$, where $H$ and $K$ are separable Hilbert spaces. Now we are ready to define gramian Schauder basic measures.

**Definition 4.1.** Let $\xi \in \mathrm{bca}(\mathfrak{A}, X)$ be such that $\xi \ll \nu$ for a finite measure $\nu$, and assume that $\xi$ has a pseudo Radon-Nikodým derivative $\xi'$ with respect to $\nu$. Then, $\xi$ is said to be **gramian Schauder basic** if the operator $T_\xi \colon \mathfrak{L}^1_*(\xi) \to X$ defined by

$$T_\xi \Phi = \int_\Theta \Phi \, d\xi, \quad \Phi \in \mathfrak{L}^1_*(\xi) \tag{4.1}$$

is one-to-one and the range $T_\xi(\mathfrak{L}^1_*(\xi)) = \{T_\xi \Phi : \Phi \in \mathfrak{L}^1_*(\xi)\}$ of $T_\xi$ is closed in $X$.

*Remark* 4.2. (1) Let $\xi \in \mathrm{bca}(\mathfrak{A}, X)$ be gramian Schauder basic. Then the operator $T_\xi$ defined by (4.1) is bounded with $\|T_\xi\| \le 1$ since for $\Phi \in \mathfrak{L}^1_*(\xi)$ it holds that

$$\|T_\xi \Phi\|_X = \|\xi_\Phi(\Theta)\|_X \le \|\xi_\Phi\|_o(\Theta) = \|\Phi\|_{o,\xi}.$$

Since $T_\xi$ is one-to-one with a closed range, $T_\xi$ is bounded below and has a bounded inverse $T_\xi^{-1} \colon T_\xi(\mathfrak{L}^1_*(\xi)) \to \mathfrak{L}^1_*(\xi)$.

(2) If $\xi \in \mathrm{bca}(\mathfrak{A}, X)$ is gramian Schauder basic, then $\xi$ is Schauder basic. In fact, suppose $\xi$ is gramian Schauder basic. Then there exists a constant $\alpha > 0$ such that

$$\|T_\xi \Phi\|_X \ge \alpha \|\Phi\|_{o,\xi}, \quad \Phi \in \mathfrak{L}^1_*(\xi).$$

This implies that

$$\|T_\xi f\|_X \ge \alpha \|f\|_{o,\xi} \ge \alpha \|f\|_{s,\xi}, \quad f \in L^0(\Theta).$$

Since $\|T_\xi f\|_X \le \|f\|_{s,\xi}$ for $f \in L^1(\xi)$ we see that $T_\xi \colon L^1(\xi) \to X$ has a bounded inverse and hence $\xi$ is Schauder basic.

The following lemma may be interesting.

**Lemma 4.3.** *Let* $\xi \in \mathrm{bca}(\mathfrak{A}, X)$ *be gramian Schauder basic with the operator* $T_\xi$ *defined by* (4.1). *Then the operator* $T_\xi$ *restricted to* $L^1_*(\xi)$ *is one-to-one with a closed range.*

*Proof.* $T_\xi$ is obviously one-to-one since $L^1_*(\xi) \subset \mathfrak{L}^1_*(\xi)$. To see that the range $T_\xi(L^1_*(\xi))$ is closed, let $\{x_n\}_{n=1}^\infty \subset T_\xi(L^1_*(\xi))$ be a Cauchy sequence such that $x_n = T_\xi f_n$ with $f_n \in L^1_*(\xi)$ for $n \geq 1$. Then $\{x_n\}_{n=1}^\infty$ is Cauchy in $X$, so that there exists a unique $x \in X$ such that $\|x_n - x\|_X \to 0$. Since $T_\xi(L^1_*(\xi)) \subset T_\xi(\mathfrak{L}^1_*(\xi))$ and the latter is closed, there exists some $\Phi \in \mathfrak{L}^1_*(\xi)$ such that $x = T_\xi \Phi$. Hence it follows that

$$\|x_n - x\|_X = \|T_\xi f_n - T_\xi \Phi\|_X = \|T_\xi (f_n - \Phi)\|_X \to 0$$

and that

$$\|T_\xi^{-1}(x_n - x)\|_{o,\xi} = \|f_n - \Phi\|_{o,\xi} \to 0.$$

Since $L^1_*(\xi)$ is closed in $\mathfrak{L}^1_*(\xi)$ by Proposition 3.7 we have $\Phi \in L^1_*(\xi)$. Thus the range $T_\xi(L^1_*(\xi))$ is closed. □

Let $F \in \mathrm{ca}(\mathfrak{A}, T^+(H))$ be a $T^+(H)$-valued measure, where $T^+(H) = \{a \in T(H) : a \geq 0\}$. Then, $\nu(\cdot) = |F|(\cdot) = \|F(\cdot)\|_\tau$ is a dominating measure for $F$ and the ordinary Radon-Nikodým derivative $F' = dF/d\nu$ exists in $L^1(\Theta, \nu; T(H))$ since $T(H)$ is a separable dual space (cf. [3]). Let $\Phi$ and $\Psi$ be two $O(H)$-valued $\mathfrak{A}$-measurable functions on $\Theta$. Then, a pair $(\Phi, \Psi)$ is said to be **$F$-integrable** if $\Phi F'^{1/2}$ and $\Psi F'^{1/2}$ are $S(H)$-valued functions and the $T(H)$-valued function $(\Phi F'^{1/2})(\Psi F'^{1/2})^*$ is Bochner integrable with respect to $\nu$. Here, $S(H)$ is the set of all Hilbert-Schmidt class operators on $H$. In this case we write

$$[\Phi, \Psi]_F = \int_\Theta \Phi \, dF \Psi^* = \int_\Theta \left(\Phi F'^{1/2}\right)\left(\Psi F'^{1/2}\right)^* d\nu \in T(H). \tag{4.2}$$

Then, the space $L^2(F)$ is defined by

$$L^2(F) = \{\Phi : \Phi \text{ is } O(H)\text{-valued, } \mathfrak{A}\text{-measurable, and } (\Phi, \Phi) \text{ is } F\text{-integrable}\}$$

with the norm defined by

$$\|\Phi\|_F = \|[\Phi, \Phi]_F\|_\tau^{1/2}, \quad \Phi \in L^2(F).$$

As was shown by Mandrekar and Salehi [8] $L^2(F)$ is a normal Hilbert $B(H)$-module with the gramian given by (4.2). Moreover, the set $L^0(\Theta; B(H))$ of $\mathfrak{A}$-simple $B(H)$-valued functions on $\Theta$ is dense in $L^2(F)$.

*Remark* 4.4. For $\xi \in \mathrm{cagos}(\mathfrak{A}, X)$ let

$$F_\xi(\cdot) = [\xi(\cdot), \xi(\cdot)].$$

Then, $F_\xi \in \mathrm{ca}(\mathfrak{A}, T^+(H))$ and $L^2(F_\xi)$ is formed with $\nu_\xi(\cdot) = \|F_\xi(\cdot)\|_\tau \in \mathrm{ca}(\mathfrak{A}, \mathbb{R}^+)$ and $F'_\xi = dF_\xi/d\nu_\xi \in L^1(\Theta, \nu_\xi; T(H))$. If $\xi$ has a pseudo Radon-Nikodým derivative, then it holds that $\mathfrak{L}^1(\xi) = \mathfrak{L}^1_*(\xi) = L^2(F_\xi)$ and $\|\Phi\|_\xi = \|\Phi\|_{F_\xi}$ for all $\Phi$ in these spaces.

Note that an orthogonally scattered measure is Schauder basic [2]. Similarly a gramian orthogonally scattered measure is gramian Schauder basic as described below.

**Proposition 4.5.** *If $\xi \in \mathrm{cagos}(\mathfrak{A}, X)$ is an $X$-valued gramian orthogonally scattered measure with a pseudo Radon-Nikodým derivative $\xi'$ with respect to a dominating measure, then it is gramian Schauder basic.*

*Proof.* Let $T_\xi$ be defined by (4.1). Then, for $\Phi \in \mathfrak{L}^1_*(\xi) = L^2(F_\xi)$ it holds that

$$[T_\xi\Phi, T_\xi\Phi] = \left[\int_\Theta \Phi\, d\xi, \int_\Theta \Phi\, d\xi\right] = \int_\Theta \Phi\, dF_\xi \Phi^* = [\Phi, \Phi]_{F_\xi}.$$

Hence, $T_\xi$ is gramian unitary, so that it is one-to-one and of norm 1. Also it is clear that $T_\xi(L^2(F_\xi)) = \mathfrak{S}(\xi)$, the closed submodule of $X$ generated by the set $\{\xi(A) : A \in \mathfrak{A}\}$. Thus the range of $T_\xi$ is closed. Therefore $\xi$ is gramian Schauder basic. □

Another basic fact is the following.

**Proposition 4.6.** *Let $\xi \in \mathrm{bca}(\mathfrak{A}, X)$ be gramian Schauder basic with a pseudo Radon-Nikodým derivative $\xi' = d\xi/d\nu$, $\nu$ being a dominating measure, and $S \in A(X)$ be a bounded module map with a bounded inverse $S^{-1} \in A(X)$. Then, $\eta = S\xi \in \mathrm{bca}(\mathfrak{A}, X)$ is gramian Schauder basic with a pseudo Radon-Nikodým derivative $\eta' = d\eta/d\nu = S\xi'$. Moreover, it holds that $T_\eta = ST_\xi$ and $\mathfrak{L}^1_*(\eta) = \mathfrak{L}^1_*(\xi)$.*

*Proof.* Let $T_\xi\colon \mathfrak{L}^1_*(\xi) \to X$ be defined by (4.1). Then, $T_\xi$ is one-to-one and its range $T_\xi(\mathfrak{L}^1_*(\xi)) = \mathfrak{S}(\xi)$ is closed. Obviously $\eta = S\xi$ is of bounded operator semivariation since $\|\eta\|_o(\Theta) \le \|S\|\|\xi\|_o(\Theta)$. It follows from Corollary 2.3.5 of [7] that the pseudo Radon-Nikodým derivative $\eta' = d\eta/d\nu$ of $\eta$ with respect to $\nu$ exists and $\eta' = S\xi'$. Hence $\mathfrak{L}^1_*(\eta)$ is defined.

To show that $\mathfrak{L}^1_*(\eta) = \mathfrak{L}^1_*(\xi)$ first let $\Phi \in \mathfrak{L}^1_*(\xi)$. Then $\Phi\xi' \in X$ $\nu$-a.e. and there exists a sequence $\{\Phi_n\}_{n=1}^\infty \subset L^0(\Theta; B(H))$ of $B(H)$-valued $\mathfrak{A}$-simple functions such that

(a) $\|\Phi_n\xi' - \Phi\xi'\|_X \to 0$ $\nu$-a.e.;

(b) $\left\{\int_A \Phi_n\, d\xi\right\}_{n=1}^\infty$ is a Cauchy sequence in $X$ for every $A \in \mathfrak{A}$.

Hence, we have that

$$\begin{aligned}\|\Phi_n\eta' - S(\Phi\xi')\|_X &= \|\Phi_n S\xi' - S(\Phi\xi')\|_X \\ &= \|S(\Phi_n\xi') - S(\Phi\xi')\|_X \\ &\le \|S\|\|\Phi_n\xi' - \Phi\xi'\|_X \to 0.\end{aligned}$$

Thus, $\{\Phi_n\eta'\}_{n=1}^{\infty}$ is a Cauchy sequence in $X$ $\nu$-a.e. If we define

$$\Phi\eta' = \lim_{n\to\infty} \Phi_n\eta' = S(\Phi\xi'),$$

then $\Phi\eta' \in X$ $\nu$-a.e. and $\|\Phi_n\eta' - \Phi\eta'\|_X \to 0$ $\nu$-a.e. Moreover, $\left\{\int_A \Phi_n\,d\eta\right\}_{n=1}^{\infty}$ is a Cauchy sequence for every $A \in \mathfrak{A}$ since

$$\int_A \Phi_n\,d\eta = \int_A \Phi_n\,d(S\xi) = S\int_A \Phi_n\,d\xi.$$

Consequently $\Phi \in \mathfrak{L}^1_*(\eta)$. Therefore $\mathfrak{L}^1_*(\xi) \subseteq \mathfrak{L}^1_*(\eta)$. The reverse inclusion is obtained by considering $\xi = S^{-1}\eta$, so that $\mathfrak{L}^1_*(\eta) = \mathfrak{L}^1_*(\xi)$.

Note that for $\Phi \in \mathfrak{L}^1_*(\eta) = \mathfrak{L}^1_*(\xi)$,

$$\int_A \Phi\,d\eta = \int_A \Phi\,d(S\xi) = S\int_A \Phi\,d\xi, \quad A \in \mathfrak{A}.$$

That is, $T_\eta = ST_\xi$. Hence $T_\eta$ is one-to-one and obviously $T_\eta(\mathfrak{L}^1_*(\eta))$ is closed. Thus $\eta$ is gramian Schauder basic. □

We say that two measures $\xi, \eta \in \mathrm{bca}(\mathfrak{A}, X)$ are **gramian similar** if there exists a bounded module map $S \in A(X)$ with a bounded inverse such that $\eta = S\xi$. The operator $S$ is called a **gramian similarity operator**. Then, Proposition 4.6 asserts that if two measures in $\mathrm{bca}(\mathfrak{A}, X)$ are gramian similar and one is gramian Schauder basic, then the other is also gramian Schauder basic.

In a same way, two measures $\xi, \eta \in \mathrm{ca}(\mathfrak{A}, K)$ are said to be **similar** if there exists a bounded operator $S \in B(K)$ with a bounded inverse such that $\eta = S\xi$, where $S$ is called a **similarity operator**. It follows from Proposition 3.3 that any measure in $\mathrm{ca}(\mathfrak{A}, K)$ is Schauder basic if and only if it is similar to an orthogonally scattered measure in $\mathrm{caos}(\mathfrak{A}, K)$.

The following proposition is a part of our main theorem in this section that characterises gramian Schauder basic measures, where the gramian in $X$ is denoted by $[\,\cdot\,,\cdot\,]_X$.

**Proposition 4.7.** *Let $\xi \in \mathrm{bca}(\mathfrak{A}, X)$ have a pseudo Radon-Nikodým derivative $\xi' = d\xi/d\nu$ for some dominating measure $\nu$. Then the following conditions are equivalent.*

(1) *There exist another normal Hilbert* $B(H)$*-module* $Y$*, a* $Y$*-valued gramian orthogonally scattered measure* $\eta \in \mathrm{cagos}(\mathfrak{A}, Y)$ *and a bounded module map* $S \colon \mathfrak{S}(\eta) \to \mathfrak{S}(\xi)$ *with a bounded inverse such that* $\xi = S\eta$*.*

(2) *There exist a* $T^+(H)$*-valued measure* $F \in \mathrm{ca}(\mathfrak{A}, T^+(H))$ *and a positive constant* $\alpha > 0$ *such that*

$$\frac{1}{\alpha}[\Phi, \Phi]_F \leq [T_\xi \Phi, T_\xi \Phi]_X \leq \alpha[\Phi, \Phi]_F, \quad \Phi \in L^0(\Theta; B(H)).$$

*Proof.* (1) $\Rightarrow$ (2). Assume the existence of such $Y$, $\eta$ and $S$. Let $\Phi = \sum_{i=1}^n a_i 1_{A_i} \in L^0(\Theta; B(H))$ with $a_1, \ldots, a_n \in B(H)$ and $\{A_1, \ldots, A_n\} \in \Pi(\Theta)$. Then we see that

$$\int_\Theta \Phi \, dF_\eta \Phi^* = \sum_{i=1}^n a_i F_\eta(A_i) a_i^* = [\Phi, \Phi]_{F_\eta}$$

and

$$\begin{aligned}
[T_\xi \Phi, T_\xi \Phi]_X &= \left[\int_\Theta \Phi \, d\xi, \int_\Theta \Phi \, d\xi\right]_X = \left[\sum_{i=1}^n a_i \xi(A_i), \sum_{j=1}^n a_j \xi(A_j)\right]_X \\
&= \left[\sum_{i=1}^n a_i S\eta(A_i), \sum_{j=1}^n a_j S\eta(A_j)\right]_X = \left[S\sum_{i=1}^n a_i \eta(A_i), S\sum_{j=1}^n a_j \eta(A_j)\right]_X \\
&\leq \|S\|^2 \left[\sum_{i=1}^n a_i \eta(A_i), \sum_{j=1}^n a_j \eta(A_j)\right]_Y = \|S\|^2 \left[\int_\Theta \Phi \, d\eta, \int_\Theta \Phi \, d\eta\right]_Y \\
&= \|S\|^2 [\Phi, \Phi]_{F_\eta},
\end{aligned}$$

or $[T_\xi \Phi, T_\xi \Phi]_X \leq \|S\|^2 [\Phi, \Phi]_{F_\eta}$.

Similarly we see that $[\Phi, \Phi]_{F_\eta} \leq \|S^{-1}\|^2 [T_\xi \Phi, T_\xi \Phi]_X$. Hence, by letting $\alpha = \max\{\|S\|^2, \|S^{-1}\|^2\}$ and $F = F_\eta$ we have the desired inequalities.

(2) $\Rightarrow$ (1). Assume the existence of $F$ and $\alpha$. Define a mapping $[\,\cdot\,, \cdot\,]_o \colon \mathfrak{S}(\xi) \times \mathfrak{S}(\xi) \to T(H)$ as follows. For $\Phi = \sum_{i=1}^m a_i 1_{A_i}, \Psi = \sum_{j=1}^n b_j 1_{B_j} \in L^0(\Theta; B(H))$ let $x = T_\xi \Phi, y = T_\xi \Psi \in \mathfrak{S}(\xi)$ and set

$$[x, y]_o = [T_\xi \Phi, T_\xi \Psi]_o = \sum_{i=1}^m \sum_{j=1}^n a_i F(A_i \cap B_j) b_j^*.$$

Then, we see that

$$[\Phi, \Psi]_F = \int_\Theta \Phi \, dF \Psi^* = \sum_{i=1}^m \sum_{j=1}^n a_i F(A_i \cap B_j) b_j^* = [x, y]_o.$$

By assumption it holds that

$$\frac{1}{\alpha}[x,x]_o \leq [T_\xi\Phi, T_\xi\Phi]_X \leq \alpha[x,x]_o,$$

or

$$\frac{1}{\alpha}[x,x]_X \leq [x,x]_o \leq \alpha[x,x]_X.$$

Thus there exists a bounded positive module map $S\colon \mathfrak{S}(\xi) \to \mathfrak{S}(\xi)$ with a bounded inverse such that

$$[Sx,y]_X = [x,y]_o, \quad x,y \in \mathfrak{S}(\xi)$$

(cf. Kakihara [6, pp. 25–26]). Consider $S^{1/2}\colon \mathfrak{S}(\xi) \to \mathfrak{S}(\xi)$ which is a bounded module map with a bounded inverse and define $\eta(A) = S^{1/2}\xi(A)$ for $A \in \mathfrak{A}$. Then observe that for $A, B \in \mathfrak{A}$

$$\begin{aligned}[\eta(A),\eta(B)]_X &= [S^{1/2}\xi(A), S^{1/2}\xi(B)]_X = [S\xi(A),\xi(B)]_X \\ &= [\xi(A),\xi(B)]_o = F(A\cap B),\end{aligned}$$

which implies that $\eta \in \mathrm{cagos}(\mathfrak{A}, \mathfrak{S}(\xi)) \subset \mathrm{cagos}(\mathfrak{A}, X)$. Therefore (1) holds. □

*Remark* 4.8. In (1) of the above proposition, since $S\colon \mathfrak{S}(\xi) \to \mathfrak{S}(\eta)$ is a bounded module map with a bounded inverse, there exists a gramian unitary operator $U_0\colon \mathfrak{S}(\eta) \to \mathfrak{S}(\xi)$. Consider the normal Hilbert $B(H)$-module $\widetilde{Y} = \mathfrak{S}(\eta) \oplus \mathfrak{S}(\xi)^\#$, where $\mathfrak{S}(\xi)^\# = \{x \in X : [x,y] = 0, y \in \mathfrak{S}(\xi)\}$, the gramian orthogonal complement of $\mathfrak{S}(\xi)$. Define $U, \widetilde{S}\colon \widetilde{Y} \to X$ by

$$Uy = \begin{cases} U_0 y & \text{if } y \in \mathfrak{S}(\eta), \\ y & \text{if } y \in \mathfrak{S}(\xi)^\#, \end{cases} \qquad \widetilde{S}y = \begin{cases} Sy & \text{if } y \in \mathfrak{S}(\eta), \\ y & \text{if } y \in \mathfrak{S}(\xi)^\#. \end{cases}$$

Then $U$ is gramian unitary and $\widetilde{S}$ is a bounded module map with a bounded inverse. Note that $\widetilde{\eta}$ given by $\widetilde{\eta} = U\eta$ is in $\mathrm{cagos}(\mathfrak{A}, X)$. If we define $T = \widetilde{S}U^{-1}$, then we see that $T \in A(X)$ with a bounded inverse and

$$T\widetilde{\eta} = \widetilde{S}U^{-1}U\eta = S\eta = \xi.$$

Thus, $\xi$ is gramian similar to a gramian orthogonally scattered measure.

Let $\xi \in \mathrm{ca}(\mathfrak{A}, X)$, $\phi \in H$ and define $\xi_\phi(\cdot)$ by

$$\xi_\phi(A) = \langle \xi(A), \phi\rangle_H, \quad A \in \mathfrak{A} \tag{4.3}$$

(cf. Proposition 2.1 and (2.3)). Then, $\xi_\phi$ is a $K$-valued measure, i.e., $\xi_\phi \in \mathrm{ca}(\mathfrak{A}, K)$.

After defining scalarly Schauder basic measures in the following we can obtain some necessary and sufficient conditions for an $X$-valued measure to be gramian Schauder basic.

**Definition 4.9.** Let $\xi \in \mathrm{bca}(\mathfrak{A}, X)$. We say that $\xi$ is **scalarly Schauder basic** if $\xi_\phi$ is Schauder basic for each $\phi \in H$, where $\xi_\phi$ is defined by (4.3). That is, for each $\phi \in H$ there exists a constant $\alpha_\phi > 0$ such that

$$\|T_{\xi_\phi} f\|_K \geq \alpha_\phi \|f\|_{s,\xi_\phi}, \quad f \in L^1(\xi_\phi),$$

where, as in (3.3), $T_{\xi_\phi}: L^1(\xi_\phi) \to K$ is defined by

$$T_{\xi_\phi} f = \int_\Theta f \, d\xi_\phi, \quad f \in L^1(\xi_\phi).$$

$\xi$ is said to be **uniformly scalarly Schauder basic** if there exists an $\alpha > 0$ such that for each $\phi \in H$ it holds that

$$\|T_{\xi_\phi} f\|_K \geq \alpha \|f\|_{s,\xi_\phi}, \quad f \in L^1(\xi_\phi).$$

Let $\xi \in \mathrm{bca}(\mathfrak{A}, X)$ have a pseudo Radon-Nikodým derivative with respect to a dominating measure. If $\{\phi_n\}_{n=1}^\infty$ is an orthonormal basis of $H$, then it follows from (2.3) that

$$\xi(\cdot) = \sum_{n=1}^\infty \xi_{\phi_n}(\cdot) \otimes \phi_n,$$

where the series converges in the norm $\|\cdot\|_X$. Moreover, for $\phi \in H$ it holds that

$$T_{\xi_\phi} f = \int_\Theta f \, d\xi_\phi = \int_\Theta f \langle d\xi, \phi \rangle_H = \langle T_\xi f, \phi \rangle_H, \quad f \in L^1(\xi_\phi). \tag{4.4}$$

Hence, we can consider that $L^1(\xi_\phi) \subseteq \mathfrak{L}^1_*(\xi)$.

If $\xi \in \mathrm{bca}(\mathfrak{A}, X)$ is scalarly Schauder basic, then it follows from Proposition 3.3 that for any $\phi \in H$ there exists an operator $S_\phi \in B(K)$ with a bounded inverse $S_\phi^{-1} \in B(K)$ such that

$$\eta_\phi = S_\phi \xi_\phi \in \mathrm{caos}(\mathfrak{A}, K).$$

In other words, $\xi_\phi$ is similar to an orthogonally scattered measure $\eta_\phi$ by a similarity operator $S_\phi$. Using these notations we can prove the following theorem.

**Theorem 4.10.** *Let $\xi \in \mathrm{bca}(\mathfrak{A}, X)$ have a pseudo Radon-Nikodým derivative with respect to a suitable dominating measure $\nu$. Then the following conditions are equivalent.*

(1) *$\xi$ is gramian Schauder basic.*

(2) *$\xi$ is uniformly scalarly Schauder basic.*

(3) *For each nonzero $\phi \in H$ the measure $\xi_\phi$ is similar to some orthogonally scattered measure $\eta_\phi \in \mathrm{caos}(\mathfrak{A}, K)$ with a similarity operator $S_\phi \in B(K)$. Moreover, there exist constants $\beta, \gamma > 0$ such that $\beta \leq \|S_\phi\| \leq \gamma$ for all nonzero $\phi \in H$.*

(4) *$\xi$ is gramian similar to some $\eta \in$* cagos$(\mathfrak{A}, X)$.

(5) *There exist a $T^+(H)$-valued measure $F \in$* ca$(\mathfrak{A}, T^+(H))$ *and a positive constant $\alpha > 0$ such that*

$$\frac{1}{\alpha}[\Phi, \Phi]_F \le [T_\xi \Phi, T_\xi \Phi]_X \le \alpha[\Phi, \Phi]_F, \quad \Phi \in L^0(\Theta; B(H)).$$

*Proof.* (1) ⇒ (2). Suppose that $\xi$ is gramian Schauder basic, so that there is some $\alpha > 0$ such that $\|T_\xi \Phi\|_X \ge \alpha \|\Phi\|_{o,\xi}$ for $\Phi \in \mathfrak{L}^1_*(\xi)$. Let $X_\xi = T_\xi(\mathfrak{L}^1_*(\xi)) \subseteq X$ be the range of $T_\xi$. Then there exists a closed subspace $K_\xi \subseteq K$ such that $X_\xi = K_\xi \otimes H$. Let $\phi \in H$ be arbitrary with $\|\phi\|_H = 1$ and $\{\phi_n\}_{n=1}^\infty \subseteq H$ be an orthonormal basis of $H$ such that $\phi_1 = \phi$. It follows from (2.3) that we can identify $X_\xi$ as

$$\begin{aligned} X_\xi &= \left\{ (f_1, f_2, \ldots) : f_n \in K_\xi, n \ge 1, \sum_{n=1}^\infty \|f_n\|_K^2 < \infty \right\} \\ &= \bigoplus_{n=1}^\infty K_\xi \otimes \phi_n = \ell^2(K_\xi). \end{aligned} \tag{4.5}$$

Since $T_\xi \colon \mathfrak{L}^1_*(\xi) \to X_\xi$ is one-to-one and onto, so is $T_\xi^{-1} \colon X_\xi \to \mathfrak{L}^1_*(\xi)$. By $T_{\xi_\phi} f = \langle T_\xi f, \phi \rangle_H$ for $f \in L^1(\xi_\phi)$ (cf. (4.4)) and considering the closed subspace $K_\xi \otimes \phi_1 \subset X_\xi$, we see that $T_{\xi_{\phi_1}} = T_{\xi_\phi}$ is one-to-one and its range $T_{\xi_\phi}(L^1(\xi_\phi)) = K_\xi \otimes \phi$ is closed. Thus $\xi_\phi$ is Schauder basic. Since $\phi \in H$ is an arbitrary vector of norm 1 we see that $\xi$ is scalarly Schauder basic.

To see that $\xi$ is uniformly scalarly Schauder basic, let $\phi \in H$ be of norm 1 and $\{\phi_n\}_{n=1}^\infty$ be an orthonormal basis of $H$ with $\phi_1 = \phi$. Note that any $\Phi \in \mathfrak{L}^1_*(\xi)$ can be written uniquely in a way that

$$\Phi = T_\xi^{-1}(\langle T_\xi \Phi, \phi_1 \rangle_H, \langle T_\xi \Phi, \phi_2 \rangle_H, \ldots)$$

by (4.5). For $f \in L^1(\xi_\phi)$ let $\widetilde{f} = T_\xi^{-1}(T_{\xi_\phi} f, 0, 0, \ldots) \in \mathfrak{L}^1_*(\xi)$. Then we see that $T_\xi \widetilde{f} = (T_{\xi_\phi} f, 0, 0, \ldots) \in X_\xi$ and

$$\|T_{\xi_\phi} f\|_K = \|T_\xi \widetilde{f}\|_X \ge \alpha \|\widetilde{f}\|_{o,\xi} \ge \alpha \|f\|_{s,\xi_\phi}.$$

This implies that $\xi$ is uniformly scalarly Schauder basic.

(2) ⇒ (3). Assume that $\xi$ is uniformly scalarly Schauder basic, so that there exists a constant $\alpha > 0$ such that $\|T_{\xi_\phi} f\|_K \ge \alpha \|f\|_{s,\xi_\phi}$ for $\phi \in H$ and $f \in L^1(\xi_\phi)$. Since, for each nonzero $\phi \in H$, $\xi_\phi \in$ ca$(\mathfrak{A}, K)$ is Schauder basic it is similar to an orthogonally scattered measure $\eta_\phi \in$ caos$(\mathfrak{A}, K)$ by a similarity operator $S_\phi \in B(K)$ with a bounded inverse such that $\eta_\phi = S_\phi \xi_\phi$. Since $T_{\eta_\phi} = S_\phi T_{\xi_\phi}$, $\|T_{\xi_\phi}\| \le 1$ and $\|T_{\eta_\phi}\| = 1$, we see that

$$1 = \|T_{\eta_\phi}\| = \|S_\phi T_{\xi_\phi}\| \le \|S_\phi\| \|T_{\xi_\phi}\| \le \|S_\phi\|.$$

If we use $S_\phi = T_{\eta_\phi} T_{\xi_\phi}^{-1}$, then we have that $\|S_\phi\| \le \|T_{\xi_\phi}^{-1}\|$, and hence $1 \le \|S_\phi\| \le \|T_{\xi_\phi}^{-1}\|$. Since $\|T_{\xi_\phi}\| \ge \alpha$ for every $\phi \in H$, there exists some $\gamma > 0$ such that $\|T_{\xi_\phi}^{-1}\| \le \gamma$ for $\phi \in H$. Thus (3) holds with this $\gamma$ and $\beta = 1$.

(3) $\Rightarrow$ (4). Assume (3) is true. Then, for each nonzero $\phi \in H$, $\xi_\phi$ is similar to an $\eta_\phi = S_\phi \xi_\phi \in \mathrm{caos}(\mathfrak{A}, K)$ with a similarity operator $S_\phi \in B(K)$, where there are two positive constants $\beta, \gamma > 0$ independent of $\phi$ such that $\beta \le \|S_\phi\| \le \gamma$. Let $\{\phi_n\}_{n=1}^\infty$ be an orthonormal basis of $H$ and define $\eta$ and $S$ respectively by

$$\eta(A) = \sum_{n=1}^\infty \eta_{\phi_n}(A) \otimes \phi_n, \qquad A \in \mathfrak{A},$$
$$Sx = \sum_{n=1}^\infty S_{\phi_n} \langle x, \phi_n \rangle_H \otimes \phi_n, \qquad x \in X.$$

Then we see that for $A \in \mathfrak{A}$,

$$\begin{aligned} \|\eta(A)\|_X^2 &= \left\| \sum_{n=1}^\infty S_{\phi_n} \xi_{\phi_n}(A) \otimes \phi_n \right\|_X^2 = \sum_{n=1}^\infty \|S_{\phi_n} \xi_{\phi_n}(A)\|_K^2 \\ &\le \sum_{n=1}^\infty \|S_{\phi_n}\|^2 \|\xi_{\phi_n}(A)\|_K^2 \le \gamma^2 \sum_{n=1}^\infty \|\xi_{\phi_n}(A)\|_K^2 = \gamma^2 \|\xi(A)\|_X^2, \end{aligned}$$

and hence $\eta$ is well-defined. Moreover, $\eta$ is gramian orthogonally scattered, which follows from Proposition 2.1. Similarly, $S$ is a well-defined bounded operator in $A(X)$ such that $\|S\| \le \gamma$. Also it is seen that $S^{-1}$ is given by

$$S^{-1}x = \sum_{n=1}^\infty S_{\phi_n}^{-1} \langle x, \phi_n \rangle_H \otimes \phi_n, \quad x \in X,$$

which is bounded. Since $\eta = S\xi$, we conclude that $\xi$ is gramian similar to $\eta$.

(4) $\Rightarrow$ (1) is seen from Propositions 4.5 and 4.6.

(4) $\Leftrightarrow$ (5) follows from Proposition 4.7. □

The above theorem is the main theorem of this paper and will be used to characterize gramian uniformly bounded linearly stationary processes in the next section.

## 5. Gramian uniformly bounded linearly stationary processes

Let $(\Omega, \mathfrak{F}, \mu)$ be a probability measure space and

$$L_0^2(\Omega) = L_0^2(\Omega, \mathfrak{F}, \mu) = \left\{ f \in L^2(\Omega) : \int_\Omega f(\omega)\, \mu(d\omega) = 0 \right\}.$$

A second order stochastic process $\{x(t)\}$ on a locally compact abelian group $G$ is a mapping $x(\cdot)\colon G \to L_0^2(\Omega)$. For such processes uniformly bounded linearly stationary ones are defined and studied by Tjøstheim and Thomas [12] and Niemi [9, 10], where their characterizations are given. Abreu and Salehi [2] showed an equivalence condition using Schauder basic measures.

If we consider Hilbert space valued second order stochastic processes, then

$$X = L_0^2(\Omega; H) = \left\{ x : \Omega \to H, \int_\Omega \|x(\omega)\|_H^2 \,\mu(d\omega) < \infty, \int_\Omega x(\omega)\,\mu(d\omega) = 0 \right\}$$

is an appropriate space. Note that $X = L_0^2(\Omega) \otimes H = S(L_0^2(\Omega), H)$, so that $X$ is a normal Hilbert $B(H)$-module with a gramian $[\,\cdot\,,\cdot\,]$ defined by

$$([x, y]\phi, \psi)_H = \int_\Omega (x(\omega), \psi)_H (\phi, y(\omega))_H \,\mu(d\omega)$$

for $x, y \in X$ and $\phi, \psi \in H$. This justifies to study Hilbert space valued second order stochastic processes in a normal Hilbert $B(H)$-module setting, where we assume that $H$ and $L_0^2(\Omega)$ are separable. $X$-valued gramian uniformly bounded linearly stationary processes on $G$ were defined in [5, 6]. We shall characterize this class of processes using gramian Schauder basic measures.

To do so we need some terminologies on $X$-valued processes (cf. [6, Chapter 4]). So let $\widetilde{x} = \{x(t)\}$ be an $X$-valued process on a locally compact abelian group $G$, i.e., $x(\cdot)\colon G \to X$. The **operator covariance function** $\Gamma(s, t)$ of $\{x(t)\}$ is defined by

$$\Gamma(s, t) = [x(s), x(t)], \quad s, t \in G.$$

An $X$-valued process $\{x(t)\}$ is said to be **operator stationary** if it is norm continuous and its operator covariance function $\Gamma$ can be written as

$$\Gamma(s, t) = \widetilde{\Gamma}(st^{-1}), \quad s, t \in G,$$

for some weakly continuous $T(H)$-valued function $\widetilde{\Gamma}$. In this case there exists a regular gramian orthogonally scattered measure $\xi \in \mathrm{cagos}(\mathfrak{B}, X)$, called the **representing measure**, such that

$$x(t) = \int_{\widehat{G}} \langle t, \chi\rangle \,\xi(d\chi), \quad t \in G, \tag{5.1}$$

where $\widehat{G}$ is the dual group of $G$ with the Borel $\sigma$-algebra $\mathfrak{B}$ and $\langle\,\cdot\,,\cdot\,\rangle$ is the duality pair. In this section we consider **regular** measures in $\mathrm{ca}(\mathfrak{B}, X)$, $\mathrm{bca}(\mathfrak{B}, X)$, etc. For instance, we denote by $\mathrm{rcagos}(\mathfrak{B}, X)$ the set of all regular measures in $\mathrm{cagos}(\mathfrak{B}, X)$. $\{x(t)\}$ is said to be **weakly operator harmonizable** if its operator covariance function is expressed as

$$\Gamma(s, t) = \iint_{\widehat{G}^2} \langle s, \chi\rangle \overline{\langle t, \chi'\rangle}\, M(d\chi, d\chi'), \quad s, t \in G$$

for some $T(H)$-valued regular bimeasure $M$ on $\mathfrak{B} \times \mathfrak{B}$ of bounded operator semivariation (cf. [6, Section 3.4]). As is well-known, $\{x(t)\}$ is weakly operator harmonizable if and only if it is represented as (5.1) by some $\xi \in \mathrm{rbca}(\mathfrak{B}, X)$ of bounded operator semivariation, also called the representing measure.

The **modular time domain** $\mathcal{H}(\widetilde{x})$ of $\{x(t)\}$ is the closed submodule of $X$ spanned by $\{x(t) : t \in G\}$, i.e., $\mathcal{H}(\widetilde{x}) = \mathfrak{S}\{x(t) : t \in G\}$. If we let

$$\mathfrak{S}_1(\widetilde{x}) = \left\{ \sum_{j=1}^{n} a_j x(t_j) : a_j \in B(H), t_j \in G, 1 \le j \le n, n \in \mathbb{N} \right\},$$

then $\mathfrak{S}_1(\widetilde{x})$ is dense in $\mathcal{H}(\widetilde{x})$. $\{x(t)\}$ is said to admit a **shift operator group** if, for each $h \in G$, an operator $V_h$ on $\mathfrak{S}_1(\widetilde{x})$ given by

$$V_h \left( \sum_{j=1}^{n} a_j x(t_j) \right) = \sum_{j=1}^{n} a_j x(t_j h) \tag{5.2}$$

is well-defined. In this case, every $V_h$ is densely defined and $\{V_h\}_{h \in G}$ forms a group of module maps on $\mathfrak{S}_1(\widetilde{x})$. If every $V_h$ is bounded, let us denote its extension to $\mathcal{H}(\widetilde{x})$ by the same letter $V_h$, so that $V_h \in A(\mathcal{H}(\widetilde{x}))$ for $h \in G$, where $A(\mathcal{H}(\widetilde{x}))$ is the set of all bounded module maps on $\mathcal{H}(\widetilde{x})$. When $\{V_h\}_{h \in G}$ is uniformly bounded, i.e., there exists a constant $\alpha > 0$ such that $\|V_h\| \le \alpha$ for $h \in G$, then we say that $\{x(t)\}$ is **gramian uniformly bounded linearly stationary** (**gramian UBLS**). Note that this is equivalent to

$$\left[ \sum_{i=1}^{n} a_i x(t_i h), \sum_{j=1}^{n} a_j x(t_j h) \right] \le \alpha^2 \left[ \sum_{i=1}^{n} a_i x(t_i), \sum_{j=1}^{n} a_j x(t_j) \right]$$

for $a_i \in B(H)$, $t_i, h \in G$, $1 \le i \le n$, $n \in \mathbb{N}$.

We use connections between an $X$-valued process and $L_0^2(\Omega)$-valued processes it induces. Thus let $\{x(t)\}$ be an $X$-valued process on $G$. Then, for $\phi \in H$, $\{x_\phi(t)\} = \{\langle x(t), \phi \rangle_H\}$ is an $L_0^2(\Omega)$-valued process. Here, the interpretation of $\langle x(t), \phi \rangle_H$ is given by $\langle x(t), \phi \rangle_H(\omega) = (x(t)(\omega), \phi)_H$ for $\omega \in \Omega$. Stationarity, weak harmonizability and uniformly bounded linear stationarity are introduced in a similar way as for $X$-valued processes. Let $\gamma_\phi$ be the (scalar) covariance function of $\{x_\phi(t)\}$, i.e.,

$$\gamma_\phi(s,t) = (x_\phi(s), x_\phi(t))_2, \quad s,t \in G,$$

where $(\,\cdot\,,\cdot\,)_2$ is the inner product in $L_0^2(\Omega)$. Then, $\{x_\phi(t)\}$ is said to be **stationary** if

$$\gamma_\phi(s,t) = \widetilde{\gamma}_\phi(st^{-1}), \quad s,t \in G$$

for some continuous function $\widetilde{\gamma}_\phi$. In this case, there exists an $L_0^2(\Omega)$-valued regular orthogonally scattered measure $\xi_\phi \in \mathrm{rcaos}(\mathfrak{B}, L_0^2(\Omega))$ such that

$$x_\phi(t) = \int_{\widehat{G}} \langle t, \chi \rangle \, \xi_\phi(d\chi), \quad t \in G. \tag{5.3}$$

$\{x_\phi(t)\}$ is said to be **weakly harmonizable** if its covariance function $\gamma_\phi$ is expressed as

$$\gamma_\phi(s,t) = \iint_{\widehat{G}^2} \langle s, \chi \rangle \overline{\langle t, \chi' \rangle} \, m(d\chi, d\chi'), \quad s, t \in G$$

for some $\mathbb{C}$-valued bounded regular bimeasure $m$ on $\mathfrak{B} \times \mathfrak{B}$ (cf. [6, Section 3.4]). Also it is known that $\{x_\phi(t)\}$ is weakly harmonizable if and only if it is represented as (5.3) with $\xi_\phi \in \mathrm{rca}(\mathfrak{B}, L_0^2(\Omega))$. In these cases, the measure $\xi_\phi$ is called the representing measure. $\{x_\phi(t)\}$ is said to be **uniformly bounded linearly stationary** (**UBLS**) if there exists some $\alpha_\phi > 0$ such that $\|V_{\phi,h}\| \le \alpha_\phi$ for all $h \in G$, where the operator $V_{\phi,h}$ is defined on the time domain $\mathcal{H}(\widetilde{x}_\phi)$ of $\widetilde{x}_\phi = \{x_\phi(t)\}$, the closed subspace of $L_0^2(\Omega)$ generated by $\{x_\phi(t) : t \in G\}$:

$$V_{\phi,h}\langle x(t), \phi \rangle_H = \langle x(th), \phi \rangle_H = \langle V_h x(t), \phi \rangle_H, \quad t \in G, \tag{5.4}$$

$V_h$ being defined by (5.2). In this case, $\{V_{\phi,h}\}_{h \in G}$ is forming a shift operator group.

The next proposition follows from Abreu and Salehi [2] and Niemi [9].

**Proposition 5.1.** *Let $\{p(t)\}$ be a continuous $L_0^2(\Omega)$-valued process on $G$. Then, the following conditions are equivalent for $\{p(t)\}$.*

(1) *$\{p(t)\}$ is UBLS.*

(2) *$\{p(t)\}$ has a **stationary similarity**, i.e., there exist an $L_0^2(\Omega)$-valued stationary process $\{q(t)\}$ on $G$ and a bounded operator $L \in B(L_0^2(\Omega))$ with a bounded inverse such that $q(t) = Lp(t)$ for $t \in G$.*

(3) *$\{p(t)\}$ is weakly harmonizable with the representing measure $\xi \in \mathrm{rca}(\mathfrak{B}, L_0^2(\Omega))$ such that there exist an orthogonally scattered measure $\eta \in \mathrm{rcaos}(\mathfrak{B}, L_0^2(\Omega))$ and a bounded operator $L \in B(L_0^2(\Omega))$ with a bounded inverse such that $\eta = L\xi$.*

(4) *$\{p(t)\}$ is weakly harmonizable with a Schauder basic representing measure $\xi \in \mathrm{rca}(\mathfrak{B}, L_0^2(\Omega))$.*

We need a few more definitions and notations. An $X$-valued process $\{x(t)\}$ is said to be **scalarly uniformly bounded linearly stationary** (**scalarly UBLS**) if, for each $\phi \in H$, $\{\langle x(t), \phi \rangle_H\}$ is UBLS. $\{x(t)\}$ is said to be **uniformly scalarly uniformly bounded linearly stationary** (**uniformly scalarly UBLS**) if there exists some $\alpha > 0$ such that

$\|V_{\phi,h}\| \leq \alpha$ for all $\phi \in H$ and $h \in G$, where $V_{\phi,h}$ is defined by (5.4). Two processes $\{x(t)\}$ and $\{y(t)\}$ are said to be **gramian similar** if there exists a bounded module map $S \in A(X)$ with a bounded inverse such that $x(t) = Sy(t)$ for $t \in G$.

Now gramian UBLS processes are characterized as follows.

**Theorem 5.2.** *For an $X$-valued process $\{x(t)\}$ on $G$, the following conditions are equivalent.*

(1) $\{x(t)\}$ *is gramian UBLS.*

(2) $\{x(t)\}$ *is uniformly scalarly UBLS.*

(3) $\{x(t)\}$ *is gramian similar to an operator stationary process.*

*If in* (3) *the representing measure of the operator stationary process has a pseudo Radon-Nikodým derivative, then* (1), (2) *and* (3) *are equivalent to* (4)*:*

(4) $\{x(t)\}$ *is weakly operator harmonizable with a gramian Schauder basic representing measure.*

*Proof.* (1) ⇔ (3) was proved in [5] and (3) ⇔ (4) follows from Theorem 4.10.

(1) ⇔ (2) can be seen from Proposition 5.1 and the following equalities: for $h, t \in G$,

$$V_h x(t) = \sum_{n=1}^{\infty} V_{\phi_n,h} x_{\phi_n}(t) \otimes \phi_n, \quad \|V_h x(t)\|_X^2 = \sum_{n=1}^{\infty} \|V_{\phi_n,h} x_{\phi_n}(t)\|^2,$$

where $\{\phi_n\}_{n=1}^{\infty}$ is an orthonormal basis of $H$. □

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