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A Menon-type Identity with Multiplicative and Additive Characters

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Abstract. This paper studies Menon-type identities involving both multiplicative characters and additive characters. In the paper, we shall give the explicit formula of the following sum

$$\sum_{\substack{a\in\mathbb{Z}_n^*\\ b_1,\dots,b_k\in\mathbb{Z}_n}} \gcd(a-1,b_1,\dots,b_k,n)\chi(a)\lambda_1(b_1)\cdots\lambda_k(b_k),$$

where for a positive integer n, $\mathbb{Z}_n^*$ is the group of units of the ring $\mathbb{Z}_n=\mathbb{Z}/n\mathbb{Z}$, gcd represents the greatest common divisor, χ is a Dirichlet character modulo n, and for a nonnegative integer k, $\lambda_1,\dots,\lambda_k$ are additive characters of $\mathbb{Z}_n$. Our formula further extends the previous results by Sury [13], Zhao-Cao [17] and Li-Hu-Kim [4].

1. Introduction

In 1965, P. K. Menon [7] proved the following beautiful identity:

$$\sum_{a\in\mathbb{Z}_n^*} \gcd(a-1,n)=\varphi(n)\tau(n), \tag{1.1}$$

where for a positive integer n, $\mathbb{Z}_n^*$ is the group of units of the ring $\mathbb{Z}_n=\mathbb{Z}/n\mathbb{Z}$, gcd represents the greatest common divisor, φ is the Euler's totient function and $\tau(n)$ is the number of positive divisors of n.

The Menon's identity (1.1) is very interesting and appealing. Many mathematicians made contributions on it. It has been proved by B. Sury [13] that

$$\sum_{\substack{a\in\mathbb{Z}_n^*\\ b_1,\dots,b_k\in\mathbb{Z}_n}} \gcd(a-1,b_1,\dots,b_k,n)=\varphi(n)\sigma_k(n), \tag{1.2}$$

where $\sigma_k(n)=\sum_{d|n} d^k$ by using the Cauchy-Frobenius-Burnside lemma. It is also interesting to note that Miguel [8,9] extended identities (1.1) and (1.2) from $\mathbb{Z}$ to any residually finite Dedekind domain.

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Recently, Zhao and Cao [17] derived the following elegant Menon-type identity with a Dirichlet character

$$\sum_{a\in\mathbb{Z}_n^*} \gcd(a-1,n)\chi(a) = \varphi(n)\tau\left(\frac{n}{d}\right), \tag{1.3}$$

where χ is a Dirichlet character modulo n and d is the conductor of χ.

From the point of view of Fourier analysis on finite Abelian groups, Zhao and Cao's results in fact give the explicit expression of Fourier transformation of the function $f(a) = \gcd(a-1,n)$ on the Abelian group $(\mathbb{Z}/n\mathbb{Z})^*$. Therefore, the identity (1.3) is not only graceful but also gives more information.

In [4], Li, Hu and Kim further extended identities (1.2) and (1.3). They obtained the following identity with Dirichlet character χ:

$$\sum_{\substack{a\in\mathbb{Z}_n^*\\ b_1,\ldots,b_k\in\mathbb{Z}_n}} \gcd(a-1,b_1,\ldots,b_k,n)\chi(a) = \varphi(n)\sigma_k\left(\frac{n}{d}\right), \tag{1.4}$$

where d is the conductor of χ and k is a nonnegative integer.

For other related works on Menon's identity, see [1–3, 5, 6, 10–12, 14–16] and references therein.

Denote

$$S_{\chi,\underline{\lambda}}(n,k) = \sum_{\substack{a\in\mathbb{Z}_n^*\\ b_1,\ldots,b_k\in\mathbb{Z}_n}} \gcd(a-1,b_1,\ldots,b_k,n)\chi(a)\lambda_1(b_1)\cdots\lambda_k(b_k), \tag{1.5}$$

where $\lambda_1,\ldots,\lambda_k$ are additive characters of $\mathbb{Z}_n$ and $\underline{\lambda}$ represents the vector $(\lambda_1,\ldots,\lambda_k)$. For $1\le i\le k$, each λ_i can be uniquely written as

$$\lambda_i(b) = \exp(2\pi\sqrt{-1}w_i b/n), \quad 0\le w_i\le n-1,\ w_i\in\mathbb{Z} \tag{1.6}$$

where $b\in\mathbb{Z}_n$ and $\sqrt{-1}$ is the square root of -1 whose imaginary part is positive. Denote the order of λ_i by d_i, that is,

$$d_i = \frac{n}{\gcd(w_i,n)}. \tag{1.7}$$

Theorem 1.1. *Let n be a positive integer and χ be a Dirichlet character modulo n whose conductor is d. Assume k is a nonnegative integer. Let $\lambda_1,\ldots,\lambda_k$ be additive characters of $\mathbb{Z}_n$, explicitly given in* (1.6)*. Let $d_1,\ldots,d_k$ as in* (1.7) *be the orders of $\lambda_1,\ldots,\lambda_k$, respectively. Then, we have the following identity:*

$$\sum_{\substack{a\in\mathbb{Z}_n^*\\ b_1,\ldots,b_k\in\mathbb{Z}_n}} \gcd(a-1,b_1,\ldots,b_k,n)\chi(a)\lambda_1(b_1)\cdots\lambda_k(b_k) = \varphi(n)\sigma_k\left(\frac{n}{\operatorname{lcm}(d,d_1,\ldots,d_k)}\right)$$

where lcm *represents the least common multiple. Equivalently, it can also be written as*

$$\sum_{\substack{a\in\mathbb{Z}_n^*\\ b_1,\ldots,b_k\in\mathbb{Z}_n}} \gcd(a-1,b_1,\ldots,b_k,n)\chi(a)\lambda_1(b_1)\cdots\lambda_k(b_k)=\varphi(n)\sigma_k\left(\gcd\left(\frac{n}{d},w_1,\ldots,w_k\right)\right).$$

From the point of view of Fourier analysis, Theorem 1.1 gives the explicit expression of Fourier coefficients of the function $f(a,b_1,\ldots,b_k)=\gcd(a-1,b_1,\ldots,b_k,n)$ on the Abelian group $(\mathbb{Z}/n\mathbb{Z})^*\times(\mathbb{Z}/n\mathbb{Z})^k$.

Remark 1.2. If additive characters $\lambda_1,\ldots,\lambda_k$ are trivial, then Theorem 1.1 reduces to identity (1.4). If both additive characters $\lambda_1,\ldots,\lambda_k$ and multiplicative character χ are trivial, Theorem 1.1 reduces to Sury's identity (1.2). If $k=0$, Theorem 1.1 reduces to Zhao and Cao's identity (1.3).

The rest of paper is organized as follows. In Section 2, we prove Theorem 1.1 in the special case of n being a prime power. The general case is treated in Section 3 by combining prime power cases with the Chinese remainder theorem.

2. Prime power case

In this section, we assume $n=p^m$, where p is a prime number and m is a positive integer. Let χ be a Dirichlet character modulo n with conductor d. Since $d\mid n$, we denote $d=p^t$, where $0\le t\le m$. Let $\lambda_1,\ldots,\lambda_k$ be additive characters of $\mathbb{Z}_n$ with orders $d_1,\ldots,d_k$, respectively. For $1\le i\le k$, since $d_i\mid n$, we denote $d_i=p^{v_i}$, where $0\le v_i\le m$.

Since $n=p^m$ is a prime power, the whole subgroups of $\mathbb{Z}_n$ form a chain:

$$0=p^m\mathbb{Z}_n\subset p^{m-1}\mathbb{Z}_n\subset\cdots\subset p\mathbb{Z}_n\subset\mathbb{Z}_n.$$

Clearly, for $0\le s\le m$, $\#(p^s\mathbb{Z}_n)=p^{m-s}$, where # denote the cardinality of sets.

In the following, we adopt the similar method as in [4] to calculate $S_{\chi,\underline{\lambda}}(p^m,k)$. From (1.5), we obtain

$$\begin{aligned}
&\sum_{\substack{a\in\mathbb{Z}_n^*\\ b_1,\ldots,b_k\in\mathbb{Z}_n}} \gcd(a-1,b_1,\ldots,b_k,n)\chi(a)\lambda_1(b_1)\cdots\lambda_k(b_k)\\
&=\sum_{s=0}^{m}\sum_{\substack{\gcd(b_1,\ldots,b_k,n)=p^s\\ b_1,\ldots,b_k\in\mathbb{Z}_n}}\sum_{a\in\mathbb{Z}_n^*}\gcd(a-1,p^s)\chi(a)\lambda_1(b_1)\cdots\lambda_k(b_k)\\
&=\sum_{s=0}^{m}\left(\sum_{a\in\mathbb{Z}_n^*}\gcd(a-1,p^s)\chi(a)\right)\left(\sum_{\substack{\gcd(b_1,\ldots,b_k,n)=p^s\\ b_1,\ldots,b_k\in\mathbb{Z}_n}}\lambda_1(b_1)\cdots\lambda_k(b_k)\right).
\end{aligned}\tag{2.1}$$

Therefore, we need to compute

$$\sum_{a\in\mathbb{Z}_n^*}\gcd(a-1,p^s)\chi(a)\quad\text{and}\quad\sum_{\substack{\gcd(b_1,\ldots,b_k,n)=p^s\\ b_1,\ldots,b_k\in\mathbb{Z}_n}}\lambda_1(b_1)\cdots\lambda_k(b_k)$$

explicitly. The first summation is already treated in [4]. We quote it here as Lemma 2.1. The second summation is computed in Lemma 2.3.

Lemma 2.1. [4, Lemma 2.2] *Let $n=p^m$ and χ be a Dirichlet character modulo n with conductor p^t, where $0\le t\le m$. Let s be an integer such that $0\le s\le m$. Then we obtain*

$$\sum_{a\in\mathbb{Z}_n^*}\gcd(a-1,p^s)\chi(a)=\begin{cases}(s-t+1)(p^m-p^{m-1}) & \text{if } s\ge t,\\ 0 & \text{otherwise.}\end{cases}$$

Note that in Lemma 2.1, if $s=m$, this is just Lemma 3.1 of [17].

The following lemma is important to prove Lemma 2.3. It is a standard fact on characters of finite Abelian groups. For convenience of readers, we give a concrete proof here.

Lemma 2.2. *Let $n=p^m$ and λ be an additive character of $\mathbb{Z}_n$ with order p^v. Then, for $0\le s\le m$, we have*

$$\sum_{b\in p^s\mathbb{Z}_n}\lambda(b)=p^{m-s}[s\ge v],$$

where $[s\ge v]$ is the Iverson bracket, that is,

$$[s\ge v]=\begin{cases}1 & \text{if } s\ge v,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. For $b\in p^s\mathbb{Z}_n$, write $b=p^sb'$ with some $b'\in\mathbb{Z}_n$. Then $\lambda(b)=\lambda^{p^s}(b')$. If $s\ge v$, then λ^{p^s} is a trivial character. In this case, $\lambda(b)=1$ for every $b\in p^s\mathbb{Z}_n$. Therefore, we obtain

$$\sum_{b\in p^s\mathbb{Z}_n}\lambda(b)=\#(p^s\mathbb{Z}_n)=p^{m-s}.$$

Otherwise, λ^{p^s} is nontrivial on $\mathbb{Z}_n$. Hence, there exists some $b_0=p^sb_0'\in p^s\mathbb{Z}_n$ such that $\lambda(b_0)=\lambda^{p^s}(b_0')\ne 1$. We have

$$\sum_{b\in p^s\mathbb{Z}_n}\lambda(b)=\sum_{b\in p^s\mathbb{Z}_n}\lambda(b+b_0)=\lambda(b_0)\sum_{b\in p^s\mathbb{Z}_n}\lambda(b).$$

As a result, we obtain

$$\sum_{b\in p^s\mathbb{Z}_n}\lambda(b)=0.$$

□

Lemma 2.3. *Let $n = p^m$ be a prime power and $0 \leq s \leq m$ be an integer. Assume $k \geq 0$ is an integer. Let $\lambda_1, \ldots, \lambda_k$ be additive characters of $\mathbb{Z}_n$ with orders $p^{v_1}, \ldots, p^{v_k}$, respectively. Denote $v = \max\{v_1, \ldots, v_k\}$. Then for $0 \leq s \leq m-1$,*

$$\sum_{\substack{b_1,\ldots,b_k \in \mathbb{Z}_n \\ \gcd(b_1,\ldots,b_k,p^m)=p^s}} \lambda_1(b_1)\cdots\lambda_k(b_k) = p^{(m-s)k}[s \geq v] - p^{(m-s-1)k}[s+1 \geq v],$$

where $[s \geq v]$ is the Iverson bracket. Otherwise, for $s = m$, it is equal to 1.

Proof. The case $k = 0$ is obvious. Thus, we assume $k \geq 1$. Clearly,

$$\sum_{b_1,\ldots,b_k \in p^s\mathbb{Z}_n} \lambda_1(b_1)\cdots\lambda_k(b_k) = \prod_{i=1}^{k} \sum_{b_i \in p^s\mathbb{Z}_n} \lambda_i(b_i). \tag{2.2}$$

Substituting Lemma 2.2 into (2.2), we get that

$$\sum_{b_1,\ldots,b_k \in p^s\mathbb{Z}_n} \lambda_1(b_1)\cdots\lambda_k(b_k) = \prod_{i=1}^{k} p^{m-s}[s \geq v_i] = p^{(m-s)k}[s \geq v]. \tag{2.3}$$

Note that $p^s \mid \gcd(b_1, \ldots, b_k, p^m)$ if and only if $b_1, \ldots, b_k \in p^s\mathbb{Z}_n$ holds. Therefore, for $0 \leq s \leq m-1$,

$$\gcd(b_1,\ldots,b_k,p^m) = p^s \iff (b_1,\ldots,b_k) \in (p^s\mathbb{Z}_n)^k - (p^{s+1}\mathbb{Z}_n)^k.$$

Hence, we obtain

$$\begin{aligned} &\sum_{\substack{b_1,\ldots,b_k \in \mathbb{Z}_n \\ \gcd(b_1,\ldots,b_k,p^m)=p^s}} \lambda_1(b_1)\cdots\lambda_k(b_k) \\ &= \sum_{b_1,\ldots,b_k \in p^s\mathbb{Z}_n} \lambda_1(b_1)\cdots\lambda_k(b_k) - \sum_{b_1,\ldots,b_k \in p^{s+1}\mathbb{Z}_n} \lambda_1(b_1)\cdots\lambda_k(b_k). \end{aligned} \tag{2.4}$$

It then follows from (2.3) and (2.4) that

$$\sum_{\substack{b_1,\ldots,b_k \in \mathbb{Z}_n \\ \gcd(b_1,\ldots,b_k,p^m)=p^s}} \lambda_1(b_1)\cdots\lambda_k(b_k) = p^{(m-s)k}[s \geq v] - p^{(m-s-1)k}[s+1 \geq v].$$

Thus, the case $0 \leq s \leq m-1$ is done.

Clearly, for $s = m$, one can readily check that

$$\gcd(b_1,\ldots,b_k,p^m) = p^s \iff (b_1,\ldots,b_k) \in (p^s\mathbb{Z}_n)^k.$$

In this case, there is only one summation term $\lambda_1(p^m)\cdots\lambda_k(p^m)$, which is equal to 1. This concludes the proof. □

Finally, we prove the following result, which is a special case of Theorem 1.1.

Theorem 2.4. *Let $n = p^m$ be a prime power and χ be a Dirichlet character whose conductor is $d = p^t$. Assume k is a nonnegative integer. Let λ_i be an additive character of $\mathbb{Z}_n$ with order $d_i = p^{v_i}$ such that $0 \le v_i \le m$, where $1 \le i \le k$. Then, the following identity holds*

$$\sum_{\substack{a\in\mathbb{Z}_n^*\\ b_1,\ldots,b_k\in\mathbb{Z}_n}} \gcd(a-1,b_1,\ldots,b_k,n)\chi(a)\lambda_1(b_1)\cdots\lambda_k(b_k) = \varphi(n)\sigma_k\left(\frac{n}{\operatorname{lcm}(d,d_1,\ldots,d_k)}\right),$$

where $\operatorname{lcm}(d,d_1,\ldots,d_k)$ *is the least common multiple of* $d,d_1,\ldots,d_k$.

Proof. By equation (2.1), $S_{\chi,\underline{\lambda}}(p^m,k)$ equals to

$$\sum_{s=0}^{m}\left(\sum_{a\in\mathbb{Z}_n^*}\gcd(a-1,p^s)\chi(a)\right)\left(\sum_{\substack{b_1,\ldots,b_k\in\mathbb{Z}_n\\ \gcd(b_1,\ldots,b_k,p^m)=p^s}}\lambda_1(b_1)\cdots\lambda_k(b_k)\right). \tag{2.5}$$

Substituting Lemma 2.1 into (2.5), we get

$$S_{\chi,\underline{\lambda}}(p^m,k) = \sum_{s=t}^{m}(s-t+1)(p^m-p^{m-1})\left(\sum_{\substack{b_1,\ldots,b_k\in\mathbb{Z}_n\\ \gcd(b_1,\ldots,b_k,p^m)=p^s}}\lambda_1(b_1)\cdots\lambda_k(b_k)\right). \tag{2.6}$$

Denote $v = \max\{v_1,\ldots,v_k\}$. Then substituting Lemma 2.3 into (2.6), we have that $S_{\chi,\underline{\lambda}}(p^m,k)$ equals to

$$\begin{aligned}
&\varphi(p^m)\left(\sum_{s=t}^{m-1}(s-t+1)\left(p^{(m-s)k}[s\ge v]-p^{(m-s-1)k}[s+1\ge v]\right)+(m-t+1)\right)\\
&=\varphi(p^m)\left(\sum_{s=t}^{m}(s-t+1)p^{(m-s)k}[s\ge v]-\sum_{s=t}^{m-1}(s-t+1)p^{(m-s-1)k}[s+1\ge v]\right)\\
&=\varphi(p^m)\left(\sum_{s=t}^{m}(s-t+1)p^{(m-s)k}[s\ge v]-\sum_{s=t+1}^{m}(s-t)p^{(m-s)k}[s\ge v]\right).
\end{aligned}$$

The last equality is obtained by substituting $s+1$ with s in the posterior summation. It is easy to see that

$$\begin{aligned}
S_{\chi,\underline{\lambda}}(p^m,k) &= \varphi(p^m)\sum_{s=t}^{m}p^{(m-s)k}[s\ge v]\\
&=\varphi(p^m)\sum_{s=\max\{t,v\}}^{m}p^{(m-s)k}\\
&=\varphi(p^m)\sum_{s=0}^{m-\max\{t,v\}}p^{sk}.
\end{aligned}$$

Further, the last equality is obtained by substituting $m-s$ with s. Therefore,

$$S_{\chi,\underline{\lambda}}(p^m,k)=\varphi(p^m)\sigma_k\left(\frac{p^m}{p^{\max\{t,v\}}}\right)=\varphi(p^m)\sigma_k\left(\frac{p^m}{\operatorname{lcm}(p^t,p^{v_1},\dots,p^{v_k})}\right),$$

which concludes the proof. □

3. The general case

In this section, we will prove the main theorem. First, we show that $S_{\chi,\underline{\lambda}}(n,k)$ is multiplicative with respect to n by the Chinese remainder theorem. Then, using multiplicative property, we prove Theorem 1.1 by combining prime power cases, which are already treated in Section 2.

Let $n=n_1n_2$ be the product of positive integers n_1 and n_2 such that $\gcd(n_1,n_2)=1$. By the Chinese remainder theorem, we have the ring isomorphism: $\mathbb{Z}_n\simeq\mathbb{Z}_{n_1}\oplus\mathbb{Z}_{n_2}$, which induces the multiplicative group isomorphism: $\mathbb{Z}_n^*\simeq\mathbb{Z}_{n_1}^*\times\mathbb{Z}_{n_2}^*$. Therefore, each Dirichlet character modulo n can be uniquely written as $\chi=\chi'\cdot\chi''$, where χ, χ' and χ'' are Dirichlet characters modulo n, n_1 and n_2, respectively. Similarly, any additive character λ of $\mathbb{Z}_n$ can be uniquely written as $\lambda=\lambda'\cdot\lambda''$, where λ' and λ'' are additive characters of $\mathbb{Z}_{n_1}$ and $\mathbb{Z}_{n_2}$, respectively. Explicitly, we obtain that

$$\chi(c \bmod n)=\chi'(c \bmod n_1)\cdot\chi''(c \bmod n_2) \tag{3.1}$$

and

$$\lambda(c \bmod n)=\lambda'(c \bmod n_1)\cdot\lambda''(c \bmod n_2) \tag{3.2}$$

for any integer c such that $\gcd(c,n)=1$. For $1\le i\le k$, we denote $\lambda_i=\lambda_i'\cdot\lambda_i''$ with the same meaning as above.

To simplify notations, for $a\in\mathbb{Z}_n$, we let $a'\in\mathbb{Z}_{n_1}$ and $a''\in\mathbb{Z}_{n_2}$ denote the image of a in $\mathbb{Z}_{n_1}$ and $\mathbb{Z}_{n_2}$, respectively, i.e.,

$$a'\equiv a \mod n_1 \quad\text{and}\quad a''\equiv a \mod n_2.$$

Let d, d' and d'' be the conductors of χ, χ' and χ'', respectively. It is well known that $d=d'd''$. For $1\le i\le k$, let d_i, d_i' and d_i'' be the orders of λ_i, λ_i' and λ_i'', respectively. Since d_i' and d_i'' are coprime to each other, we have $d_i=d_i'\cdot d_i''$, where $1\le i\le k$. Denote the vectors $(\lambda_1',\dots,\lambda_k')$ and $(\lambda_1'',\dots,\lambda_k'')$ by $\underline{\lambda}'$ and $\underline{\lambda}''$, respectively.

The following lemma shows that $S_{\chi,\underline{\lambda}}(n,k)$ is multiplicative with respect to n.

Lemma 3.1. *With the above notations we have*

$$S_{\chi,\underline{\lambda}}(n,k)=S_{\chi',\underline{\lambda}'}(n_1,k)\cdot S_{\chi'',\underline{\lambda}''}(n_2,k).$$

Proof. From (1.5), (3.1) and (3.2), we have

$$\begin{aligned}
&\sum_{\substack{a\in\mathbb{Z}_n^*\\ b_1,\dots,b_k\in\mathbb{Z}_n}} \gcd(a-1,b_1,\dots,b_k,n)\chi(a)\lambda_1(b_1)\cdots\lambda_k(b_k)\\
&= \sum_{\substack{a\in\mathbb{Z}_n^*\\ b_1,\dots,b_k\in\mathbb{Z}_n}} \gcd(a-1,b_1,\dots,b_k,n_1)\gcd(a-1,b_1,\dots,b_k,n_2)\\
&\qquad\qquad \times \chi'(a)\chi''(a)\lambda_1'(b_1)\lambda_1''(b_1)\cdots\lambda_k'(b_k)\lambda_k''(b_k)\\
&= \sum_{\substack{a'\in\mathbb{Z}_{n_1}^*\\ b_1',\dots,b_k'\in\mathbb{Z}_{n_1}}} \gcd(a'-1,b_1',\dots,b_k',n_1)\chi'(a')\lambda_1'(b_1')\cdots\lambda_k'(b_k')\\
&\quad \times \sum_{\substack{a''\in\mathbb{Z}_{n_2}^*\\ b_1'',\dots,b_k''\in\mathbb{Z}_{n_2}}} \gcd(a''-1,b_1'',\dots,b_k'',n_2)\chi''(a'')\lambda_1''(b_1'')\cdots\lambda_k''(b_k'').
\end{aligned}$$

The last equality is obtained by the Chinese remainder theorem. Indeed, as $(a,b_1,\dots,b_k)$ runs over $\mathbb{Z}_n^*\times(\mathbb{Z}_n)^k$, $(a',b_1',\dots,b_k',a'',b_1'',\dots,b_k'')$ runs over $\mathbb{Z}_{n_1}^*\times(\mathbb{Z}_{n_1})^k\times\mathbb{Z}_{n_2}^*\times(\mathbb{Z}_{n_2})^k$, too. Therefore, we have

$$S_{\chi,\underline{\lambda}}(n,k) = S_{\chi',\underline{\lambda}'}(n_1,k)\cdot S_{\chi'',\underline{\lambda}''}(n_2,k). \qquad \square$$

Remark 3.2. The proof of Lemma 3.1 is similar to that of Lemma 3.1 in [4]. Also see the proof of Theorem 1.1 and Theorem 1.2 in [17].

Proof of Theorem 1.1. We prove the first identity by induction on $\omega(n)$, where $\omega(n)$ is the number of distinct prime factors of n.

If $\omega(n)=1$, i.e., n is a prime power, this is proved in Theorem 2.4. Assume it is true for $\omega(n)=u-1$, where $u\geq 2$ is an integer. Now we consider the case $\omega(n)=u$.

Let p^m be a prime power, exactly dividing n. Denote $n_1=p^m$ and $n_2=n/p^m$. Then $\gcd(n_1,n_2)=1$.

Factor $\chi=\chi'\cdot\chi''$, where χ' and χ'' are Dirichlet characters modulo n_1 and n_2 with conductors d' and d'', respectively. Similarly, for $1\leq i\leq k$, decompose $\lambda_i=\lambda_i'\cdot\lambda_i''$ where λ_i' and λ_i'' are additive characters of $\mathbb{Z}_{n_1}$ and $\mathbb{Z}_{n_2}$ with orders d_i' and d_i''.

We note that

$$d=d'd'',\quad \gcd(d',d'')=1 \quad\text{and}\quad d_i=d_i'd_i'',\quad \gcd(d_i',d_i'')=1, \tag{3.3}$$

where $1\leq i\leq k$. Denote the vectors $(\lambda_1',\dots,\lambda_k')$ and $(\lambda_1'',\dots,\lambda_k'')$ by $\underline{\lambda}'$ and $\underline{\lambda}''$, respectively. By Theorem 2.4 and the assumption, we have

$$S_{\chi',\underline{\lambda}'}=\varphi(n_1)\left(\frac{n_1}{\operatorname{lcm}(d',d_1',\dots,d_k')}\right) \quad\text{and}\quad S_{\chi'',\underline{\lambda}''}=\varphi(n_2)\left(\frac{n_2}{\operatorname{lcm}(d'',d_1'',\dots,d_k'')}\right). \tag{3.4}$$

Combining Lemma 3.1 and (3.4), we get

$$\begin{aligned} S_{\chi,\underline{\lambda}}(n,k) &= S_{\chi',\underline{\lambda}'}(n_1,k)S_{\chi'',\underline{\lambda}''}(n_2,k) \\ &= \varphi(n_1)\varphi(n_2)\sigma_k\left(\frac{n_1}{\operatorname{lcm}(d',d'_1,\ldots,d'_k)}\right)\sigma_k\left(\frac{n_2}{\operatorname{lcm}(d'',d''_1,\ldots,d''_k)}\right). \end{aligned}$$

Since arithmetic functions φ, σ_k and lcm are multiplicative, by (3.3), we get the desired result

$$S_{\chi,\underline{\lambda}}(n,k) = \varphi(n)\sigma_k\left(\frac{n}{\operatorname{lcm}(d,d_1,\ldots,d_k)}\right).$$

The second identity can be justified as follows:

$$\begin{aligned} \frac{n}{\operatorname{lcm}(d,d_1,\ldots,d_k)} &= \frac{n}{\operatorname{lcm}(n/(n/d), n/\gcd(w_1,n),\ldots,n/\gcd(w_k,n))} \\ &= \frac{n}{n/\gcd(n/d,\gcd(w_1,n),\ldots,\gcd(w_k,n))} \\ &= \gcd(n/d,w_1,\ldots,w_k). \end{aligned}$$

This completes the proof of Theorem 1.1. □

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