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The Number of Cusps of Complete Riemannian Manifolds with Finite Volume

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Abstract. In this paper, we count the number of cusps of complete Riemannian manifolds M with finite volume. When M is a complete smooth metric measure spaces, we show that the number of cusps in bounded by the volume V of M if some geometric conditions hold true. Moreover, we use the nonlinear theory of the p-Laplacian to give a upper bound of the number of cusps on complete Riemannian manifolds. The main ingredients in our proof are a decay estimate of volume of cusps and volume comparison theorems.

1. Introduction

Let E be an end of a Riemannian manifold M^n and $\lambda_1(M)$ be the first Dirichlet eigenvalue of the Laplacian on M. It is well-known that information of $\lambda_1(M)$ tells us some geometric properties of the manifold. For example, if $\lambda_1(M) > 0$ then M must have infinite volume, or if $\lambda_1(E) > 0$ then either E has finite volume, namely E is a cusp; or E is non-parabolic end with volume of exponent growth. In [2], Cheng considered complete manifolds (M^n, g) of dimension n with $\mathrm{Ric}_M \geq -(n-1)$ and gave an upper bound of $\lambda_1(M)$:

$$\lambda_1(M) \leq \frac{(n-1)^2}{4}.$$

Later, Li and Wang showed in [5] that if $\mathrm{Ric}_M \geq -(n-1)$ and $\lambda_1(M)$ is maximal then either M has only one end or; M is a topological cylinder with certain warped metric product. Since $\lambda_1(M)$ is maximal, M must have infinite volume. Hence, Li-Wang's result says that we can count the number of ends of complete Riemannian manifold M with $\mathrm{Ric}_M \geq -(n-1)$ provided $\lambda_1(M)$ obtains its maximal value. In this case, M has at most two ends.

Interestingly, when M is a complete Riemannian manifold of finite volume, Li and Wang proved in [8] that one can count ends (cusps) via the bottom of Neumann spectrum

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$\mu_1(M)$ defined by

$$\mu_1(M) = \inf_{\phi \in H^1(M), \int_M \phi = 0} \frac{\int_M |\nabla\phi|^2}{\int \phi^2}.$$

Note that $\mu_1(M)$ plays the role of a generalized first non-zero Neumann eigenvalue, although $\mu_1(M)$ might not necessarily be an eigenvalue. However, if M is compact then $\mu_1(M)$ is itself a positive eigenvalue called the first (Neumann) eigenvalue of the Laplacian (see [9, 14]). As in [8, 14], variational principle implies

$$\mu_1(M) \le \max\{\lambda_1(\Omega_1), \lambda_1(\Omega_2)\}$$

for any two disjoint domains Ω_1 and Ω_2 of M, where $\lambda_1(\Omega_1)$ and $\lambda_1(\Omega_2)$ are their first Dirichlet eigenvalues respectively. Li and Wang counted cusps on complete manifolds with finite volume as follows.

Theorem 1.1. [8] *Let M^n be a complete Riemannian manifold with Ricci curvature bounded from below by $\mathrm{Ric}_M \ge -(n-1)$. Assume that M has finite volume given by V, and*

$$\mu_1(M) \ge \frac{(n-1)^2}{4}.$$

Let us denote $N(M)$ to be the number of ends (cusps) of M. Then there exists a constant $C(n) > 0$ depending only on n, such that,

$$N(M) \le C(n)\left(\frac{V}{V_o(1)}\right)^2 \ln\frac{V}{V_o(1)}$$

where $V_o(1)$ denotes the volume of the unit ball centered at any point $o \in M$.

Due to Cheng's upper bound of $\lambda_1(M)$ and variational characteristic of $\mu_1(M)$, one can see that in Theorem 1.1, $\mu_1(M)$ is maximal. This means that instead of $\lambda_1(M)$, we can use $\mu_1(M)$ to count the number of cusps of complete Riemannian manifolds with finite volume. We would like to mention that to count cusps of complete manifolds, Li and Wang used a decay volume estimate and a volume comparison theorem.

Motivated by the beauty of Theorem 1.1, in this paper, we want to estimate the number of cusps on smooth metric measure spaces $(M^n, g, e^{-f}dv)$ with finite f-volume V_f. Recall that a smooth metric measure space $(M^n, g, e^{-f}dv)$ is a complete Riemannian manifold (M^n, g) of dimension n with $f \in \mathcal{C}^\infty(M)$ is a smooth weighted function and $e^{-f}dv$ is the weighted volume. Here dv is the volume form with respect to the metric g. On $(M, g, e^{-f}dv)$, we consider the weighted Laplacian

$$\Delta_f \cdot = \Delta \cdot - \langle \nabla f, \nabla \cdot \rangle$$

which is a self-adjoint operator. Associated to the weighted Laplacian, we define the Bakry-Émery curvature by

$$\mathrm{Ric}_f := \mathrm{Ric}_M + \mathrm{Hess}\, f$$

where Ric_M is the Ricci curvature of M and $\mathrm{Hess}\, f$ is the Hessian of f. Following the same strategy as in [8], we first give a decay estimate for the weighted volume and use a volume comparison theorem in [11] to prove the next theorem.

Theorem 1.2. *Let $(M^n, g, e^f dv)$ be a complete smooth metric measure space with Ricci curvature bounded from below by*

$$\mathrm{Ric}_f \geq -(n-1).$$

Assume for some nonnegative constants α such that

$$|\nabla f|(x) \leq \alpha$$

for $x \in M$ and that M has finite volume given by V_f, moreover

$$\mu_1(M) \geq \frac{(n-1+\alpha)^2}{4}.$$

Let us denote $N(M)$ to be the number of ends (cusps) of M. Then there exists a constant $C(n) > 0$ depending only on n, such that,

$$N(M) \leq C(n) \left(\frac{V_f}{V_f(B(o,1))} \right)^2 \ln \left(\frac{V_f}{V_f(B(o,1))} \right)$$

where $V_f(B(o,1))$ denotes the f-volume of the unit ball centered at any fixed point $o \in M$.

On the other hand, our second aim in this paper is to count the number of cusps on complete Riemannian manifold (M^n, g) with finite volume via the nonlinear theory of p-Laplacian for $1 \leq p < \infty$. On a complete Riemannian manifold, for any $u \in W^{1,p}_{\mathrm{loc}}$, the p-Laplacian denoted by Δ_p acting on u as follows:

$$\Delta_p u = \mathrm{div}(|\nabla u|^{p-2} \nabla u).$$

If λ satisfies

$$\Delta_p u = -\lambda |u|^{p-2} u$$

then λ is said to be an eigenvalue of p-Laplacian and u is called an eigenfunction with respect to λ. As in [9, 14], we define

$$\mu_{1,p}(M) = \inf \left\{ \frac{\int_M |\nabla \phi|^p}{\int \phi^p} ; \phi \in W^{1,p}(M), \phi \neq 0 \text{ and } \int_M |\phi|^{p-1} \phi = 0 \right\}.$$

For any couple of nonempty disjoint open subsets Ω_1, Ω_2 of M, Veron proved, in his paper [14] (see also [9]), the following result

$$\mu_{1,p}(M) \leq \max\{\lambda_{1,p}(\Omega_1), \lambda_{1,p}(\Omega_2)\}$$

where $\lambda_{1,p}(\Omega_1)$ and $\lambda_{1,p}(\Omega_2)$ are their first Dirichlet p-eigenvalues, respectively. As in [12,13] we know that if M satisfies $\mathrm{Ric}_M \geq -(n-1)$ then $\lambda_{1,p}$ is bounded from upper by

$$\lambda_{1,p}(M) \leq \left(\frac{n-1}{p}\right)^p.$$

For further discussion on p-Laplacian and its eigenvalues, we refer the reader to [9,12–14] and the references therein. Now, we can count the number of cusps as follows.

Theorem 1.3. *Let (M^n, g) be a complete with finite volume given by V. Assume that the Ricci curvature is bounded from below by*

$$\mathrm{Ric}_M \geq -(n-1)$$

and

$$\mu_{1,p}(M) \geq \frac{(n-1)^p}{p^p}.$$

Let us denote $N(M)$ to be the number of ends (cusps) of M. Then there exists a constant $C(n,p) > 0$ depending only on n and p such that,

$$N(M) \leq C(n,p)\left(\frac{V}{V_o(1)}\right)^2 \ln^{\alpha(p)}\left(\frac{V}{V_o(1)}\right)$$

where $\alpha(p) = \max\{1, p/2\}$ and $V_o(1)$ denotes the volume of the unit ball centered at a fixed point $o \in M$.

As noticed in the previous part, two main ingredients in our proof are a decay estimate for volume and a relative comparison volume theorem. Moreover, it is also worth to note that $\mu_{1,p}(M)$ in Theorem 1.3 is maximal. When $\lambda_{1,p}(M)$ is maximal, it is proved in [12,13] that M has at most two ends. In fact, in [12,13], the authors pointed out that either M must have only one end or; M is a topological cylinder provided that $\mathrm{Ric}_M \geq -(n-1)$ and $\lambda_{1,p}$ is maximal.

The paper is organized as follows. In Section 2, we introduce an estimate of volume decay rate on smooth metric measure spaces which can be considered as a generalization of volume decay rate in [7,8]. In Section 3, we will count the number of cusps on complete smooth metric measure spaces with finite volume. Finally, we use the nonlinear theory of p-Laplacian in Section 4 to estimate the number of cusps on Riemannian manifolds. Several complete manifolds of finite volume are investigated in this section.

2. Smooth metric measure spaces with weighted Poincaré inequality

Let $(M^n, g, e^{-f}dv)$ be a smooth metric measure space with a weighted positive function $\rho \in \mathcal{C}(M)$. We define the ρ-metric by

$$ds_\rho^2 = \rho\, ds^2.$$

Thanks to this metric, we can define the ρ-distance function to be

$$r_\rho(x, y) = \inf_\gamma \ell_\rho(\gamma),$$

where the infimum is taken over all smooth curves γ joining x and y, and ℓ is the length of γ with respect to ds_ρ^2. For a fixed point $o \in M$, we denote $r_\rho(x) = r_\rho(o, x)$ to be the ρ-distance to o. As in [7], we know that $|\nabla r_\rho|^2(x) = \rho(x)$.

Throughout this article, we denote

$$B_\rho(o, R) = \{x \in M \mid r_\rho(o, x) < R\}$$

to be the geodesic ball centered at $o \in M$ with radius R. We also denote the geodesic ball

$$B(o, R) = \{x \in M \mid r(o, x) < R\}$$

to be the set of points in M that has distance less than R from point o with respect to the background metric ds_M^2. To simplify the notation, sometime, we will suppress the dependency of o and write $B_\rho(R) = B_\rho(o, R)$ and $B(R) = B(o, R)$. Finally, suppose that E is an end of M, we denote $E_\rho(R) = B_\rho(R) \cap E$.

Lemma 2.1. *Let $(M, ds_\rho^2, e^{-f}dv)$ be a complete smooth metric measure space. Suppose E is an end of M satisfying there exists a nonnegative function $\rho(x)$ defined on E with the property that*

$$\int_E \rho\phi^2 e^{-f} \le \int_E |\nabla\phi|^2 e^{-f} - \int_E \mu\phi^2 e^{-f}$$

for any compactly supported function $\phi \in C_c^\infty(E)$ and μ a function defined on E. Let u be a nonnegative function defined on E such that the differential inequality

$$\Delta_f u \ge -\mu u$$

holds true. If u has the growth condition

$$\int_{E_\rho(R)} \rho u^2 e^{-2r_\rho} e^{-f} = o(R)$$

as $R \to \infty$, then it must satisfy the decay estimate

$$\int_{E_\rho(R+1)\setminus E_\rho(R)} \rho u^2 e^{-f} \le C(1 + (R - R_0)^{-1})e^{-2R} \int_{E_\rho(R_0+1)\setminus E_\rho(R_0)} e^{2r_\rho} u^2 e^{-f}$$

for some constant $C > 0$ and for all $R \ge 2(R_0 + 1)$.

Proof. To prove Lemma 2.1, we will combine both arguments in [7,8]. Let $\phi(r_\rho(x))$ be a nonnegative cut-off function with support in E, where $r_\rho(x)$ is the ρ-distance to the fixed point p. Then for any function $h(r_\rho(x))$, integration by parts implies

$$\begin{aligned}
&\int_E |\nabla(\phi e^{2h} u)|^2 e^{-f} \\
&= \int_E |\nabla(\phi e^h)|^2 u^2 e^{-f} + \int_E (\phi e^h)^2 |\nabla u|^2 e^{-f} + 2\int_E (\phi e^h) u \langle \nabla(\phi e^h), \nabla u\rangle e^{-f} \\
&= \int_E |\nabla(\phi e^h)|^2 u^2 e^{-f} + \int_E \phi^2 |\nabla u|^2 e^{2h} e^{-f} + \frac{1}{2}\int_E \langle \nabla(\phi^2 e^{2h}), \nabla u^2\rangle e^{-f} \\
&= \int_E |\nabla(\phi e^h)|^2 u^2 e^{-f} + \int_E \phi^2 |\nabla u|^2 e^{2h} e^{-f} - \frac{1}{2}\int_E \phi^2 \Delta_f(u^2) e^{2h} e^{-f} \qquad (2.1)\\
&= \int_E |\nabla(\phi e^h)|^2 u^2 e^{-f} - \int_E \phi^2 u(\Delta_f u) e^{2h} e^{-f} \\
&\le \int_E |\nabla \phi|^2 u^2 e^{2h} e^{-f} + 2\int_E \phi \langle \nabla\phi, \nabla h\rangle u^2 e^{2h} e^{-f} \\
&\quad + \int_E \phi^2 |\nabla h|^2 u^2 e^{2h} e^{-f} + \int_E \phi^2 \mu u^2 e^{2h} e^{-f}.
\end{aligned}$$

On the other hand, by assumption, we have

$$\int_E \rho\phi^2 e^{-f} \le \int_E |\nabla\phi|^2 e^{-f} - \int_E \mu\phi^2 e^{-f},$$

hence using (2.1), we obtain

$$\int_E \rho\phi^2 u^2 e^{2h} e^{-f} \le \int_E |\nabla\phi|^2 u^2 e^{2h} e^{-f} + 2\int_E \phi e^{2h} \langle \nabla\phi, \nabla h\rangle u^2 e^{-f} + \int_E \phi^2 |\nabla h|^2 u^2 e^{2h} e^{-f}. \qquad (2.2)$$

From now, we divide the proof into three steps.

Step 1*:* We claim that for any $0 < \delta < 1$, there exists a constant $0 < C_1 < \infty$ such that,

$$\int_E \rho e^{2\delta r_\rho} u^2 e^{-f} \le C_1.$$

Indeed, let us now choose

$$\phi(r_\rho(x)) = \begin{cases} r_\rho(x) - R_0 & \text{on } E_\rho(R_0+1) \setminus E_\rho(R_0), \\ 1 & \text{on } E_\rho(R) \setminus E_\rho(R_0+1), \\ R^{-1}(2R - r_\rho(x)) & \text{on } E_\rho(2R) \setminus E_\rho(R), \\ 0 & \text{on } E \setminus E_\rho(2R). \end{cases}$$

It is easy to see that

$$|\nabla\phi|^2(x) = \begin{cases} \rho(x) & \text{on } E_\rho(R_0+1) \setminus E_\rho(R_0), \\ R^{-2}\rho(x) & \text{on } E_\rho(2R) \setminus E_\rho(R), \\ 0 & \text{on } (E_\rho(R) \setminus E_\rho(R_0+1)) \cup (E \setminus E_\rho(2R)). \end{cases}$$

Moreover, we also choose

$$h(r_\rho(x)) = \begin{cases} \delta r_\rho(x) & \text{for } r_\rho \le K/(1+\delta), \\ K - r_\rho(x) & \text{for } r_\rho \ge K/(1+\delta) \end{cases}$$

for some fixed $K > (R_0+1)(1+\delta)$. When $R \ge K/(1+\delta)$, we have

$$|\nabla h|^2(x) = \begin{cases} \delta^2 \rho(x) & \text{for } r_\rho \le K/(1+\delta), \\ \rho(x) & \text{for } r_\rho \ge K/(1+\delta) \end{cases}$$

and

$$\langle \nabla\phi, \nabla h\rangle(x) = \begin{cases} \delta\rho(x) & \text{on } E_\rho(R_0+1) \setminus E_\rho(R_0), \\ R^{-1}\rho(x) & \text{on } E_\rho(2R) \setminus E_\rho(R), \\ 0 & \text{otherwise.} \end{cases}$$

Substituting these into (2.2), we infer

$$\begin{aligned} \int_E \rho\phi^2 u^2 e^{2h} e^{-f} \le {} & \int_{E_\rho(R_0+1)\setminus E_\rho(R_0)} \rho u^2 e^{2h} e^{-f} + R^{-2} \int_{E_\rho(2R)\setminus E_\rho(R)} \rho u^2 e^{2h} e^{-f} \\ & + 2\delta \int_{E_\rho(R_0+1)\setminus E_\rho(R_0)} \rho u^2 e^{2h} e^{-f} + 2R^{-1} \int_{E_\rho(2R)\setminus E_\rho(R)} \rho u^2 e^{2h} e^{-f} \\ & + \delta^2 \int_{E_\rho(\frac{K}{1+\delta})\setminus E_\rho(R_0)} \rho\phi^2 u^2 e^{2h} e^{-f} + \int_{E_\rho(2R)\setminus E_\rho(\frac{K}{1+\delta})} \rho\phi^2 u^2 e^{2h} e^{-f}. \end{aligned}$$

Therefore, by rearrangement of the above inequality, we obtain

$$\begin{aligned} & \int_{E_\rho(\frac{K}{1+\delta})\setminus E_\rho(R_0+1)} \rho u^2 e^{2h} e^{-f} \\ & \le \int_{E_\rho(\frac{K}{1+\delta})} \rho\phi^2 u^2 e^{2h} e^{-f} \\ & \le \int_{E_\rho(R_0+1)\setminus E_\rho(R_0)} \rho u^2 e^{2h} e^{-f} + R^{-2} \int_{E_\rho(2R)\setminus E_\rho(R)} \rho u^2 e^{2h} e^{-f} \\ & \quad + 2\delta \int_{E_\rho(R_0+1)\setminus E_\rho(R_0)} \rho u^2 e^{2h} e^{-f} + 2R^{-1} \int_{E_\rho(2R)\setminus E_\rho(R)} \rho u^2 e^{2h} e^{-f} \\ & \quad + \delta^2 \int_{E_\rho(\frac{K}{1+\delta})\setminus E_\rho(R_0)} \rho u^2 e^{2h} e^{-f}. \end{aligned}$$

Thus

$$\begin{aligned} & (1-\delta^2) \int_{E_\rho(\frac{K}{1+\delta})\setminus E_\rho(R_0+1)} \rho u^2 e^{2h} e^{-f} \\ & \le (1+\delta)^2 \int_{E_\rho(R_0+1)\setminus E_\rho(R_0)} \rho u^2 e^{2h} e^{-f} + R^{-2} \int_{E_\rho(2R)\setminus E_\rho(R)} \rho u^2 e^{2h} e^{-f} \\ & \quad + 2R^{-1} \int_{E_\rho(2R)\setminus E_\rho(R)} \rho u^2 e^{2h} e^{-f}. \end{aligned}$$

Due to the definition of h and the assumption on the growth condition of u, we see that the last two terms on the right-hand side tend to zero as $R \to \infty$. Hence, we have the estimate

$$\int_{E_\rho(\frac{K}{1+\delta})\setminus E_\rho(R_0+1)} \rho u^2 e^{2\delta r_\rho} e^{-f} \leq \frac{(1+\delta)^2}{1-\delta^2} \int_{E_\rho(R_0+1)\setminus E_\rho(R_0)} \rho u^2 e^{2\delta r_\rho} e^{-f}.$$

Note that the right-hand side does not depend on K, by letting $K \to \infty$ we conclude that

$$\int_{E\setminus E_\rho(R_0+1)} \rho u^2 e^{2\delta r_\rho} e^{-f} \leq C_1, \tag{2.3}$$

where

$$C_1 = \frac{(1+\delta)^2}{1-\delta^2} \int_{E(R_0+1)\setminus E(R_0)} u^2 e^{2\delta r_\rho} e^{-f}.$$

Step 2*:* We want to prove that, there exists a constant $C_2 > 0$ such that

$$\int_{E_\rho(R)} \rho u^2 e^{2r_\rho} e^{-f} \leq C_2 R. \tag{2.4}$$

To do this, our first aim is to improve (2.3) by setting $h = r_\rho$ in the previous arguments. Now, suppose that $h = r_\rho$, by (2.2), we infer

$$-2\int_E \phi e^{2r_\rho} \langle \nabla\phi, \nabla r_\rho\rangle u^2 e^{-f} \leq \int_E |\nabla\phi|^2 u^2 e^{2r_\rho} e^{-f}.$$

For $R_0 < R_1 < R$, we choose

$$\phi(x) = \begin{cases} \frac{r_\rho(x)-R_0}{R_1-R_0} & \text{on } E_\rho(R_1)\setminus E_\rho(R_0), \\ \frac{R-r_\rho(x)}{R-R_1} & \text{on } E_\rho(R)\setminus E_\rho(R_1). \end{cases}$$

Plugging ϕ in the above inequality, we obtain

$$\begin{aligned} &\frac{2}{R-R_1}\int_{E_\rho(R)\setminus E_\rho(R_1)} \left(\frac{R-r_\rho(x)}{R-R_1}\right) \rho u^2 e^{2r_\rho} e^{-f} \\ &\leq \frac{1}{(R_1-R_0)^2}\int_{E_\rho(R_1)\setminus E_\rho(R_0)} \rho u^2 e^{2r_\rho} e^{-f} + \frac{1}{(R-R_1)^2}\int_{E_\rho(R)\setminus E_\rho(R_1)} \rho u^2 e^{2r_\rho} e^{-f} \\ &\quad + \frac{2}{(R_1-R_0)^2}\int_{E_\rho(R_1)\setminus E_\rho(R_0)} (r_\rho - R_0)\rho u^2 e^{2r_\rho} e^{-f}. \end{aligned}$$

Observe that for any $0 < t < R - R_1$, the following inequality

$$\frac{2t}{(R-R_1)^2}\int_{E_\rho(R-t)\setminus E_\rho(R_1)} \rho u^2 e^{2r_\rho} e^{-f} \leq \frac{2}{(R-R_1)^2}\int_{E_\rho(R)\setminus E_\rho(R_1)} (R-r_\rho)\rho u^2 e^{2r_\rho} e^{-f}$$

holds true. Therefore, we conclude that

$$\begin{aligned}\frac{2t}{(R-R_1)^2}\int_{E_\rho(R-t)\setminus E_\rho(R_1)}&\rho u^2 e^{2r_\rho}e^{-f}\\ \le\left(\frac{2}{R_1-R_0}+\frac{1}{(R_1-R_0)^2}\right)&\int_{E_\rho(R_1)\setminus E_\rho(R_0)}\rho u^2 e^{2r_\rho}e^{-f}\\ +\frac{1}{(R-R_1)^2}&\int_{E_\rho(R)\setminus E_\rho(R_1)}\rho u^2 e^{2r_\rho}e^{-f}.\end{aligned}\tag{2.5}$$

Now, by taking $R_1 = R_0 + 1$, $t = 1$, and setting

$$g(R)=\int_{E_\rho(R)\setminus E_\rho(R_0+1)}\rho u^2 e^{2r_\rho}e^{-f},$$

the inequality (2.5) becomes

$$g(R-1)\le C_3R^2+\frac{1}{2}g(R),$$

where

$$C_3=\frac{3}{2}\int_{E_\rho(R_0+1)\setminus E_\rho(R_0)}\rho u^2 e^{2r_\rho}e^{-f}$$

is independent of R. Iterating this inequality, we obtain that for any positive integer k and $R \ge 1$

$$\begin{aligned}g(R)&\le C_3\sum_{i=1}^{k}\frac{(R+i)^2}{2^{i-1}}+2^{-k}g(R+k)\\ &\le C_3R^2\sum_{i=1}^{\infty}\frac{(1+i)^2}{2^{i-1}}+2^{-k}g(R+k)\\ &=C_4R^2+2^{-k}g(R+k)\end{aligned}$$

where

$$C_4=C_3\sum_{i=1}^{\infty}\frac{(1+i)^2}{2^{i-1}}.$$

Note that in our previous estimate (2.3), we have proved the following inequality

$$\int_E \rho u^2 e^{2\delta r_\rho}e^{-f}\le C_1$$

for any $\delta < 1$. Thus, this implies that

$$\begin{aligned}g(R+k)&=\int_{E_\rho(R+k)\setminus E_\rho(R_0+1)}\rho u^2 e^{2r_\rho}e^{-f}\\ &\le e^{2(R+k)(1-\delta)}\int_{E_\rho(R+k)\setminus E_\rho(R_0+1)}\rho u^2 e^{2\delta r_\rho}e^{-f}\\ &\le C_1e^{2(R+k)(1-\delta)}.\end{aligned}$$

Now, if we choose $2(1-\delta) < \ln 2$, then

$$2^{-k} g(R+k) \to 0$$

as $k \to \infty$. Consequently, we obtain

$$g(R) = \int_{E_\rho(R)\setminus E_\rho(R_0+1)} \rho u^2 e^{2r_\rho} e^{-f} \le C_4 R^2 \tag{2.6}$$

for all $R \ge R_0 + 1$.

Finally, using inequality (2.5) again and by choosing $R_1 = R_0 + 1$ and $t = R/2$ this time, we infer

$$R \int_{E_\rho(\frac{R}{2})\setminus E_\rho(R_0+1)} \rho u^2 e^{2r_\rho} e^{-f} \le C_5 R^2 + \int_{E_\rho(R)\setminus E_\rho(R_0+1)} \rho u^2 e^{2r_\rho} e^{-f}.$$

Observe that the second term on the right-hand side is bounded by (2.6), we have

$$\int_{E_\rho(\frac{R}{2})\setminus E_\rho(R_0+1)} \rho u^2 e^{2r_\rho} e^{-f} \le C_6 R \quad \text{for all } R \ge 2(R_0+1).$$

Therefore, the claim is proved.

Step 3*:* In this step, we will complete the proof of Lemma 2.1 by using (2.4). Indeed, letting $t = 2$ and $R_1 = R - 4$ in (2.5), we obtain

$$\begin{aligned}
&\int_{E_\rho(R-2)\setminus E_\rho(R-4)} \rho u^2 e^{2r_\rho} e^{-f} \\
&\le \left(\frac{8}{R-R_0-4} + \frac{4}{(R-R_0-4)^2} \right) \int_{E_\rho(R-4)\setminus E_\rho(R_0)} \rho u^2 e^{2r_\rho} e^{-f} \\
&\quad + \frac{1}{4} \int_{E_\rho(R)\setminus E_\rho(R-4)} \rho u^2 e^{2r_\rho} e^{-f}.
\end{aligned}$$

Thanks to (2.4), the first term on the right-hand side is estimated by

$$C_2(1 + (R-R_0-4)^{-1})$$

for $R - 4 \ge 2(R_0+1)$. Hence, by renaming R, the above inequality can be rewritten as

$$\int_{E_\rho(R-2)\setminus E_\rho(R-4)} \rho u^2 e^{2r_\rho} e^{-f} \le C_2(1 + (R-R_0)^{-1}) + \frac{1}{3} \int_{E_\rho(R)\setminus E_\rho(R-2)} \rho u^2 e^{2r_\rho} e^{-f}.$$

Iterating this inequality k times, we conclude that

$$\begin{aligned}
\int_{E_\rho(R+2)\setminus E_\rho(R)} \rho u^2 e^{2r_\rho} e^{-f} &\le C_2(1 + (R-R_0)^{-1}) \sum_{i=0}^{k-1} 3^{-i} \\
&\quad + 3^{-k} \int_{E_\rho(R+2(k+1))\setminus E_\rho(R+2k)} \rho u^2 e^{2r_\rho} e^{-f}.
\end{aligned}$$

However, using (2.4) again, we deduce that the second term is bounded by

$$3^{-k}\int_{E_\rho(R+2(k+1))\setminus E_\rho(R+2k)}\rho u^2e^{2r_\rho}e^{-f}\le C_2 3^{-k}(R+2(k+1))$$

which tends to 0 as $k\to\infty$. This implies

$$\int_{E_\rho(R+2)\setminus E_\rho(R)}\rho u^2e^{2r_\rho}e^{-f}\le C(1+(R-R_0)^{-1})$$

for some constant $C>0$ independent of R, and the lemma follows. □

Remark 2.2. Recently, in [10], Munteanu et al. introduced a parabolic version of decay estimate for weighted volume and used it to investigate Poisson equation on complete smooth metric measure spaces.

Corollary 2.3. *Let E be an end of a complete smooth metric measure space $(M,g,e^{-f}dv)$. Suppose that $\lambda_{1,f}(E)>0$, i.e.,*

$$\lambda_{1,f}\int_E\varphi^2e^{-f}\le\int_E|\nabla\varphi|^2e^{-f}$$

for any compactly supported function $\varphi\in\mathcal{C}_0^\infty(E)$. Let u be a nonnegative function defined on E such that

$$(\Delta_f+\mu)u\ge 0$$

for some constant μ satisfying $\lambda_{1,f}-\mu>0$. If u has the growth condition

$$\int_{E(R)}u^2e^{-2ar}e^{-f}=o(R)$$

as $R\to\infty$, where $a=\sqrt{\lambda_{1,f}-\mu}$, then u must satisfy the decay estimate

$$\int_{E(R+1)\setminus E(R)}u^2e^{-f}\le C(1+(R-R_0)^{-1})e^{-2aR}\int_{E(R_0+1)\setminus E(R_0)}e^{2ar}u^2e^{-f}$$

for some constant $C>0$ depending on f, and a.

Proof. By variational principle for $\lambda_{1,f}$, we have

$$(\lambda_{1,f}-\mu)\int_E\varphi^2e^{-f}\le\int_E|\varphi|^2e^{-f}-\int_E\mu\varphi^2e^{-f}.$$

Let $\rho=a^2$, the distance function with respect to the complete metric $\rho\,ds^2$ is given by

$$r_\rho(x)=ar(x).$$

Now, we can apply Lemma 2.1 to complete the proof. □

Note that if $u \equiv 1$, we obtain the following decay estimate.

Corollary 2.4. *Suppose that E is an f-parabolic end of smooth metric measure space $(M, g, e^{-f} dv)$ with $\lambda_{1,f}(E) > 0$. Denote by $V_{E,f}(R)$ the weighted volume of $E(R)$, then the following decay estimate*

$$\begin{aligned} &V_{E,f}(R+1) - V_{E,f}(R) \\ &\leq C(\lambda_{1,f}(E)) \left(1 + \frac{1}{R - R_0}\right) e^{2\sqrt{\lambda_{1,f}}(R-R_0)} (V_{E,f}(R_0+1) - V_{E,f}(R_0)) \end{aligned}$$

holds true. Here R_0 is a given positive number and $C(\lambda_{1,f}(E))$ is some positive constant depending on $\lambda_{1,f}(E)$.

It is worth to mention that in [1], Buckley and Koskela also proved earlier a version of decay estimate for f-volume in a more general setting.

3. Counting cusps on smooth metric measure spaces of finite f-volume

As what we mentioned in the introduction part, in order to estimate the number of cusps we must have a decay estimate of the f-volume proved in Section 2 and a volume comparison theorem. Hence, first we introduce a volume comparison result given by Wei and Wylie in [15].

Lemma 3.1. [15, Theorem 1.2] *Let $(M^n, g, e^{-f} dv)$ be a complete smooth metric measure space with $\mathrm{Ric}_f \geq -(n-1)$. Suppose there exists a nonnegative constant α such that the weighted function satisfies*

$$|\nabla f|(x) \leq \alpha$$

for all $x \in M$. Then there exists a constant $C > 0$ such that the volume upper bound

$$V_f(B(o, R)) \leq C e^{(n-1+\alpha)R} V_f(B(o, 1))$$

holds for all $R > 0$. Here $V_f(B(o, R))$ stands for the weighted volume of $B(o, R)$.

Now, we will combine Lemma 3.1 and Corollary 2.4 to count the number of cusps. The first result is as follows.

Theorem 3.2. *Let $(M^n, g, e^{-f} dv)$ be a smooth metric measure space with $\mathrm{Ric}_f \geq -(n-1)$. Suppose there exists a nonnegative constant α such that the weighted function satisfies*

$$|\nabla f|(x) \leq \alpha$$

for all $x \in M$. If M has finite f-volume given by V_f, and

$$\lambda_{1,f}(M \setminus B(o, R_0)) \geq \frac{(n-1+\alpha)^2}{4}$$

for some $R_0 > 0$*, then*

$$N(M) \leq Ce^{(n-1+\alpha)R_0} \frac{V_f}{V_f(B(o,1))}.$$

Here $C > 0$ *is a constant depending on* $\lambda_{1,f}$*.*

Proof. To simplify the notation, we denote $V_{o,f}(R)$ to be the weighted volume of the geodesic ball $B(o,R)$, then for all $R > 2(R_0+1)$, we have

$$\begin{aligned} &V_{o,f}(R+2) - V_{o,f}(R) \\ &\leq C\left(1 + \frac{1}{R-R_0}\right) e^{(n-1+\alpha)(R_0-R)}(V_{o,f}(R_0+1) - V_{o,f}(R_0)). \end{aligned} \tag{3.1}$$

Here we use Corollary 2.4.

On the other hand, if $y \in \partial B(o,R+1)$ then $B(o,1) \subset B(y,R+2)$. Hence, we use Lemma 3.1 to obtain

$$V_{o,f}(1) \leq V_{y,f}(R+2) \leq C_1 e^{(n-1+\alpha)R} V_{y,f}(1). \tag{3.2}$$

Suppose that $M \backslash B(o,R)$ has $N(R)$ unbounded components, then there exist $N(R)$ number of points $\{y_i \in \partial B(o,R+1)\}$ such that $B(y_i,1) \cap B(y_j,1) = \emptyset$ for $i \neq j$. In particular, applying (3.2) to each of the y_i and combining with (3.1), we have

$$\begin{aligned} N(R) C_1^{-1} e^{-(n-1+\alpha)R} V_{o,f}(1) &\leq \sum_{i=1}^{N(R)} V_{y_i,f}(1) \\ &\leq V_{o,f}(R+2) - V_{o,f}(R) \\ &\leq C\left(1 + \frac{1}{R-R_0}\right) e^{(n-1+\alpha)(R_0-R)}(V_{o,f}(R_0+1) - V_{o,f}(R_0)). \end{aligned}$$

This implies that

$$N(R) \leq CC_1(1 + (R-R_0)^{-1}) e^{(n-1+\alpha)R_0}(V_{o,f}(R_0+1) - V_{o,f}(R_0)) V_{o,f}^{-1}(1).$$

Note that $N(R)$ is the number of ends of M with respect to $B(o,R)$, letting $R \to \infty$, we complete the proof. □

Next, we will derive a weighted version of a Li-Wang's result in [8] to estimate $\lambda_{1,f}(B_o(R))$ of a geodesic ball centered at o with radius R in terms of the weighted volume of the ball. It is worth to mention that as in [8], we do not require any curvature assumptions on M.

Lemma 3.3. *Let* $(M, g, e^f dv)$ *be a complete smooth metric measure space. Then for any* $0 < \delta < 1$*,* $R > 2$ *and* $o \in M$*, we have*

$$\lambda_{1,f}(B(o,R)) \leq \frac{1}{4\delta^2(R-1)^2} \left(\ln\left(\frac{V_f(B(o,R))}{V_f(B(o,1))}\right) + \ln\left(\frac{81}{1-\delta}\right)\right)^2.$$

Proof. Observe that
$$\frac{4}{R^2} \leq \frac{1}{4\delta^2(R-1)^2} \ln^2\left(\frac{81}{1-\delta}\right).$$
Hence, we may assume $\lambda_{1,f}(B(o,R)) \geq 4/R^2$.

To simplify the notation, let us use $\lambda_{1,f}$ to denote $\lambda_{1,f}(B(o,R))$. By the variational characteristic of $\lambda_{1,f}(B(o,R))$, we have

$$\begin{aligned}
&\lambda_{1,f} \int_M \phi^2 \exp(-2\delta\sqrt{\lambda_{1,f}}r)e^{-f} \\
&\leq \int_M |\nabla(\phi \exp(-\delta\sqrt{\lambda_{1,f}}r))|^2 e^{-f} \\
&= \int_M |\nabla\phi|^2 \exp(-2\delta\sqrt{\lambda_{1,f}}r)e^{-f} - 2\delta\sqrt{\lambda_{1,f}} \int_M \phi \exp(-2\delta\sqrt{\lambda_{1,f}}r)\langle\nabla\phi, \nabla r\rangle e^{-f} \\
&\quad + \delta^2\lambda_{1,f} \int_M \phi^2 \exp(-2\delta\sqrt{\lambda_{1,f}}r)e^{-f}
\end{aligned}$$

for any nonnegative Lipschitz function ϕ with support in $B(o,R)$. Consequently,

$$\begin{aligned}
&\lambda_{1,f}(1-\delta^2) \int_M \phi^2 \exp(-2\delta\sqrt{\lambda_{1,f}}r)e^{-f} \\
&\leq \int_M |\nabla\phi|^2 \exp(-2\delta\sqrt{\lambda_{1,f}}r)e^{-f} - 2\delta\sqrt{\lambda_{1,f}} \int_M \phi \exp(-2\delta\sqrt{\lambda_{1,f}}r)\langle\nabla\phi, \nabla r\rangle e^{-f}.
\end{aligned}$$

In particular, for $R > 2$, we choose

$$\phi = \begin{cases} 1 & \text{on } B(o, R - \lambda_{1,f}^{1/2}), \\ \sqrt{\lambda_{1,f}}(R-r) & \text{on } B(o,R) \setminus B(o, R - \lambda_{1,f}^{1/2}), \\ 0 & \text{on } M \setminus B(o,R), \end{cases}$$

then $\phi = 1$ on $B(o,1)$ since $R - \lambda_{1,f}^{1/2} \geq R/2 > 1$. Plugging ϕ in the above inequality, we obtain

$$\begin{aligned}
&(1-\delta^2)\lambda_{1,f} \exp(-2\delta\sqrt{\lambda_{1,f}})V_f(B(o,1)) \\
&\leq (1-\delta^2)\lambda_{1,f} \int_M \phi^2 \exp(-2\delta\sqrt{\lambda_{1,f}}r)e^{-f} \\
&= \int_M |\nabla\phi|^2 e^{-2\delta\sqrt{\lambda_{1,f}}r} e^{-f} - 2\delta\sqrt{\lambda_{1,f}} \int_M \phi e^{-2\delta\sqrt{\lambda_{1,f}}r} \langle\nabla\phi, \nabla r\rangle e^{-f} \\
&\leq \lambda_{1,f} \int_{B(o,R)\setminus B(o,R-\lambda_{1,f}^{1/2})} e^{-2\delta\sqrt{\lambda_{1,f}}r} e^{-f} + 2\delta\lambda_{1,f} \int_{B(o,R)\setminus B(o,R-\lambda_{1,f}^{1/2})} \phi e^{-2\delta\sqrt{\lambda_{1,f}}r} e^{-f} \\
&\leq (1+2\delta)\lambda_{1,f} e^{-2\delta(\sqrt{\lambda_{1,f}}R-1)} V_f(B(o,R)).
\end{aligned}$$

Here we use $\sqrt{\lambda_{1,f}}(R-r) \leq 1$ on $B(o,R)\setminus B(o,R-\lambda_{1,f}^{1/2})$ in the last inequality. Therefore,

$$e^{2\delta\sqrt{\lambda_{1,f}}(R-1)} \leq \frac{(1+2\delta)e^{2\delta}}{1-\delta^2} \frac{V_f(B(o,R))}{V_f(B(o,1))} \leq \frac{27}{1-\delta} \frac{V_f(B(o,R))}{V_f(B(o,1))}.$$

This implies

$$2\delta\sqrt{\lambda_{1,f}}(R-1) \leq \ln\left(\frac{27}{1-\delta}\right) + \ln\left(\frac{V_f(B(o,R))}{V_f(B(o,1))}\right).$$

The lemma follows by rewriting this inequality. □

Note that $\lambda_{1,f}(M) = \lim_{R\to\infty} \lambda_{1,f}(B(o,R))$. If we first let R go to infinity and then δ go to 1 in the estimate of Lemma 3.3, we have the following result.

Corollary 3.4. *Let $(M^n, g, e^f dv)$ be a complete smooth metric measure space and $\lambda_{1,f}(M)$ be the first weighted eigenvalue. Then*

$$\lambda_{1,f}(M) \leq \frac{1}{4}\left(\liminf_{R\to\infty} \frac{\ln V_f(B(o,R))}{R}\right)^2.$$

Now, we will give a proof of Theorem 1.2.

Proof of Theorem 1.2. Note that $o \in M$ is a fixed point. For any $0 < \delta < 1$, let

$$R_0 = \frac{1}{(n-1+\alpha)\delta}\left(\ln\left(\frac{81}{1-\delta}\right) + \ln\left(\frac{V_f}{V_f(B(o,1))}\right)\right) + 3.$$

Thanks to Lemma 3.3, we have

$$\lambda_{1,f}(B(o,R_0)) \leq \frac{\delta^2(n-1+\alpha)^2}{4\delta^2} \frac{\left(\ln\left(\frac{V_f(B(o,R))}{V_f(B(o,1))}\right) + \ln\left(\frac{81}{1-\delta}\right)\right)^2}{\left(\ln\left(\frac{81}{1-\delta}\right) + \ln\left(\frac{V_f}{V_f(B(o,1))}\right)\right)^2} \leq \frac{(n-1+\alpha)^2}{4}.$$

On the other hand, by the variational principle, we have

$$\mu_1(M) \leq \max\{\lambda_{1,f}(B(o,R_0)), \lambda_{1,f}(M \setminus B(o,R_0))\}.$$

Hence, combining this inequality with assumption regarding to $\mu_1(M)$, we infer

$$\lambda_{1,f}(M \setminus B(o,R_0)) \geq \frac{(n-1+\alpha)^2}{4}.$$

So Theorem 3.2 implies

$$N(M) \leq C(n) V_f V_f^{-1}(B(o,1)) \exp((n-1+\alpha)R_0).$$

To finish the proof, we first choose

$$\delta = 1 - \frac{1}{\ln(V_f V_f^{-1}(B_o(1)))}$$

then replace the value of R_0, δ in the last inequality. So we are done. □

As we mentioned in the introduction part, two main ingredients in our proof of Theorem 1.2 are the decay estimate of the volume and the volume comparison result. Therefore, using Corollary 2.4 and the relative volume comparison theorems in [4], we can count the number of cusps of the following manifold.

Theorem 3.5. *Let M be a complete noncompact 16-dimensional manifold with holonomy group $\mathrm{Spin}(9)$ with finite volume given by V, and $\mu_1(M) \geq 121$. Then there exists a constant $C > 0$ such that*

$$N(M) \leq C \left(\frac{V}{V_o(1)} \right)^2 \ln \left(\frac{V}{V_o(1)} \right)$$

where $V_o(1)$ denotes the (non-weighted) volume of the unit ball centered at any point $o \in M$. Here $\mu_1(M)$ is defined by the (non-weighted) Reileigh quotient

$$\mu_1(M) = \inf_{\phi \in H^1(M), \int_M \phi = 0} \frac{\int_M |\nabla \phi|^2}{\int_M \phi^2}.$$

4. Counting cusps via the p-Laplacian

In this section, we will use the nonlinear theory of p-Laplacian to estimate the number of cusps of Riemannian manifolds. Again, our strategy is to use a (nonlinear) decay estimate of the volume and corresponding volume comparison theorem. Therefore, let us recall the following nonlinear version regarding to the rates of volume decay.

Theorem 4.1. [1] *Let E be an end of a complete Riemannian manifold (M^n, g) with respect to $\overline{B}(o, R_0)$. If the first eigenvalues of the p-Laplacian $\lambda_{1,p}(E) > 0$, ($1 \leq p < \infty$) and M has finite volume V, then the following decay estimate*

$$V_E(R+1) - V_E(R) \leq CVe^{-p\lambda_{1,p}^{1/p}(R-R_0)}$$

holds true for all $R > R_0 + 2$. Here C is a constant depending only on p.

Proof. As in [1], we can assume $\lambda_{1,p} = 1$ and let $E(R) = E \cap B(o, R)$ and $V(R)$ stands for the volume of $E(R)$. Let $V(\infty)$ be the volume of E. Since M has finite volume so does E. Hence by the proof of Theorem 0.1 in [1, page 279], E must be p-parabolic. For $R > R_0 + 2$, it is proved in [1, page 278] that

$$e^{p(R-1)}(V(R+1) - V(R-1)) \leq 2^p e^{p(R_0+1)} V(R_0 + 1) + \epsilon$$

for any $\epsilon > 0$ fixed. Note that $V(R_0 + 1) \leq V$, this implies

$$V(R+1) - V(R-1) \leq 2^p e^{2p} e^{-p(R-R_0)} V.$$

Consequently,

$$V_E(R+1) - V_E(R) \le CVe^{-p\lambda_{1,p}^{1/p}(R-R_0)}.$$

Here $C = 2^p e^{2p}$ is a constant depending only on p, we noted that $\lambda = 1$. The proof is complete. □

We note that the decay estimate of Theorem 4.1 can be considered as a generalization of Corollary 2.4. However, we would like to mention that the results in Section 2 may hold for non-parabolic ends. Therefore, they are of independent interest. Now, we can estimate the number of cusps of smooth metric measure spaces as follows.

Theorem 4.2. *Let (M^n, g) be a smooth metric measure space with* $\mathrm{Ric} \ge -(n-1)$. *If M has finite volume given by V, and*

$$\lambda_{1,p}(M \setminus B(o, R_0)) \ge \frac{(n-1)^p}{p^p}, \quad p \ge 1$$

for some $R_0 > 0$, then

$$N(M) \le Ce^{(n-1)R_0} \frac{V}{V_o(1)}.$$

Here $C > 0$ is a constant depending on $\lambda_{1,p}$.

Proof. It is worth to notice that the conclusion of Theorem 4.1 and (3.1) are of the same type. Therefore, by using the decay estimate in Theorem 4.1 and the volume comparison theorem (see Lemma 3.1), we can repeat the proof of Theorem 3.2 to derive the conclusion. Since they are almost the same, we omit the detail. □

Next, we will estimate $\lambda_{1,p}$ on the ball $B(o, R)$.

Lemma 4.3. *Let (M^n, g) be a Riemannian manifold. Then for any $0 < \delta < 1$, $1 \le p < \infty$ and $o \in M$, we have*

$$\lambda_{1,p}(B(o,R)) \le \frac{1}{\delta^p (R-1)^p} \left(2 + \frac{\alpha(p)}{p} \ln \frac{1}{1-\delta} + \frac{1}{p} \ln \frac{V_o(R)}{V_o(1)}\right)^p$$

where $\alpha(p) = \max\{1, p/2\}$.

Proof. Observe that

$$\frac{2^p}{R^p} \le \frac{1}{\delta^p (R-1)^p} \left(2 + \frac{\alpha(p)}{p} \ln \frac{1}{1-\delta}\right)^p.$$

Hence, we may assume $\lambda_{1,p}(B(o,R)) \ge 2^p/R^p$.

To simplify the notation, let us use $\lambda_{1,p}$ to denote $\lambda_{1,p}(B(o,R))$. By the variational characteristic of $\lambda_{1,p}(B(o,R))$, we have

$$\begin{aligned}
&\lambda_{1,p}\int_M \phi^p \exp(-p\delta\sqrt[p]{\lambda_{1,p}}r) \\
&\leq \int_M |\nabla(\phi\exp(-\delta\sqrt[p]{\lambda_{1,p}}r))|^p \\
&= \int_M e^{-p\delta\sqrt[p]{\lambda_{1,p}}r}|\nabla\phi - \delta\sqrt[p]{\lambda_{1,p}}\phi\nabla r|^p \\
&= \int_M e^{-p\delta\sqrt[p]{\lambda_{1,p}}r}\left(|\nabla\phi|^2 + 2\delta\sqrt[p]{\lambda_{1,p}}|\nabla\phi||\nabla r| + \delta^2\lambda_{1,p}^{2/p}\phi^2\right)^{p/2}
\end{aligned} \tag{4.1}$$

for any nonnegative Lipschitz function ϕ with support in $B(o,R)$. We have two cases.

Case 1*:* $p \geq 2$. Observe that $x^{p/2}$ is a convex function, we have the following basic inequality

$$(A+B)^{p/2} = \left((1-\delta^2)\frac{A}{1-\delta^2} + \delta^2\frac{B}{\delta^2}\right)^{p/2} \leq (1-\delta^2)^{1-p/2}A^{p/2} + \delta^{2-p}B^{p/2}$$

for any $A, B \geq 0$. Hence, by (4.1), we have

$$\begin{aligned}
&\lambda_{1,p}\int_M \phi^p\exp(-p\delta\sqrt[p]{\lambda_{1,p}}r) \\
&\leq (1-\delta^2)^{1-p/2}\int_M e^{-p\delta\sqrt[p]{\lambda_{1,p}}r}\left(|\nabla\phi|^2 + 2\phi\delta\sqrt[p]{\lambda_{1,p}}|\nabla\phi||\nabla r|\right)^{p/2} \\
&\quad + \delta^2\lambda_{1,p}\int_M e^{-p\delta\sqrt[p]{\lambda_{1,p}}r}\phi^p.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\lambda_{1,p}\int_M \phi^p\exp(-p\delta\sqrt[p]{\lambda_{1,p}}r) \\
&\leq \frac{1}{(1-\delta^2)^{p/2}}\int_M e^{-p\delta\sqrt[p]{\lambda_{1,p}}r}\left(|\nabla\phi|^2 + 2\phi\delta\sqrt[p]{\lambda_{1,p}}|\nabla\phi||\nabla r|\right)^{p/2} \\
&\leq \frac{1}{(1-\delta)^{p/2}}\int_M e^{-p\delta\sqrt[p]{\lambda_{1,p}}r}\left(|\nabla\phi|^2 + 2\phi\delta\sqrt[p]{\lambda_{1,p}}|\nabla\phi||\nabla r|\right)^{p/2}.
\end{aligned}$$

Here we use $0 < \delta < 1$ in the last inequality.

Case 2*:* $1 \leq p < 2$. Since $0 < p/2 < 1$, we have for $A, B \geq 0$,

$$(A+B)^{p/2} \leq A^{p/2} + B^{p/2}.$$

Therefore, by (4.1), we obtain

$$\begin{aligned}
\lambda_{1,p}\int_M \phi^p\exp(-p\delta\sqrt[p]{\lambda_{1,p}}r) &\leq \int_M e^{-p\delta\sqrt[p]{\lambda_{1,p}}r}\left(|\nabla\phi|^2 + 2\phi\delta\sqrt[p]{\lambda_{1,p}}|\nabla\phi||\nabla r|\right)^{p/2} \\
&\quad + \delta^2\lambda_{1,p}\int_M e^{-p\delta\sqrt[p]{\lambda_{1,p}}r}\phi^p.
\end{aligned}$$

Since $0<\delta<1$, this implies,

$$\lambda_{1,p}\int_M \phi^p\exp(-p\delta\sqrt[p]{\lambda_{1,p}}r)\le\frac{1}{1-\delta}\int_M e^{-p\delta\sqrt[p]{\lambda_{1,p}}r}\left(|\nabla\phi|^2+2\phi\delta\sqrt[p]{\lambda_{1,p}}|\nabla\phi||\nabla r|\right)^{p/2}.$$

In conclusion, we obtain in both cases that

$$\begin{aligned}&\lambda_{1,p}\int_M \phi^p\exp(-p\delta\sqrt[p]{\lambda_{1,p}}r)\\ &\le\frac{1}{(1-\delta)^{\alpha(p)}}\int_M e^{-p\delta\sqrt[p]{\lambda_{1,p}}r}\left(|\nabla\phi|^2+2\phi\delta\sqrt[p]{\lambda_{1,p}}|\nabla\phi||\nabla r|\right)^{p/2}.\end{aligned}$$

Now, for $R>2$, we choose

$$\phi=\begin{cases}1 & \text{on } B(o,R-\lambda_{1,f}^{-1/p}),\\ \sqrt[p]{\lambda_{1,p}}(R-r) & \text{on } B(o,R)\setminus B(o,R-\lambda_{1,f}^{-1/p}),\\ 0 & \text{on } M\setminus B(o,R),\end{cases}$$

then $\phi=1$ on $B(o,1)$ since $R-\lambda_{1,p}^{-1/p}\ge R/2>1$. Plugging ϕ in the above inequality, we obtain

$$\begin{aligned}\lambda_{1,p}\exp(-p\delta\lambda_{1,p}^{1/p})V_o(1)&\le\lambda_{1,p}\int_M\phi^p\exp(-p\delta\sqrt[p]{\lambda_{1,f}}r)\\ &\le\frac{\lambda_{1,p}}{(1-\delta)^{\alpha(p)}}\int_{B(o,R)\setminus B(o,R-\lambda_{1,p}^{1/p})}(1+2\phi\delta)^{p/2}e^{-p\delta\sqrt[p]{\lambda_{1,p}}r}\\ &\le\frac{\lambda_{1,p}}{(1-\delta)^{\alpha(p)}}3^{p/2}e^{-p\delta(\lambda_{1,p}^{1/p}R-1)}V_o(R).\end{aligned}$$

Here we use $0\le\phi\le1$ in the third inequality. Therefore,

$$e^{p\delta\lambda_{1,p}^{1/p}(R-1)}\le\frac{3^{p/2}e^{p\delta}}{(1-\delta)^{\alpha(p)}}\frac{V_o(R)}{V_o(1)}\le\frac{2^pe^p}{(1-\delta)^{\alpha(p)}}\frac{V_o(R)}{V_o(1)}.$$

This implies

$$p\delta\lambda_{1,p}^{1/p}(R-1)\le p(\ln2+1)+\alpha(p)\ln\frac{1}{1-\delta}+\ln\left(\frac{V_o(R)}{V_o(1)}\right).$$

The lemma follows by rewriting this inequality. □

Corollary 4.4. *Let M be a complete smooth metric measure spaces and $\lambda_{1,p}$ is the first eigenvalue of the p-Laplacian. Then*

$$\lambda_{1,p}\le\left(\frac{1}{p}\liminf_{R\to\infty}\frac{\ln V_o(R)}{R}\right)^p,\quad p\ge1.$$

Proof. We use Lemma 4.3 to give the proof. Indeed, by the conclusion of Lemma 4.3, we first let $R \to \infty$ then let $\delta \to 1$, we obtain

$$\lambda_{1,p} \leq \left(\frac{1}{p} \liminf_{R\to\infty} \frac{\ln V_o(R)}{R}\right)^p, \quad p \geq 1.$$

The proof is complete. □

Now we give a proof of Theorem 1.3.

Proof of Theorem 1.3. Note that $o \in M$ is a fixed point. Let

$$R_0 = \frac{p}{(n-1)\delta}\left(2 + \frac{\alpha(p)}{p} \ln \frac{1}{1-\delta} + \frac{1}{p} \ln \frac{V}{V_o(1)}\right) + 3.$$

Thanks to Lemma 4.3, we infer

$$\lambda_{1,p}(B(o, R_0)) \leq \frac{(n-1)^p}{p^p}.$$

On the other hand, by the variational principle, we have

$$\mu_{1,p}(M) \leq \max\{\lambda_{1,p}(B(o, R_0)), \lambda_{1,p}(M \setminus B(o, R_0))\}.$$

Hence, combining this inequality with assumption regarding to $\mu_{1,p}(M)$, we obtain

$$\lambda_{1,f}(M \setminus B(o, R_0)) \geq \frac{(n-1)^p}{p^p}.$$

So Theorem 4.2 implies

$$N(M) \leq Ce^{(n-1)R_0} \frac{V}{V_o(1)}.$$

To finish the proof, we first choose $\delta = 1 - \frac{1}{\ln(V \cdot V_o(1)^{-1})}$ then replace the value of R_0, δ in the last inequality. So we are done. □

Similarly, using the volume comparison theorems in [3, 4, 6], we have the following theorems.

Theorem 4.5. *Let M^m be a complete Kähler manifold of complex dimension m with finite volume. Assume that M has holomorphic bisectional curvatures satisfying*

$$R_{i\bar{i}j\bar{j}} \geq -(1 + \delta_{ij})$$

for all unitary frame $\{e_1, \ldots, e_m\}$. If

$$\mu_{1,p}(M) \geq \left(\frac{2m}{p}\right)^p, \quad p \geq 1,$$

then there exists a constant $C(m,p) > 0$ *depending only on* m *and* p *such that*

$$N(M) \le C(m,p) \left(\frac{V}{V_o(1)}\right)^2 \ln^{\alpha(p)} \left(\frac{V}{V_o(1)}\right)$$

where $\alpha(p) = \max\{1, p/2\}$ *and* $V_o(1)$ *denotes the volume of the unit ball centered at any point* $o \in M$.

Theorem 4.6. *Let* (M^{4m}, g) *be a complete quarternionic Kähler of real dimension* $4m$ *with finite volume given by* V. *Assume that its scalar curvature satisfies the bound*

$$S_M \ge -16m(m+2)$$

and

$$\mu_{1,p}(M) \ge \frac{(2(2m+1))^p}{p^p}, \quad p \ge 1.$$

Let us denote $N(M)$ *to be the number of ends (cusps) of* M. *Then there exists a constant* $C(m,p) > 0$ *depending only on* m *and* p *such that*

$$N(M) \le C(n,p) \left(\frac{V}{V_o(1)}\right)^2 \ln^{\alpha(p)} \left(\frac{V}{V_o(1)}\right)$$

where $\alpha(p) = \max\{1, p/2\}$ *and* $V_o(1)$ *denotes the volume of the unit ball centered at a fixed point* $o \in M$.

Theorem 4.7. *Let* M *be a complete noncompact* 16*-dimensional manifold with holonomy group* Spin(9) *with finite volume given by* V. *Assume that*

$$\mu_{1,p}(M) \ge \frac{22^p}{p^p}, \quad p \ge 1.$$

Let us denote $N(M)$ *to be the number of ends (cusps) of* M. *Then there exists a constant* $C(p) > 0$ *depending only on* p *such that*

$$N(M) \le C(p) \left(\frac{V}{V_o(1)}\right)^2 \ln^{\alpha(p)} \left(\frac{V}{V_o(1)}\right)$$

where $\alpha(p) = \max\{1, p/2\}$ *and* $V_o(1)$ *denotes the volume of the unit ball centered at a fixed point* $o \in M$.

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