

An Approach to ϵ -duality Theorems for Nonconvex Semi-infinite Multiobjective Optimization Problems

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Abstract. Using a scheme for solving multiobjective optimization problems via a system of corresponding scalar problems, approximate optimality conditions for a nonconvex semi-infinite multiobjective optimization problem are established. As a new approach, the scheme is developed to study the approximate duality theorems of the problem via a pair of primal-dual scalar problems. Several ϵ -duality theorems are given. Furthermore, the existence theorem for almost quasi weakly ϵ -Pareto solutions of the primal problem, and the existence theorem for quasi weakly ϵ -Pareto solutions of the dual problem are established without any constraint qualification.

1. Introduction

Scalarizing methods are important tools to find optimal solutions in multiobjective optimization programming. The relevance of using scalarizing methods to solve multiobjective optimization problems is that, scalar problems can have more effective means of finding optima than vector problems can. The reader is referred to the papers [27,34], where surveys of methods for multiobjective optimization are reviewed and to the books [2,12,13,25], the papers [4,24,33] for the deeper insight into this topic. In this paper, we are interested in a scalarizing method for solving multiobjective optimization problems introduced in the Chankong-Haines's book [5]. This scheme was presented since 1983 and it has attracted much attention by several authors for years. The reader is referred to the papers such as [3,4,26,27,35] for details. The criteria of this scheme is to solve a vector optimization problem via a system of corresponding scalar problems. Hence, it could approach to the dual problem of a given multiobjective optimization problem via pairs of primal-dual scalar problems.

In this paper, we deal with approximate Pareto solutions for a multiobjective optimization nonconvex problem based on the method, and study its dual problem via the scheme. Related to approximate solutions in multiobjective optimization problems, there are many

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papers published for years [1, 8, 14, 15, 18–21, 32]. In our results, we are interested in the concepts of quasi ϵ -Pareto solutions and almost quasi (weakly) ϵ -Pareto solutions. We note that the concepts of a quasi ϵ -solution and an almost quasi ϵ -solution for a single objective nonconvex problem were considered in [31]. Afterward, it was developed for multiobjective problems [32]. There are several papers concerned with these concepts [11, 22, 30, 38, 41].

Let us consider the following nonconvex semi-infinite multiobjective optimization problem:

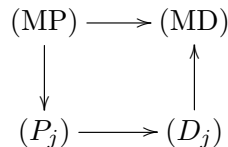
$$(MP) \quad \text{Minimize } f(x) := (f_1(x), f_2(x), \dots, f_m(x)) \text{ subject to } g_t(x) \leq 0, t \in T, x \in C,$$

where $f_i: \mathbb{R}^n \rightarrow \mathbb{R}$ and g_t are locally Lipschitz functions for $i \in M := \{1, 2, \dots, m\}$ and $t \in T$, T is an index set (possibly infinite), and C is a closed convex subset of \mathbb{R}^n . The optimization problems which have a number of infinite constraints are attracted, recently, by several authors [6, 7, 36–41]. We also note that the results on approximate Pareto solutions and approximate duality theorems for (MP) via a non-weighted sum scalarizing method were established recently in [38]. Due to the scheme introduced by Chankong-Haimes, for $j \in M$ and given $y \in C$, we associate to (MP) the following scalar problem.

$$(P_j(y)) \quad \begin{aligned} &\text{Minimize } f_j(x) \text{ subject to } f_k(x) \leq f_k(y), \\ &k \in M^j := M \setminus \{j\}, g_t(x) \leq 0, t \in T, x \in C, \end{aligned}$$

where f_j, f_k, g_t and C are as above with $k \in M^j$ and $t \in T$. In this research, the scheme will be developed to establish relationships between approximate optimality solutions of (MP) and $(P_j(y))$.

As far as we know, the scheme is usually used for finding exact solutions of multiobjective optimization problems and there is no work dealing with dual problems of vector optimization problems. The aim of this research is to propose an approach for establishing the relations between a vector optimization problem and its dual problem via a pair of primal-dual scalar problems corresponding Chankong-Haimes scheme. So, the relations between (MP) and its dual problem will be investigated via the following diagram



where (MD) is the dual problem of (MP), (D_j) is the dual problem of (P_j) , (P_j) and (D_j) are the scalar problems corresponding to (MP) and (MD), respectively. When approaching the problem from this direction, there exists a question: How to formulate the problem (D_j) such that it is a dual problem of (P_j) and it can be suitable for (MD) in Chankong-Haimes scheme, simultaneously. This problem will be discussed in Section 4.

Our results are divided in two parts. In the first one, we deal with approximate Pareto solutions of (MP). Some approximate optimality conditions for (MP) are given. The obtained results also describe the relationships between approximate Pareto solutions of (MP) and approximate solutions of $(P_j(y))$. In particular, we prove that there exists an almost quasi approximate Pareto solution for (MP) without assuming any constraint qualification. In the second part, we propose an approximate dual problem of (MP) in Wolfe type. To obtain new results related to the dual problem (MD), we give some theorems to describe the relationships between $(P_j(y))$ and (D_j) , between (D_j) and (MD). In this case, the problem (D_j) is formulated in Wolfe type with a small modification in comparison with the original one. Then, duality relationships, between (MP) and (MD), can be investigated via a pair primal-dual scalar problems. We also prove that there exists a quasi weakly approximate Pareto solution for (MD) without assuming any constraint qualification.

The rest of the paper is organized as follows. The next section is devoted to preliminaries and lemmas for describing the criteria of the method. In the third section, we present approximate optimality conditions for (MP) and $(P_j(y))$ and the relationships between the problems. In the last section, approximate duality theorems for (D_j) and (MD) and relationships between the problems are established. Finally, relations between (MP) and (MD) are exploited via a pair of scalar problems. An example is given and applications into fractional semi-infinite multiobjective optimization problems are discussed.

2. Preliminaries

Most of the following basic concepts are concerned with nonsmooth analysis theory. They can be found in the following books [9, 10, 16].

A function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be a locally Lipschitz function if for any $x \in \mathbb{R}^n$ there exists a positive constant K and a neighborhood N of x such that

$$|f(y) - f(z)| \leq K\|y - z\|, \quad \forall y, z \in N(x).$$

For a sequence $(g_t)_{t \in T}$, $g_t: \mathbb{R}^n \rightarrow \mathbb{R}$, we say that $(g_t)_{t \in T}$ is locally Lipschitz with respect to x uniformly in t if there exists a neighborhood $N(x)$ and a constant $K > 0$ such that

$$|g_t(y) - g_t(z)| \leq K\|y - z\|, \quad \forall y, z \in N(x) \text{ and } \forall t \in T.$$

In this research, the constraint functions g_t , $t \in T$, of (MP) are locally Lipschitz with respect to x uniformly in t .

The generalized directional derivative of f at x in the direction $d \in X$ is defined by (see [9])

$$f^\circ(x; d) := \limsup_{\substack{h \rightarrow 0 \\ t \downarrow 0}} \frac{f(x + h + td) - f(x + h)}{t}$$

and the Clarke’s subdifferential of f at x , denoted by $\partial^c f(x)$, is

$$\partial^c f(x) := \{u \in \mathbb{R}^n \mid u(d) \leq f^\circ(x; d), \forall d \in \mathbb{R}^n\}.$$

For $d \in \mathbb{R}^n$, if the limit

$$\lim_{t \downarrow 0} \frac{f(x + td) - f(x)}{t}$$

exists then it is called the directional derivative of f at x in the direction d and is denoted by $f'(x; d)$. The function f is said to be quasidifferentiable or regular at x (in the sense of Clarke) if $f'(x; d)$ exists and equals to $f^\circ(x; d)$ for each $d \in \mathbb{R}^n$ (see [9, 10]). The function f is said to be regular if it is regular at each point of \mathbb{R}^n .

For a closed subset D of \mathbb{R}^n , the tangent cone to D is defined by

$$T_D(x) = \{h \in \mathbb{R}^n \mid d_D^\circ(x; h) = 0\},$$

where d_D denotes the distance function to D . The normal cone to D is defined by

$$N_D(x) = \{u \in \mathbb{R}^n \mid u(h) \leq 0, \forall h \in T_D(x)\}.$$

When D is convex, the normal cone to D coincides with the one in the sense of convex analysis:

$$N_D(x) = \{u \in \mathbb{R}^n \mid u(y - x) \leq 0, \forall y \in D\}.$$

The following definitions are due to Hiriart-Urruty [23].

Definition 2.1. A locally Lipschitz function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be pseudoconvex if for all $x, y \in \mathbb{R}^n$,

$$f^\circ(x; y - x) \geq 0 \implies f(y) \geq f(x).$$

Remark 2.2. From Definition 2.1, if there exists $u \in \partial^c f(x)$ such that $u(y - x) \geq 0$ then $f(y) \geq f(x)$.

Definition 2.3. A locally Lipschitz function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be quasiconvex if for all $x, y \in \mathbb{R}^n$,

$$f(y) \leq f(x) \implies f^\circ(x; y - x) \leq 0.$$

We note that if f is pseudoconvex on C then f is quasiconvex on C . Moreover, a quasiconvex function is characterized by the convexity of its lower-level sets (see [16]).

Definition 2.4. Let C be a subset of \mathbb{R}^n and let $\epsilon > 0$. A locally Lipschitz function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be ϵ -pseudoconvex at $x \in C$ if

$$d \in \mathbb{R}^n, x + d \in C, f^\circ(x; d) + \sqrt{\epsilon}\|d\| \geq 0 \implies f(x + d) + \sqrt{\epsilon}\|d\| \geq f(x).$$

The function f is said to be ϵ -pseudoconvex on C if f is ϵ -pseudoconvex at every $x \in C$.

We also note that, in Definition 2.4, if f is supposed further to be regular at x then f is said to be ϵ -semiconvex at x (see [31,41]).

We now recall some concepts of approximate solutions for a single objective optimization problem. Let us consider the following scalar optimization problem.

$$(P) \quad \text{Minimize } f(x) \text{ subject to } g_t(x) \leq 0, t \in T, x \in C,$$

where $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is locally Lipschitz function and the functions $g_t, t \in T$, and C are as above. The feasible set of (P) is denoted by

$$F := \{x \in C \mid g_t(x) \leq 0, t \in T\}.$$

Let $\alpha > 0$. We denote by F^α the α -feasible set of (P),

$$F^\alpha := \{x \in C \mid g_t(x) \leq \sqrt{\alpha}, t \in T\}.$$

The following notions come from [31].

Definition 2.5. Let $\alpha > 0$. A point $z \in F^\alpha$ is said to be

- (i) an almost α -solution of (P) if $f(z) \leq f(x) + \alpha$ for all $x \in F$;
- (ii) an almost α -quasisolution of (P) if $f(z) \leq f(x) + \sqrt{\alpha}\|x - z\|$ for all $x \in F$;
- (iii) an almost regular α -solution of (P) if it is an almost α -solution and an almost α -quasisolution of (P).

The word “almost” is removed if $z \in F$.

We note that the existence theorem for almost approximate solutions of nonconvex programming problems was given in [31,38,41].

The following linear space is proposed firstly in [28] and is used for semi-infinite programming [17]:

$$\mathbb{R}^{(T)} := \{\lambda = (\lambda_t)_{t \in T} \mid \lambda_t = 0 \text{ for all } t \in T \text{ but only finitely many } \lambda_t \neq 0\}.$$

The nonnegative cone of $\mathbb{R}^{(T)}$ is denoted by

$$\mathbb{R}_+^{(T)} = \{\lambda = (\lambda_t)_{t \in T} \in \mathbb{R}^{(T)} \mid \lambda_t \geq 0, t \in T\}.$$

With $\lambda \in \mathbb{R}^{(T)}$, its supporting set

$$T(\lambda) = \{t \in T \mid \lambda_t \neq 0\}$$

is a finite subset of T .

Whenever $T(\lambda) = \emptyset$, we assume $\sum_{t \in T} \lambda_t \vartheta_t = 0$ for any sequence $\{\vartheta_t\}_{t \in T}$. For $\lambda \in \mathbb{R}^{(T)}$, we define

$$\|\lambda\|_1 := \sum_{t \in T(\lambda)} |\lambda_t|.$$

Let $x \in \mathbb{R}^n$. We need the following condition

$$(A) \quad \exists d \in T_C(x) : g_t^\circ(x; d) < 0 \text{ for all } t \in I(x) := \{t \in T \mid g_t(x) = 0\}.$$

We use B^* to denote a closed unit ball in \mathbb{R}^n . According to Theorems 4.1 and 4.2 presented in [41] (where the problem (P) is defined on a Banach space), we derive the following theorems for the case of the involved functions are defined on \mathbb{R}^n and the index set T is compact. The proofs can be omitted.

Theorem 2.6. *Let $\alpha > 0$ and let z be an α -quasisolution for (P). Assume that the condition (A) holds for z . Then there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that*

$$0 \in \partial^c f(z) + \sum_{t \in T} \lambda_t \partial^c g_t(z) + N_C(z) + \sqrt{\alpha} B^*, \quad g_t(z) = 0, \quad \forall t \in T(\lambda).$$

Definition 2.7. A pair $(z, \lambda) \in C \times \mathbb{R}_+^{(T)}$ is called a generalized KKT pair up to $\alpha > 0$ if the following condition holds

$$(2.1) \quad 0 \in \partial^c f(z) + \sum_{t \in T} \lambda_t \partial^c g_t(z) + N_C(x_\alpha) + \sqrt{\alpha} B^*, \quad g_t(z) \geq 0, \quad \forall t \in T(\lambda).$$

It was called a strict pair up to α if (2.1) is satisfied with $g_t(z) > 0$ for all $t \in T(\lambda)$, it is equivalent to $\lambda_t = 0$ if $g_t(z) \leq 0$.

Theorem 2.8. *Let $\alpha > 0$. There exists an almost regular α -solution $z \in F^\alpha$ for (P) and $\lambda \in \mathbb{R}_+^{(T)}$ such that (z, λ) is a strict generalized KKT pair up to α .*

For the problem (MP), its feasible set is also denoted by F . Let $\epsilon = (\epsilon_1, \dots, \epsilon_m) \in \mathbb{R}_{++}^m$ with \mathbb{R}_{++}^m denoting the positive orthant of \mathbb{R}^m . Set

$$\theta := \min_{i \in M} \epsilon_i$$

and

$$F^\theta := \{x \in C \mid g_t(x) \leq \sqrt{\theta}, t \in T\}.$$

The following concepts are due to the ones presented in [30] (see also [38]).

Definition 2.9. A point $z \in F^\theta$ is said to be

- (i) an almost ϵ -Pareto solution of (MP) if there exists no $x \in F$ such that

$$f_i(x) \leq f_i(z) - \epsilon_i, \quad i \in M$$

with at least one strict inequality;

- (ii) an almost ϵ -Pareto quasisolution of (MP) if there exists no $x \in F$ such that

$$f_i(x) + \sqrt{\epsilon_i} \|x - z\| \leq f_i(z), \quad i \in M$$

with at least one strict inequality;

- (iii) an almost regular ϵ -Pareto solution of (MP) if it is an almost ϵ -Pareto solution of (MP) and is an almost ϵ -Pareto quasisolution of (MP).

If $z \in F$ then we have the concepts of ϵ -Pareto solution, ϵ -Pareto quasisolution, and regular ϵ -Pareto solution of (MP), respectively.

Definition 2.10. A point $z \in F^\theta$ is said to be

- (i) an almost weakly ϵ -Pareto solution of (MP) if there exists no $x \in F$ such that

$$f_i(x) < f_i(z) - \epsilon_i, \quad i \in M;$$

- (ii) an almost weakly ϵ -Pareto quasisolution of (MP) if there exists no $x \in F$ such that

$$f_i(x) + \sqrt{\epsilon_i} \|x - z\| < f_i(z), \quad i \in M;$$

- (iii) an almost regular weakly ϵ -Pareto solution of (MP) if it is an almost weakly ϵ -Pareto solution of (MP) and is an almost weakly ϵ -Pareto quasisolution of (MP).

If $z \in F$ then we have the concepts of weakly ϵ -Pareto solution, weakly ϵ -Pareto quasisolution, and regular weakly ϵ -Pareto solution of (MP), respectively.

The criteria of Chankong-Haijmes scheme (see also [4, 27]) applied for (MP) is that: A feasible point z of (MP) is a Pareto solution of (MP) if and only if it is an optimal solution of $(P_j(z))$ for each $j \in M$.

For a problem $(P_j(y))$, $j \in M$, its feasible set is denoted by $F_j(y)$. Set $M^j := M \setminus \{j\}$. The method above is developed for finding approximate Pareto solutions of (MP) as follows. Given $y \in C$, $j \in M$ and $\epsilon_j > 0$, the ϵ_j -feasible set of $(P_j(y))$ is defined by

$$F_j^{\epsilon_j}(y) := \{x \in C \mid f_k(x) \leq f_k(y) + \sqrt{\epsilon_j}, k \in M^j; g_t(x) \leq \sqrt{\epsilon_j}, t \in T\}.$$

From the criteria of Chankong-Haijmes scheme discussed above, it is easy to derive the following lemmas.

Lemma 2.11. *Let $\epsilon \in \mathbb{R}_{++}^m$.*

- (i) *A point z is an almost ϵ -Pareto solution of (MP) if and only if z is an almost ϵ_j -solution of $(P_j(z))$ for each $j \in M$.*
- (ii) *A point z is an almost ϵ -Pareto quasisolution of (MP) if and only if z is an almost ϵ_j -quasisolution of $(P_j(z))$ for each $j \in M$.*
- (iii) *A point z is an almost regular ϵ -Pareto solution of (MP) if and only if z is an almost regular ϵ_j -solution of $(P_j(z))$ for each $j \in M$.*

Lemma 2.12. *Let $\epsilon \in \mathbb{R}_{++}^m$ and let $j \in M$ be such that $\epsilon_j = \min_{i \in M} \epsilon_i$.*

- (i) *A point z is an almost weakly ϵ -Pareto solution of (MP) if and only if z is an almost ϵ_j -solution of $(P_j(z))$.*
- (ii) *A point z is an almost weakly ϵ -Pareto quasisolution of (MP) if and only if z is an almost ϵ_j -quasisolution of $(P_j(z))$.*
- (iii) *A point z is an almost regular weakly ϵ -Pareto solution of (MP) if and only if z is an almost regular ϵ_j -solution of $(P_j(z))$.*

Remark 2.13. In Lemmas 2.11 and 2.12, the conclusions are valid if we remove the word “almost”.

Example 2.14. The following example illustrates relations between approximate solutions of (MP) and approximate solutions of scalar problems corresponding to Chankong-Haimes method

$$(MP1) \quad \text{Min } f(x) = (x, x^2 - x) \text{ subject to } g(x) = tx^2 - x \leq 0, t \in [0, 1], x \in [-1, 1].$$

Let $f_1(x) = x$ and $f_2(x) = x^2 - x$. Observe that the feasible set of (MP1) is $[0, 1]$. Given $\epsilon = (1/2, 1/2)$, it is clear that $z = 1/2$ is ϵ -Pareto solution of (MP1).

The two scalar problems corresponding to (MP1) in Chankong-Haimes method are formulated as follows:

$$(P_1(z))$$

$$\text{Min } f_1(x) = x \text{ subject to } x^2 - x \leq f_2(z) = -1/4, tx^2 - x \leq 0, t \in [0, 1], x \in [-1, 1]$$

and

$$(P_2(z))$$

$$\text{Min } f_2(x) = x^2 - x \text{ subject to } x \leq f_1(z) = 1/2, tx^2 - x \leq 0, t \in [0, 1], x \in [-1, 1].$$

For the problem $(P_1(z))$, with a simple computation, its feasible set is $F_1 = \{1/2\}$. On the other hand, we also have $\inf (P_1(z)) = 0$. Moreover, we can see $f(z) \leq \inf (P_1(z)) + \epsilon_1$.

Hence, $z = 1/2$ is an ϵ_1 -solution of $(P_1(z))$. For the problem $(P_2(z))$, its feasible set is $F_2 = \{x \mid 0 \leq x \leq 1/2\}$ and its optimal value is $\inf (P_2(z)) = -1/4$. We can see that $f(z) \leq \inf (P_2(z)) + \epsilon_2$. Thus, $z = 1/2$ is also an ϵ_2 -solution of $(P_2(z))$.

3. ϵ -optimality conditions

We now establish ϵ -optimality conditions for (MP). The following condition is associated to the problem $(P_j(z))$.

Let $x \in \mathbb{R}^n$, $I(x) = \{t \in T \mid g_t(x) = 0\}$, $H_j(x) = \{k \in M^j \mid f_k(x) = f_k(z)\}$, $M^j := M \setminus \{j\}$ and $\bar{T}(x) = I(x) \cup H_j(x)$.

$$(A_j) \quad \exists d \in T_C(x) : \begin{cases} g_t^0(x; d) < 0 & \text{for all } t \in I(x), \\ f_k^0(x; d) < 0 & \text{for all } k \in H_j(x). \end{cases}$$

By applying Theorem 2.6 for the problem $(P_j(z))$, we can obtain the following result. The proof is omitted.

Proposition 3.1. *Let $\epsilon_j > 0$ and let z be an ϵ_j -quasisolution for $(P_j(z))$. Assume that the condition (A_j) holds for z . Then there exist $\mu_k \geq 0$, $k \in M^j$ and $\lambda \in \mathbb{R}_+^{(T)}$ such that*

$$(3.1) \quad \begin{aligned} 0 \in \partial^c f_j(z) + \sum_{k \in M^j} \mu_k \partial^c f_k(z) + \sum_{t \in T} \lambda_t \partial^c g_t(z) + N_C(z) + \sqrt{\epsilon_j} B^*, \\ g_t(z) = 0, \quad \forall t \in T(\lambda). \end{aligned}$$

We need the following proposition. The proof is omitted.

Proposition 3.2. *Let $\epsilon_j > 0$ and let $z \in F_j(z)$. Assume that the function f_j is ϵ_j -pseudoconvex, the functions f_k and g_t are quasiconvex where $k \in M^j$ and $t \in T$. If there exist $\mu_k \geq 0$, $k \in M^j$ and $\lambda \in \mathbb{R}_+^{(T)}$ such that*

$$\begin{aligned} 0 \in \partial^c f_j(z) + \sum_{k \in M^j} \mu_k \partial^c f_k(z) + \sum_{t \in T} \lambda_t \partial^c g_t(z) + N_C(z) + \sqrt{\epsilon_j} B^*, \\ g_t(z) = 0, \quad \forall t \in T(\lambda), \end{aligned}$$

then z is an ϵ_j -quasisolution for $(P_j(z))$.

In the following theorem, a necessary condition for an ϵ -Pareto quasisolution of (MP) can be obtained if there exists $j \in M$ such that the condition (A_j) holds.

Theorem 3.3. *Let $\epsilon \in \mathbb{R}_+^m$. Let $z \in F$ be an ϵ -Pareto quasisolution of (MP). If there exists $j \in M$ such that the condition (A_j) holds for z then there exist $\bar{\mu}_k \geq 0$, $k \in M$, $\sum_{k \in M} \bar{\mu}_k = 1$ and $\bar{\lambda} \in \mathbb{R}_+^{(T)}$ such that*

$$0 \in \sum_{k \in M} \bar{\mu}_k \partial^c f_k(z) + \sum_{t \in T} \bar{\lambda}_t \partial^c g_t(z) + N_C(z) + \sqrt{\epsilon_j} B^*, \quad g_t(z) = 0, \quad \forall t \in T(\bar{\lambda}).$$

Proof. Note that if z is an ϵ -Pareto quasisolution of (MP) then, by Lemma 2.11 and Remark 2.13, z is an ϵ_j -quasisolution ($P_j(z)$). Since the condition (\mathcal{A}_j) holds for z , by Proposition 3.1, there exist $\mu_k \geq 0$, $k \in M^j$ and $\lambda \in \mathbb{R}_+^{(T)}$ such that

$$0 \in \partial^c f_j(z) + \sum_{k \in M^j} \mu_k \partial^c f_k(z) + \sum_{t \in T} \lambda_t \partial^c g_t(z) + N_C(z) + \sqrt{\epsilon_j} B^*,$$

$$g_t(z) = 0, \quad \forall t \in T(\lambda),$$

where B^* is an unit ball in \mathbb{R}^n . This implies that

(3.2)

$$0 \in \frac{1}{1 + \sum_{k \in M^j} \mu_k} \left(\partial^c f_j(z) + \sum_{k \in M^j} \mu_k \partial^c f_k(z) + \sum_{t \in T} \lambda_t \partial^c g_t(z) + N_C(z) + \sqrt{\epsilon_j} B^* \right),$$

and $g_t(z) = 0, \forall t \in T(\lambda)$. Set

$$\bar{\mu}_j = \frac{1}{1 + \sum_{k \in M^j} \mu_k}, \quad \bar{\mu}_k = \frac{\mu_k}{1 + \sum_{k \in M^j} \mu_k}, \quad \bar{\lambda}_t = \frac{\lambda_t}{1 + \sum_{k \in M^j} \mu_k}.$$

Note that

$$\frac{1}{1 + \sum_{k \in M^j} \mu_k} N_C(z) \subset N_C(z) \quad \text{and} \quad \frac{1}{1 + \sum_{k \in M^j} \mu_k} \sqrt{\epsilon_j} B^* \subset \sqrt{\epsilon_j} B^*.$$

From (3.2), we deduce that

$$0 \in \sum_{k \in M} \bar{\mu}_k \partial^c f_k(z) + \sum_{t \in T} \bar{\lambda}_t \partial^c g_t(z) + N_C(z) + \sqrt{\epsilon_j} B^*, \quad g_t(z) = 0, \quad \forall t \in T(\bar{\lambda}).$$

It is easy to check that $\sum_{k \in M} \bar{\mu}_k = 1$. □

The reader is referred to [41], where a strict KKT condition is proposed corresponding to an almost approximate solution of a single objective nonconvex optimization problem with an infinite number of constraints, and is referred to [11,29] where various type of constraint qualifications are required to study KKT conditions for multiobjective optimization problems with inequality constraints.

We now prove that there exists an almost weakly ϵ -Pareto quasisolution for the problem (MP) without assuming any constraint qualification. To start with, firstly, we give the definition of a so-called generalized j -KKT pair for (MP). It is modified from the one in [41]. Then we prove that there exists an almost regular ϵ_j -solution for ($P_j(z)$).

Definition 3.4. Let $\epsilon \in \mathbb{R}_{++}^m$. A pair $(z, \lambda) \in C \times \mathbb{R}_+^{(T)}$ is said to be a generalized j -KKT pair up to ϵ_j if the following condition is satisfied

(3.3)

$$0 \in \partial^c f_j(z) + \sum_{t \in T} \lambda_t \partial^c g_t(z) + N_C(z) + \sqrt{\epsilon_j} B^*, \quad g_t(z) \geq 0, \quad \forall t \in T(\lambda).$$

It is called strict if (3.3) holds with $g_t(z) > 0$ for all $t \in T(\lambda)$, equivalent to $\lambda_t = 0$ if $g_t(z) \leq 0$.

Lemma 3.5. *Let $\epsilon \in \mathbb{R}_{++}^m$. For each $j \in M$, there exists an almost regular ϵ_j -solution z for $(P_j(z))$ and $\bar{\lambda} \in \mathbb{R}_+^{(T)}$ such that the pair $(z, \bar{\lambda})$ satisfies the condition (3.3) with $g_t(z) > 0$ for all $t \in T(\bar{\lambda})$.*

Proof. Let us consider the function G_t defined by

$$G_t = \begin{cases} f_t - f_t(z) & \text{if } t \in M^j, \\ g_t & \text{if } t \in T \end{cases}$$

and $\bar{T} = T \cup M^j$. For $\epsilon_j > 0$, applying Theorem 2.8 to the problem

$$\min\{f_j(x) \mid x \in C, G_t(x) \leq 0, t \in \bar{T}\}.$$

There exists an almost regular ϵ_j -solution $z \in F_j^{\epsilon_j}(z)$ and $\bar{\lambda} \in \mathbb{R}_+^{(\bar{T})}$ such that $(z, \bar{\lambda})$ satisfies strictly the generalized KKT pair up to ϵ_j , i.e.,

$$(3.4) \quad \begin{aligned} 0 \in \partial^c f_j(z) + \sum_{t \in M^j} \bar{\lambda}_t \partial^c G_t(z) + \sum_{t \in T} \bar{\lambda}_t \partial^c G_t(z) + N_C(z) + \sqrt{\epsilon_j} B^*, \\ G_t(z) > 0, \quad t \in T(\bar{\lambda}). \end{aligned}$$

Since $G_t(z) = 0$ for all $t \in M^j$ and $(z, \bar{\lambda})$ satisfies strictly a generalized KKT pair up to ϵ_j , by Definition 2.7, we get $\bar{\lambda}_t = 0$ for all $t \in M^j$. This and (3.4) imply

$$(3.5) \quad 0 \in \partial^c f_j(z) + \sum_{t \in T} \bar{\lambda}_t \partial^c g_t(z) + N_C(z) + \sqrt{\epsilon_j} B^*, \quad g_t(z) > 0, \quad \forall t \in T(\bar{\lambda}).$$

The desired result is proved. □

Corollary 3.6. *Let $\epsilon \in \mathbb{R}_{++}^m$ and let $j \in M$ be such that $\epsilon_j = \min\{\epsilon_i, i \in M\}$. Then there exist an almost regular weakly ϵ -Pareto solution z for (MP) and $\bar{\lambda} \in \mathbb{R}_+^{(T)}$ such that $(z, \bar{\lambda})$ satisfies strictly a generalized j -KKT pair up to ϵ_j .*

Proof. Indeed, by combining Lemmas 2.12 and 3.5, one can guarantee that there exists an almost regular weakly ϵ -Pareto solution z for (MP). □

From the corollary above, we propose a sufficient condition for an almost weakly ϵ -Pareto quasisolution for (MP) as follows.

Theorem 3.7. *Let $\epsilon \in \mathbb{R}_{++}^m$ and let $j \in M$ be such that $\epsilon_j = \min\{\epsilon_k, k \in M\}$. Suppose that the function f_j is ϵ_j -pseudoconvex, the functions g_t are quasiconvex for all $t \in T$, and the pair $(z, \lambda) \in C \times \mathbb{R}_+^{(T)}$ satisfies the generalized j -KKT condition up to ϵ_j given by (3.3).*

- (i) *If $z \in F_j^{\epsilon_j}(z)$, then z is an almost weakly ϵ -Pareto quasisolution of (MP).*
- (ii) *If $z \in F_j(z)$, then z is a weakly ϵ -Pareto quasisolution of (MP).*

Proof. (i) Let $(z, \lambda) \in C \times \mathbb{R}_+^{(T)}$ be such that the condition (3.3) is satisfied. Then there exist $u_j \in \partial^c f_j(z)$, $v_t \in \partial^c g_t(z)$, $t \in T$, $w \in N_C(z)$ and $b \in B^*$ such that

$$u_j + \sum_{t \in T} \lambda_t v_t + w + \sqrt{\epsilon_j} b = 0.$$

Hence,

$$u_j(x - z) + \sum_{t \in T} \lambda_t v_t(x - z) + \sqrt{\epsilon_j} \|x - z\| = -w(x - z) \geq 0, \quad \forall x \in C.$$

On the other hand, since $g_t(z) \geq 0$ for all $t \in T(\lambda)$, and g_t are quasiconvex functions, we can deduce that $v_t(x - z) \leq 0$ for all $t \in T$ and for all $x \in F$. Consequently,

$$(3.6) \quad u_j(x - z) + \sqrt{\epsilon_j} \|x - z\| \geq 0, \quad \forall x \in F.$$

If $z \in F_j^{\epsilon_j}(z)$ then $z \in F^\theta$. We claim that z is an almost weakly ϵ -Pareto quasisolution for (MP). On the contrary, if z is not an almost weakly ϵ -Pareto quasisolution of (MP), then there exists $x \in F$ such that

$$f_i(x) + \sqrt{\epsilon_i} \|x - z\| < f_i(z), \quad \forall i \in M.$$

This implies that

$$f_j(x) + \sqrt{\epsilon_j} \|x - z\| < f_j(z).$$

Since f_j is ϵ_j -pseudoconvex, the inequality above implies

$$f_j^\circ(z; x - z) + \sqrt{\epsilon_j} \|x - z\| < 0.$$

Hence,

$$u_j(x - z) + \sqrt{\epsilon_j} \|x - z\| < 0, \quad \forall u_j \in \partial^c f_j(z),$$

a contradiction to (3.6).

(ii) When $z \in F_j(z)$, the conclusion can be derived similarly. □

4. ϵ -duality theorems

Let us define the vector Lagrange function associated to (MP) by

$$\mathcal{L}(y, \lambda) = (L_1(y, \lambda), \dots, L_m(y, \lambda))$$

where $L_j(y, \lambda) = f_j(y) + \sum_{t \in T} \lambda_t g_t(y)$. For $\epsilon \in \mathbb{R}_+^m$, the approximate dual problem of (MP) in Wolfe type is formulated as follows:

$$(MD) \quad \begin{aligned} \text{Max } \mathcal{L}(y, \lambda), \quad & 0 \in \sum_{j \in M} \tau_j \partial^c f_j(y) + \sum_{t \in T} \lambda_t \partial^c g_t(y) + N_C(y) + \sum_{j \in M} \tau_j \sqrt{\epsilon_j} B^*, \\ \tau_j \geq 0, \quad & \sum_{j \in M} \tau_j = 1, \quad y \in C, \quad \lambda \in \mathbb{R}_+^{(T)}. \end{aligned}$$

Definition 4.1. A feasible point $(\bar{z}, \bar{\lambda}, \bar{\tau})$ of (MD) is said to be

- (i) an ϵ -Pareto solution of (MD) if there exists no feasible point (y, λ, τ) of (MD) such that

$$L_i(\bar{z}, \bar{\lambda}) \leq L_i(y, \lambda) - \epsilon_i, \quad i \in M$$

with at least one strict inequality;

- (ii) an ϵ -Pareto quasisolution of (MD) if there exists no feasible point (y, λ, τ) of (MD) such that

$$L_i(y, \lambda) - \sqrt{\epsilon_i} \|y - \bar{z}\| - \sqrt{\epsilon_i} \|\bar{\lambda} - \lambda\|_1 \geq L_i(\bar{z}, \bar{\lambda}), \quad i \in M$$

with at least one strict inequality;

- (iii) a regular ϵ -Pareto solution of (MD) if it is an ϵ -Pareto solution of (MD) and is an ϵ -Pareto quasisolution of (MD).

In the spirit of Definition 2.10, the corresponding notions of weakly ϵ -Pareto solutions of (MD) are obtained.

To investigate duality relationships between (MP) and (MD), following the diagram introduced in the first section, we need a pair of primal-dual scalar problems (P_j) and (D_j) such that (P_j) is associated to (MP) and (D_j) is associated to (MD). Let us take a glance at Lemma 3.5. If we focus on almost ϵ_j -quasisolutions of (P_j) then the obtained generalized ϵ_j -KKT conditions do not contain the constraints f_k for $k \in M^j$ (compare (3.1) with (3.5)). Motivated by this observation, we can propose a formulation for an approximate dual problem (D_j) in Wolfe type with a small modification. The small modification respect to (D_j) is that, it has no relation to the constraint functions f_k , $k \in M^j$, of (P_j) . In this case, (D_j) will be proved to be compatible with (MD). It is formulated as follows:

$$(D_j) \quad \begin{aligned} & \text{Max } L_j(y, \lambda) = f_j(y) + \sum_{t \in T} \lambda_t g_t(y), \\ & 0 \in \partial^c f_j(y) + \sum_{t \in T} \lambda_t \partial^c g_t(y) + N_C(y) + \sqrt{\epsilon_j} B^*, \quad (y, \lambda) \in C \times \mathbb{R}_+^{(T)}. \end{aligned}$$

4.1. Duality relationships between $(P_j(y))$ and (D_j)

Let us denote by F_j^D the feasible set of (D_j) .

Definition 4.2. Let $(\bar{z}, \bar{\lambda})$ be a feasible solution of (D_j) . The point $(\bar{z}, \bar{\lambda})$ is called

- (i) an ϵ_j -solution for (D_j) if

$$L_j(\bar{z}, \bar{\lambda}) \geq L_j(y, \lambda) - \epsilon_j, \quad \forall (y, \lambda) \in F_j^D;$$

(ii) an ϵ_j -quasisolution for (D_j) if

$$L_j(\bar{z}, \bar{\lambda}) \geq L_j(y, \lambda) - \sqrt{\epsilon_j} \|\bar{z} - y\| - \sqrt{\epsilon_j} \|\bar{\lambda} - \lambda\|_1, \quad \forall (y, \lambda) \in F_j^D;$$

(iii) a regular ϵ_j -solution for (D_j) if it is both an ϵ_j -solution and an ϵ_j -quasisolution.

The following theorem describes the approximate weak duality between $(P_j(y))$ and (D_j) .

Theorem 4.3. *Let x and (y, λ) be the feasible points of $(P_j(y))$ and (D_j) , respectively. Suppose that $L_j(\cdot, \lambda)$ is ϵ_j -pseudoconvex and $f_j, g_t, t \in T$, are regular on C . Then,*

$$f_j(x) \geq L_j(y, \lambda) - \sqrt{\epsilon_j} \|x - y\|.$$

Proof. Since (y, λ) is a feasible solution of (D_j) , there exist $u \in \partial^c f_j(y), v_t \in \partial^c g_t(y), w \in N_C(y)$ and $b \in B^*$ such that

$$0 = u + \sum_{t \in T} \lambda_t v_t + w + \sqrt{\epsilon_j} b.$$

Since $w(x - y) \leq 0$ for all $x \in F$, it is easy to deduce that

$$(4.1) \quad u(x - y) + \sum_{t \in T} \lambda_t v_t(x - y) + \sqrt{\epsilon_j} \|x - y\| \geq 0, \quad \forall x \in C.$$

Since $f_j, g_t, t \in T$, are regular on C and $L_j(\cdot, \lambda)$ is ϵ_j -pseudoconvex, from (4.1), we obtain

$$L_j(x, \lambda) + \sqrt{\epsilon_j} \|x - y\| \geq L_j(y, \lambda), \quad \forall x \in C.$$

On the other hand,

$$L_j(x, \lambda) = f_j(x) + \sum_{t \in T} \lambda_t g_t(x) \leq f_j(x), \quad \forall x \in F.$$

The desired result follows. □

Theorem 4.4. *For problem (D_j) , suppose that the $L_j(\cdot, \lambda)$ is ϵ_j -pseudoconvex and $f_j, g_t, t \in T$, are regular on C . Then, there exists an ϵ_j -quasisolution for (D_j) .*

Proof. By Lemma 3.5, there exists an almost ϵ_j -quasisolution \bar{z} for $(P_j(\bar{z}))$ and $\bar{\lambda} \in \mathbb{R}_+^{(T)}$ such that $(\bar{z}, \bar{\lambda})$ is a strict generalized j -KKT pair up to ϵ_j , i.e.,

$$(4.2) \quad 0 \in \partial^c f_j(\bar{z}) + \sum_{t \in T} \bar{\lambda}_t \partial^c g_t(\bar{z}) + N_C(\bar{z}) + \sqrt{\epsilon_j} B^*, \quad g_t(\bar{z}) > 0, \quad \forall t \in T(\bar{\lambda}).$$

Hence, $(\bar{z}, \bar{\lambda})$ is a feasible point of (D_j) . It needs to prove that $(\bar{z}, \bar{\lambda})$ is an ϵ_j -quasisolution for (D_j) , i.e., to prove that

$$L_j(\bar{z}, \bar{\lambda}) \geq L_j(y, \lambda) - \sqrt{\epsilon_j} \|\bar{z} - y\| - \sqrt{\epsilon_j} \|\bar{\lambda} - \lambda\|_1, \quad \forall (y, \lambda) \in F_j^D.$$

We claim that

$$(4.3) \quad L_j(\bar{z}, \bar{\lambda}) \geq L_j(\bar{z}, \lambda) - \sqrt{\epsilon_j} \|\bar{\lambda} - \lambda\|_1, \quad \forall \lambda \in \mathbb{R}_+^{(T)},$$

i.e.,

$$\sum_{t \in T} (\bar{\lambda}_t - \lambda_t) g_t(\bar{z}) \geq -\sqrt{\epsilon_j} \sum_{t \in T} |\bar{\lambda}_t - \lambda_t|.$$

Indeed, set $T = T(\bar{\lambda}) \cup (T \setminus T(\bar{\lambda}))$. If $t \in T(\bar{\lambda})$ then $g_t(\bar{z}) > 0$. Hence,

$$(\bar{\lambda}_t - \lambda_t) g_t(\bar{z}) \geq -|\bar{\lambda}_t - \lambda_t| g_t(\bar{z}).$$

Since $\bar{z} \in F_j^{\epsilon_j}(\bar{z})$, $g_t(\bar{z}) \leq \sqrt{\epsilon_j}$. We get

$$(4.4) \quad (\bar{\lambda}_t - \lambda_t) g_t(\bar{z}) \geq -|\bar{\lambda}_t - \lambda_t| g_t(\bar{z}) \geq -|\bar{\lambda}_t - \lambda_t| \sqrt{\epsilon_j}.$$

If $t \in T \setminus T(\bar{\lambda})$ then, by the strict generalized j -KKT pair up to ϵ_j (4.2), we have $\bar{\lambda}_t = 0$ and $g_t(\bar{z}) \leq 0$. Hence,

$$(\bar{\lambda}_t - \lambda_t) g_t(\bar{z}) = -\lambda_t g_t(\bar{z}) \geq 0.$$

Thus,

$$(4.5) \quad (\bar{\lambda}_t - \lambda_t) g_t(\bar{z}) \geq -|\bar{\lambda}_t - \lambda_t| \sqrt{\epsilon_j}.$$

From (4.4) and (4.5) we can deduce (4.3). On the other hand, for a feasible point (y, λ) of (D_j) , we get

$$0 \in \partial^c f_j(y) + \sum_{t \in T} \lambda_t \partial^c g_t(y) + N_C(y) + \sqrt{\epsilon_j} B^*.$$

Using an argument similar to the one in the proof of Theorem 4.3, we get

$$L_j(x, \lambda) + \sqrt{\epsilon_j} \|x - y\| \geq L_j(y, \lambda), \quad \forall x \in C.$$

Hence,

$$(4.6) \quad L_j(\bar{z}, \lambda) \geq L_j(y, \lambda) - \sqrt{\epsilon_j} \|\bar{z} - y\|.$$

The desired conclusion follows by combining (4.6) and (4.3). □

Remark 4.5. Suppose that the $L_j(\cdot, \lambda)$ is ϵ_j -pseudoconvex and $f_j, g_t, t \in T$, are regular on C . If \bar{z} is an almost ϵ_j -quasisolution for $(P_j(\bar{z}))$ and $\bar{\lambda} \in \mathbb{R}_+^{(T)}$ such that $(\bar{z}, \bar{\lambda})$ is a generalized j -KKT pair up to ϵ_j , then $(\bar{z}, \bar{\lambda})$ is an ϵ_j -quasisolution of (D_j) .

Theorem 4.6. *Let $(\bar{z}, \bar{\lambda})$ be a feasible solution of (D_j) . Suppose that f_j is ϵ_j -pseudoconvex, g_t are quasiconvex for $t \in T$. Suppose further that $g_t(x) \leq g(z), t \in T$, for all $x \in F$ and f_j, g_t are regular on C for $t \in T$.*

- (i) If $\bar{z} \in F_j(\bar{z})$, then \bar{z} is an ϵ_j -quasisolution of $(P_j(\bar{z}))$.
- (ii) If $\bar{z} \in F_j^{\epsilon_j}(\bar{z})$, then \bar{z} is an almost ϵ_j -quasisolution of $(P_j(\bar{z}))$.

Proof. Let $(\bar{z}, \bar{\lambda})$ be a feasible solution of (D_j) . We get

$$0 \in \partial^c f_j(\bar{z}) + \sum_{t \in T} \bar{\lambda}_t \partial^c g_t(\bar{z}) + N_C(\bar{z}) + \sqrt{\epsilon_j} B^*.$$

Then there exist $u_j \in \partial^c f_j(\bar{z})$, $v_t \in \partial^c g_t(\bar{z})$, $t \in T$, $w \in N_C(\bar{z})$ and $b \in B^*$ such that

$$u_j(x - \bar{z}) + \sum_{t \in T} \bar{\lambda}_t v_t(x - \bar{z}) + \sqrt{\epsilon_j} \|x - \bar{z}\| = -w(x - \bar{z}) \geq 0, \quad \forall x \in C.$$

Since g_t are quasiconvex and $g_t(x) \leq g_t(\bar{z})$ for all $t \in T$ and for all $x \in F$, we deduce that

$$u_j(x - \bar{z}) + \sqrt{\epsilon_j} \|x - \bar{z}\| \geq 0, \quad \forall x \in F.$$

- (i) By the property of ϵ_j -pseudoconvexity of f_j , we get

$$f(z) \leq f(x) + \sqrt{\alpha} \|x - \bar{z}\|$$

for all $x \in F$. If $\bar{z} \in F_j(\bar{z})$ then \bar{z} is an ϵ_j -quasisolution of $(P_j(\bar{z}))$.

- (ii) If $\bar{z} \in F_j^{\epsilon_j}(\bar{z})$ then the conclusion (ii) follows. □

4.2. Relationships between (D_j) and (MD)

Proposition 4.7. *Suppose that the function $L_j(\cdot, \lambda)$ is ϵ_j -pseudoconvex. Let $\bar{\tau} \in \mathbb{R}_+^m$ be such that $\bar{\tau}_j = 1$ and $\bar{\tau}_k = 0$ for $k \neq j$.*

- (i) *If $(\bar{z}, \bar{\lambda})$ is an ϵ_j -solution of (D_j) then $(\bar{z}, \bar{\lambda}, \bar{\tau})$ is a weakly ϵ -Pareto solution of (MD).*
- (ii) *If $(\bar{z}, \bar{\lambda})$ is an ϵ_j -quasisolution of (D_j) then $(\bar{z}, \bar{\lambda}, \bar{\tau})$ is a weakly ϵ -Pareto quasisolution of (MD).*
- (iii) *If $(\bar{z}, \bar{\lambda})$ is a regular ϵ_j -solution of (D_j) then $(\bar{z}, \bar{\lambda}, \bar{\tau})$ is a regular weakly ϵ -Pareto quasisolution of (MD).*

Proof. (i) Let $(\bar{z}, \bar{\lambda})$ be an ϵ_j -solution of (D_j) . Hence,

$$0 \in \partial^c f_j(\bar{z}) + \sum_{t \in T} \bar{\lambda}_t \partial^c g_t(\bar{z}) + N_C(\bar{z}) + \sqrt{\epsilon_j} B^*$$

and

$$(4.7) \quad L_j(z, \bar{\lambda}) \geq L_j(y, \lambda) - \epsilon_j$$

for all feasible point (y, λ) of (D_j) . Let $\bar{\tau} \in \mathbb{R}_+^m$ be such that $\bar{\tau}_k = 0$ if $k \neq j$ and $\bar{\tau}_k = 1$ for $k = j$. Then $(\bar{z}, \bar{\lambda}, \bar{\tau})$ is a feasible point of (MD). We claim that $(\bar{z}, \bar{\lambda}, \bar{\tau})$ is a weakly ϵ -Pareto solution for (MD). On the contrary, if

$$L_k(y, \lambda) - \epsilon_k > L_k(\bar{z}, \bar{\lambda}), \quad \forall k \in M$$

holds for some feasible point (y, λ, τ) of (MD), then, this inequality contradicts (4.7).

(ii) Let $(\bar{z}, \bar{\lambda})$ be a quasi ϵ_j -solution of (D_j) . Hence,

$$0 \in \partial^c f_j(\bar{z}) + \sum_{t \in T} \bar{\lambda}_t \partial^c g_t(\bar{z}) + N_C(\bar{z}) + \sqrt{\epsilon_j} B^*$$

and

$$L_j(\bar{z}, \bar{\lambda}) \geq L_j(y, \lambda) - \sqrt{\epsilon_j} \|\bar{z} - y\| - \sqrt{\epsilon_j} \|\bar{\lambda} - \lambda\|_1, \quad \forall (y, \lambda) \in F_j^D.$$

Let $\bar{\tau} \in \mathbb{R}_+^m$ be such that $\bar{\tau}_k = 0$ if $k \neq j$ and $\bar{\tau}_k = 1$ for $k = j$. Then $(\bar{z}, \bar{\lambda}, \bar{\tau})$ is a feasible point of (MD). If $(\bar{z}, \bar{\lambda}, \bar{\tau})$ is not a weakly ϵ -Pareto quasisolution of (MD) then there exists a feasible point (x, λ, τ) of (MD) such that, for any $i \in M$,

$$L_i(x, \lambda) - \sqrt{\epsilon_i} \|x - \bar{z}\| - \sqrt{\epsilon_i} \|\bar{\lambda} - \lambda\|_1 > L_i(\bar{z}, \bar{\lambda}).$$

Using an argument similar to the one above, we can easily derive a contradiction.

(iii) The conclusion can be derived by combining (i) and (ii). □

Proposition 4.8. *Suppose that the functions $L_j(\cdot, \lambda)$ are ϵ_j -pseudoconvex for $j \in M$. Let $\bar{\tau} \in \mathbb{R}_{++}^m$ be such that $\sum_{j \in M} \bar{\tau}_j = 1$.*

- (i) *If $(\bar{z}, \bar{\lambda})$ is an ϵ_j -solution of (D_j) for each $j \in M$ then $(\bar{z}, \bar{\lambda}, \bar{\tau})$ is an ϵ -Pareto solution of (MD).*
- (ii) *If $(\bar{z}, \bar{\lambda})$ is an ϵ_j -quasisolution of (D_j) for each $j \in M$ then $(\bar{z}, \bar{\lambda}, \bar{\tau})$ is an ϵ -Pareto quasisolution of (MD).*
- (iii) *If $(\bar{z}, \bar{\lambda})$ is a regular ϵ_j -solution of (D_j) for each $j \in M$ then $(\bar{z}, \bar{\lambda}, \bar{\tau})$ is a regular ϵ -Pareto solution of (MD).*

Proof. (i) Suppose that $(\bar{z}, \bar{\lambda})$ is an ϵ_j -solution of (D_j) for each $j \in M$. Then,

$$0 \in \partial^c f_j(\bar{z}) + \sum_{t \in T} \bar{\lambda}_t \partial^c g_t(\bar{z}) + N_C(\bar{z}) + \sqrt{\epsilon_j} B^*, \quad j \in M$$

and

$$(4.8) \quad L_j(\bar{z}, \bar{\lambda}) \geq L_j(y, \lambda) - \epsilon_j, \quad \forall (y, \lambda) \in F_j^D.$$

Hence, there exist $u_j \in \partial^c f_j(\bar{z})$, $v_t^j \in \partial^c g_t(\bar{z})$, $t \in T$, $w^j \in N_C(\bar{z})$, $b^j \in B^*$ such that

$$0 = u^j + \sum_{t \in T} \bar{\lambda}_t v_t^j + w^j + \sqrt{\epsilon_j} b^j, \quad j \in M.$$

For $\bar{\tau} \in \mathbb{R}_{++}^m$ and $\sum_{j \in M} \bar{\tau}_j = 1$, from each equality above, we get

$$0 = \bar{\tau}_j u^j + \sum_{t \in T} \bar{\lambda}_t \bar{\tau}_j v_t^j + \bar{\tau}_j w^j + \bar{\tau}_j \sqrt{\epsilon_j} b^j, \quad j \in M.$$

Adding the equalities above side by side, we obtain

$$(4.9) \quad 0 = \sum_{j \in M} \bar{\tau}_j u^j + \sum_{j \in M} \sum_{t \in T} \bar{\lambda}_t \bar{\tau}_j v_t^j + \sum_{j \in M} \bar{\tau}_j w^j + \sum_{j \in M} \bar{\tau}_j \sqrt{\epsilon_j} b^j.$$

Since $N_C(\bar{z})$ and $\partial^c g_t(\bar{z})$ are convex, there exists $w \in N_C(\bar{z})$ such that $w = \sum_{j \in M} \bar{\tau}_j w^j$ and there exists $v_t \in \partial^c g_t(\bar{z})$ such that $v_t = \sum_{j \in M} \bar{\tau}_j v_t^j$. The equality (4.9) becomes

$$0 = \sum_{j \in M} \bar{\tau}_j u^j + \sum_{t \in T} \bar{\lambda}_t v_t + w + \sum_{j \in M} \bar{\tau}_j \sqrt{\epsilon_j} b^j,$$

i.e.,

$$0 \in \sum_{j \in M} \bar{\tau}_j \partial^c f_j(\bar{z}) + \sum_{t \in T} \bar{\lambda}_t \partial^c g_t(\bar{z}) + N_C(\bar{z}) + \sum_{j \in M} \bar{\tau}_j \sqrt{\epsilon_j} B^*.$$

Thus, $(\bar{z}, \bar{\lambda}, \bar{\tau})$ is a feasible solution of (MD). If $(\bar{z}, \bar{\lambda}, \bar{\tau})$ is not an ϵ -Pareto solution of (MD) then there exists a feasible point (y, λ, τ) of (MD) such that

$$\mathcal{L}(y, \lambda) - \epsilon \geq \mathcal{L}(\bar{z}, \bar{\lambda})$$

and $L_k(y, \lambda) - \epsilon_k > L_k(\bar{z}, \bar{\lambda})$ with some $k \in M$, a contradiction to (4.8). This guarantees that $(\bar{z}, \bar{\lambda}, \bar{\tau})$ is an ϵ -Pareto solution of (MD).

(ii) Suppose that $(\bar{z}, \bar{\lambda})$ is an ϵ_j -quasisolution of (D_j) for each $j \in M$. Then,

$$0 \in \partial^c f_j(\bar{z}) + \sum_{t \in T} \bar{\lambda}_t \partial^c g_t(\bar{z}) + N_C(\bar{z}) + \sqrt{\epsilon_j} B^*,$$

and for all feasible point (y, λ) of (D_j) we get

$$(4.10) \quad L_j(\bar{z}, \bar{\lambda}) \geq L_j(y, \lambda) - \sqrt{\epsilon_j} \|\bar{z} - y\| - \sqrt{\epsilon_j} \|\bar{\lambda} - \lambda\|_1, \quad \forall (y, \lambda) \in F_j^D, \forall i \in M.$$

Using an argument similar to the one as above, we get $(\bar{z}, \bar{\lambda}, \bar{\tau})$ is a feasible solution of (MD). If $(\bar{z}, \bar{\lambda}, \bar{\tau})$ is not an ϵ -Pareto quasisolution of (MD) then there exists a feasible point (y, λ, τ) of (MD) such that

$$L_i(y, \lambda) - \sqrt{\epsilon_i} \|x - \bar{z}\| - \sqrt{\epsilon_i} \|\bar{\lambda} - \lambda\|_1 \geq L_i(\bar{z}, \bar{\lambda}), \quad 1 \leq i \leq m,$$

with at least one strict inequality. Suppose that the strict inequality holds with $i = k$. We get a contradiction to (4.10).

(iii) The last conclusion can be obtained by combining (i) and (ii). □

Theorem 4.9. *Let $(\bar{z}, \bar{\lambda}, \bar{\tau})$ be a feasible solution of (MD), where $\bar{\tau} \in \mathbb{R}_{++}^m$ and $\sum_{j \in M} \bar{\tau}_j = 1$.*

- (i) *If $(\bar{z}, \bar{\lambda}, \bar{\tau})$ is an ϵ -Pareto solution of (MD) then $(\bar{z}, \bar{\lambda})$ is an ϵ -solution of (D_j) for each $j \in M$.*
- (ii) *If $(\bar{z}, \bar{\lambda}, \bar{\tau})$ is an ϵ -Pareto quasisolution of (MD), then $(\bar{z}, \bar{\lambda})$ is an ϵ_j -quasisolution of (D_j) for each $j \in M$.*
- (iii) *If $(\bar{z}, \bar{\lambda}, \bar{\tau})$ is a regular ϵ -Pareto solution of (MD) then $(\bar{z}, \bar{\lambda})$ is regular ϵ_j -solution of (D_j) for each $j \in M$.*

Proof. First, we will prove that if $(\bar{z}, \bar{\lambda}, \bar{\tau})$ is a feasible solution of (MD) then $(\bar{z}, \bar{\lambda})$ is a feasible solution of (D_j) for every $j \in M$.

If $(\bar{z}, \bar{\lambda}, \bar{\tau})$ is a feasible solution for (MD) with $0 < \bar{\tau} \in \mathbb{R}^m$ and $\sum_{j \in M} \bar{\tau}_j = 1$, we get

$$(4.11) \quad 0 \in \sum_{j \in M} \bar{\tau}_j \partial^c f_j(\bar{z}) + \sum_{t \in T} \bar{\lambda}_t \partial^c g_t(\bar{z}) + N_C(\bar{z}) + \sum_{j \in M} \bar{\tau}_j \sqrt{\epsilon_j} B^*.$$

Let K be the set of index $k \in M$ such that $(\bar{z}, \bar{\lambda})$ is not feasible solution for (D_k) and assume that K is not empty. Then,

$$0 \notin \partial^c f_k(\bar{z}) + \sum_{t \in T} \bar{\lambda}_t \partial^c g_t(\bar{z}) + N_C(\bar{z}) + \sqrt{\epsilon_k} B^*, \quad k \in K$$

and

$$0 \in \partial^c f_i(\bar{z}) + \sum_{t \in T} \bar{\lambda}_t \partial^c g_t(\bar{z}) + N_C(\bar{z}) + \sqrt{\epsilon_i} B^*, \quad \forall i \in M \setminus K.$$

Hence,

$$(4.12) \quad 0 \notin \bar{\tau}_k \partial^c f_k(\bar{z}) + \bar{\tau}_k \sum_{t \in T} \bar{\lambda}_t \partial^c g_t(\bar{z}) + \bar{\tau}_k N_C(\bar{z}) + \bar{\tau}_k \sqrt{\epsilon_k} B^*, \quad \forall k \in K,$$

and we get

$$(4.13) \quad 0 \in \bar{\tau}_i \partial^c f_i(\bar{z}) + \bar{\tau}_i \sum_{t \in T} \bar{\lambda}_t \partial^c g_t(\bar{z}) + \bar{\tau}_i N_C(\bar{z}) + \bar{\tau}_i \sqrt{\epsilon_i} B^*, \quad \forall i \in M \setminus K.$$

From (4.12) and (4.13), by adding side by side, we obtain

$$0 \notin \sum_{j \in M} \bar{\tau}_j \partial^c f_j(\bar{z}) + \sum_{t \in T} \bar{\lambda}_t \partial^c g_t(\bar{z}) + N_C(\bar{z}) + \sum_{j \in M} \bar{\tau}_j \sqrt{\epsilon_j} B^*,$$

a contradiction to (4.11).

- (i) If $(\bar{z}, \bar{\lambda})$ is not an ϵ_j -solution for (D_j) then there exists (y, λ) such that

$$L_j(\bar{z}, \bar{\lambda}) < L_j(y, \lambda) - \epsilon_j,$$

a contradiction to (i) of Definition 4.1.

(ii) If $(\bar{z}, \bar{\lambda})$ is not an ϵ_j -quasisolution for (D_j) then there exists (y, λ) such that

$$L_j(\bar{z}, \bar{\lambda}) < L_j(y, \lambda) - \sqrt{\epsilon_j} \|\bar{z} - y\| - \sqrt{\epsilon_j} \|\bar{\lambda} - \lambda\|_1,$$

a contradiction to (ii) of Definition 4.1.

(iii) The conclusion follows by (i) and (ii). □

Theorem 4.10. *Suppose that $L_j(\cdot, \lambda)$ is ϵ_j -pseudoconvex and f_j, g_t are regular on C for $t \in T$. For $j \in M$, suppose that there exists $(\bar{z}, \bar{\lambda}) \in C \times \mathbb{R}_+^{(T)}$ satisfying the strict generalized j -KKT condition up to ϵ_j given by (3.3). Then, $(\bar{z}, \bar{\lambda}, \bar{\tau})$ is a weakly ϵ -Pareto quasisolution of (MD) where $\bar{\tau}_j = 1$ and $\bar{\tau}_k = 0$ for $k \neq j$.*

Proof. Since $(\bar{z}, \bar{\lambda})$ satisfies the condition (3.3), $(\bar{z}, \bar{\lambda})$ is a feasible point of (D_j) . Moreover, it is easy to check that $(\bar{z}, \bar{\lambda}, \bar{\tau})$ is a feasible solution of (MD) where $\bar{\tau}_j = 1$ and $\bar{\tau}_k = 0$ for $k \neq j$. To prove that $(\bar{z}, \bar{\lambda}, \bar{\tau})$ is a weakly ϵ -Pareto solution of (MD), we will prove that $(\bar{z}, \bar{\lambda})$ is an ϵ_j -solution for (D_j) . Indeed, since $(\bar{z}, \bar{\lambda}) \in C \times \mathbb{R}_+^{(T)}$ satisfying the strict generalized j -KKT condition up to ϵ_j , using the argument similarly to the one in the proof of Theorem 4.4, we can verify that $(\bar{z}, \bar{\lambda})$ is an ϵ_j -quasisolution of (D_j) . The conclusion follows by Proposition 4.7. □

Theorem 4.11. *Suppose that there exists $j \in M$ such that the function $L_j(\cdot, \lambda)$ is ϵ_j -pseudoconvex and the functions g_t are regular for $t \in T$. Then there exists a quasi weakly ϵ -Pareto solution of (MD).*

Proof. Let us consider the problem (D_j) corresponding to (MD) and the problem $(P_j(y))$ corresponding to (D_j) . By Lemma 3.5, there exists an almost regular ϵ_j -solution \bar{z} for $(P_j(\bar{z}))$ and $\bar{\lambda} \in \mathbb{R}_+^{(T)}$ such that the pair $(\bar{z}, \bar{\lambda})$ is a strict generalized j -KKT pair up to ϵ_j . By Corollary 4.5, $(\bar{z}, \bar{\lambda})$ is an ϵ_j -quasisolution of (D_j) . The desired result derives by applying Proposition 4.7. □

4.3. Duality relationships between (MP) and (MD)

Theorem 4.12. *Suppose that f_j and $L_j(\cdot, \lambda)$ are ϵ_j -pseudoconvex for each $j \in M$, and g_t are quasiconvex for all $t \in T$. Let x and (y, λ, τ) be feasible solutions of (MP) and (MD), respectively. Then, the following cannot hold:*

$$(4.14) \quad (f_1(x), \dots, f_m(x)) < (L_1(y) - \sqrt{\epsilon_1} \|x - y\|, \dots, L_m(y) - \sqrt{\epsilon_m} \|x - y\|).$$

Proof. Let (y, λ, τ) be a feasible solution of (MD). There exist $v_t \in \partial^c g_t(y)$, $t \in T$, $w \in N_C(y)$, $u_j \in \partial^c f_j(y)$, $b \in B^*$ such that

$$\sum_{t \in T} \lambda_t v_t + w + \sum_{j \in M} \tau_j [u_j + \sqrt{\epsilon_j} b] = 0.$$

Hence,

$$(4.15) \quad \sum_{t \in T} \lambda_t v_t(x - y) + \sum_{j \in M} \tau_j [u_j(x - y) + \sqrt{\epsilon_j} \|x - y\|] \geq 0, \quad \forall x \in C.$$

Suppose contrary to (4.14) that

$$f_i(x) < L_i(y) - \sqrt{\epsilon_i} \|x - y\|, \quad i \in M,$$

i.e.,

$$f_i(x) + \sqrt{\epsilon_i} \|x - y\| < f_i(y) + \sum_{t \in T} \lambda_t g_t(y), \quad i \in M.$$

For $x \in F$, we get $\lambda_t g_t(x) \leq 0$ for all $t \in T$. Then

$$f_i(x) + \sum_{t \in T} \lambda_t g_t(x) + \sqrt{\epsilon_i} \|x - y\| < f_i(y) + \sum_{t \in T} \lambda_t g_t(y), \quad i \in M.$$

Since $L_i(\cdot, \lambda)$ is ϵ_i -pseudoconvex, the inequality above implies that

$$L_i^\circ(y; x - y) + \sqrt{\epsilon_i} \|x - y\| < 0, \quad i \in M.$$

Hence, $u_i(x - y) + \sum_{t \in T} \lambda_t v_t(x - y) + \sqrt{\epsilon_i} \|x - y\| < 0, i \in M$. Thus,

$$\sum_{t \in T} \lambda_t v_t(x - y) + \sum_{i \in M} \tau_i [u_i(x - y) + \sqrt{\epsilon_i} \|x - y\|] < 0.$$

This and (4.15) imply a contradiction. □

We now propose some theorems to describe the relationships between (MP) and (MD) via a pair of primal-dual scalar problems.

Theorem 4.13. *Suppose that $L_j(\cdot, \lambda)$ is ϵ_j -pseudoconvex and f_j, g_t are regular on C for $t \in T$. Let $\epsilon \in \mathbb{R}_{++}^m$. Suppose further that \bar{z} is an almost ϵ -Pareto quasisolution of (MP) and there exist $j \in M, \bar{\lambda} \in \mathbb{R}_+^{(T)}$, such that $(\bar{z}, \bar{\lambda})$ is a strict generalized j -KKT pair up to ϵ_j then $(\bar{z}, \bar{\lambda}, \bar{\tau})$ is a weakly ϵ -Pareto quasisolution of (MD) where $\tau_j = 1$ and $\tau_k = 0$ with $k \neq j$.*

Proof. Let $\epsilon \in \mathbb{R}_{++}^m$. Suppose that \bar{z} is an almost ϵ -Pareto quasisolution of (MP). Hence, \bar{z} is an almost ϵ_j -quasisolution for $(P_j(\bar{z}))$ for each $j \in M$. Since, there exist $j \in M$ and $\bar{\lambda}$ such that $(\bar{z}, \bar{\lambda})$ is a strict generalized j -KKT pair up to ϵ_j , by using an argument similar to the one in the proof of Theorem 4.4, we get $(\bar{z}, \bar{\lambda})$ is an ϵ_j -quasisolution for (D_j) . The desired result follows by Proposition 4.7. □

Theorem 4.14. *Let $\epsilon > 0, \epsilon \in \mathbb{R}^m$ and let $(\bar{z}, \bar{\lambda}, \bar{\tau})$ be a quasi ϵ -Pareto solution for (MD), where $\bar{\tau} \in \mathbb{R}_{++}^m$ and $\sum_{j \in M} \bar{\tau}_j = 1$.*

- (i) If $\bar{z} \in F_j(\bar{z})$ then \bar{z} is a weakly ϵ -Pareto quasisolution of (MP).
- (ii) For $\epsilon_j = \min\{\epsilon_k, k \in M\}$, if $\bar{z} \in F_j^{\epsilon_j}(\bar{z})$ then \bar{z} is an almost weakly ϵ -Pareto quasisolution of (MP).

Proof. Let $\epsilon > 0, \epsilon \in \mathbb{R}^m$. Suppose that $(\bar{z}, \bar{\lambda}, \bar{\tau})$ is an ϵ -Pareto quasisolution for (MD), where $0 < \bar{\tau} \in \mathbb{R}^m$ and $\sum_{j \in M} \bar{\tau}_j = 1$. By Theorem 4.9, $(\bar{z}, \bar{\lambda})$ is an ϵ_j -quasisolution of (D_j) for each $j \in M$.

If $\bar{z} \in F_j(\bar{z})$ ($\bar{z} \in F_j^{\epsilon_j}(\bar{z})$), by Theorem 4.6, it is an (almost) ϵ_j -quasisolution of $(P_j(\bar{z}))$. For $\epsilon_j = \min\{\epsilon_k \mid k \in M\}$, the conclusions can be derived by Theorem 3.7. □

4.4. A modification of Chankong-Haimes method for a dual problem of (MP)

Note that, if we deal with exact solution of (MP) then a dual problem of (MP) in Wolfe type is formulated by

$$\begin{aligned}
 \text{(MD*)} \quad & \text{Max } \mathcal{L}(y, \lambda), \quad 0 \in \sum_{j \in M} \tau_j \partial^c f_j(y) + \sum_{t \in T} \lambda_t \partial^c g_t(y) + N_C(y), \\
 & \tau_j \geq 0, \quad \sum_{j \in M} \tau_j = 1, \quad y \in C, \quad \lambda \in \mathbb{R}_+^{(T)}.
 \end{aligned}$$

Hence, the problem $(D_j), j = 1, 2$, becomes

$$\begin{aligned}
 \text{(D}_j^*) \quad & \text{Max } L_j(y, \lambda) = f_j(y) + \sum_{t \in T} \lambda_t g_t(y), \\
 & 0 \in \partial^c f_j(y) + \sum_{t \in T} \lambda_t \partial^c g_t(y) + N_C(y), \quad (y, \lambda) \in C \times \mathbb{R}_+^{(T)}.
 \end{aligned}$$

The problem (D_j^*) is not compatible with (P_j) under the dual scheme of Wolfe type. But, there exist connections between (D_j^*) and (MD^*) , directly. Approximate solutions of (MD^*) can be obtained via (D_j^*) in the spirit of Chankong-Haimes method. We propose some more propositions with the proofs omitted because they are similar to the proofs of Propositions 4.7(i), 4.8(i) and Theorem 4.9(i), respectively.

Proposition 4.15. *Suppose that the function $L_j(\cdot, \lambda)$ is ϵ_j -pseudoconvex. Let $\bar{\tau} \in \mathbb{R}_+^m$ be such that $\bar{\tau}_j = 1$ and $\bar{\tau}_k = 0$ for $k \neq j$. If $(\bar{z}, \bar{\lambda})$ is an ϵ_j -solution of (D_j^*) then $(\bar{z}, \bar{\lambda}, \bar{\tau})$ is a weakly ϵ -Pareto solution of (MD^*) .*

Proposition 4.16. *Suppose that the functions $L_j(\cdot, \lambda)$ are ϵ_j -pseudoconvex for $j \in M$. Let $\bar{\tau} \in \mathbb{R}_+^m$ be such that $\sum_{j \in M} \bar{\tau}_j = 1$. If $(\bar{z}, \bar{\lambda})$ is an ϵ_j -solution of (D_j^*) for each $j \in M$ then $(\bar{z}, \bar{\lambda}, \bar{\tau})$ is an ϵ -Pareto solution of (MD^*) .*

Proposition 4.17. *Let $(\bar{z}, \bar{\lambda}, \bar{\tau})$ be a feasible solution of (MD^*) , where $\bar{\tau} \in \mathbb{R}_+^m$ and $\sum_{j \in M} \bar{\tau}_j = 1$. If $(\bar{z}, \bar{\lambda}, \bar{\tau})$ is an ϵ -Pareto solution of (MD^*) then $(\bar{z}, \bar{\lambda})$ is an ϵ -solution of (D_j^*) for each $j \in M$.*

Example 4.18. The following example illustrates Proposition 4.15:

$$(MP_1) \quad \text{Min } f(x) = (x, x^2 - x) \text{ subject to } g(x) = x^2 - x \leq 0.$$

The dual problem of (MP_1) in Wolfe type, following (MD) , is

$$(MD_1^*) \quad \begin{aligned} &\text{Max } L(y, \lambda) = (y + \lambda(y^2 - y), y^2 - y + \lambda(y^2 - y)) \\ &\text{subject to } 0 = \tau_1 + \tau_2(2y - 1) + \lambda(2y - 1), \\ &\tau_1, \tau_2 \geq 0, \quad \tau_1 + \tau_2 = 1, \quad y \in \mathbb{R}, \quad \lambda \in \mathbb{R}_+. \end{aligned}$$

Given $\epsilon = (\epsilon_1, \epsilon_2) = (1/4, 1/4)$, it is easy to check that the following functions are ϵ_j -pseudoconvex ($j = 1, 2$), respectively:

$$L_1(y, \lambda) = y + \lambda(y^2 - y), \quad L_2(y, \lambda) = y^2 - y + \lambda(y^2 - y).$$

The problems (D_1^*) and (D_2^*) formulated due to (D_j^*) are as follows:

$$(D_1^*) \quad \text{Maximize } L_1(y, \lambda) = y + \lambda(y^2 - y) \text{ subject to } 0 = 1 + \lambda(2y - 1), \quad y \in \mathbb{R}, \quad \lambda \in \mathbb{R}_+,$$

$$(D_2^*) \quad \begin{aligned} &\text{Maximize } L_2(y, \lambda) = y^2 - y + \lambda(y^2 - y) \\ &\text{subject to } 0 = 2y - 1 + \lambda(2y - 1), \quad y \in \mathbb{R}, \quad \lambda \in \mathbb{R}_+. \end{aligned}$$

For (D_1^*) , it is easy to check that $(y, \lambda) = (0, 1)$ is an exact solution and the optimal value of the problem is 0. With a simple computation, we can see that each pair $(z, \bar{\lambda})$, where $z \in [-(1 + \sqrt{5})/4, (-1 + \sqrt{5})/4]$ and $\bar{\lambda} = 1/(1 - 2z)$, is a $1/4$ -solution of (D_1^*) . In this case, the quadruple $(z, \bar{\lambda}, \tau_1, \tau_2)$ where $z \in [-(1 + \sqrt{5})/4, (-1 + \sqrt{5})/4]$, $\bar{\lambda} = 1/(1 - 2z)$, $\tau_1 = 1$, $\tau_2 = 0$ is a weakly ϵ -Pareto solution of (MD_1^*) .

Also, for (D_2^*) , we can check that $(y, \lambda) = (1/2, 0)$ is an exact solution and the optimal value is $-1/4$. Each pair $(z, \bar{\lambda})$, where $z = 1/2$ and $\bar{\lambda} \in [0, 1]$, is a $1/4$ -solution of (D_2^*) . Hence, the quadruple $(z, \bar{\lambda}, \tau_1, \tau_2)$ where $z = 1/2$, $\bar{\lambda} \in [0, 1]$, $\tau_1 = 0$, $\tau_2 = 1$ is a weakly ϵ -Pareto solution of (MD_1^*) .

5. Conclusions

We proposed an approach to investigate ϵ -duality theorems for a class of semi-infinite multiobjective optimization problems via a pair of primal-dual scalar problems. This approach is based on a development of Chankong-Haijmes method [5] and it differs from the weighted-sum method for solving multiobjective optimization problems introduced in [38]. The relations among four problems, the primal-dual multiobjective problems and the primal-dual scalar problems, are established. Then, we give ϵ -duality theorems due to the symmetric diagram proposed in Section 1. At the end of the paper, we show that

approximate Pareto solution of the dual problem of (MP) can be obtained directly by the spirit of Chankong-Haimes method applied for the dual problem of (MP). As an application, we state that this scheme can be applied for fractional semi-infinite multiobjective optimization problems by the use of a well known Dinkelbach transformation (to change a fractional optimization problem to a nonfractional optimization problem). We leave this for our further research.

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