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Parametrized Multilinear Littlewood-Paley Operators on Hardy Spaces

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Abstract. In this paper, we study the parametrized multilinear Marcinkiewicz integral μ^ρ and the multilinear Littlewood-Paley g_λ^*-function. We proved that if the kernel Ω associated to parametrized multilinear Marcinkiewicz integral μ^ρ is homogeneous of degree zero and satisfies the Lipschitz continuous condition, or the kernel K associated to the multilinear Littlewood-Paley g_λ^*-function satisfies the Hörmander condition, then they are bounded from $H^{p_1} \times \cdots \times H^{p_m}$ to L^p with $mn/(mn+\gamma) < p_1, \ldots, p_m \leq 1$ and $1/p = 1/p_1 + \cdots + 1/p_m$.

1. Introduction and main results

In 1960, Hörmander [10] first studied the parametrized Marcinkiewicz integral μ^ρ which is connected closely with the Marcinkiewicz integral defined and studied by Stein [17] in 1958. He proved that μ^ρ is L^p bounded if the kernel Ω satisfies some Dini type conditions. Later, Sakamoto and Yabuta [15] showed that the parametrized Lusin area integral S^ρ and the parametrized Littlewood-Paley g_λ^*-function are bounded on L^p provided the kernel Ω satisfies the Lipschitz conditions.

It was well known that the multilinear operator was first studied by Coifman and Meyer [4, 5]. After that, the multilinear theory has been paid great attention, and it develops rapidly in recent years. Especially, Christ and Journé [2], Kenig and Stein [11], Grafakos and Torres [9] and Lerner et al. [13] have made great contributions to this field. On the other hand, in 1972, Fefferman and Stein [6] studied the classical Calderón-Zygmund operators on the real Hardy space $H^p(\mathbb{R}^n)$ $(0 < p < 1)$. Later, Grafakos and Kalton [7] investigated the boundedness of the nonconvolution multilinear Calderón-Zygmund operators on Hardy spaces. Recently, Li et al. [14] extended the results of [7] to the weighted case. Shi et al. [16] introduced the multilinear Littlewood-Paley g_λ^* function

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with the kernel K satisfies the Hörmander condition and proved the strong weighted estimates and the weak type boundedness of g_λ^*.

There is a natural question about multilinear Littlewood-Paley operators, that is, are these operators bounded on Hardy spaces? The aim of this paper is to investigate the boundedness of the multilinear Littlewood-Paley operators on Hardy spaces.

Throughout this paper, for any set E, χ_E will be used to denote the characteristic function of the set E. The letter C will always denote a positive constant that may vary at each occurrence but is independent of the essential variable.

In order to state our results, we first introduce some definitions and notations.

Definition 1.1. Let Ω be a function defined on $(\mathbb{R}^n)^m$ with the following properties:

(i) Ω is homogeneous of degree 0, i.e.,

$$\Omega(\lambda y) = \Omega(y), \tag{1.1}$$

where $y = (y_1, \ldots, y_m) \in (\mathbb{R}^n)^m$.

(ii) Ω is Lipschitz continuous on $(S^{n-1})^m$, i.e., there are $0 < \gamma < 1$ and $C > 0$ such that for any $\xi = (\xi_1, \ldots, \xi_m), \eta = (\eta_1, \ldots, \eta_m) \in (\mathbb{R}^n)^m$,

$$|\Omega(\xi) - \Omega(\eta)| \le C|\xi' - \eta'|^\gamma, \tag{1.2}$$

where $y' = (y_1, \ldots, y_m)' = (y_1, \ldots, y_m)/(|y_1| + \cdots + |y_m|)$.

For any $\vec{f} = (f_1, \ldots, f_m) \in \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n)$, $t > 0$, we define

$$K^\rho(y_1, \ldots, y_m) = \frac{\Omega(y_1, \ldots, y_m)\chi_{(B(0,1))^m}(y_1, \ldots, y_m)}{(|y_1| + \cdots + |y_m|)^{m(n-\rho)}},$$
$$K_t^\rho(y_1, \ldots, y_m) = \frac{1}{t^{mn}} K^\rho\left(\frac{y_1}{t}, \ldots, \frac{y_m}{t}\right)$$

and

$$\begin{aligned} G_t^\rho(\vec{f})(y) &= K_t^\rho * (f_1 \otimes \cdots \otimes f_m)(y) \\ &= \frac{1}{t^{m\rho}} \int_{(B(y,t))^m} \frac{\Omega(y - z_1, \ldots, y - z_m)}{(|y - z_1| + \cdots + |y - z_m|)^{m(n-\rho)}} \prod_{i=1}^m f_i(z_i)\, dz_i, \end{aligned}$$

where $B(x,t) = \{y \in \mathbb{R}^n : |y - x| \le t\}$.

Then the parametrized multilinear Marcinkiewicz integral μ^ρ is defined by

$$\mu^\rho(\vec{f})(x) = \left(\int_0^\infty |G_t^\rho(\vec{f})(x)|^2 \, \frac{dt}{t}\right)^{1/2}.$$

In this paper, we assume that for $1 < q_1, \ldots, q_m < \infty$ satisfying $1/q = 1/q_1 + \cdots + 1/q_m$, there exists a constant $C > 0$ such that

$$\|\mu^\rho(\vec{f})\|_{L^q} \le C \prod_{i=1}^m \|f_i\|_{L^{q_i}}. \tag{1.3}$$

In fact, considering the bilinear case, the parametrized bilinear Marcinkiewicz integral μ^ρ-function can be rewritten as a 4-linear Fourier multiplier with symbol

$$m^\rho(\xi_1, \xi_2, \zeta_1, \zeta_2) = \int_0^\infty \widehat{\psi^\rho}(t\xi_1, t\xi_2)\widehat{\psi^\rho}(t\zeta_1, t\zeta_2)\,\frac{dt}{t}.$$

If assume the kernel is sufficiently smooth, then by the results of Grafakos-Miyachi-Tomita [8], we know that the parametrized bilinear Marcinkiewicz integral μ^ρ-function does satisfy the boundedness in (1.3), which shows our assumption (1.3) is reasonable.

The first result in this paper is as follows.

Theorem 1.2. *Let $\gamma > 0$, $\rho \in (0, (n+\gamma)/m)$. Suppose Ω satisfies* (1.1) *and* (1.2), *if $f_i \in H^{p_i}(\mathbb{R}^n)$, where $p_i \in (mn/(mn+\gamma), 1]$ for $i = 1, \ldots, m$, then for any p with $1/p = 1/p_1 + \cdots + 1/p_m$, there exists a constant $C > 0$ such that*

$$\|\mu^\rho(\vec{f})\|_{L^p(\mathbb{R}^n)} \le C \prod_{i=1}^m \|f_i\|_{H^{p_i}(\mathbb{R}^n)}.$$

To obtain the boundedness of multilinear Littlewood-Paley g_λ^* function, we also introduce another kind of multilinear standard kernel, we say a function K defined on $(\mathbb{R}^n)^m$ is a multilinear standard kernel, if K satisfies the following two conditions:

$$|K(y_1, \ldots, y_m)| \le \frac{C}{(1 + \sum_{i=1}^m |y_i|)^{mn+\delta}} \tag{1.4}$$

for some $\delta > 0$, and all $(y_1, \ldots, y_m) \in (\mathbb{R}^n)^m$,

$$|K(y_1, \ldots, y_i + z, \ldots, y_m) - K(y_1, \ldots, y_i, \ldots, y_m)| \le \frac{C|z|^\gamma}{(1 + \sum_{i=1}^m |y_i|)^{mn+\delta+\gamma}} \tag{1.5}$$

for some $\gamma > 0$, $i = 1, 2, \ldots, m$, $2|z| \le \max_{i=1,\ldots,m}\{|y_i|\}$ and all $(y_1, \ldots, y_m) \in (\mathbb{R}^n)^m$.

For any $\vec{f} = (f_1, \ldots, f_m) \in \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n)$, $t > 0$ and $z \notin \bigcap_{i=1}^m \operatorname{supp} f_i$, we will denote by G_t as

$$G_t(\vec{f})(z) = \frac{1}{t^{mn}} \int_{(\mathbb{R}^n)^m} K\left(\frac{z - y_1}{t}, \ldots, \frac{z - y_m}{t}\right) \prod_{i=1}^m f_i(y_i)\, dy_i.$$

The multilinear Littlewood-Paley g_λ^* function is defined by

$$g_\lambda^*(\vec{f})(x) = \left[\iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t + |x - z|}\right)^{n\lambda} |G_t(\vec{f})(z)|^2\, \frac{dz\, dt}{t^{n+1}}\right]^{1/2}, \quad \lambda > 1.$$

We point out that this kind of Littlewood-Paley g_λ^* function was introduced by Shi et al. [16], in which they obtain the strong weighted estimates and the weak type boundedness of g_λ^* for $p_i \geq 1$. Our second result is as follows:

Theorem 1.3. *Let $\lambda > 2m$ and $\gamma \in (0, n(\lambda - 2m)/2)$. If $f_i \in H^{p_i}(\mathbb{R}^n)$, where $p_i \in (mn/(mn+\gamma), 1]$ for $i = 1, \ldots, m$, then for any p with $1/p = 1/p_1 + \cdots + 1/p_m$, there exists a constant $C > 0$ such that*

$$\|g_\lambda^*(\vec{f})\|_{L^p(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{H^{p_i}(\mathbb{R}^n)}.$$

Remark 1.4. If we define the parametrized multilinear Littlewood-Paley $g_\lambda^{*,\rho}$-function by Ω, our method doesn't work. It leaves open that whether the parametrized multilinear Littlewood-Paley $g_\lambda^{*,\rho}$-function is bounded on Hardy spaces.

The article is organized as follows. The proofs of Theorems 1.2 and 1.3 will be shown in Section 2. Throughout this paper, if $f \leq Cg$, we denote by $f \lesssim g$ for short.

2. Proofs of Theorems 1.2 and 1.3

Proof of Theorem 1.2. We will prove Theorem 1.2 by using the atomic decomposition of H^p. See Coifman [3] and Latter [12] for detail. At first, we give the definition of atoms in $\mathbb{R}^n$.

Definition 2.1. Let $0 < p \leq 1$, $1 < q \leq \infty$. A function a is called an L^q-atom for $H^p(\mathbb{R}^n)$ if there exists a cube Q such that

(i) $\operatorname{supp} a \subset Q$,

(ii) $\|a\|_{L^q} \leq |Q|^{1/q-1/p}$,

(iii) $\int_Q a(x)\,dx = 0$.

Latter [12] proved the following atomic decomposition in $\mathbb{R}^n$.

Theorem 2.2. [12] *Let $0 < p \leq 1$, $1 < q \leq \infty$. A distribution f is in $H^p(\mathbb{R}^n)$ if and only if there exists a sequence of L^q atoms a_i for $H^p(\mathbb{R}^n)$ and a sequence of non-negative real numbers λ_i such that*

$$f = \sum_{i=0}^\infty \lambda_i a_i,$$

in the sense of distributions and

$$A\|f\|_{H^p(\mathbb{R}^n)} \leq \left(\sum_{i=0}^\infty \lambda_i^p\right)^{1/p} \leq B\|f\|_{H^p(\mathbb{R}^n)},$$

where A, B are constants which depend only on n and p.

Since finite sums of atoms are dense in H^p, we will work with such sums and we will obtain estimates independent of the number of terms in each sum. The general case will follow by a simple density argument. By the atomic decomposition of H^p, we split each f_i, $i = 1, \ldots, m$, as a finite sum of L^∞-atoms a_{i,k_i} for H^{p_i}. This means there exist cubes Q_{i,k_i}, such that

$$\begin{gathered}\operatorname{supp} a_{i,k_i} \subset Q_{i,k_i}, \\ \|a_{i,k_i}\|_{L^\infty} \le |Q_{i,k_i}|^{-1/p_i}, \\ \int_{Q_{i,k_i}} a_{i,k_i}(x)\,dx = 0.\end{gathered} \tag{2.1}$$

For a cube Q, let $\widetilde{Q}$ be the cube with the same center as Q and $8\sqrt{n}$ its side length. Using multilinearity, we write

$$\mu^\rho(f_1, \ldots, f_m)(x) \le \sum_{k_1,\ldots,k_m} \lambda_{1,k_1} \cdots \lambda_{m,k_m} \mu^\rho(a_{1,k_1}, \ldots, a_{m,k_m})(x).$$

Set

$$|\mu^\rho(f_1, \ldots, f_m)(x)| \le I_1 + I_2,$$

where

$$\begin{aligned} I_1 &= \sum_{k_1,\ldots,k_m} |\lambda_{1,k_1}| \cdots |\lambda_{m,k_m}| |\mu^\rho(a_{1,k_1}, \ldots, a_{m,k_m})(x)| \chi_{\widetilde{Q}_{1,k_1} \cap \cdots \cap \widetilde{Q}_{m,k_m}}, \\ I_2 &= \sum_{k_1,\ldots,k_m} |\lambda_{1,k_1}| \cdots |\lambda_{m,k_m}| |\mu^\rho(a_{1,k_1}, \ldots, a_{m,k_m})(x)| \chi_{\widetilde{Q}^c_{1,k_1} \cup \cdots \cup \widetilde{Q}^c_{m,k_m}}. \end{aligned}$$

Now, let us begin to discuss I_1. For fixed $k_1, \ldots, k_m$, assume that

$$\widetilde{Q}_{1,k_1} \cap \cdots \cap \widetilde{Q}_{m,k_m} \ne \emptyset,$$

since otherwise there is nothing needs to be proved.

Suppose that Q_{1,k_1} has the smallest size among all these cubes. We take a cube $G_{k_1,\ldots,k_m}$ such that

$$\widetilde{Q}_{1,k_1} \cap \cdots \cap \widetilde{Q}_{m,k_m} \subset G_{k_1,\ldots,k_m} \subset \widetilde{G}_{k_1,\ldots,k_m} \subset \widetilde{\widetilde{Q}}_{1,k_1} \cap \cdots \cap \widetilde{\widetilde{Q}}_{m,k_m}$$

and

$$|G_{k_1,\ldots,k_m}| \ge C|Q_{1,k_1}|.$$

By Hölder's inequality and [1, Theorem 1.2], we have

$$\frac{1}{|G_{k_1,\ldots,k_m})} \int_{G_{k_1,\ldots,k_m}} |\mu^\rho(a_{1,k_1}, \ldots, a_{m,k_m})(x)|\,dx$$

$$\begin{aligned}
&\lesssim |G_{k_1,\ldots,k_m}|^{-1/2}\|\mu^\rho(a_{1,k_1},\ldots,a_{m,k_m})\|_{L^2}\\
&\lesssim |G_{k_1,\ldots,k_m}|^{-1/2}\|a_{1,k_1}\|_{L^2}\prod_{i=2}^m\|a_{i,k_i}\|_{L^\infty}\\
&\lesssim |G_{k_1,\ldots,k_m}|^{-1/2}|Q_{1,k_1}|^{1/2-1/p_1}\prod_{i=2}^m|Q_{i,k_i}|^{-1/p_i}\\
&\lesssim \left(\prod_{i=1}^m|Q_{i,k_i}|^{-1/p_i}\right).
\end{aligned}$$

Then we need the following lemma.

Lemma 2.3. [7] *Let $0<p\le 1$. Then there is a constant $C=C(p,n)$ such that for all finite collection $\{Q_k\}_{k=1}^m$ of cubes in $\mathbb{R}^n$ and all nonnegative functions $g_k\in L^p$ with* $\operatorname{supp} g_k\subset Q_k$ *we have*

$$\left\|\sum_{k=1}^m g_k\right\|_{L^p}\le C\left\|\sum_{k=1}^m\left(\frac{1}{|Q_k|}\int_{Q_k}g_k(x)\,dx\right)\chi_{\widetilde{Q}_k}\right\|_{L^p}.$$

By Lemma 2.3 and Hölder's inequality, we have

$$\begin{aligned}
\|I_1\|_{L^p}&\lesssim\left\|\sum_{k_1,\ldots,k_m}|\lambda_{1,k_1}|\cdots|\lambda_{m,k_m}|\prod_{i=1}^m|Q_{i,k_i}|^{-1/p_i}\chi_{\widetilde{\widetilde{Q}}_{1,k_1}}\cdots\chi_{\widetilde{\widetilde{Q}}_{m,k_m}}\right\|_{L^p}\\
&\lesssim\left\|\prod_{i=1}^m\left(\sum_{k_i}|\lambda_{i,k_i}||Q_{i,k_i}|^{-1/p_i}\chi_{\widetilde{\widetilde{Q}}_{i,k_i}}\right)\right\|_{L^p}\\
&\lesssim\prod_{i=1}^m\left\|\left(\sum_{k_i}|\lambda_{i,k_i}||Q_{i,k_i}|^{-1/p_i}\chi_{\widetilde{\widetilde{Q}}_{i,k_i}}\right)\right\|_{L^{p_i}}\\
&\lesssim\prod_{i=1}^m\left(\sum_{k_i}|\lambda_{i,k_i}|^{p_i}|Q_{i,k_i}|^{-1}\left|\widetilde{\widetilde{Q}}_{i,k_i}\right|\right)^{1/p_i}\\
&\lesssim\prod_{i=1}^m\left(\sum_{k_i}|\lambda_{i,k_i}|^{p_i}\right)^{1/p_i}\\
&\lesssim\prod_{i=1}^m\|f_i\|_{H^{p_i}}.
\end{aligned}\tag{2.2}$$

Then, we estimate $I_2(x)$. Let A be a nonempty subset of $\{1,\ldots,m\}$, and we denote the cardinality of A by $|A|$. Then $1\le|A|\le m$. Set $A^c=\{1,\ldots,m\}\setminus A$. For $A=\{1,\ldots,m\}$, we define

$$\left(\bigcap_{i\in A}\widetilde{Q}_{i,k_i}^c\right)\cap\left(\bigcap_{i\in A^c}\widetilde{Q}_{i,k_i}\right)=\bigcap_{i\in A}\widetilde{Q}_{i,k_i}^c.$$

Then we have

$$\widetilde{Q}_{1,k_1}^c \cup \cdots \cup \widetilde{Q}_{m,k_m}^c = \bigcup_{A\subset\{1,\dots,m\}} \left(\left(\bigcap_{i\in A} \widetilde{Q}_{i,k_i}^c \right) \cap \left(\bigcap_{i\in A^c} \widetilde{Q}_{i,k_i} \right) \right).$$

For fixed $1 \le r \le m$, without loss of generality, we consider the particular case, that is, by permuting the indices, we assume $x \in E_r$, where

$$E_r = (\widetilde{Q}_{1,k_1}^c \cap \cdots \cap \widetilde{Q}_{r,k_r}^c) \cap (\widetilde{Q}_{r+1,k_{r+1}} \cap \cdots \cap \widetilde{Q}_{m,k_m}).$$

Denoting the center of Q_{i,k_i} by c_{i,k_i} and the sidelength of Q_{i,k_i} by l_{i,k_i}, let us estimate I_2. For fixed E_r, $x \in E_r$, $y_i \in Q_{i,k_i}$, $1 \le i \le r$, we have

$$|x - y_i| \sim |x - c_{i,k_i}|.$$

There exists $i_0 \in \{1, \dots, r\}$ such that

$$|x - c_{i_0,k_{i_0}}| = \max\{|x - c_{i,k_i}| : i \in \{1, \dots, r\}\}.$$

Then, if $t < |x - c_{i_0,k_{i_0}}| - l_{i_0,k_{i_0}}$, then $B(x,t) \cap Q_{i_0,k_{i_0}} = \emptyset$. Thus, using the vanish condition of atom, we have

$$\begin{aligned}
&|\mu^\rho(a_{1,k_1}, \dots, a_{m,k_m})(x)| \\
&= \Bigg[\int_0^\infty \Bigg| \int_{(B(x,t))^m} \frac{\Omega(x-y_1, \dots, x-y_m)}{\left(\sum_{i=1}^m |x-y_i|\right)^{m(n-\rho)}} \\
&\qquad \times a_{1,k_1}(y_1) \cdots a_{m,k_m}(y_m)\, dy_1 \cdots dy_m \Bigg|^2 \frac{dt}{t^{1+2m\rho}} \Bigg]^{1/2} \\
&= \Bigg[\int_{|x-c_{i_0,k_{i_0}}|-l_{i_0,k_{i_0}}}^\infty \Bigg| \int_{(B(x,t))^{m-r}} \int_{\mathbb{R}^{rn}} \frac{\Omega(x-y_1, \dots, x-y_m)}{\left(\sum_{i=1}^r |x-c_{i,k_i}| + \sum_{i=r+1}^m |x-y_i|\right)^{m(n-\rho)}} \\
&\qquad \times a_{1,k_1}(y_1) \cdots a_{m,k_m}(y_m)\, dy_1 \cdots dy_m \Bigg|^2 \frac{dt}{t^{1+2m\rho}} \Bigg]^{1/2} \\
&= \Bigg[\int_{|x-c_{i_0,k_{i_0}}|-l_{i_0,k_{i_0}}}^\infty \Bigg| \int_{(B(x,t))^{m-r}} \int_{\mathbb{R}^{rn}} \\
&\qquad \times \frac{\Omega(x-y_1, \dots, x-y_m) - \Omega(x-c_{1,k_1}, \dots, x-y_m)}{\left(\sum_{i=1}^r |x-c_{i,k_i}| + \sum_{i=r+1}^m |x-y_i|\right)^{m(n-\rho)}} \\
&\qquad \times a_{1,k_1}(y_1) \cdots a_{m,k_m}(y_m)\, dy_1 \cdots dy_m \Bigg|^2 \frac{dt}{t^{1+2m\rho}} \Bigg]^{1/2} \\
&\lesssim \prod_{i=1}^m |Q_{i,k_i}|^{-1/p_i} \Bigg[\int_{|x-c_{i_0,k_{i_0}}|-l_{i_0,k_{i_0}}}^\infty \Bigg(\int_{(B(x,t))^{m-r}} \int_{Q_{1,k_1}\times\cdots\times Q_{r,k_r}} \\
&\qquad \times \frac{|\Omega(x-y_1, \dots, x-y_m) - \Omega(x-c_{1,k_1}, \dots, x-y_m)|}{\left(\sum_{i=1}^r |x-c_{i,k_i}| + \sum_{i=r+1}^m |x-y_i|\right)^{m(n-\rho)}}\, dy_1 \cdots dy_m \Bigg)^2 \frac{dt}{t^{1+2m\rho}} \Bigg]^{1/2}.
\end{aligned}$$

From (1.2) and the fact that Q_{1,k_1} is the smallest cube among all the cubes, we see that

$$
\begin{aligned}
&|\Omega(x-y_1,\dots,x-y_m)-\Omega(x-c_{1,k_1},\dots,x-y_m)| \\
&\lesssim \left|\frac{(x-y_1,\dots,x-y_m)}{\sum_{i=1}^m |x-y_i|}-\frac{(x-c_{1,k_1},\dots,x-y_m)}{|x-c_{1,k_1}|+\sum_{i=2}^m |x-y_i|}\right|^{\gamma} \\
&\lesssim \frac{l_{1,k_1}^{\gamma}}{\left(\sum_{i=1}^m |x-y_i|\right)^{\gamma}}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&|\mu^{\rho}(a_{1,k_1},\dots,a_{m,k_m})(x)| \\
&\lesssim \prod_{i=1}^m |Q_{i,k_i}|^{-1/p_i}\Bigg[\int_{|x-c_{i_0,k_{i_0}}|-l_{i_0,k_{i_0}}}^{\infty}\Bigg(\int_{(\mathbb{R}^n)^{m-r}}\int_{Q_{1,k_1}\times\cdots\times Q_{r,k_r}} \\
&\qquad\times \frac{l_{1,k_1}^{\gamma}}{\left(\sum_{i=1}^r |x-c_{i,k_i}|+\sum_{i=r+1}^m |x-y_i|\right)^{m(n-\rho)+\gamma}}\,dy_1\cdots dy_m\Bigg)^2\frac{dt}{t^{1+2m\rho}}\Bigg]^{1/2} \\
&\lesssim \prod_{i=1}^r |Q_{i,k_i}|^{1-1/p_i}\prod_{i=r+1}^m |Q_{i,k_i}|^{-1/p_i} \\
&\qquad\times \int_{(\mathbb{R}^n)^{m-r}} \frac{l_{1,k_1}^{\gamma}}{\left(\sum_{i=1}^r |x-c_{i,k_i}|+\sum_{i=r+1}^m |x-y_i|\right)^{m(n-\rho)+\gamma}}\,dy_{r+1}\cdots dy_m \\
&\qquad\times \left[\int_{|x-c_{i_0,k_{i_0}}|-l_{i_0,k_{i_0}}}^{\infty}\frac{dt}{t^{1+2m\rho}}\right]^{1/2} \\
&\lesssim \prod_{i=1}^r |Q_{i,k_i}|^{1-1/p_i}\prod_{i=r+1}^m |Q_{i,k_i}|^{-1/p_i}\frac{l_{1,k_1}^{\gamma}}{\left(\sum_{i=1}^r |x-c_{i,k_i}|\right)^{rn+\gamma}} \\
&\lesssim \prod_{i=1}^r \frac{l_{i,k_i}^{n-n/p_i+\gamma/r}}{\left(|x-c_{i,k_i}|+l_{i,k_i}\right)^{n+\gamma/r}}\prod_{i=r+1}^m |l_{i,k_i}|^{-n/p_i} \\
&\lesssim \prod_{i=1}^m \frac{l_{i,k_i}^{n-n/p_i+\gamma/r}}{\left(|x-c_{i,k_i}|+l_{i,k_i}\right)^{n+\gamma/r}}
\end{aligned}
$$

where we use

$$
\int_{\mathbb{R}^n}\frac{1}{(b+|y-c|)^M}\,dy \le b^{n-M}
$$

if $M>n$, b is any positive number and $c\in\mathbb{R}^n$ in the third inequality.

Summing over all possible $1\le r\le m$ and all possible combinations of subset of $\{1,\dots,m\}$ of size r we obtain the pointwise estimate

$$
|\mu^{\rho}(a_{1,k_1},\dots,a_{m,k_m})(x)| \lesssim \prod_{i=1}^m \frac{l_{i,k_i}^{n-n/p_i+\gamma/m}}{\left(|x-c_{i,k_i}|+l_{i,k_i}\right)^{n+\gamma/m}}
$$

for all $x \in \widetilde{Q}^c_{1,k_1} \cup \cdots \cup \widetilde{Q}^c_{m,k_m}$.

Hence, by the fact $p_i > mn/(mn+\gamma)$, we see that

$$\begin{aligned}
\|I_2\|_{L^p} &\lesssim \Big\| \sum_{k_1,\ldots,k_m} |\lambda_{1,k_1}|\cdots|\lambda_{m,k_m}| \prod_{i=1}^m \frac{l_{i,k_i}^{n-n/p_i+\gamma/m}}{\big(|x-c_{i,k_i}|+l_{i,k_i}\big)^{n+\gamma/m}} \Big\|_{L^p} \\
&\lesssim \prod_{i=1}^m \Big\| \sum_{k_i} |\lambda_{i,k_i}| \frac{l_{i,k_i}^{n-n/p_i+\gamma/m}}{\big(|x-c_{i,k_i}|+l_{i,k_i}\big)^{n+\gamma/m}} \Big\|_{L^{p_i}} \\
&\lesssim \prod_{i=1}^m \Big\{ \int_{\mathbb{R}^n} \Big| \sum_{k_i} |\lambda_{i,k_i}| \frac{l_{i,k_i}^{n-n/p_i+\gamma/m}}{\big(|x-c_{i,k_i}|+l_{i,k_i}\big)^{n+\gamma/m}} \Big|^{p_i} dx \Big\}^{1/p_i} \\
&\lesssim \prod_{i=1}^m \Big\{ \sum_{k_i} |\lambda_{i,k_i}|^{p_i} l_{i,k_i}^{np_i-n+\gamma p_i/m} \int_{\mathbb{R}^n} \frac{1}{\big(|x-c_{i,k_i}|+l_{i,k_i}\big)^{np_i+\gamma p_i/m}} dx \Big\}^{1/p_i} \\
&\lesssim \prod_{i=1}^m \Big\{ \sum_{k_i} |\lambda_{i,k_i}|^{p_i} \Big\}^{1/p_i} \\
&\lesssim \prod_{i=1}^m \|f_i\|_{H^{p_i}},
\end{aligned}$$

which combined with (2.2), completes the proof of Theorem 1.2. □

Proof of Theorems 1.3. Use the same notations as in the proof of Theorems 1.2. Similar to the proof of Theorems 1.2, it suffices to show that, for all $x \in \widetilde{Q}^c_{1,k_1} \cup \cdots \cup \widetilde{Q}^c_{m,k_m}$,

$$|g^*_\lambda(a_{1,k_1},\ldots,a_{m,k_m})(x)| \lesssim \prod_{i=1}^m \frac{l_{i,k_i}^{n-n/p_i+\gamma/m}}{\big(|x-c_{i,k_i}|+l_{i,k_i}\big)^{n+\gamma/m}}. \tag{2.3}$$

Assume $x \in E_r$, where

$$E_r = (\widetilde{Q}^c_{1,k_1} \cap \cdots \cap \widetilde{Q}^c_{r,k_r}) \cap (\widetilde{Q}_{r+1,k_{r+1}} \cap \cdots \cap \widetilde{Q}_{m,k_m}),$$

and assume the side length of the cube Q_{1,k_1} is the smallest among the side length of the cube Q_{i,k_i}, $1 \le i \le r$.

Denoting the center of Q_{i,k_i} by c_{i,k_i} and the sidelength of Q_{i,k_i} by l_{i,k_i}, let us estimate I_2. For fixed E_r, $x \in E_r$, $y_i \in Q_{i,k_i}$, $1 \le i \le r$, we have

$$|x-y_i| \sim |x-c_{i,k_i}|.$$

There exists $i_0 \in \{1,\ldots,r\}$ such that

$$|x-c_{i_0,k_{i_0}}| = \max\{|x-c_{i,k_i}| : i \in \{1,\ldots,r\}\}.$$

For $x \in E_r$ and $z \in \mathbb{R}^n$, let

$$A(x,z) := \Big\{(y_1,\dots,y_m) \in (\mathbb{R}^n)^m : 2|y_1 - c_{1,k_1}| \le \max_{1\le i\le m} |x+z-y_i|\Big\}.$$

Thus, using the vanish condition of atom, we have

$$\begin{aligned}
&|g_\lambda^*(a_{1,k_1},\dots,a_{m,k_m})(x)| \\
&= \bigg\{\int_0^\infty\int_{\mathbb{R}^n} \Big(\frac{t}{t+|z|}\Big)^{n\lambda} \bigg|\int_{(\mathbb{R}^n)^m} \\
&\quad \times \Big[K\Big(\frac{x+z-y_1}{t},\dots,\frac{x+z-y_m}{t}\Big) - K\Big(\frac{x+z-c_{1,k_1}}{t},\dots,\frac{x+z-y_m}{t}\Big)\Big] \\
&\quad \times a_{1,k_1}(y_1)\cdots a_{m,k_m}(y_m)\,dy_1\cdots dy_m\bigg|^2 \frac{dz\,dt}{t^{2mn+n+1}}\bigg\}^{1/2} \\
&\le \bigg\{\int_0^\infty\int_{\mathbb{R}^n} \Big(\frac{t}{t+|z|}\Big)^{n\lambda} \bigg|\int_{A} \\
&\quad \times \Big[K\Big(\frac{x+z-y_1}{t},\dots,\frac{x+z-y_m}{t}\Big) - K\Big(\frac{x+z-c_{1,k_1}}{t},\dots,\frac{x+z-y_m}{t}\Big)\Big] \\
&\quad \times a_{1,k_1}(y_1)\cdots a_{m,k_m}(y_m)\,dy_1\cdots dy_m\bigg|^2 \frac{dz\,dt}{t^{2mn+n+1}}\bigg\}^{1/2} \\
&\quad + \bigg\{\int_0^\infty\int_{\mathbb{R}^n} \Big(\frac{t}{t+|z|}\Big)^{n\lambda} \bigg|\int_{A^c} \\
&\quad \times \Big[K\Big(\frac{x+z-y_1}{t},\dots,\frac{x+z-y_m}{t}\Big) - K\Big(\frac{x+z-c_{1,k_1}}{t},\dots,\frac{x+z-y_m}{t}\Big)\Big] \\
&\quad \times a_{1,k_1}(y_1)\cdots a_{m,k_m}(y_m)\,dy_1\cdots dy_m\bigg|^2 \frac{dz\,dt}{t^{2mn+n+1}}\bigg\}^{1/2} \\
&=: \mathrm{I} + \mathrm{II}.
\end{aligned}$$

From (1.5), we see that

$$\begin{aligned}
\mathrm{I} &\lesssim \bigg\{\int_0^\infty\int_{\mathbb{R}^n} \Big(\frac{t}{t+|z|}\Big)^{n\lambda} \bigg|\int_{(\mathbb{R}^n)^m} \frac{|y_1-c_{1,k_1}|^\gamma/t^\gamma}{\big(1+\sum_{i=1}^m \frac{|x+z-y_i|}{t}\big)^{mn+\delta+\gamma}} \\
&\quad \times a_{1,k_1}(y_1)\cdots a_{m,k_m}(y_m)\,dy_1\cdots dy_m\bigg|^2 \frac{dz\,dt}{t^{2mn+n+1}}\bigg\}^{1/2} \\
&\lesssim \bigg\{\int_0^\infty\int_{\mathbb{R}^n} \Big(\frac{t}{t+|z|}\Big)^{n\lambda} \bigg|\int_{(\mathbb{R}^n)^m} \frac{|y_1-c_{1,k_1}|^\gamma}{\big(t+\sum_{i=1}^m |x+z-y_i|\big)^{mn+\delta+\gamma}} \\
&\quad \times a_{1,k_1}(y_1)\cdots a_{m,k_m}(y_m)\,dy_1\cdots dy_m\bigg|^2 \frac{dz\,dt}{t^{n+1-2\delta}}\bigg\}^{1/2},
\end{aligned}$$

which, together with, (2.1) and the fact

$$\int_{\mathbb{R}^n} \frac{1}{(b+|y-c|)^M}\,dy \le b^{n-M},$$

further implies that

$$
\begin{aligned}
\mathrm{I} &\lesssim l_{1,k_1}^{\gamma} \prod_{i=1}^{m} l_{i,k_i}^{-n/p_i} \bigg\{ \int_0^\infty \int_{\mathbb{R}^n} \Big(\frac{t}{t+|z|}\Big)^{n\lambda} \\
&\quad \times \bigg| \int_{Q_{1,k_1}\times\cdots\times Q_{r,k_r}} \frac{1}{\big(t+\sum_{i=1}^r |x+z-y_i|\big)^{nr+\delta+\gamma}}\, dy_1\cdots dy_r \bigg|^2 \frac{dz\,dt}{t^{n+1-2\delta}} \bigg\}^{1/2} \\
&\lesssim l_{1,k_1}^{\gamma} \prod_{i=1}^{m} l_{i,k_i}^{-n/p_i} \bigg\{ \int_0^{|x-c_{i_0,k_{i_0}}|} \int_{\mathbb{R}^n} \Big(\frac{t}{t+|z|}\Big)^{n\lambda} \\
&\quad \times \bigg| \int_{Q_{1,k_1}\times\cdots\times Q_{r,k_r}} \frac{1}{\big(t+\sum_{i=1}^r |x+z-y_i|\big)^{nr+\delta+\gamma}}\, dy_1\cdots dy_r \bigg|^2 \frac{dz\,dt}{t^{n+1-2\delta}} \bigg\}^{1/2} \\
&\quad + l_{1,k_1}^{\gamma} \prod_{i=1}^{m} l_{i,k_i}^{-n/p_i} \bigg\{ \int_{|x-c_{i_0,k_{i_0}}|}^\infty \int_{\mathbb{R}^n} \Big(\frac{t}{t+|z|}\Big)^{n\lambda} \\
&\quad \times \bigg| \int_{Q_{1,k_1}\times\cdots\times Q_{r,k_r}} \frac{1}{\big(t+\sum_{i=1}^r |x+z-y_i|\big)^{nr+\delta+\gamma}}\, dy_1\cdots dy_r \bigg|^2 \frac{dz\,dt}{t^{n+1-2\delta}} \bigg\}^{1/2} \\
&=: l_{1,k_1}^{\gamma} \prod_{i=1}^{m} l_{i,k_i}^{-n/p_i} (\mathrm{I}_1+\mathrm{I}_2).
\end{aligned}
$$

To show

$$
\mathrm{I} \lesssim \prod_{i=1}^{m} \frac{l_{i,k_i}^{n-n/p_i+\gamma/m}}{\big(|x-c_{i,k_i}|+l_{i,k_i}\big)^{n+\gamma/m}}, \tag{2.4}
$$

by the fact that Q_{1,k_1} is the smallest cube among $\{Q_{i,k_i}\}$, $1\le i\le r$, it suffices to show that

$$
\mathrm{I}_1+\mathrm{I}_2 \lesssim \frac{\prod_{i=1}^r l_{i,k_i}^n}{\big(|x-c_{i_0,k_{i_0}}|\big)^{nr+\gamma}}. \tag{2.5}
$$

Let

$$
\Theta_1 := \left\{ z\in\mathbb{R}^n : |z| \le \frac{1}{2}|x-c_{i_0,k_{i_0}}| \right\},
$$

$$
\Theta_2 := \left\{ z\in\mathbb{R}^n : \frac{1}{2}|x-c_{i_0,k_{i_0}}| < |z| < 4|x-c_{i_0,k_{i_0}}| \right\}
$$

and

$$
\Theta_3 := \{ z\in\mathbb{R}^n : |z| \ge 4|x-c_{i_0,k_{i_0}}| \}.
$$

When $t\in(0,|x-c_{i_0,k_{i_0}}|)$ and $z\in\Theta_1\cup\Theta_3$, we have

$$
t+\sum_{i=1}^r |x+z-y_i| \gtrsim \sum_{i=1}^r |x+z-y_i| \gtrsim |x-c_{i_0,k_{i_0}}|.
$$

Thus,

$$
\begin{aligned}
&\bigg\{\int_0^{|x-c_{i_0,k_{i_0}}|}\int_{\Theta_1\cup\Theta_3}\Big(\frac{t}{t+|z|}\Big)^{n\lambda}\\
&\quad\times\bigg|\int_{Q_{1,k_1}\times\cdots\times Q_{r,k_r}}\frac{1}{\big(t+\sum_{i=1}^r|x+z-y_i|\big)^{nr+\delta+\gamma}}\,dy_1\cdots dy_r\bigg|^2\frac{dz\,dt}{t^{n+1-2\delta}}\bigg\}^{1/2}\\
&\lesssim\frac{\prod_{i=1}^r l_{i,k_i}^n}{\big(|x-c_{i_0,k_{i_0}}|\big)^{nr+\gamma+\delta}}\bigg\{\int_0^{|x-c_{i_0,k_{i_0}}|}\int_{\Theta_1\cup\Theta_3}\Big(\frac{t}{t+|z|}\Big)^{n\lambda}\frac{dz\,dt}{t^{n+1-2\delta}}\bigg\}^{1/2}\\
&\lesssim\frac{\prod_{i=1}^r l_{i,k_i}^n}{\big(|x-c_{i_0,k_{i_0}}|\big)^{nr+\gamma+\delta}}\bigg\{\int_0^{|x-c_{i_0,k_{i_0}}|}\frac{dt}{t^{1-2\delta}}\bigg\}^{1/2}\\
&\lesssim\frac{\prod_{i=1}^r l_{i,k_i}^n}{\big(|x-c_{i_0,k_{i_0}}|\big)^{nr+\gamma}}.
\end{aligned}\tag{2.6}
$$

On the other hand, since $\gamma<\frac{n}{2}(\lambda-2m)$, there exists some $\epsilon>0$ such that $n(\lambda-2m)-2\gamma-2\epsilon>0$, thus

$$
\begin{aligned}
&\bigg\{\int_0^{|x-c_{i_0,k_{i_0}}|}\int_{\Theta_2}\Big(\frac{t}{t+|z|}\Big)^{n\lambda}\\
&\quad\times\bigg|\int_{Q_{1,k_1}\times\cdots\times Q_{r,k_r}}\frac{1}{\big(t+\sum_{i=1}^r|x+z-y_i|\big)^{nr+\delta+\gamma}}\,dy_1\cdots dy_r\bigg|^2\frac{dz\,dt}{t^{n+1-2\delta}}\bigg\}^{1/2}\\
&\lesssim\prod_{i=1,i\neq i_0}^r l_{i,k_i}^n\bigg\{\int_0^{|x-c_{i_0,k_{i_0}}|}\int_{\Theta_2}\Big(\frac{t}{|x-c_{i_0,k_{i_0}}|}\Big)^{n\lambda}\\
&\quad\times\bigg|\int_{Q_{i_0,k_{i_0}}}\frac{1}{t^{n(r-1/2)+\delta+\gamma+\epsilon}}\frac{1}{|x+z-y_{i_0}|^{n/2-\epsilon}}\,dy_{i_0}\bigg|^2\frac{dz\,dt}{t^{n+1-2\delta}}\bigg\}^{1/2}\\
&\lesssim\frac{\prod_{i=1}^r l_{i,k_i}^n}{\big(|x-c_{i_0,k_{i_0}}|\big)^{n\lambda/2}}\bigg\{\int_0^{|x-c_{i_0,k_{i_0}}|}\frac{t^{n\lambda}}{t^{2nr+2\gamma+2\epsilon+1}}\,dt\bigg\}^{1/2}\\
&\quad\times\bigg\{\int_{\Theta_2}\bigg|\int_{Q_{i_0,k_{i_0}}}\frac{1}{|x+z-y_{i_0}|^{n/2-\epsilon}}\,dy_{i_0}\bigg|^2dz\bigg\}^{1/2}\\
&\lesssim\frac{\prod_{i=1}^r l_{i,k_i}^n}{\big(|x-c_{i_0,k_{i_0}}|\big)^{nr+\gamma}}.
\end{aligned}\tag{2.7}
$$

When $t\in[|x-c_{i_0,k_{i_0}}|,\infty)$, we have

$$t+\sum_{i=1}^r|x+z-y_i|\gtrsim t.$$

Thus,

$$
\begin{aligned}
&\Big\{\int_{|x-c_{i_0,k_{i_0}}|}^{\infty}\int_{\mathbb{R}^n}\Big(\frac{t}{t+|z|}\Big)^{n\lambda} \\
&\quad\times\Big|\int_{Q_{1,k_1}\times\cdots\times Q_{r,k_r}}\frac{1}{\big(t+\sum_{i=1}^{r}|x+z-y_i|\big)^{nr+\delta+\gamma}}\,dy_1\cdots dy_r\Big|^2\frac{dz\,dt}{t^{n+1-2\delta}}\Big\}^{1/2} \\
&\lesssim\prod_{i=1}^{r} l_{i,k_i}^{n}\Big\{\int_{|x-c_{i_0,k_{i_0}}|}^{\infty}\int_{\mathbb{R}^n}\Big(\frac{t}{t+|z|}\Big)^{n\lambda}\frac{dz\,dt}{t^{2nr+2\gamma+n+1}}\Big\}^{1/2} \\
&\lesssim\frac{\prod_{i=1}^{r} l_{i,k_i}^{n}}{\big(|x-c_{i_0,k_{i_0}}|\big)^{nr+\gamma}},
\end{aligned}
\tag{2.8}
$$

where we use

$$\int_{\mathbb{R}^n}\Big(\frac{t}{t+|z|}\Big)^{n\lambda}dz\lesssim t^n$$

as $\lambda>1$.

Combining (2.6), (2.7) and (2.8), we show that (2.5) holds and hence (2.4).

Now we estimate II. From (1.4), and $2l_{1,k_1}\geq 2|y_1-c_{1,k_1}|\geq\max_{1\leq i\leq m}|x+z-y_i|$, we see that

$$
\begin{aligned}
\mathrm{II}\lesssim&\Big\{\int_0^{\infty}\int_{\mathbb{R}^n}\Big(\frac{t}{t+|z|}\Big)^{n\lambda}\Big|\int_{(\mathbb{R}^n)^m}\frac{l_{1,k_1}^{\gamma}}{\big(1+\sum_{i=1}^{m}\frac{|x+z-y_i|}{t}\big)^{mn+\delta}\big(\sum_{i=1}^{m}|x+z-y_i|\big)^{\gamma}} \\
&\times a_{1,k_1}(y_1)\cdots a_{m,k_m}(y_m)\,dy_1\cdots dy_m\Big|^2\frac{dz\,dt}{t^{2mn+n+1}}\Big\}^{1/2}.
\end{aligned}
$$

Then, by the same method as in the proof of I, we have

$$\mathrm{II}\lesssim\prod_{i=1}^{m}\frac{l_{i,k_i}^{n-n/p_i+\gamma/m}}{\big(|x-c_{i,k_i}|+l_{i,k_i}\big)^{n+\gamma/m}},$$

which, together with (2.4), further completes the proof of (2.3) and hence Theorem 1.3. □

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References

[1] X. Chen, Q. Xue and K. Yabuta, *On multilinear Littlewood-Paley operators*, Nonlinear Anal. **115** (2015), 25–40.

[2] M. Christ and J.-L. Journé, *Polynomial growth estimates for multilinear singular integral operators*, Acta Math. **159** (1987), no. 1-2, 51–80.

[3] R. R. Coifman, *A real variable characterization of H^p*, Studia Math. **51** (1974), 269–274.

[4] R. R. Coifman and Y. Meyer, *On commutators of singular integrals and bilinear singular integrals*, Trans. Amer. Math. Soc. **212** (1975), 315–331.

[5] ———, *Commutateurs d'intégrales singulières et opérateurs multilin'eaires*, Ann. Inst. Fourier (Grenoble) **28** (1978), no. 3, 177–202.

[6] C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), no. 3-4, 137–193.

[7] L. Grafakos and N. Kalton, *Multilinear Calderón-Zygmund operators on Hardy spaces*, Collect. Math. **52** (2001), no. 2, 169–179.

[8] L. Grafakos, A. Miyachi and N. Tomita, *On multilinear Fourier multipliers of limited smoothness*, Canad. J. Math. **65** (2013), no. 2, 299–330.

[9] L. Grafakos and R. H. Torres, *Multilinear Calderón-Zygmund theory*, Adv. Math. **165** (2002), no. 1, 124–164.

[10] L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math. **104** (1960), 93–140.

[11] C. E. Kenig and E. M. Stein, *Multilinear estimates and fractional integration*, Math. Res. Lett. **6** (1999), no. 1, 1–15.

[12] R. H. Latter, *A characterization of $H^p(\mathbb{R}^n)$ in terms of atoms*, Studia Math. **62** (1978), no. 1, 92–101.

[13] A. K. Lerner, S. Ombrosi, C. Pérez, R. H. Torres and R. Trujillo-González, *New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory*, Adv. Math. **220** (2009), no. 4, 1222–1264.

[14] W. Li, Q. Xue and K. Yabuta, *Multilinear Calderón-Zygmund operators on weighted Hardy spaces*, Studia Math. **199** (2010), no. 1, 1–16.

[15] M. Sakamoto and K. Yabuta, *Boundedness of Marcinkiewicz functions*, Studia Math. **135** (1999), no. 2, 103–142.

[16] S. Shi, Q. Xue and K. Yabuta, *On the boundedness of multilinear Littlewood-Paley g_λ^* function*, J. Math. Pures Appl. **101** (2014), no. 3, 394–413.

[17] E. M. Stein, *On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz*, Trans. Amer. Math. Soc. **88** (1958), 430–466.

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