Existence and Boundedness of Second-order Karush-Kuhn-Tucker Multipliers for Set-valued Optimization with Variable Ordering Structures

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Abstract. In this paper we investigate second-order Karush-Kuhn-Tucker multipliers for both local nondominated and local minimal points of set-valued optimization with variable ordering structures. We prove calculus rules of second-order contingent derivatives of index $\gamma \in \{0,1\}$ and use them to establish improved Karush-Kuhn-Tucker multiplier rules of nonclassical forms which involve separately such derivatives of the objective, constraint and ordering maps. The equivalence between the nonemptiness and boundedness of the multiplier sets in these rules and second-order constraint qualifications of the Kurcyusz-Robinson-Zowe and Mangasarian-Fromovitz types is demonstrated.

1. Introduction

Let X, Y and Z be real Banach spaces. In this paper we address the following set-valued vector optimization problem with variable ordering structure:

$$\text{Min}_{\mathcal{C}(x)}F(x) \quad \text{such that} \quad G(x) \cap (-D) \neq \emptyset,$$

where $F\colon X \rightrightarrows Y$, $G\colon X \rightrightarrows Z$, $\mathcal{C}\colon X \rightrightarrows Y$ are nonempty-valued and D is a closed convex cone with nonempty interior in Z. Suppose that $\mathcal{C}(x)$ is a closed convex cone in Y for each $x \in X$. Then, $\mathcal{C}$ defines a variable partial order on Y by

$$y_1 \leq_{\mathcal{C}(x)} y_2 \quad \Longleftrightarrow \quad y_2 - y_1 \in \mathcal{C}(x).$$

A vector optimization problem with such a variable ordering is usually referred to as a problem with variable ordering structure. This type of problem was introduced by Yu and Berstreeser in [5, 38]. In this framework, a point being local nondominated means that it is the best with respect to the order of any point in a neighborhood. This concept is more general than the local Pareto point and is crucial in many applications to medical image registration, decision making, location theory, game theory etc, see [5, 14, 15, 39] and the

references therein. Besides the concept of nondominated point, another optimality notion was considered by Chen and Yang [6]. Namely, a point is local minimal if it is the best in a neighborhood with respect to only the order at this point. In the literature, there have been only a few contributions to necessary optimality conditions for vector optimization with variable ordering structures. Based on advanced tools of variational analysis and generalized differentiation, Bao and Mordukhovich [4] established some necessary conditions for nondominated points of sets and nondominated solutions of constrained multiobjective optimization problems with respect to general variable ordering structures. For unconstrained set-valued optimization, Durea et al. [12] employed the contingent derivative and Mordukhovich generalized differentiation objects to obtain generalized Fermat multiplier rules. Eichfelder and Ha in [16] considered generalized Fermat and Lagrange multiplier rules for set-valued optimization problems with respect to the Bishop-Phelps variable ordering structure. All these papers dealt with only first-order conditions. To the best of our knowledge, there are no publications devoted to second-order optimality conditions, while second-order conditions are always of a great interest because they refine the first-order ones by second-order information which is very helpful for recognizing solutions as well as for designing numerical algorithms to compute them.

Besides optimality conditions, boundedness of Karush-Kuhn-Tucker (KKT) multiplier sets is also important, e.g., for studies of stability and numerical algorithms (see [1,18] and the references therein). As far as we know, there have been only first-order considerations of this boundedness for the fixed ordering case so far. Namely, in [17] Gauvin considered the nonemptiness and boundedness of first-order KKT multiplier sets for finite-dimensional scalar problems with smooth data under the Mangasarian-Fromovitz constraint qualification. For vector problems in finite dimensions, Dutta and Lalitha [13] got corresponding results for nonsmooth problems and Li and Zhang [28] investigated this topic in terms of upper convexificators of locally Lipschitz functions. For vector problems in infinite dimensions, Durea et al. [10] showed that the Mangasarian-Fromovitz constraint qualification ensures the boundedness of first-order KKT multipliers for single-valued optimization with smooth data. They also studied corresponding results for a set-valued problem in terms of Mordukhovich coderivatives. All the aforementioned contributions to the boundedness topic are only for first-order multiplier sets and in cases with nonvariable ordering structures.

Motivated by the preceding discussions, in this paper we consider the existence and boundedness of second-order KKT multiplier sets for problems with variable ordering structures. After Section 2 presenting preliminary facts, in Section 3 we establish calculus rules for second-order contingent derivatives of index $\gamma \in \{0, 1\}$. We obtain explicit formulas for the sum and product of these derivatives by both imposing the known defi-

nition of pseudo-Lipschitz property and proto-differentiability and proposing new notions of directional metric subregularity of index γ, directional compactness and Shi contingent derivative. Section 4 contains main results. First, we investigate second-order KKT necessary conditions of nonclassical forms for both nondominated points and minimal points of a set-valued problem with generalized inequality constraints. Thanks to the usage of the calculus rules obtained in Section 3, in our KKT multiplier rules, the derivatives of the objective, constraint and ordering maps are involved separately, and we can impose constraint qualifications in terms of only the constraint map (these facts make the rules sharper). Regarding the boundedness of multipliers, we propose a relaxed second-order Mangasarian-Fromovitz constraint qualification and prove the equivalence between this qualification, the second-order Kurcyusz-Robinson-Zowe (KRZ) qualification and the nonemptiness and boundedness of KKT multiplier sets when the aforementioned assumptions for our calculus rules are satisfied.

2. Preliminaries

Throughout the paper, if not otherwise stated, let X, Y and Z be real Banach spaces, $\mathbb{N}$, $\mathbb{R}^n$ and $\mathbb{R}^n_+$ be the set of the natural numbers, an n-dimensional vector space and its nonnegative orthant, respectively. B_X denotes the open unit ball of X and $B_X(x, r)$ the open ball with center x and radius r. For $M \subseteq X$, $\operatorname{int} M$, $\operatorname{cl} M$ and $\operatorname{bd} M$ stand for its interior, closure and boundary, respectively, of M. The distance from x to M is $d(x, M) := \inf\{\|x - a\| \mid a \in M\}$, with the convention $d(x, \emptyset) = \infty$. The cone generated by M is $\operatorname{cone} M := \{\lambda x \mid \lambda \geq 0, x \in M\}$. X^* stands for the topological dual of X and $\langle \cdot, \cdot \rangle$ for the canonical pairing of any pair of dual spaces. For a cone $C \subseteq Y$, the dual cone is $C^* := \{y^* \in Y^* \mid \langle y^*, c \rangle \geq 0, \forall c \in C\}$. We denote $D(z_0) := \operatorname{cone}(D + z_0)$. Then, for $z_0 \in -D$, $[D(z_0)]^* = N(-D, z_0)$, the normal cone of $-D$ at z_0. Furthermore, if D is a convex cone, $N(-D, z_0) = \{d^* \in D^* \mid \langle d^*, z_0 \rangle = 0\}$.

We recall that $D \subseteq Z$ is called dually compact (see [40]) if there exists a compact set $M \subseteq Z$ such that

$$D^* \subseteq \left\{ z^* \in Z^* \;\middle|\; \|z^*\| \leq \sup_{z \in M} \langle z^*, z \rangle \right\}.$$

Here are several basic properties of dually compact cones (see [10, 40]):

- a cone with nonempty interior in a normed space is dually compact (in general, the converse does not hold, see Example 2.1 in [10]);

- if D is dually compact, every w^*-convergent-to-0 sequence $\{d_n\} \subseteq D^*$ also converges strongly to 0.

For a set-valued map $F\colon X \rightrightarrows Y$, the domain and graph are, respectively, $\operatorname{dom} F :=$ $\{x \in X \mid F(x) \neq \emptyset\}$ and $\operatorname{gph} F := \{(x,y) \in X \times Y \mid y \in F(x)\}$. F is said to be pseudo-Lipschitz at $(x_0, y_0) \in \operatorname{gph} F$ (see [2]) if there exist neighborhoods U of x_0, V of y_0 and $L > 0$ such that $F(x) \cap V \subseteq F(x') + L\|x - x'\| \operatorname{cl} B_Y$, $\forall x, x' \in U$. (This property is also called the Aubin property.) F is called compact at x_0 (see [32]) if, for $(x_n, y_n) \in \operatorname{gph} F$ with $x_n \to x_0$, there exist $(x_{n_k}, y_{n_k}) \to (x_0, y)$ for some $y \in F(x_0)$. F is said to be second-order directionally compact with index γ at (x_0, y_0) with respect to (u, v) in direction $x \in X$ (see [37]) if for any $(t_n, r_n) \downarrow (0,0)$, $\frac{t_n}{r_n} \to \gamma$ and $x_n \to x$, every sequence $y_n \in Y$ satisfying $y_0 + t_n v + \frac{1}{2} t_n r_n y_n \in F(x_0 + t_n u + \frac{1}{2} t_n r_n x_n)$ has a convergent subsequence.

Definition 2.1. Let $\Phi\colon X \rightrightarrows Y$ and $(x_0, y_0) \in \operatorname{gph} \Phi$. Φ is called metrically regular at (x_0, y_0) if there exist $\alpha > 0$, neighborhoods U of x_0 and V of y_0 such that, for all $(x, y) \in U \times V$,

$$d(x, \Phi^{-1}(y)) \leq \alpha d(y, \Phi(x)).$$

By fixing $y = y_0$, we obtain a weaker property called metric subregularity. For developments of linear and nonlinear models of regularity with applications, the reader is referred to $[2, 7, 8, 20, 23, 30, 34, 35]$ and the references therein. Here, we use the following directional metric subregularity.

Definition 2.2. Let $\Phi\colon X \rightrightarrows Y$, $(x_0, y_0) \in \operatorname{gph} \Phi$, $S \subseteq X$ and $u \in X$. Then, Φ is said to be directionally metrically subregular at (x_0, y_0) in direction u with respect to S if there are a neighborhood U of x_0, $\alpha \geq 0$ and $r > 0$ such that, for all $t \in (0, r)$ and $v \in B_X(u, r)$ with $x_0 + tv \in S \cap U$,

$$d(x_0 + tv, \Phi^{-1}(y_0) \cap S) \leq \alpha d(y_0, \Phi(x_0 + tv)).$$

We extend this notion relative to a second-order direction w besides the direction u as follows.

Definition 2.3. Let $\Phi\colon X \rightrightarrows Y$, $(x_0, y_0) \in \operatorname{gph} \Phi$, $S \subseteq X$ and $u, w \in X$. Then, Φ is said to be (u, w)-directionally metrically subregular of index γ at (x_0, y_0) with respect to S if there are a neighborhood U of x_0 and $\alpha \geq 0$ such that, for any $(t_n, r_n, w_n) \to (0^+, 0^+, w)$ with $\frac{t_n}{r_n} \to \gamma$ and $x_0 + t_n u + \frac{1}{2} t_n r_n w_n \in S \cap U$,

$$d(x_0 + t_n u + \tfrac{1}{2} t_n r_n w_n, \Phi^{-1}(y_0) \cap S) \leq \alpha d(y_0, \Phi(x_0 + t_n u + \tfrac{1}{2} t_n r_n w_n)).$$

If this property is satisfied for all $w \in X$, we replace (u, w) by u in the saying.

Next, we recall notions of tangent cones and second-order tangent sets.

Definition 2.4. Let $M \subseteq X$, $x_0, u \in X$ and $\gamma \in \{0, 1\}$.

(i) The contingent cone, adjacent cone and interior cone of M at x_0 are

$$T(M, x_0) := \{u \in X \mid \exists\, t_n \downarrow 0, \exists\, u_n \to u, \forall\, n \in \mathbb{N}, x_0 + t_n u_n \in M\},$$

$$T^\flat(M, x_0) := \{u \in X \mid \forall\, t_n \downarrow 0, \exists\, u_n \to u, \forall\, n \in \mathbb{N}, x_0 + t_n u_n \in M\},$$

$$IT(M, x_0) := \{u \in X \mid \forall\, t_n \downarrow 0, \forall\, u_n \to u, \forall\, n \text{ large}, x_0 + t_n u_n \in M\}.$$

(ii) The second-order contingent set of index γ of M at (x, u) is

$$T_\gamma^2(M, x, u)$$
$$:= \{w \in X \mid \exists\, (t_n, r_n) \downarrow (0,0), \tfrac{t_n}{r_n} \to \gamma, \exists\, w_n \to w, x + t_n u + \tfrac{1}{2} t_n r_n w_n \in M, \forall\, n \in \mathbb{N}\}.$$

(iii) The second-order adjacent set of index γ of M at (x, u) is

$$T_\gamma^{\flat 2}(M, x, u)$$
$$:= \{w \in X \mid \forall\, (t_n, r_n) \downarrow (0,0), \tfrac{t_n}{r_n} \to \gamma, \exists\, w_n \to w, x + t_n u + \tfrac{1}{2} t_n r_n w_n \in M, \forall\, n \in \mathbb{N}\}.$$

(iv) The second-order interior tangent set of index γ of M at (x, u) is

$$IT_\gamma^2(M, x, u)$$
$$:= \{w \in X \mid \forall\, (t_n, r_n) \downarrow (0,0), \tfrac{t_n}{r_n} \to \gamma, \forall\, w_n \to w, x + t_n u + \tfrac{1}{2} t_n r_n w_n \in M, \forall\, n \text{ large}\}.$$

Note that, if $x_0 \notin \mathrm{cl}\, M$, then all the above tangent sets are empty; and if $u \notin T(M, x_0)$, then all the second-order tangent sets are empty. Hence, we always assume conditions like $x_0 \in \mathrm{cl}\, M$, $u \in T(M, x_0)$. When $\gamma = 1$, the sets $T_1^2(M, x, u)$, $T_1^{\flat 2}(M, x, u)$ and $IT_1^2(M, x, u)$ are said to be the second-order contingent, adjacent and interior tangent sets, respectively. They are closed sets, but not necessarily cones. If M is convex, then $T_1^{\flat 2}(M, x_0, u)$ is convex, while $T^2(M, x_0, u)$ may not be convex. When $\gamma = 0$, the sets $T_0^2(M, x, u)$, $T_0^{\flat 2}(M, x, u)$ and $IT_0^2(M, x, u)$ are cones and called the asymptotic second-order contingent, adjacent and interior tangent cones, respectively. The cones $T_0^2(M, x_0, u)$, $T_0^{\flat 2}(M, x_0, u)$ were proposed by Penot [33]. If X is a reflexive Banach space and $u \in T(M, x_0)$, then either $T_1^2(M, x_0, u)$ or $T_0^2(M, x_0, u)$ is nonempty. Some known properties of second-order tangent sets are collected in the following (see more in [7, 19, 21, 24, 33]).

Proposition 2.5. *Let $M \subseteq X$, $x_0, u \in X$ and $\gamma \in \{0, 1\}$.*

(i) $T_\gamma^2(M, x_0, 0) = T(M, x_0)$ *and* $T_\gamma^{\flat 2}(M, x_0, 0) = T^\flat(M, x_0)$.

(ii) $IT_\gamma^2(M, x_0, u) \subseteq T_\gamma^{\flat 2}(M, x_0, u) \subseteq T_\gamma^2(M, x_0, u) \subseteq \mathrm{clcone}(\mathrm{cone}(M - x_0) - u)$.

Let, in addition, M be convex and $u \in T(M, x_0)$. Then, the following assertions hold.

(iii) $T(T(M, x_0), u) = \mathrm{clcone}(\mathrm{cone}(M - x_0) - u)$, *and hence* $T_1^2(M, x_0, u) \subseteq T(T(M, x_0), u)$. *Additionally, if* $0 \in T_1^2(M, x_0, u)$, *then* $T_1^2(M, x_0, u) = T(T(M, x_0), u)$. *If* $T_0^2(M, x_0, u) \neq \emptyset$, *then* $T_0^2(M, x_0, u) = T(T(M, x_0), u)$, *and hence* $T_1^2(M, x_0, u) \subseteq T_0^2(M, x_0, u)$.

(iv) If $T_\gamma^{\flat 2}(M, x_0, u) \neq \emptyset$, then $IT_\gamma^{\flat 2}(M, x_0, u) = \operatorname{int} T_\gamma^{\flat 2}(M, x_0, u)$, $\operatorname{cl} IT_\gamma^{\flat 2}(M, x_0, u) = T_\gamma^{\flat 2}(M, x_0, u)$ and

$$T_\gamma^{\flat 2}(M, x_0, u) + T(T(M, x_0), u) \subseteq T_\gamma^{\flat 2}(M, x_0, u).$$

Definition 2.6. [2, 22] Let $F: X \rightrightarrows Y$, $x \in X$, $(x_0, y_0) \in \operatorname{gph} F$, $(u, v) \in X \times Y$ and $\gamma \in \{0, 1\}$.

(i) The contingent derivative of F at (x_0, y_0) is a set-valued map $DF(x_0, y_0): X \rightrightarrows Y$ defined by

$$DF(x_0, y_0)(x)$$
$$:= \{y \in Y \mid \exists\, t_n \downarrow 0, (x_n, y_n) \to (x, y), y_0 + t_n y_n \in F(x_0 + t_n x_n), \forall\, n \in \mathbb{N}\}.$$

(ii) The second-order contingent derivative of index γ of F at (x_0, y_0) in direction (u, v) is a set-valued map $D_\gamma^2 F(x_0, y_0, u, v): X \rightrightarrows Y$ defined by

$$D_\gamma^2 F(x_0, y_0, u, v)(x)$$
$$:= \{y \in Y \mid \exists\, (t_n, r_n) \downarrow (0, 0), \tfrac{t_n}{r_n} \to \gamma, \exists\, (x_n, y_n) \to (x, y),$$
$$y_0 + t_n v + \tfrac{1}{2} t_n r_n y_n \in F(x_0 + t_n u + \tfrac{1}{2} t_n r_n x_n), \forall\, n \in \mathbb{N}\}.$$

(iii) The second-order adjacent contingent derivative of index γ of F at (x_0, y_0) in direction (u, v) is a set-valued map $D_\gamma^{\flat 2} F(x_0, y_0, u, v): X \rightrightarrows Y$ defined by

$$D_\gamma^{\flat 2} F(x_0, y_0, u, v)(x)$$
$$:= \{y \in Y \mid \forall\, (t_n, r_n) \downarrow (0, 0), \tfrac{t_n}{r_n} \to \gamma, \exists\, (x_n, y_n) \to (x, y),$$
$$y_0 + t_n v + \tfrac{1}{2} t_n r_n y_n \in F(x_0 + t_n u + \tfrac{1}{2} t_n r_n x_n), \forall\, n \in \mathbb{N}\}.$$

Remark 2.7. (i) It is easy to see that, for every $x \in X$,

$$D_\gamma^{\flat 2} F(x_0, y_0, u, v)(x) \subseteq D_\gamma^2 F(x_0, y_0, u, v)(x).$$

In general the reverse inclusion does not hold.

(ii) Obviously, $\operatorname{gph} DF(x_0, y_0) = T(\operatorname{gph} F, (x_0, y_0))$, $\operatorname{gph} D_\gamma^2 F(x_0, y_0, u, v) = T_\gamma^2(\operatorname{gph} F, (x_0, y_0), (u, v))$ and $\operatorname{gph} D_\gamma^{\flat 2} F(x_0, y_0, u, v) = T_\gamma^{\flat 2}(\operatorname{gph} F, (x_0, y_0), (u, v))$. Then, it follows from Proposition 2.5(i) that, for $x \in X$, $D_\gamma^2 F(x_0, y_0, 0, 0)(x) = DF(x_0, y_0)(x)$.

(iii) (see [26]) We say that F is second-order proto-differentiable at (x_0, y_0) in direction (u, v) if $D_\gamma^{\flat 2} F(x_0, y_0, u, v) = D_\gamma^2 F(x_0, y_0, u, v)$. If $\operatorname{gph} F$ is convex and $(u, v) \in \operatorname{gph} F - (x_0, y_0)$, one has $T_\gamma^2(\operatorname{gph} F, (x_0, y_0), (u, v)) = T_\gamma^{\flat 2}(\operatorname{gph} F, (x_0, y_0), (u, v))$ (see [27]) and then, by Remark 2.7(i), F is second-order proto-differentiable at (x_0, y_0) in direction (u, v).

(iv) If $\operatorname{gph} F$ is convex, $(0,0) \in T^2_\gamma(\operatorname{gph} F, (x_0, y_0), (u, v))$ and $D^2_\gamma F(x_0, y_0, u, v)$ is compact at x, then F is second-order directionally compact with index γ at (x_0, y_0) with respect to (u, v) in the direction x. Indeed, because $\operatorname{gph} F$ is convex and $(0,0) \in T^2_\gamma(\operatorname{gph} F, (x_0, y_0), (u, v))$, by Proposition 2.5(iii), one has

$$T^2_\gamma(\operatorname{gph} F, (x_0, y_0), (u, v)) = T(T(\operatorname{gph} F, (x_0, y_0)), (u, v))$$
$$= \operatorname{cl}(\operatorname{cone}(\operatorname{cone}(\operatorname{gph} F - (x_0, y_0)) - (u, v))).$$

For any sequences $(t_n, r_n) \downarrow (0, 0)$ with $\frac{t_n}{r_n} \to \gamma$, $x_n \to x$ and y_n with

$$y_0 + t_n v + \tfrac{1}{2} t_n r_n y_n \in F(x_0 + t_n u + \tfrac{1}{2} t_n r_n x_n),$$

one has $(x_0 + t_n u + \tfrac{1}{2} t_n r_n x_n, y_0 + t_n v + \tfrac{1}{2} t_n r_n y_n) \in \operatorname{gph} F$. This implies that

$$(x_n, y_n) \in 2 r_n^{-1} \left(t_n^{-1} (\operatorname{gph} F - (x_0, y_0)) - (u, v) \right)$$
$$\subseteq \operatorname{cl}(\operatorname{cone}(\operatorname{cone}(\operatorname{gph} F - (x_0, y_0)) - (u, v))).$$

Hence, $(x_n, y_n) \in T^2_\gamma(\operatorname{gph} F, (x_0, y_0), (u, v))$, i.e., $y_n \in D^2_\gamma F(x_0, z_0, u, v)(x_n)$. As $D^2_\gamma F(x_0, z_0, u, v)$ is compact at x, y_n has a convergent subsequence. So, F is second-order directionally compact with index γ at (x_0, y_0) with respect to (u, v) in the direction x.

3. Calculus rules of second-order derivatives

For $F \colon X \rightrightarrows Y$, $(x_0, y_0) \in \operatorname{gph} F$ and $(u, v) \in X \times Y$, we impose the following condition

$(\mathrm{A}(F, x_0, y_0, u, v))$ F is second-order proto-differentiable at (x_0, y_0) in direction (u, v) and F is pseudo-Lipschitz at (x_0, y_0).

Proposition 3.1. *Let* $F_1, F_2 \colon X \rightrightarrows Y$, $\overline{x}, u \in X$, $\overline{y}_1, \overline{y}_2, v_1, v_2 \in Y$ *and* $\gamma \in \{0, 1\}$. *If either* $(\mathrm{A}(F_1, \overline{x}, \overline{y}_1, u, v_1))$ *or* $(\mathrm{A}(F_2, \overline{x}, \overline{y}_2, u, v_2))$ *is satisfied, then for any* $x \in X$,

(i) $D^2_\gamma F_1(\overline{x}, \overline{y}_1, u, v_1)(x) + D^2_\gamma F_2(\overline{x}, \overline{y}_2, u, v_2)(x) \subseteq D^2_\gamma(F_1 + F_2)(\overline{x}, \overline{y}_1 + \overline{y}_2, u, v_1 + v_2)(x);$

(ii) $D^2_\gamma F_1(\overline{x}, \overline{y}_1, u, v_1)(x) \times D^2_\gamma F_2(\overline{x}, \overline{y}_2, u, v_2)(x) = D^2_\gamma(F_1, F_2)(\overline{x}, (\overline{y}_1, \overline{y}_2), u, (v_1, v_2))(x).$

Proof. By reasons of similarity, we assume that $(\mathrm{A}(F_1, \overline{x}, \overline{y}_1, u, v_1))$ holds.

(i) Let $y_1 \in D^2_\gamma F_1(\overline{x}, \overline{y}_1, u, v_1)(x)$ and $y_2 \in D^2_\gamma F_2(\overline{x}, \overline{y}_2, u, v_2)(x)$. Since F_1 is second-order proto-differentiable at $(\overline{x}, \overline{y}_1)$ in direction (u, v_1), there are $(t_n, r_n) \downarrow (0, 0)$, $\frac{t_n}{r_n} \to \gamma$, $(x'_n, y'_{1n}) \to (x, y_1)$ and $(x_n, y_{2n}) \to (x, y_2)$ such that

$$\overline{y}_1 + t_n v_1 + \tfrac{1}{2} t_n r_n y'_{1n} \in F_1(\overline{x} + t_n u + \tfrac{1}{2} t_n r_n x'_n),$$
$$\overline{y}_2 + t_n v_2 + \tfrac{1}{2} t_n r_n y_{2n} \in F_2(\overline{x} + t_n u + \tfrac{1}{2} t_n r_n x_n).$$

Because F_1 is pseudo-Lipschitz at $(\overline{x}, \overline{y}_1)$, there exist a neighborhood V of $\overline{y}_1$ and $L > 0$ such that, for large n,

$$F_1(\overline{x} + t_n u + \tfrac{1}{2} t_n r_n x_n') \cap V \subseteq F_1(\overline{x} + t_n u + \tfrac{1}{2} t_n r_n x_n) + \tfrac{1}{2} L t_n r_n \|x_n' - x_n\| \operatorname{cl} B_Y.$$

Hence, there is $b_n \in \operatorname{cl} B_Y$ such that, for large n,

$$\overline{y}_1 + t_n v_1 + \tfrac{1}{2} t_n r_n (y_{1n}' - L\|x_n' - x_n\| b_n) \in F_1(\overline{x} + t_n u + \tfrac{1}{2} t_n r_n x_n).$$

By setting $y_{1n} := y_{1n}' - L\|x_n' - x_n\| b_n$, one has $y_{1n} \to y_1$ and

$$\overline{y}_1 + t_n v_1 + \tfrac{1}{2} t_n r_n y_{1n} \in F_1(\overline{x} + t_n u + \tfrac{1}{2} t_n r_n x_n).$$

Adding the above relations, one has

$$\overline{y}_1 + \overline{y}_2 + t_n(v_1 + v_2) + \tfrac{1}{2} t_n r_n (y_{1n} + y_{2n}) \in (F_1 + F_2)(\overline{x} + t_n u + \tfrac{1}{2} t_n r_n x_n).$$

Hence, $y_1 + y_2 \in D_\gamma^2(F_1 + F_2)(\overline{x}, \overline{y}_1 + \overline{y}_2, u, v_1 + v_2)(x)$.

(ii) It is easy to see that $D_\gamma^2(F_1, F_2)(\overline{x}, (\overline{y}_1, \overline{y}_2), u, (v_1, v_2)) \subseteq D_\gamma^2 F_1(\overline{x}, \overline{y}_1, u, v_1) \times D_\gamma^2 F_2(\overline{x}, \overline{y}_2, u, v_2)$. For the reverse inclusion, let (y_1, y_2) be in the right-hand side. Similar to the proof of part (i), there are $(t_n, r_n) \downarrow (0,0)$, $\frac{t_n}{r_n} \to \gamma$, $(x_n, y_{1n}) \to (x, y_1)$ and $(x_n, y_{2n}) \to (x, y_2)$ such that

$$\overline{y}_1 + t_n v_1 + \tfrac{1}{2} t_n r_n y_{1n} \in F_1(\overline{x} + t_n u + \tfrac{1}{2} t_n r_n x_n),$$
$$\overline{y}_2 + t_n v_2 + \tfrac{1}{2} t_n r_n y_{2n} \in F_2(\overline{x} + t_n u + \tfrac{1}{2} t_n r_n x_n).$$

Hence,

$$(\overline{y}_1, \overline{y}_2) + t_n(v_1, v_2) + \tfrac{1}{2} t_n r_n (y_{1n}, y_{2n}) \in (F_1, F_2)(\overline{x} + t_n u + \tfrac{1}{2} t_n r_n x_n).$$

This means that $(y_1, y_2) \in D_\gamma^2(F_1, F_2)(\overline{x}, (\overline{y}_1, \overline{y}_2), u, (v_1, v_2))$ and completes the proof. $\square$

Note that Proposition 3.1(ii) is also true for $F_2 \colon X \rightrightarrows Z$ with $Z \neq Y$.

Definition 3.2. The second-order Shi contingent derivative of F at (x_0, y_0) in direction (u, v) is a set-valued map $D_S^2 F(x_0, y_0, u, v) \colon X \rightrightarrows Y$ defined by

$$D_S^2 F(x_0, y_0, u, v)(x)$$
$$:= \{y \in Y \mid \exists\, t_n \downarrow 0, \exists\, r_n > 0, \exists\, (x_n, y_n) \to (x, y), t_n r_n x_n \to 0,$$
$$y_0 + t_n v + \tfrac{1}{2} t_n r_n y_n \in F(x_0 + t_n u + \tfrac{1}{2} t_n r_n x_n), \forall\, n \in \mathbb{N}\}.$$

By imposing other assumptions (different from $(A(F, x_0, y_0, u, v)))$, we have the following rule including equality instead of the inclusion in Proposition 3.1(i).

Proposition 3.3. *Let $F_1, F_2 \colon X \rightrightarrows Y$, $\overline{x}, u \in X$, $\overline{y}_1, \overline{y}_2, v_1, v_2 \in Y$ and $\gamma \in \{0, 1\}$. Suppose that either F_1 is second-order proto-differentiable at $(\overline{x}, \overline{y}_1)$ in direction (u, v_1) or F_2 is second-order proto-differentiable at $(\overline{x}, \overline{y}_2)$ in direction (u, v_2), and the function $g \colon (X \times Y)^2 \to X$, defined by $g(a, b, c, d) = a - c$, is $((u, v_1), (u, v_2))$-directionally metrically subregular of index γ at $((\overline{x}, \overline{y}_1), (\overline{x}, \overline{y}_2), 0)$ with respect to $\operatorname{gph} F_1 \times \operatorname{gph} F_2$. Then, for any $x \in X$,*

  (i) $D_\gamma^2 F_1(\overline{x}, \overline{y}_1, u, v_1)(x) + D_\gamma^2 F_2(\overline{x}, \overline{y}_2, u, v_2)(x) \subseteq D_\gamma^2(F_1 + F_2)(\overline{x}, \overline{y}_1 + \overline{y}_2, u, v_1 + v_2)(x)$. *If additionally, either of the following conditions is fulfilled, then the inclusion becomes equality:*

    (i$_1$) *F_i is second-order directionally compact with index γ at (x_0, y_i) with respect to (u, v_i) in the direction x for $i = 1, 2$;*

    (i$_2$) *Y is finite-dimensional and*

$$D_S^2 F_1(\overline{x}, \overline{y}_1, u, v_1)(0) \cap \big(-D_S^2 F_2(\overline{x}, \overline{y}_2, u, v_2)(0) \big) = \{0\}.$$

  (ii) $D_\gamma^2 F_1(\overline{x}, \overline{y}_1, u, v_1)(x) \times D_\gamma^2 F_2(\overline{x}, \overline{y}_2, u, v_2)(x) = D_\gamma^2(F_1, F_2)(\overline{x}, (\overline{y}_1, \overline{y}_2), u, (v_1, v_2))(x)$.

Proof. We assume that F_1 is second-order proto-differentiable at $(\overline{x}, \overline{y}_1)$ in direction (u, v_1).

(i) Let $y_1 \in D_\gamma^2 F_1(\overline{x}, \overline{y}_1, u, v_1)(x)$ and $y_2 \in D_\gamma^2 F_2(\overline{x}, \overline{y}_2, u, v_2)(x)$. It follows from the proto-differentiability that there are $(t_n, r_n) \downarrow (0, 0)$, $\frac{t_n}{r_n} \to \gamma$, $(x_{1n}, y_{1n}) \to (x, y_1)$ and $(x_{2n}, y_{2n}) \to (x, y_2)$ such that

$$\overline{y}_1 + t_n v_1 + \tfrac{1}{2} t_n r_n y_{1n} \in F_1(\overline{x} + t_n u + \tfrac{1}{2} t_n r_n x_{1n}),$$
$$\overline{y}_2 + t_n v_2 + \tfrac{1}{2} t_n r_n y_{2n} \in F_2(\overline{x} + t_n u + \tfrac{1}{2} t_n r_n x_{2n}).$$

Setting

$$(a_n, b_n, c_n, d_n) = (\overline{x}, \overline{y}_1, \overline{x}, \overline{y}_2) + t_n(u, v_1, u, v_2) + \frac{1}{2} t_n r_n(x_{1n}, y_{1n}, x_{2n}, y_{2n}),$$

by the assumed subregularity of g, one gets $\alpha > 0$ such that, for large n,

$$d((a_n, b_n, c_n, d_n), g^{-1}(0) \cap \operatorname{gph} F_1 \times \operatorname{gph} F_2) \leq \alpha d(0, g(a_n, b_n, c_n, d_n)).$$

Because $g(a_n, b_n, c_n, d_n) = a_n - c_n = \frac{1}{2} t_n r_n(x_{1n} - x_{2n})$, one has

$$d((a_n, b_n, c_n, d_n), g^{-1}(0) \cap \operatorname{gph} F_1 \times \operatorname{gph} F_2) \leq \frac{1}{2} \alpha t_n r_n \|x_{1n} - x_{2n}\| + o(t_n r_n).$$

Hence, there are $((\widehat{x}_n, \widehat{y}_{1n}), (\widehat{x}_n, \widehat{y}_{2n})) \in g^{-1}(0) \cap \operatorname{gph} F_1 \times \operatorname{gph} F_2$ such that

$$\left\| \overline{x} + t_n u + \frac{1}{2} t_n r_n x_{1n} - \widehat{x}_n \right\| \leq \frac{1}{2} \alpha t_n r_n \|x_{1n} - x_{2n}\| + o(t_n r_n),$$

$$\left\| \overline{y}_1 + t_n v_1 + \frac{1}{2} t_n r_n y_{1n} - \widehat{y}_{1n} \right\| \leq \frac{1}{2} \alpha t_n r_n \|x_{1n} - x_{2n}\| + o(t_n r_n),$$

$$\left\| \overline{y}_2 + t_n v_1 + \frac{1}{2} t_n r_n y_{2n} - \widehat{y}_{2n} \right\| \leq \frac{1}{2} \alpha t_n r_n \|x_{1n} - x_{2n}\| + o(t_n r_n).$$

Consequently,

$$\overline{x}_n := \frac{\widehat{x}_n - \overline{x} - t_n u}{\frac{1}{2} t_n r_n} \to x, \quad \overline{y}_{1n} := \frac{\widehat{y}_{1n} - \overline{y}_1 - t_n v_1}{\frac{1}{2} t_n r_n} \to y_1$$

and

$$\overline{y}_{2n} := \frac{\widehat{y}_{2n} - \overline{y}_2 - t_n v_2}{\frac{1}{2} t_n r_n} \to y_2.$$

Furthermore, as $((\widehat{x}_n, \widehat{y}_{1n}), (\widehat{x}_n, \widehat{y}_{2n})) \in \mathrm{gph}\, F_1 \times \mathrm{gph}\, F_2$, one has

$$\overline{y}_1 + t_n v_1 + \tfrac{1}{2} t_n r_n \overline{y}_{1n} \in F_1(\overline{x} + t_n u + \tfrac{1}{2} t_n r_n \overline{x}_n),$$
$$\overline{y}_2 + t_n v_2 + \tfrac{1}{2} t_n r_n \overline{y}_{2n} \in F_2(\overline{x} + t_n u + \tfrac{1}{2} t_n r_n \overline{x}_n).$$

By adding the two relations, one has

$$\overline{y}_1 + \overline{y}_2 + t_n(u_1 + v_1) + \tfrac{1}{2} t_n r_n(\overline{y}_{1n} + \overline{y}_{2n}) \in (F_1 + F_2)(\overline{x} + t_n u + \tfrac{1}{2} t_n r_n \overline{x}_n).$$

Therefore, $y_1 + y_2 \in D_\gamma^2(F_1 + F_2)(\overline{x}, \overline{y}_1 + \overline{y}_2, u, v_1 + v_2)(x)$.

Now we prove the reverse inclusion under each of the additional assumptions.

(i_1) Let $y \in D_1^2(F_1 + F_2)(\overline{x}, \overline{y}_1 + \overline{y}_2, u, v_1 + v_2)(x)$, $(t_n, r_n) \downarrow (0,0)$, $\frac{t_n}{r_n} \to \gamma$, $(x_n, y_n) \to (x, y)$ such that

$$\overline{y}_1 + \overline{y}_2 + t_n(v_1 + v_2) + \tfrac{1}{2} t_n r_n y_n \in (F_1 + F_2)(\overline{x} + t_n u + \tfrac{1}{2} t_n r_n x_n).$$

Hence,

$$y_n \in \frac{F_1(\overline{x} + t_n u + \tfrac{1}{2} t_n r_n x_n) - \overline{y}_1 - t_n v_1}{\frac{1}{2} t_n r_n} + \frac{F_2(\overline{x} + t_n u + \tfrac{1}{2} t_n r_n x_n) - \overline{y}_2 - t_n v_2}{\frac{1}{2} t_n r_n}.$$

Therefore, there are $\overline{y}_{1n} \in F_1(\overline{x} + t_n u + \tfrac{1}{2} t_n r_n x_n)$, $\overline{y}_{2n} \in F_2(\overline{x} + t_n u + \tfrac{1}{2} t_n r_n x_n)$, $y_{1n} = (\tfrac{1}{2} t_n r_n)^{-1}(\overline{y}_{1n} - \overline{y}_1 - t_n v_1)$, and $y_{2n} = (\tfrac{1}{2} t_n r_n)^{-1}(\overline{y}_{2n} - \overline{y}_2 - t_n v_2)$ such that $y_n = y_{1n} + y_{2n}$ and

$$\overline{y}_1 + t_n v_1 + \tfrac{1}{2} t_n r_n y_{1n} \in F_1(\overline{x} + t_n u + \tfrac{1}{2} t_n r_n x_n),$$
$$\overline{y}_2 + t_n v_2 + \tfrac{1}{2} t_n r_n y_{2n} \in F_2(\overline{x} + t_n u + \tfrac{1}{2} t_n r_n x_n).$$

By the directional compactness of F_1 and F_2, one has (for subsequences)

$$y_{1n} \to y_1 \in D_\gamma^2 F_1(\overline{x}, \overline{y}_1, u, v_1)(x) \quad \text{and} \quad y_{2n} \to y_2 \in D_\gamma^2 F_2(\overline{x}, \overline{y}_2, u, v_2)(x).$$

(i_2) We have to show that both y_{1n} and y_{2n} obtained as in part (i) have convergent subsequences. Suppose that $\{y_{1n}\}$ has no convergent subsequence. Then, we can assume that $\|y_{1n}\| \to \infty$. We have

(3.1)
$$\frac{y_n}{\|y_{1n}\|} = \frac{y_{1n}}{\|y_{1n}\|} + \frac{y_{2n}}{\|y_{1n}\|},$$

$$\frac{y_{1n}}{\|y_{1n}\|} = \frac{\overline{y}_{1n} - \overline{y}_1 - t_n v_1}{\frac{1}{2} t_n r_n \|y_{1n}\|} \in \frac{F_1(\overline{x} + t_n u + \frac{1}{2} t_n r_n x_n) - \overline{y}_1 - t_n v_1}{\frac{1}{2} t_n r_n \|y_{1n}\|},$$

$$\frac{y_{2n}}{\|y_{1n}\|} = \frac{\overline{y}_{2n} - \overline{y}_2 - t_n v_2}{\frac{1}{2} t_n r_n \|y_{1n}\|} \in \frac{F_2(\overline{x} + t_n u + \frac{1}{2} t_n r_n x_n) - \overline{y}_2 - t_n v_2}{\frac{1}{2} t_n r_n \|y_{1n}\|}.$$

By setting $r'_n = r_n \|y_{1n}\|$, $x'_n = \|y_{1n}\|^{-1} x_n$, one has

$$\frac{y_{1n}}{\|y_{1n}\|} \in \frac{F_1(\overline{x} + t_n u + \frac{1}{2} t_n r'_n x'_n) - \overline{y}_1 - t_n v_1}{\frac{1}{2} t_n r'_n},$$

$$\frac{y_{2n}}{\|y_{1n}\|} \in \frac{F_2(\overline{x} + t_n u + \frac{1}{2} t_n r'_n x'_n) - \overline{y}_2 - t_n v_2}{\frac{1}{2} t_n r'_n}.$$

By taking a subsequence if necessary, we may assume that $\|y_{1n}\|^{-1}(y_{1n}) \to \overline{y}$ for some $\overline{y}$ of norm one. As $x'_n = \|y_{1n}\|^{-1} x_n \to 0$, we have $\overline{y} \in D_S^2 F_1(\overline{x}, \overline{y}_1, u, v_1)(0)$. Furthermore, as $\|y_{1n}\|^{-1} y_n \to 0$, by the equation (3.1), $\|y_{1n}\|^{-1}(y_{2n}) \to -\overline{y}$. Therefore,

$$\overline{y} \in D_S^2 F_1(\overline{x}, \overline{y}_1, u, v_1)(0) \cap \big(- D_S^2 F_2(\overline{x}, \overline{y}_2, u, v_2)(0) \big).$$

This is a contradiction.

(ii) Similar to Proposition 3.1, we have to check only that the left-hand side is contained in the right-hand one. Take the first part of the proof of (i) to obtain (3.1). Then, one has

$$(\overline{y}_1, \overline{y}_2) + t_n (v_1, v_2) + \tfrac{1}{2} t_n r_n (\overline{y}_{1n}, \overline{y}_{2n}) \in (F_1, F_2)(\overline{x} + t_n u + \tfrac{1}{2} t_n r_n \overline{x}_n).$$

With $\overline{y}_{1n} \to y_1, \overline{y}_{2n} \to y_2$, we conclude that $(y_1, y_2) \in D_\gamma^2(F_1, F_2)(\overline{x}, (\overline{y}_1, \overline{y}_2), u, (v_1, v_2))(x)$. The proof is complete. $\square$

Remark 3.4. (i) The condition $(A(F_1, \overline{x}, \overline{y}_1, u, v_1))$ (or $(A(F_2, \overline{x}, \overline{y}_2, u, v_2))$) in Proposition 3.1 and the subregularity of g in Proposition 3.3 are not comparable, because the condition on g depends on $\mathrm{gph}\, F_1 \times \mathrm{gph}\, F_2$, while $(A(F_1, \overline{x}, \overline{y}_1, u, v_1))$ (or $(A(F_2, \overline{x}, \overline{y}_2, u, v_2))$) only depends on F_1 (or F_2).

(ii) Proposition 3.3(i) improves Theorem 3.7(b) in [11], where the metric subregularity (of order 1 and not "directional") of g is imposed and $\gamma = 1$.

(iii) In the special case $F_2(x) = C$ for all $x \in X$, where C is a closed convex cone, for $\overline{x}, u \in X$, $\overline{y}_1, \overline{y}_2, v_1, v_2 \in Y$ with $\overline{y}_2 + v_2 \in C$, we claim for every $x \in X$ that

$$(3.2) \qquad D_\gamma^2 F(\overline{x}, \overline{y}_1, u, v_1)(x) + T_\gamma^2(C, \overline{y}_2, v_2) \subseteq D_\gamma^2 F_+(\overline{x}, \overline{y}_1 + \overline{y}_2, u, v_1 + v_2)(x),$$

where $F_+(x) := F(x) + C$. Indeed, clearly $F_2 = C$ is pseudo-Lipschitz at $(\overline{x}, \overline{y}_2)$. As $\mathrm{gph}\, F_2 = X \times C$ is convex and $\overline{y}_2 + v_2 \in C$, one has $(\overline{x}, \overline{y}_2) \in \mathrm{gph}\, F_2 - (u, v_2)$. By Remark 2.7(iii), F_2 is second-order proto-differentiable at $(\overline{x}, \overline{y}_2)$ in direction (u, v_2). Hence, condition $(A(F_2, \overline{x}, \overline{y}_2, u, v_2))$ is satisfied. By applying Proposition 3.1(i) with

$D^2_\gamma F_2(\overline{x}, \overline{y}_2, u, v_2)(x) = T^2_\gamma(C, \overline{y}_2, v_2)$, we obtain (3.2). This relation (3.2) is new. When applied to the simple case: $\overline{y}_2 = v_2 = 0$, it collapses to (because $T^2_\gamma(C, \overline{y}_2, v_2) = C$) the classical result

$$D^2_\gamma F(\overline{x}, \overline{y}_1, u, v_1)(x) + C \subseteq D^2_\gamma F_+(\overline{x}, \overline{y}_1, u, v_1)(x).$$

(iv) Instead of the assumed directional compactness in Proposition 3.3(i) for the equality, in Proposition 2 of [26], with $\gamma = 1$ only, the authors imposed a second-order lower semidifferentiability assumption. However, the following example shows that this assertion is inadequate. Let $F_1, F_2 \colon \mathbb{R} \rightrightarrows \mathbb{R}$ be defined by

$$F_1(x) = \begin{cases} \mathbb{R}_+ & \text{if } x \geq 0, \\ \{x^2\} & \text{if } x < 0, \end{cases} \qquad F_2(x) = \begin{cases} \{-1\} \cup \{-x\} & \text{if } x \geq 0, \\ \{-x\} & \text{if } x < 0. \end{cases}$$

Consider $\overline{x} = 0$, $\overline{y}_1 = \overline{y}_2 = 0$, $u = v_1 = v_2 = 0$, $x \in \mathbb{R}$ and $\gamma = 1$. It is easy to check that all the assumptions of Proposition 2 of [26] are satisfied. Straightforward calculations yield

$$(F_1 + F_2)(x) = \begin{cases} \{y \in \mathbb{R} \mid y \geq -1\} \cup \{y \in \mathbb{R} \mid y \geq -x\} & \text{if } x \geq 0, \\ \{x^2 - x\} & \text{if } x < 0, \end{cases}$$

$D^2_1 F_1(\overline{x}, \overline{y}_1, u, v_1)(x) = \mathbb{R}_+$, $D^2_1 F_2(\overline{x}, \overline{y}_2, u, v_2)(x) = \{-x\}$ and $D^2_1(F_1 + F_2)(\overline{x}, \overline{y}_1 + \overline{y}_2, u, v_1 + v_2)(x) = \mathbb{R}$. Hence, $D^2_1(F_1 + F_2)(\overline{x}, \overline{y}_1 + \overline{y}_2, u, v_1 + v_2)(x) \not\subseteq D^2_1 F_1(\overline{x}, \overline{y}_1, u, v_1)(x) + D^2_1 F_2(\overline{x}, \overline{y}_2, u, v_2)(x)$. For a reading convenience, we also mention here the inadequate reasoning by mistake in the proof in [26]. For $y \in D^2_1(F_1 + F_2)(\overline{x}, \overline{y}_1 + \overline{y}_2, u, v_1 + v_2)(x)$, $(t_n, r_n) \downarrow (0, 0)$, $\frac{t_n}{r_n} \to 1$, and $(x_n, y_n) \to (x, y)$ such that

$$\overline{y}_1 + \overline{y}_2 + t_n(v_1 + v_2) + \tfrac{1}{2} t_n r_n y_n \in (F_1 + F_2)(x_0 + t_n u + \tfrac{1}{2} t_n r_n x_n),$$

by the second-order lower semidifferentiability of F_1, one has y_{1n} such that

$$\overline{y}_1 + t_n v_1 + \tfrac{1}{2} t_n r_n y_{1n} \in F_1(x_0 + t_n u + \tfrac{1}{2} t_n r_n x_n).$$

From this one cannot derive as in [26] that $\overline{y}_2 + t_n v_2 + \tfrac{1}{2} t_n r_n (y_n - y_{1n}) \in F_2(x_0 + t_n u + \tfrac{1}{2} t_n r_n x_n)$, because the image of $(F_1 + F_2) - F_1$ may not be contained in F_2.

4. Second-order KKT multipliers

In this section, let $F \colon X \rightrightarrows Y$, $G \colon X \rightrightarrows Z$, $\mathcal{C} \colon X \rightrightarrows Y$ be nonempty-valued and D be a closed convex cone with nonempty interior in Z. Suppose that $\mathcal{C}(x)$ is a closed convex cone in Y for all $x \in X$. Then, $\mathcal{C}$ defines a variable partial order on Y by

$$y_1 \leq_{\mathcal{C}(x)} y_2 \quad \Longleftrightarrow \quad y_2 - y_1 \in \mathcal{C}(x).$$

Our set-valued vector optimization problem is

(4.1) $\mathrm{Min}_{\mathcal{C}(x)} F(x)$ such that $G(x) \cap (-D) \neq \emptyset$.

Let $\Omega := \{x \in X \mid G(x) \cap (-D) \neq \emptyset\}$ denote the feasible set and $D(z_0) := \mathrm{cone}(D + z_0)$.

Definition 4.1. Let $x_0 \in \Omega$ and $(x_0, y_0) \in \mathrm{gph}\, F$.

  (i) (see [38]) Assume that there exists a neighborhood V of x_0 such that $\mathrm{int}\, \mathcal{C}(x) \neq \emptyset$ for every $x \in V$. A point (x_0, y_0) is called a local weak nondominated point for problem (4.1) if there is a neighborhood $U \subseteq V$ of x_0 such that, for every $x \in U \cap \Omega$,

$$(F(x) - y_0) \cap (-\mathrm{int}\, \mathcal{C}(x)) = \emptyset.$$

  (ii) (see [6]) If $\mathrm{int}\, \mathcal{C}(x_0) \neq \emptyset$, (x_0, y_0) is termed a local weak minimal point for (4.1) if there is a neighborhood U of x_0 such that, for every $x \in U \cap \Omega$,

$$(F(x) - y_0) \cap (-\mathrm{int}\, \mathcal{C}(x_0)) = \emptyset.$$

In case $\mathcal{C}(x) := C$ for every x in a neighborhood of x_0, where $C \subseteq Y$ is a closed convex cone with nonempty interior, the above notions reduce to the classical weak Pareto point. In general the two notions are different, see, e.g., [12, 14, 15]. It is easy to check the following

  • if (x_0, y_0) is a local weak minimal point and $\mathrm{int}\, \mathcal{C}(x) \subseteq \mathrm{int}\, \mathcal{C}(x_0)$ for all x in a neighborhood of x_0, then it is a local weak nondominated point;

  • if (x_0, y_0) is a local weak nondominated point and $\mathrm{int}\, \mathcal{C}(x_0) \subseteq \mathrm{int}\, \mathcal{C}(x)$ for all x in a neighborhood of x_0, then it is a local weak minimal point.

A first-order necessary condition for local weak nondominated points in a primal form is as follows.

Proposition 4.2. *Let (x_0, y_0) be a local weak nondominated point of (4.1) and $z_0 \in G(x_0) \cap (-D)$. Assume that there is a neighborhood V of x_0 such that $\overline{C} := \mathrm{int} \bigcap_{x \in V} \mathcal{C}(x) \neq \emptyset$. Then, for every $u \in X$ one has*

$$D((F + \mathcal{C}), G)(x_0, (y_0, z_0))(u) \cap (-\overline{C}) \times (-\mathrm{int}\, D(z_0)) = \emptyset.$$

Proof. There is a neighborhood U of x_0 such that $U \subseteq V$ and for every $x \in U \cap \Omega$,

$$(F(x) - y_0) \cap (-\mathrm{int}\, \mathcal{C}(x)) = \emptyset.$$

Suppose there exist $u \in X$ and $(v, w) \in D((F + \mathcal{C}), G)(x_0, (y_0, z_0))(u) \cap (-\overline{C}) \times (-\mathrm{int}\, D(z_0))$. Then, one has $(t_n, u_n, v_n, w_n) \to (0^+, u, v, w)$ such that $y_0 + t_n v_n \in (F + \mathcal{C})(x_0 + t_n u_n)$

and $z_0 + t_n w_n \in G(x_0 + t_n u_n)$. As $w \in -\operatorname{int} D(z_0) = IT(-D, z_0)$, $z_0 + t_n w_n \in -D$ for large n. Hence, $G(x_0 + t_n u_n) \cap (-D) \neq \emptyset$ and so $x_0 + t_n u_n \in U \cap \Omega$ for large n. Since $y_0 + t_n v_n \in (F + \mathcal{C})(x_0 + t_n u_n)$, one has

$$(F(x_0 + t_n u_n) - y_0) \cap (-\mathcal{C}(x_0 + t_n u_n) + t_n v_n) \neq \emptyset.$$

As $v \in -\overline{C}$, one has for large n,

$$t_n v_n \in -\operatorname{int} \bigcap_{x \in V} \mathcal{C}(x) \subseteq -\operatorname{int} \mathcal{C}(x_0 + t_n u_n).$$

Because $\mathcal{C}(x_0 + t_n u_n)$ is a convex cone, $-\operatorname{int} \mathcal{C}(x_0 + t_n u_n) - \mathcal{C}(x_0 + t_n u_n) \subseteq -\operatorname{int} \mathcal{C}(x_0 + t_n u_n)$. Therefore, $(F(x_0 + t_n u_n) - y_0) \cap (-\operatorname{int} \mathcal{C}(x_0 + t_n u_n)) \neq \emptyset$ for large n. This is a contradiction. $\qquad\square$

Remark 4.3. (i) Proposition 4.2 generalizes Proposition 3.1 in [12] to constrained problems.

(ii) In the above proof, we need $\operatorname{int} D \neq \emptyset$. When $\operatorname{int} D = \emptyset$ we impose the following assumptions

- the map $H(\cdot) := G(\cdot) + D$ is directionally metrically subregular at $(x_0, 0)$ in the direction u;

- the map $F + \mathcal{C}$ is pseudo-Lipschitz at (x_0, y_0)

to get the conclusion (instead of that in Proposition 4.2)

$$D((F + \mathcal{C}), G)(x_0, (y_0, z_0))(u) \cap (-\overline{C}) \times T^\flat(-D, z_0) = \emptyset.$$

Indeed, suppose there exist $u \in X$ and $(v, w) \in D((F + \mathcal{C}), G)((x_0, y_0), z_0)(u) \cap (-\overline{C}) \times T^\flat(-D, z_0)$. Then, one has $(t_n, u_n, v_n, w_n, w_n') \to (0^+, u, v, w, w)$ such that $y_0 + t_n v_n \in (F + \mathcal{C})(x_0 + t_n u_n)$, $z_0 + t_n w_n \in G(x_0 + t_n u_n)$ and $z_0 + t_n w_n' \in -D$. Hence, $t_n(w_n - w_n') \in G(x_0 + t_n u_n) + D$, i.e., $t_n(w_n - w_n') \in H(x_0 + t_n u_n)$. Because of the assumed directional subregularity of H, there is $\alpha > 0$ such that (for large n)

$$d(x_0 + t_n u_n, H^{-1}(0)) \leq \alpha d(0, H(x_0 + t_n u_n)).$$

Therefore, one has a point $x_n' \in H^{-1}(0)$ such that $\|x_0 + t_n u_n - x_n'\| \leq \alpha t_n \|w_n - w_n'\| + o(t_n)$. Since $\|w_n - w_n'\| \to 0$, by setting $\overline{u}_n = t_n^{-1}(x_n' - x_0)$, one has $x_0 + t_n \overline{u}_n = x_n' \in H^{-1}(0)$, i.e., $x_0 + t_n \overline{u}_n \in \Omega$ and $\overline{u}_n \to u$. Because $F + \mathcal{C}$ is pseudo-Lipschitz at (x_0, y_0), there are a neighborhood V of y_0 and $L > 0$ such that

$$(F + \mathcal{C})(x_0 + t_n u_n) \cap V \subseteq (F + \mathcal{C})(x_0 + t_n \overline{u}_n) + Lt_n \|\overline{u}_n - u_n\| \operatorname{cl} B_Y.$$

There exists $e_n \in \operatorname{cl} B_Y$ such that, for large n,

$$y_0 + t_n(v_n - L\|u_n - \overline{u}_n\|e_n) \in (F + \mathcal{C})(x_0 + t_n\overline{u}_n).$$

As $v_n - L\|u_n - \overline{u}_n\|e_n \to v \in \overline{C}$, by arguing similarly as for Proposition 4.2, one obtains the result.

Now, we present a second-order condition involving the critical directions u satisfying the preceding first-order necessary condition in a critical way in the sense that $v \in D(F + \mathcal{C})(x_0, y_0)(u) \cap (-\operatorname{cl}\overline{C})$ and $w \in DG(x_0, z_0)(u) \cap (-\operatorname{cl} D(z_0))$. Note that such a critical direction w was considered in [24, 25], while in many results on second-order conditions (with only fixed ordering cones), it was not employed, for instance, w was only in $-D$ in [27] and in $\operatorname{int} D - \mathbb{R}_+ z_0$ in [9].

Proposition 4.4. *Let (x_0, y_0) be a local weak nondominated point of* (4.1), *$z_0 \in G(x_0) \cap (-D)$ and $\gamma \in \{0, 1\}$. Assume that there exists a neighborhood V of x_0 such that $\overline{C} := \operatorname{int} \bigcap_{x \in V} \mathcal{C}(x) \neq \emptyset$. Then, for every $u \in X$ with $v \in D(F + \mathcal{C})(x_0, y_0)(u) \cap (-\operatorname{cl}\overline{C})$, $w \in DG(x_0, z_0)(u) \cap (-\operatorname{cl} D(z_0))$, and $x \in X$,*

$$(4.2) \qquad D_\gamma^2((F + \mathcal{C}), G)(x_0, (y_0, z_0))(u, (v, w))(x) \cap IT(-\overline{C}, v) \times IT_\gamma^2(-D, z_0, w) = \emptyset.$$

Proof. There is a neighborhood U of x_0 such that $U \subseteq V$ and, for every $x \in U \cap \Omega$,

$$(F(x) - y_0) \cap (-\operatorname{int} \mathcal{C}(x)) = \emptyset.$$

Suppose to the contrary the existence of $x \in X$ and $(y, z) \in Y \times Z$ such that

$$(y, z) \in D_\gamma^2(F, G)(x_0, (y_0, z_0), u, (v, w))(x) \cap IT(-\overline{C}, v) \times IT_\gamma^2(-D, z_0, w).$$

Since $(y, z) \in D_\gamma^2(F + \mathcal{C}, G)(x_0, (y_0, z_0), u, (v, w))(x)$, one has $(t_n, r_n) \downarrow (0, 0)$, $t_n r_n^{-1} \to \gamma$, $x_n \to x$, $y_n \to y$ and $z_n \to z$ such that

$$y_0 + t_n v + \tfrac{1}{2} t_n r_n y_n \in (F + \mathcal{C})(x_0 + t_n u + \tfrac{1}{2} t_n r_n x_n),$$
$$z_0 + t_n w + \tfrac{1}{2} t_n r_n z_n \in G(x_0 + t_n u + \tfrac{1}{2} t_n r_n x_n).$$

As $z \in IT_\gamma^2(-D, z_0, w)$, $z_0 + t_n w + \tfrac{1}{2} t_n r_n z_n \in -D$ (for large n). Hence, $G(x_0 + t_n u + \tfrac{1}{2} t_n r_n x_n) \cap (-D) \neq \emptyset$, i.e., $x_0 + t_n u + \tfrac{1}{2} t_n r_n x_n \in \Omega \cap U$. Since $y_0 + t_n v + \tfrac{1}{2} t_n r_n y_n \in (F + \mathcal{C})(x_0 + t_n u + \tfrac{1}{2} t_n r_n x_n)$,

$$(F(x_0 + t_n u + \tfrac{1}{2} t_n r_n x_n) - y_0) \cap (-\mathcal{C}(x_0 + t_n u + \tfrac{1}{2} t_n r_n x_n) + t_n v + \tfrac{1}{2} t_n v_n y_n) \neq \emptyset.$$

Because $y \in IT(-\overline{C}, v)$, one has $t_n v + \tfrac{1}{2} t_n r_n y_n \in -\overline{C}$ for large n. Hence,

$$t_n v + \tfrac{1}{2} t_n r_n y_n \in -\operatorname{int} \mathcal{C}(x_0 + t_n u + \tfrac{1}{2} t_n r_n x_n).$$

Therefore, for large n,

$$(F(x_0 + t_n u + \tfrac{1}{2} t_n r_n x_n) - y_0) \cap (- \operatorname{int} \mathcal{C}(x_0 + t_n u + \tfrac{1}{2} t_n r_n x_n)) \neq \emptyset.$$

This contradiction completes the proof.   □

Remark 4.5. (i) Note that, by arguing similarly as for Proposition 4.4, using $\mathcal{C}(x_0)$ instead of $\overline{C}$, we obtain a second-order condition for a local weak minimal point in a primal form as follows

$$D_\gamma^2((F + \mathcal{C}(x_0)), G)(x_0, (y_0, z_0))(u, (v, w))(x) \cap IT(-\mathcal{C}(x_0), v) \times IT_\gamma^2(-D, z_0, w) = \emptyset.$$

(ii) In case int $D = \emptyset$, under the additional assumptions

- the map $H(\cdot) := G(\cdot) + D$ is (u, x)-directionally metrically subregular of index γ at $(x_0, 0)$;

- the map $F + \mathcal{C}$ is pseudo-Lipschitz at (x_0, y_0)

the conclusion corresponding to that of Proposition 4.4 is

$$D_\gamma^2((F + \mathcal{C}), G)(x_0, (y_0, z_0))(u, (v, w))(x) \cap IT(-\overline{C}, v) \times T_\gamma^{\flat 2}(-D, z_0, w) = \emptyset.$$

Indeed, suppose $(y, z) \in D_\gamma^2(F, G)(x_0, (y_0, z_0), u, (v, w))(x) \cap IT(-\overline{C}, v) \times T_\gamma^{\flat 2}(-D, z_0, w)$. Then, there are $(t_n, r_n) \downarrow (0, 0)$, $t_n r_n^{-1} \to \gamma$, $x_n \to x$ and $z_n, z_n' \to z$ such that

$$z_0 + t_n w + \tfrac{1}{2} t_n r_n z_n \in G(x_0 + t_n u + \tfrac{1}{2} t_n r_n x_n), \quad z_0 + t_n w + \tfrac{1}{2} t_n r_n z_n' \in -D.$$

Hence, $\tfrac{1}{2} t_n r_n (z_n - z_n') \in G(x_0 + t_n u + \tfrac{1}{2} t_n r_n x_n) + D$, i.e., $\tfrac{1}{2} t_n r_n (z_n - z_n') \in H(x_0 + t_n u + \tfrac{1}{2} t_n r_n x_n)$. Because of the assumed directional subregularity of H, there is $\alpha > 0$ such that, for large n,

$$d(x_0 + t_n u + \tfrac{1}{2} t_n r_n x_n, H^{-1}(0)) \leq \alpha d(0, H(x_0 + t_n u + \tfrac{1}{2} t_n r_n x_n)).$$

Therefore, for large n one has a point $x_n' \in H^{-1}(0)$ such that

$$\left\| x_0 + t_n u + \frac{1}{2} t_n r_n x_n - x_n' \right\| \leq \frac{1}{2} \alpha t_n r_n \| z_n - z_n' \| + o(t_n r_n).$$

Since $\| z_n - z_n' \| \to 0$, by setting $\overline{x}_n = (\tfrac{1}{2} t_n r_n)^{-1}(x_n' - x_0 - t_n u)$, one has $x_0 + t_n u + \tfrac{1}{2} t_n r_n \overline{x}_n = x_n' \in H^{-1}(0)$, i.e., $x_0 + t_n u + \tfrac{1}{2} t_n r_n \overline{x}_n \in \Omega$, and $\overline{x}_n \to x$. By the pseudo-Lipschitz property, the rest of this proof is similar to Remark 4.3(ii).

By the standard separation theorem and a qualification condition of the KRZ type, we obtain the following second-order KKT multiplier rule.

Theorem 4.6. *Let (x_0, y_0) be a local weak nondominated point of* (4.1), $z_0 \in G(x_0) \cap (-D)$ *and* $\gamma \in \{0, 1\}$. *Assume that there exists a neighborhood V of x_0 such that $\overline{C} :=$* $\mathrm{int} \bigcap_{x \in V} \mathcal{C}(x) \neq \emptyset$. *For every $u \in X$ with $v \in D(F + \mathcal{C})(x_0, y_0)(u) \cap (-\mathrm{cl}\,\overline{C})$ and $w \in DG(x_0, z_0)(u) \cap (-\mathrm{cl}\,D(z_0))$ such that $D_\gamma^2((F + \mathcal{C}), G)(x_0, (y_0, z_0))(u, (v, w))(X)$ is convex, there exist multipliers $(c^*, d^*) \in (\mathrm{cl}\,\overline{C})^* \times N(-D, z_0) \setminus \{(0,0)\}$ such that $\langle c^*, v \rangle = \langle d^*, w \rangle = 0$ and*

$$\langle c^*, y \rangle + \langle d^*, z \rangle \geq \sup_{d \in T_\gamma^{\flat 2}(-D, z_0, w)} \langle d^*, d \rangle$$

for all $(y, z) \in D_\gamma^2((F + \mathcal{C}), G)(x_0, (y_0, z_0))(u, (v, w))(X)$. Moreover, if the KRZ *qualification condition*

$$\{z \in Z \mid (y, z) \in D_\gamma^2((F + \mathcal{C}), G)(x_0, (y_0, z_0))(u, (v, w))(X)\} - T_\gamma^{\flat 2}(-D, z_0, w) + D(z_0) = Z$$

is satisfied, then $c^* \neq 0$.

Proof. From the equality (4.2), by the convexity assumption and the standard separation theorem, we obtain $(c^*, d^*) \in Y^* \times Z^* \setminus \{(0,0)\}$ such that

(4.3) $$\langle c^*, y \rangle + \langle d^*, z \rangle \geq \langle c^*, c \rangle + \langle d^*, d \rangle$$

for all $(y, z) \in D_\gamma^2((F + \mathcal{C}), G)(x_0, (y_0, z_0))(u, (v, w))(X)$, $c \in \mathrm{cl}\,IT(-\overline{C}, v)$ and $d \in \mathrm{cl}\,IT_\gamma^2(-D, z_0, w)$. Because $\overline{C}$ is a convex cone, $IT(-\overline{C}, v) = \mathrm{int}(\mathrm{cone}(-\overline{C} - v))$. It follows from (4.3) that $\langle c^*, c \rangle \leq 0$ for all $c \in \mathrm{cone}(-\overline{C} - v)$, and so $c^* \in [\mathrm{cone}(\overline{C} + v)]^*$. As $v \in -\mathrm{cl}\,\overline{C}$, one has $c^* \in (\mathrm{cl}\,\overline{C})^*$ and $\langle c^*, v \rangle = 0$. According to Proposition 2.5, one has $\mathrm{cl}\,IT_\gamma^{\flat 2}(-D, z_0, w) = T_\gamma^{\flat 2}(-D, z_0, w)$ and $T_\gamma^{\flat 2}(-D, z_0, w) + T(T(-D, z_0), w) \subseteq T_\gamma^{\flat 2}(-D, z_0, w)$. Then, (4.3) becomes, for all $d \in T_\gamma^{\flat 2}(-D, z_0, w)$ and $d' \in T(T(-D, z_0), w)$,

$$\langle c^*, y \rangle + \langle d^*, z \rangle \geq \langle d^*, d \rangle + \langle d^*, d' \rangle.$$

Because $T(T(-D, z_0), w)$ is a cone, $d^* \in [T(T(-D, z_0), w)]^*$, i.e., $d^* \in N(-D, z_0)$ and $\langle d^*, w \rangle = 0$. By letting $d' = 0$, we obtain

$$\langle c^*, y \rangle + \langle d^*, z \rangle \geq \sup_{d \in T_\gamma^{\flat 2}(-D, z_0, w)} \langle d^*, d \rangle.$$

Now, suppose to the contrary that the qualification condition holds but $c^* = 0$. One has

$$\langle d^*, z \rangle \geq \sup_{d \in T_\gamma^{\flat 2}(-D, z_0, w)} \langle d^*, d \rangle$$

for every $(y, z) \in D_\gamma^2((F + \mathcal{C}), G)(x_0, (y_0, z_0))(u, (v, w))(X)$. Take arbitrarily $\overline{z} \in Z$. By the qualification condition, there exist $t \geq 0$, $z \in \{z' \in Z \mid (y', z') \in D_\gamma^2((F +$

$\mathcal{C}), G)(x_0, (y_0, z_0))(u, (v, w))(X)\}$, $d_1 \in T^{\flat 2}_\gamma(-D, z_0, w)$ and $d_2 \in D$ such that $\overline{z} = z - d_1 + t(d_2 + z_0)$. Since $d^* \in D^*$ and $\langle d^*, z_0 \rangle = 0$,

$$\langle d^*, \overline{z} \rangle = \langle d^*, z \rangle - \langle d^*, d_1 \rangle + t\langle d^*, d_2 + z_0 \rangle \geq \sup_{d \in T^{\flat 2}_\gamma(-D, z_0, w)} \langle d^*, d \rangle - \langle d^*, d_1 \rangle \geq 0.$$

Hence, $d^* = 0$, a contradiction because $(c^*, d^*) \neq (0, 0)$. The proof is complete. □

Next, we apply calculus rules provided in Propositions 3.1 and 3.3 to obtain a sharper second-order conditions in terms of the derivatives of the data F, G and $\mathcal{C}$ in a separate way. Here, we can use a constraint qualification which is not a qualification condition in terms of $(F + \mathcal{C}, G)$ as in Theorem 4.6.

Theorem 4.7. *Let (x_0, y_0) be a local weak nondominated point of (4.1), $z_0 \in G(x_0) \cap (-D)$ and $\gamma \in \{0, 1\}$. Assume that there exists a neighborhood V of x_0 such that $\overline{C} := \text{int} \bigcap_{x \in V} \mathcal{C}(x) \neq \emptyset$. For every $u \in X$ with $v \in D(F + \mathcal{C})(x_0, y_0)(u) \cap (-\text{cl}\,\overline{C})$ and $w \in DG(x_0, z_0)(u) \cap (-\text{cl}\,D(z_0))$, assume that $D^2_\gamma((F + \mathcal{C}), G)(x_0, (y_0, z_0))(u, (v, w))(X)$ is convex and the following conditions are satisfied.*

(i) *$(\text{A}(G, x_0, z_0, u, w))$ holds;*

(ii) *either of the following conditions holds, for $y_1 + y_2 = y_0$ and $v_1 + v_2 = v$,*

    (a) *$(\text{A}(F, x_0, y_1, u, v_1))$ or $(\text{A}(\mathcal{C}, x_0, y_2, u, v_2))$ holds,*

    (b) *F is pseudo-Lipschitz at (x_0, y_0) or $\mathcal{C}$ is pseudo-Lipschitz at $(x_0, 0)$ and $g : (X \times Y)^2 \to X$, defined by $g(a, b, c, d) = a - c$, is $((u, v_1), (u, v_2))$-directionally metrically subregular of index γ at $((x_0, y_1), (x_0, y_2), 0)$ with respect to $\text{gph}\,F \times \text{gph}\,\mathcal{C}$.*

Then, there exist multipliers $(c^*, d^*) \in (\text{cl}\,\overline{C})^* \times N(-D, z_0) \setminus \{(0, 0)\}$ such that $\langle c^*, v \rangle = \langle d^*, w \rangle = 0$ and

$$\langle c^*, y \rangle + \langle c^*, \overline{y} \rangle + \langle d^*, z \rangle \geq \sup_{d \in T^{\flat 2}_\gamma(-D, z_0, w)} \langle d^*, d \rangle$$

for all $(y, \overline{y}, z) \in D^2_\gamma F(x_0, y_1, u, v_1)(X) \times D^2_\gamma \mathcal{C}(x_0, y_2, u, v_2)(X) \times D^2_\gamma G(x_0, z_0, u, w)(X)$. Moreover, $c^* \neq 0$ if the following constraint qualification is fulfilled:

(4.4) $$D^2_\gamma G(x_0, z_0, u, w)(X) - T^{\flat 2}_\gamma(-D, z_0, w) + D(z_0) = Z.$$

Proof. By Proposition 4.4, one has (4.2). It follows from assumption (i) and Proposition 3.1(ii) that

$$D^2_\gamma((F + \mathcal{C}), G)(x_0, (y_0, z_0))(u, (v, w))(x)$$
$$= D^2_\gamma((F + \mathcal{C})(x_0, y_0, u, v))(x) \times D^2_\gamma G(x_0, z_0, u, w)(x).$$

In view of assumption (ii)(a) or (ii)(b), respectively, we employ Proposition 3.1(i) or Proposition 3.3(i), respectively, for some y_1, y_2, v_1, v_2 satisfying $y_1 + y_2 = y_0$, $v_1 + v_2 = v$ to get

$$D_\gamma^2 F(x_0, y_1, u, v_1)(x) + D_\gamma^2 \mathcal{C}(x_0, y_2, u, v_2)(x) \subseteq D_\gamma^2((F + \mathcal{C})(x_0, y_0, u, v))(x).$$

Hence, in this case (4.2) becomes

$$\left(D_\gamma^2 F(x_0, y_1, u, v_1)(x) + D_\gamma^2 \mathcal{C}(x_0, y_2, u, v_2)(x) \right) \times D_\gamma^2 G(x_0, z_0, u, w)(x)$$
$$\cap \left(IT(-\overline{C}, v) \right) \times IT_\gamma^2(-D, z_0, w) = \emptyset.$$

Therefore, arguing similarly as for Theorem 4.6, one obtains the result. $\qquad \square$

By Remark 4.5(i), we also obtain a second-order condition for local weak minimal points as follows.

Theorem 4.8. *Let (x_0, y_0) be a local weak minimal point of* (4.1), *$z_0 \in G(x_0) \cap (-D)$ and $\gamma \in \{0, 1\}$. For every $u \in X$ with $v \in D(F + \mathcal{C})(x_0, y_0)(u) \cap (- \operatorname{cl} \mathcal{C}(x_0))$ and $w \in DG(x_0, z_0)(u) \cap (- \operatorname{cl} D(z_0))$ such that $D_\gamma^2((F + \mathcal{C}), G)(x_0, (y_0, z_0))(u, (v, w))(X)$ is convex and the assumptions of Theorem 4.7 are satisfied. Then, there exist multipliers $(c^*, d^*) \in \mathcal{C}(x_0)^* \times N(-D, z_0) \setminus \{(0, 0)\}$ such that $\langle c^*, v \rangle = \langle d^*, w \rangle = 0$ and*

(4.5)
$$\langle c^*, y \rangle + \langle c^*, \overline{y} \rangle + \langle d^*, z \rangle \geq \sup_{d \in T_\gamma^{\flat 2}(-D, z_0, w)} \langle d^*, d \rangle$$

for all $(y, \overline{y}, z) \in D_\gamma^2 F(x_0, y_1, u, v_1)(X) \times D_\gamma^2 \mathcal{C}(x_0, y_2, u, v_2)(X) \times D_\gamma^2 G(x_0, z_0, u, w)(X)$. Moreover, $c^* \neq 0$ if the following constraint qualification is fulfilled:

$$D_\gamma^2 G(x_0, z_0, u, w)(X) - T_\gamma^{\flat 2}(-D, z_0, w) + D(z_0) = Z.$$

Remark 4.9. (i) Since

$$T_\gamma^{\flat 2}(-D, z_0, w) \subseteq \operatorname{cl}[\operatorname{cone}(\operatorname{cone}(-D - z_0) - w)],$$
$$d^* \in -[T(T(-D, z_0), w)]^* = -[\operatorname{cl}(\operatorname{cone}(\operatorname{cone}(-D - z_0) - w))]^*,$$

one has

$$\sup_{d \in T_\gamma^{\flat 2}(-D, z_0, w)} \langle d^*, d \rangle \leq 0.$$

For $\gamma = 0$, $T_0^{\flat 2}(-D, z_0, w)$ is a cone and hence $\sup_{d \in T_0^2(-D, z_0, w)} \langle d^*, d \rangle = 0$. While for $\gamma = 1$, this supremum can be strictly negative, i.e., the envelope-like effect occurs. Of course, it vanishes if $0 \in A^2(-D, z_0, w)$. So, for direction w satisfying this, the multiplier rule in Theorem 4.7 (resp. Theorem 4.8) takes the classical form (with zero on the right-hand side). For example, if $w \in -D(z_0)$, then $0 \in T_1^2(-D, z_0, w)$. However, Theorem 4.7

(resp. Theorem 4.8) also considers critical directions $w \in - \operatorname{cl} D(z_0)$. For $w \in -(\operatorname{cl} D(z_0) \setminus D(z_0))$, the envelope-like effect can occur.

(ii) In case $\mathcal{C}(x) := C$, where $C \subseteq Y$ is a closed and convex cone with nonempty interior, any local weak minimal point of (4.1) reduces to a weak Pareto point, Theorem 4.8 improves Theorem 3.4 in [24], in which F and G are assumed pseudo-Lipschits and the derivative of G is an adjacent derivative. Furthermore, if $y_2 + v_2 \in C$, by Remark 3.4(iii), one has $D_\gamma^2 F(x_0, y_1, u, v_1)(x) + T_\gamma^2(C, y_2, v_2) \subseteq D_\gamma^2(F + C)(x_0, y_1 + y_2, u, v_1 + v_2)(x)$. Hence, the inequality (4.5) holds for all $(y, \overline{y}, z) \in D_\gamma^2 F(x_0, y_1, u, v_1)(X) \times T_\gamma^2(C, y_2, v_2) \times D_\gamma^2 G(x_0, z_0, u, w)(X)$. This leads to

$$\langle c^*, y \rangle + \inf_{\overline{y} \in T_\gamma^2(C, y_2, v_2)} \langle c^*, \overline{y} \rangle + \langle d^*, z \rangle \geq \sup_{d \in T_\gamma^{\flat 2}(-D, z_0, w)} \langle d^*, d \rangle$$

for all $(y, z) \in D_\gamma^2 F(x_0, y_1, u, v_1)(X) \times D_\gamma^2 G(x_0, z_0, u, w)(X)$.

Obviously, as a direct consequence of Theorem 4.7 with $(u, (v, w)) = (0, (0, 0))$, we immediately obtain

Corollary 4.10. *Let (x_0, y_0) be a local weak nondominated point of (4.1) and $z_0 \in G(x_0) \cap (-D)$. Assume that there exists a neighborhood V of x_0 such that $\overline{C} := \operatorname{int} \bigcap_{x \in V} \mathcal{C}(x) \neq \emptyset$. If $D((F + \mathcal{C}), G)(x_0, (y_0, z_0))(X)$ is convex and the following conditions are satisfied*

(i) $(\mathrm{A}(G, x_0, z_0, 0, 0))$ *holds;*

(ii) *either of the following conditions holds for $y_1 + y_2 = y_0$*

    (a) $(\mathrm{A}(F, x_0, y_1, 0, 0))$ *or* $(\mathrm{A}(\mathcal{C}, x_0, y_2, 0, 0))$ *holds;*

    (b) *F is pseudo-Lipschitz at (x_0, y_0) or $\mathcal{C}$ is pseudo-Lipschitz at $(x_0, 0)$ and $g \colon (X \times Y)^2 \to X$, defined by $g(a, b, c, d) = a - c$, is $((0, 0), (0, 0))$-directionally metrically subregular of index γ at $((x_0, y_1), (x_0, y_2), 0)$ with respect to $\operatorname{gph} F \times \operatorname{gph} \mathcal{C}$.*

Then, there exist multipliers $(c^*, d^*) \in (\operatorname{cl} \overline{C})^* \times N(-D, z_0) \setminus \{(0, 0)\}$ such that

$$\langle c^*, y \rangle + \langle c^*, \overline{y} \rangle + \langle d^*, z \rangle \geq 0$$

for all $(y, \overline{y}, z) \in DF(x_0, y_1)(X) \times D\mathcal{C}(x_0, y_2)(X) \times DG(x_0, z_0)(X)$. Moreover, $c^* \neq 0$ if the following constraint qualification is satisfied

$$DG(x_0, z_0)(X) - T_\gamma^\flat(-D, z_0) + D(z_0) = Z.$$

Remark 4.11. (i) For problems with variable ordering cones, we have found in the literature only first-order multiplier rules using the coderivative. This generalized derivative has important advantages, but the usage of its second-order is not convenient here. Hence, we

prove the new rule in Proposition 4.2 using the contingent derivative in order to develop second-order rules (note that there have not been any second-order rules in the literature). Furthermore, our first-order rule with directions being concerned explicitly helps to define critical directions for getting nonclassical second-order rules.

(ii) Note that, in terms of the limiting/Mordukhovich coderivative, the first-order conditions in [3, 4, 12] only use the coderivatives of F at (x_0, y_0) and of $\mathcal{C}$ at $(x_0, 0)$, while in Corollary 4.10 the derivatives of F at (x_0, y_1) and of $\mathcal{C}$ at (x_0, y_2) for any y_1, y_2 satisfying $y_1 + y_2 = y_0$ are employed. Hence, our results may be more applicable.

Now, we illustrate Theorem 4.7 by the following.

Example 4.12. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $Z = \mathbb{R}$, $\mathcal{C}(x) = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \geq -x^2, y_2 \geq 0\}$, $D = \mathbb{R}_+$, $F(x) = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 + y_2 \geq -x\}$ and $G(x) = \{z \in \mathbb{R} \mid z \geq x^2 - 2x\}$. Consider $x_0 = 0$, $y_0 = (0, 0)$ and $z_0 = 0$. Direct computations yield $(F + \mathcal{C})(x) = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 + y_2 \geq -x - x^2\}$, $DG(x_0, z_0)(u) = \{w \in \mathbb{R} \mid w \geq -2u\}$, and $D(F + \mathcal{C})(x_0, y_0)(u) = \{(v'_1, v'_2) \in \mathbb{R}^2 \mid v'_1 + v'_2 \geq -u\}$. For any neighborhood V of x_0, $\overline{C} = \mathrm{int}\, \mathbb{R}^2_+$. Taking $u = 1$, $v = (0, -1)$ and $w = 2$, one has $v \in D(F + \mathcal{C})(x_0, y_0)(u) \cap (-\mathrm{cl}\,\overline{C})$ and $w \in DG(x_0, z_0)(u) \cap (-\mathrm{cl}\, D(z_0))$.

We employ Theorem 4.7 with $\gamma = 0$. First, we see that G is pseudo-Lipschitz at (x_0, z_0), and $D_0^2 G(x_0, z_0, u, w)(x) = D_0^{\flat 2} G(x_0, z_0, u, w)(x) = \{z \in \mathbb{R} \mid z \geq -2x\}$. Hence, G is second-order proto-differentiable at (x_0, z_0) in direction (u, w) and then $(\mathrm{A}(G, x_0, z_0, u, w))$ holds. We can check that $D_\gamma^2 G(x_0, z_0, u, w)(X) = Z$. Hence, the constraint qualification (4.4) is satisfied. Secondly, we check assumption (ii) of Theorem 4.7 with $y_1 = y_0$, $y_2 = 0$, $v_1 = v$ and $v_2 = 0$. It is easy to see that $(\mathrm{A}(F, x_0, y_1, u, v_1))$ holds with

$$D_0^2 F(x_0, y_1, u, v_1)(x) = D_0^2 F(x_0, y_0, u, v)(x) = \{(y'_1, y'_2) \in \mathbb{R}^2 \mid y'_1 + y'_2 \geq -x\},$$

and $D_0^2 \mathcal{C}(x_0, y_2, u, v_2)(x) = D_0^2 \mathcal{C}(x_0, 0, u, 0)(x) = \mathbb{R}^2_+$. Then, the convexity condition is also fulfilled.

Take $x = 1$, $y = (-1, 0) \in D_0^2 F(x_0, y_1, u, v_1)(x)$, $\overline{y} = (0, 0) \in D_0^2 \mathcal{C}(x_0, y_2, u, v_2)(x)$ and $z = -2 \in D_0^2 G(x_0, z_0, u, w)(x)$. To check the necessary condition given in this theorem with $(\mathrm{cl}\,\overline{C})^* = \mathbb{R}^2_+$, we discuss all $c^* = (c_1, c_2) \in \mathbb{R}^2_+ \setminus \{(0, 0)\}$ and $d^* \in N(-D, z_0)$ with $\langle c^*, v \rangle = 0$ and $\langle d^*, w \rangle = 0$. We have $c_1 > 0$, $c_2 = 0$ and $d^* = 0$. Then, for any $c^* = (c_1, c_2) \in \mathbb{R}^2_+$ with $c_1 > 0$ and $d^* = 0$, $\langle c^*, y + \overline{y} \rangle + \langle d^*, z \rangle = -c_1 < 0$. According to Theorem 4.7, (x_0, y_0) is not a local weak nondominated point of problem (4.1).

A few first-order conditions were established in [3, 4, 12, 16] (in terms of coderivatives). Theorem 4.2 in [4] is for multiobjective optimization problems with single-valued objectives subject to geometric constraints, Theorem 1 in [3] is for set-valued optimization with set constraints. Hence, they cannot be in use for Example 4.12. Theorems 4.8 and 4.10 in [12] are for unconstrained problems and cannot be applied either. Theorem 4.11 in [16] is for

constrained problems but is inapplicable either, because here G is not metrically regular at (x_0, z_0) as required.

In the rest of our paper, we discuss constraint qualifications and the boundedness of multiplier sets in Theorem 4.7. We restrict ourselves to the case $\gamma = 0$ leaving the case $\gamma = 1$ to a possible coming paper because it requires different techniques. Since $T_0^{\flat 2}(-D, z_0, w)$ is a cone, (4.4) is implied by the following constraint qualification

$$(4.6) \qquad D_0^2 G(x_0, z_0, u, w)(X) + D(z_0) = Z.$$

For the special case with $G = g$, a differentiable single-valued map, and $u = w = 0$, (4.6) becomes $g'(x_0)(X) + D(z_0) = Z$, which is the first-order Kurcyusz-Robinson-Zowe constraint qualification (see [41]). Our constraint qualification in Theorems 4.7 and 4.8 extends this to the second-order case.

Next, we give a sufficient condition for the qualification (4.6). If the graph of $D_0^2 G(x_0, z_0, u, w)$ is closed and convex, $0 \in \operatorname{core} D_0^2 G(x_0, z_0, u, w)(X)$ (core$(\cdot)$ stands for the algebraic interior of a set $(\cdot)$), then the qualification (4.6) is satisfied. Indeed, the graph of Φ defined by $\Phi(x) := D_0^2 G(x_0, z_0, u, w)(x) + D(z_0)$ is closed and convex, $0 \in \operatorname{core} \Phi(X)$ and $\Phi(X)$ is a convex set. By the Robinson-Ursescu open mapping theorem (see [8, 34, 36]), for $\overline{x} \in X$ with $0 \in \Phi(\overline{x})$, there exists $\epsilon > 0$ such that $\epsilon B(0, 1) \subseteq \Phi(\overline{x} + B(0, 1)) \subseteq D_0^2 G(x_0, z_0, u, w)(X) + D(z_0)$. It is easy to check that $t D_0^2 G(x_0, z_0, u, w)(x) = D_0^2 G(x_0, z_0, u, w)(tx)$ for all $t \geq 0$ and $x \in X$. As $D(z_0)$ is a cone, one has $t\epsilon B(0, 1) \subseteq t D_0^2 G(x_0, z_0, u, w)(X) + t D(z_0) \subseteq D_0^2 G(x_0, z_0, u, w)(X) + D(z_0)$ for all $t \geq 0$, i.e., (4.6) holds.

For problem (4.1), we propose a relaxed second-order Mangasarian-Fromovitz constraint qualification as follows.

(MFCQ) (Relaxed Mangasarian-Fromovitz constraint qualification) There exists $\overline{x} \in X$ such that
$$D_0^2 G(x_0, z_0, u, w)(\overline{x}) \cap IT(-D, z_0) \neq \emptyset.$$

For the special case with $G = g$, a differentiable single-valued map, and $u = w = 0$, this is weaker than $dg(x_0, u) \in -\operatorname{int} D$, the well-known first-order Mangasarian-Fromovitz constraint qualification (see [10, 17, 29]), because $-\operatorname{int} D \subseteq IT(-D, z_0)$.

Observe that if (c^*, d^*) satisfies (4.5), then $(\lambda c^*, \lambda d^*)$ does for all $\lambda > 0$. Hence, we fix c^* and consider the following Karush-Kuhn-Tucker multiplier set

$$\Lambda(x_0, y_0, z_0, c^*)$$
$$:= \{ d^* \in N(-D, z_0) \mid \langle c^*, y \rangle + \langle c^*, \overline{y} \rangle + \langle d^*, z \rangle \geq 0,$$
$$\forall (y, \overline{y}, z) \in D_0^2 F(x_0, y_1, u, v_1)(X) \times D_0^2 \mathcal{C}(x_0, y_2, u, v_2)(X) \times D_0^2 G(x_0, z_0, u, w)(X) \}.$$

The following theorem includes a necessary and sufficient condition for the nonemptiness and boundedness of the set $\Lambda(x_0, y_0, z_0, c^*)$ and also shows relations between some second-order constraint qualifications and KKT multiplier sets.

Theorem 4.13. *Let (x_0, y_0) be a local weak nondominated point of (4.1), $z_0 \in G(x_0) \cap (-D)$ and $\gamma = 0$. Assume that there exists a neighborhood V of x_0 such that $\overline{C} := \text{int} \bigcap_{x \in V} C(x) \neq \emptyset$ and, for every $u \in X$ with $v \in D(F + C)(x_0, y_0)(u) \cap (- \text{cl} \overline{C})$, $w \in DG(x_0, z_0)(u) \cap (- \text{cl} D(z_0))$, and the assumptions of Theorem 4.7 hold. Assume further that the closed unit ball of Z^* is w^*-sequentially compact. Then, the following assertions are equivalent:*

  (a) *(4.6) holds;*

  (b) *(MFCQ) holds;*

  (c) *there exists $c^* \in (\text{cl} \overline{C})^* \setminus \{0\}$ such that $\Lambda(x_0, y_0, z_0, c^*)$ is nonempty and bounded.*

Proof. (b) $\Rightarrow$ (a). By (MFCQ), for $\overline{z} \in D_0^2 G(x_0, z_0, u, w)(\overline{x}) \cap IT(-D, z_0)$, $z \in Z$, and large $n \in \mathbb{N}$, one has $-z_0 - \frac{1}{n}(\overline{z} - \frac{1}{n}z) \in D$. As $D_0^2 G(x_0, z_0, u, w)$ is strictly positively homogeneous, $n\overline{z} \in D_0^2 G(x_0, z_0, u, w)(n\overline{x})$. Then,

$$z = n\overline{z} + n^2 \left(-z_0 - \frac{1}{n}\overline{z} + \frac{1}{n^2}z + z_0 \right) \in D_0^2 G(x_0, z_0, u, w)(X) + D(z_0).$$

Because $z \in Z$ is arbitrary, we have (4.6).

  (a) $\Rightarrow$ (c). By Theorem 4.7, one has $c^* \in (\text{cl} \overline{C})^* \setminus \{0\}$ such that $\Lambda(x_0, y_0, z_0, c^*)$ is nonempty. Suppose that $\Lambda(x_0, y_0, z_0, c^*)$ is unbounded. Then, there exist $d_n^* \in \Lambda(x_0, y_0, z_0, c^*)$ with $\|d_n^*\| \to \infty$. Take arbitrarily $\overline{z} \in Z$. By the constraint qualification (4.6), there are $t \geq 0$, $z \in D_0^2 G(x_0, z_0, u, w)(X)$ and $d \in D$ such that $z = \overline{z} - t(d + z_0)$. Consequently, Theorem 4.7 says that, for all $y \in D_0^2 F(x_0, y_1, u, v_1)(X)$ and $\overline{y} \in D_0^2 \mathcal{C}(x_0, y_2, u, v_2)(X)$,

$$\langle c^*, y \rangle + \langle c^*, \overline{y} \rangle + \langle d_n^*, \overline{z} - t(d + z_0) \rangle = \langle c^*, y \rangle + \langle c^*, \overline{y} \rangle + \langle d_n^*, z \rangle \geq 0,$$

that is

$$\langle c^*, y \rangle + \langle c^*, \overline{y} \rangle + \langle d_n^*, \overline{z} \rangle \geq \langle d_n^*, t(d + z_0) \rangle.$$

Since $d \in D$ and $\langle d_n^*, z_0 \rangle = 0$, one has $\langle c^*, y \rangle + \langle c^*, \overline{y} \rangle + \langle d_n^*, \overline{z} \rangle \geq 0$ and, by dividing by $\|d_n^*\|$,

$$(4.7) \qquad \langle \|d_n^*\|^{-1} c^*, y \rangle + \langle \|d_n^*\|^{-1} c^*, \overline{y} \rangle + \langle \|d_n^*\|^{-1} d_n^*, \overline{z} \rangle \geq 0.$$

Because the closed unit ball of Z^* is w^*-sequentially compact, we assume that $\|d_n^*\|^{-1} d_n^* \xrightarrow{w^*} d^*$. Since $\text{int} D \neq \emptyset$, D is dually compact (according to [40]). If $d^* = 0$, the dual

compactness of D implies that $\|d_n^*\|^{-1}d_n^* \to 0$ in norm, which contradicts the fact that $\big\|\|d_n^*\|^{-1}d_n^*\big\| = 1$. Hence, $d^* \neq 0$. By fixing y, $\overline{y}$ and letting $n \to \infty$ in (4.7), one has $\langle d^*, \overline{z}\rangle \geq 0$. By the arbitrariness of $\overline{z}$, one gets the contradiction that $d^* = 0$.

(c) $\Rightarrow$ (b). Suppose that (MFCQ) does not hold, i.e., $D_0^2 G(x_0, z_0, u, w)(X) \cap IT(-D, z_0) = \emptyset$. By the standard separation theorem, we obtain $d^* \in Z^* \setminus \{0\}$ such that $\langle d^*, z\rangle \geq \langle d^*, d\rangle$ for all $z \in D_0^2 G(x_0, z_0, u, w)(X)$ and $d \in \mathrm{cl}\, IT(-D, z_0)$. Hence, $d^* \in N(-D, z_0)$ and $\langle d^*, z\rangle \geq 0$ for all $z \in D_0^2 G(x_0, z_0, u, w)(X)$. Taking $\overline{d}^* \in \Lambda(x_0, y_0, z_0, c^*)$ we have, for every $\alpha > 0$,

$$\langle \overline{d}^* + \alpha d^*, z_0\rangle = \langle \overline{d}^*, z_0\rangle + \alpha\langle d^*, z_0\rangle = 0$$

and, for all $(y, \overline{y}, z) \in D_0^2 F(x_0, y_1, u, v_1)(X) \times D_0^2 \mathcal{C}(x_0, y_2, u, v_2)(X) \times D_0^2 G(x_0, z_0, u, w)(X)$,

$$\langle c^*, y\rangle + \langle c^*, \overline{y}\rangle + \langle \overline{d}^* + \alpha d^*, z\rangle = \langle c^*, y\rangle + \langle c^*, \overline{y}\rangle + \langle \overline{d}^*, z\rangle + \alpha\langle d^*, z\rangle \geq 0.$$

Consequently, $\overline{d}^* + \alpha d^* \in \Lambda(x_0, y_0, z_0, c^*)$ for all $\alpha > 0$ and so $\Lambda(x_0, y_0, z_0, c^*)$ is unbounded, a contradiction. The proof is complete. $\square$

Remark 4.14. Up to our knowledge, the boundedness of second-order KKT multipliers, even for the fixed ordering structure case, has not been considered in the literature. Note that Theorem 4.13 also asserts that the KRZ constraint qualification (4.6) is equivalent to the (MFCQ) when the second-oder KKT multiplier set is nonempty and bounded.

We observe several contributions to the boundedness of first-order KKT multiplier sets. For vector optimization, the papers [10, 17] studied a case with Fréchet differentiable maps, and [13, 28, 31] with Lipschitz maps. Note also that in our KKT multiplier sets, we consider $d^* \in N(-D, z_0)$, i.e., $d^* \in D^*$ and $\langle d^*, z_0\rangle = 0$; while in [10, 28] the authors only considered $d^* \in D^*$.

The following examples illustrate Theorem 4.13.

Example 4.15. Let $X = Z = \mathbb{R}^2$, $Y = \mathbb{R}$, $\mathcal{C}(x_1, x_2) = \{y \in \mathbb{R} \mid y \geq x_1^2\}$, $D = \mathbb{R}_+^2$,

$$F(x_1, x_2) = \begin{cases} \{y \in \mathbb{R} \mid y \geq 0\} & \text{if } x_1 x_2 \geq 0, \\ \emptyset & \text{if } x_1 x_2 < 0, \end{cases}$$

and

$$G(x_1, x_2) = \begin{cases} \{(x_1, x_2), (-x_1, -x_2)\} & \text{if } x_1 x_2 \geq 0, \\ \emptyset & \text{if } x_1 x_2 < 0. \end{cases}$$

Consider $x_0 = (0, 0)$, $y_0 = 0$ and $z_0 = (0, 0)$. We check that (x_0, y_0) is a local weak nondominated point. Let $V = U = (10^{-1}, 10^{-1}) \times (10^{-1}, 10^{-1})$. Then, for every $x \in U \cap \Omega$, $(F(x) - y_0) \cap (-\mathrm{int}\, \mathcal{C}(x)) = \emptyset$. Direct calculations provide that

$$(F + \mathcal{C})(x) = \begin{cases} \{y \in \mathbb{R} \mid y \geq x_1^2\} & \text{if } x_1 x_2 \geq 0, \\ \emptyset & \text{if } x_1 x_2 < 0, \end{cases}$$

and for $u \in X$,

$$D(F + \mathcal{C})(u) = \begin{cases} \{v \in \mathbb{R} \mid v \geq 0\} & \text{if } u_1 u_2 \geq 0, \\ \emptyset & \text{if } u_1 u_2 < 0, \end{cases}$$

$$DG(x_0, z_0)(u) = \begin{cases} \{(u_1, u_2), (-u_1, -u_2)\} & \text{if } u_1 u_2 \geq 0, \\ \emptyset & \text{if } u_1 u_2 < 0. \end{cases}$$

One has $\overline{C} := \text{int} \left(\bigcap_{x \in V} \mathcal{C}(x) \right) = \{y \in \mathbb{R} \mid y > 10^{-2}\}$. For $u = (1, 1)$, $v = 0$ and $w = (-1, -1)$, one has $v \in D(F + \mathcal{C})(x_0, y_0)(u) \cap (-\text{cl}\,\overline{C})$ and $w \in DG(x_0, z_0)(u) \cap (-\text{cl}\,D(z_0))$. Now we check Theorem 4.13 with $y_1 = y_0$, $y_2 = 0$, $v_1 = v$, and $v_2 = 0$. Straightforward calculations give that

$$D_0^2 F(x_0, y_1, u, v_1)(x) = \{y \in \mathbb{R} \mid y \geq 0\},$$

$$D_0^{\flat 2} \mathcal{C}(x_0, 0, u, 0)(x) = D_0^2 \mathcal{C}(x_0, 0, u, 0)(x) = \{y \in \mathbb{R} \mid y \geq 0\},$$

$$D_0^{\flat 2} G(x_0, z_0, u, w)(x) = D_0^2 G(x_0, z_0, u, w)(x) = \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1 \geq -x_1, z_2 \geq -x_2\}.$$

It is easy to see that $\mathcal{C}$ is pseudo-Lipschitz at $(x_0, 0)$, G is pseudo-Lipschitz at (x_0, z_0), all $(\text{A}(G, x_0, z_0, u, w))$, $(\text{A}(\mathcal{C}, x_0, y_2, u, v_2))$, and the assumed convexity are satisfied. Furthermore, the closed unit ball of Z^* is w^*-sequentially compact.

As $D_0^2 G(x_0, z_0, u, w)(X) + D(z_0) = \mathbb{R}^2$, the constraint qualification (4.6) holds. We see that $D_0^2 G(x_0, z_0, u, w)(1, 1) \cap IT(-D, z_0) \neq \emptyset$ and hence (MFCQ) holds. By taking $c^* \in (\text{cl}\,\overline{C})^* \setminus \{0\}$, i.e., $c^* > 0$, one has

$$\Lambda(x_0, y_0, z_0, c^*) = \{(0, 0)\},$$

which is a nonempty and bounded set. So, all the assumptions and conclusions of Theorem 4.13 hold.

Example 4.16. Let $X = Y = \mathbb{R}^2$, $Z = \mathbb{R}$, $\mathcal{C}(x) \equiv \mathbb{R}_+^2$, $D = \mathbb{R}_+$

$$F(x) = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 = |x_1|, y_2 \geq -x_2\} \quad \text{and} \quad G(x) := \{z \in \mathbb{R} \mid z \geq x_2 |x_1|\}.$$

Consider $x_0 = (0, 0)$, $y_0 = (0, 0)$ and $z_0 = 0$. It is easy to check that (x_0, y_0) is a local weak nondominated point of (4.1). For $(u, v, w) = ((1, 1), (0, -1), 0)$, by direct computations, one has $v \in D(F + \mathcal{C})(x_0, y_0)(u) \cap (-\text{cl}\,\mathbb{R}_+^2)$ and $w \in DG(x_0, z_0)(u) \cap (-\text{cl}\,D(z_0))$. Now we apply Theorem 4.13 with $y_1 = y_0$, $y_2 = 0$, $v_1 = v$ and $v_2 = 0$. We have $D_0^2 F(x_0, y_1, u, v_1)(x) = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \geq 0, y_2 \geq -x_2\}$, $D_0^2 \mathcal{C}(x_0, 0, u, 0)(x) = \mathbb{B}_+^2$ and $D_0^2 G(x_0, z_0, u, w)(x) = \{z \in \mathbb{R} \mid z \geq 0\}$. We can check that all the assumptions of Theorem 4.13 stated before the conclusion about the equivalence are satisfied. If we choose $c^* = (c_1, 0) \in (\text{cl}\,\overline{C})^* = \mathbb{R}_+^2$ and $d^* \in D^* = \mathbb{R}_+$ with $c_1 > 0$, then $\langle c^*, v \rangle = \langle d^*, w \rangle = 0$ and

$$\langle c^*, y \rangle + \langle c^*, \overline{y} \rangle + \langle d^*, z \rangle \geq 0$$

for all $(y, \overline{y}, z) \in D_0^2 F(x_0, y_1, u, v_1)(X) \times D_0^2 \mathcal{C}(x_0, 0, u, 0)(X) \times D_0^2 G(x_0, z_0, u, w)(X)$. But, the second-order KKT multiplier set $\Lambda(x_0, y_0, z_0, c^*) = \mathbb{R}_+$ is unbounded. The cause is that the constraint qualification (4.6) does not hold:

$$D_0^2 G(x_0, z_0, u, w)(X) + D(z_0) = \mathbb{R}_+.$$

Perspectives. The results of this paper can be developed in several directions. First, equivalent relations between the nonemptiness and boundedness of multiplier sets and constraint qualifications (counterparts of Theorem 4.13) can be expectedly obtained also for the case $\gamma = 1$ (different techniques may be needed because of the nonclassical form of the multiplier rules). Secondly, particular cases of the variable ordering cones $\mathcal{C}(\cdot)$ such as the Bishop-Phelps or Laurent cones may be interesting and deserve careful studies. Further, we think that, instead of weak solutions, other types ones such as Pareto or proper (in the sense of Henig, Borwein, Benson, etc) solutions are also important objects of considerations.

Finally, we note that second-order optimality conditions in multiobjecttive optimization have been also applied to algorithm design and convergence analysis (besides to recognizing optimal solutions), concerning Newton-type algorithms for unconstrained problems. However, we have not found any contribution to this topic for problems with variable ordering structures. So, this may be a promising theme for future research.

Acknowledgments

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