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On Asymptotic Behavior of Generalized Li Coefficients

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Abstract. In this paper, we consider the asymptotic behaviour of τ-Li coefficients for the wide class of L-functions that contains the Selberg class, the class of all automorphic L-functions, the Rankin-Selberg L-functions, as well as products of suitable shifts of the mentioned functions. We consider both archimedean and non-archimedean contribution to the τ-Li coefficients, both separately, and their joint contribution to the coefficients. We also derive the behavior of the coefficients in the case the $\tau/2$-Riemann hypothesis holds, which is the generalization of the Riemann hypothesis for the class under consideration. Finally, we conclude with some examples and numerics.

1. Introduction

Li [10] defined the Li coefficients for the Riemann zeta function as

$$\lambda_n = \frac{1}{(n-1)!}\frac{d^n}{ds^n}[s^{n-1}\log\xi(s)]_{s=1},$$

where $\xi(s)$ is the completed Riemann zeta function. He also gave a simple equivalence criterion for the Riemann hypothesis: The hypothesis is true if and only if these coefficients are non-negative for every positive integer n. This criterion was generalized for an arbitrary (assuming certain convergence conditions) complex multiset of numbers by Bombieri and Lagarias [2]. Voros [16] proved that it is sufficient to study the asymptotic behavior of the coefficients: The Riemann hypothesis is true if and only if $\lambda_n \sim n\log n$ as $n \to \infty$.

The Li coefficients have been generalized in essentially two ways: by introducing a new parameter in its definition, implying treatment of strips instead of lines in the corresponding Li criterion, and by generalizing these coefficients to various sets of functions.

For example, Lagarias generalized the Li coefficients to L-functions attached to an irreducible cuspidal unitary automorphic representation [9]. In this work, the generalization to the functions in the Selberg class, extended Selberg class and modified Selberg class are important. These have been earlier considered by, for example, Smajlović [15] and Smajlović and Odžak [13, 14].

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Freitas [7] introduced a new parameter τ in the definition of Li coefficients and thus extended Li's criterion to get a new criterion to consider whether all the zeros of the Riemann zeta function lie within the strip $1-\tau/2 \leq \operatorname{Re} s \leq \tau/2$ inside the critical strip. The coefficients investigated by Freitas are called τ-Li coefficients. τ-Li coefficients attached to the functions from the Selberg class, modified Selberg class and the class $\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$ (introduced in [6]) are treated in Droll [5] and Ernvall-Hytönen et al. [6].

In this paper, we consider the asymptotic behavior of the τ-Li coefficients attached to the functions belonging to the class $\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$. Since all the functions in this class do not satisfy the Riemann hypothesis, it is of special interest to consider zeros in strips as opposed to zeros lying on a given line. Of course, the results also apply for zeros on lines, by taking an appropriate value of τ. The class $\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$ has slightly different axioms than the typical Selberg class. In some sense the conditions are more strict than, for example, for the extended Selberg class, but on the other hand, some functions belonging to the class $\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$ do not belong to the extended Selberg class. In $\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$ we assume that the Dirichlet series representing function is convergent for $\operatorname{Re} s > \sigma_0$ for some $\sigma_0 > 0$ instead of $\operatorname{Re} s > 1$ like for the Selberg and extended Selberg class. This modification allows us to consider shifts of functions from the Selberg class. The other major change is the removal of the assumption of the Ramanujan-Pettersson conjecture. This allows us to consider automorphic L-functions. Also, we allow the existence of finitely many poles (as opposed to the case of the Selberg class, where only the possible pole at $s=1$ is allowed).

The paper is organized as follows: In Section 2, we give basic properties for the functions in the class $\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$, and recall some results proved in [6] for the τ-Li coefficients of the functions in $\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$. In Section 3, we prove the asymptotic expansions for the archimedean and non-archimedean contributions of τ-Li coefficients. We will then recall the notion of the $\tau/2$-Riemann hypothesis, and give a condition under which the $\tau/2$-Riemann hypothesis holds in terms of the asymptotic behavior of the τ-Li coefficients. Namely, we extend Voros's criterion for the functions in the class $\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$ with zeros in some strips. At the end of the section, we consider the important special case of a product of shifts of the Riemann zeta function. Finally, Section 4 is devoted to some examples and results obtained in our numerical investigations.

2. Preliminaries

2.1. Class $\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$

The class $\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$ is introduced in [6] as a modification of the Selberg class in order to be able to apply the results of the paper unconditionally to automorphic L-functions and to various products of functions from the Selberg class. For example, the automorphic

L-functions are believed to belong to the Selberg class but the Ramanujan-Petersson conjecture is known only in very few cases. Axioms defining the class $\mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$ are modifications of the four axioms of the Selberg class, while the Ramanujan-Petersson conjecture is not assumed. We have also modified the range of convergence.

Let σ_0 and σ_1 be real numbers such that $\sigma_0 \geq \sigma_1 > 0$. The class $\mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$ is the class of functions F satisfying the following four axioms:

(i) (Dirichlet series) The function F possesses a Dirichlet series representation

$$F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s},$$

which converges absolutely for $\operatorname{Re} s > \sigma_0$.

(ii) (Analytic continuation) There exist at most finitely many non-negative integers $m_1, \ldots, m_N$ and complex numbers $s_1, \ldots, s_N$ such that the function $\prod_{i=1}^{N}(s-s_i)^{m_i} F(s)$ is an entire function of finite order.

(iii) (Functional equation) The function F satisfies the functional equation

$$\xi_F(s) = w\overline{\xi_F(\sigma_1 - \bar{s})},$$

where the completed function ξ_F is defined as
(2.1)
$$\xi_F(s) = F(s) Q_F^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) \prod_{i=1}^{2M+\delta(\sigma_1)} (s-s_i)^{m_i} \prod_{i=2M+1+\delta(\sigma_1)}^{N} (s-s_i)^{m_i}(\sigma_1 - s - \overline{s_i})^{m_i},$$

where $|w| = 1$, $Q_F > 0$, $r \geq 0$, $\lambda_j > 0$, $\mu_j \in \mathbb{C}$, $j = 1, \ldots, r$. Here we assume that the poles of the function F are arranged so that the first $2M + \delta(\sigma_1)$ poles $(0 \leq 2M + \delta(\sigma_1) \leq N)$ are such that $s_{2j-1} + \bar{s}_{2j} = \sigma_1$, for $j = 1, \ldots, M$, where $\delta(\sigma_1) = 1$ if $\sigma_1/2$ is a pole of F in which case $s_{2M+\delta(\sigma_1)} = \sigma_1/2$; otherwise $\delta(\sigma_1) = 0$.

(v) (Euler sum) The logarithmic derivative of the function F possesses a Dirichlet series representation

$$\frac{F'}{F}(s) = -\sum_{n=2}^{\infty} \frac{c_F(n)}{n^s}$$

converging absolutely for $\operatorname{Re} s > \sigma_0$.

This definition of the class $\mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$ gives a possibility to generalize the notion of the trivial and non-trivial zeros of the functions from the class $\mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$. The zeros of $\xi_F(s)$ are called the non-trivial zeros of $F(s)$. All the other zeros of $F(s)$ are called trivial zeros, and they arise from the poles of the gamma functions appearing in (2.1).

In the sequel, the set of non-trivial zeros of $F(s)$ is denoted by $Z(F)$. By the functional equation and the Euler product representation, all the non-trivial zeros lie in the critical strip $\sigma_1 - \sigma_0 \leq \operatorname{Re} s \leq \sigma_0$.

In [6] it is proved that $F(s) = \prod_{i=1}^{K} \zeta(s-\alpha_i)\zeta(s+\alpha_i) \in \mathcal{S}^{\sharp\flat}(\sigma_0, 1)$, where ζ denotes classical Riemann zeta function, the numbers α_i are arbitrary complex constants, and $\sigma_0 = \max_{1\leq i\leq K}\{|\operatorname{Re}\alpha_i| + 1\}$. These functions are used for our numerical computations.

2.2. Some properties of the class $\mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$

We need some additional properties for the functions in $\mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$ in addition to the properties proved in [6, 12]. Basically, we need to prove an approximate formula for the logarithmic derivative of functions in the class under consideration and the approximate formula for the distribution of its zeros.

Lemma 2.1. *Let $F \in \mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$ such that $0 \notin Z(F)$ and $s = \sigma + iT$, for an arbitrary $T > 2$, and $\sigma_1 - \sigma_0 - 2 \leq \sigma \leq \sigma_0 + 2$. Then*

$$\frac{F'}{F}(s) = \sum_{\substack{\rho\in Z(F)\\ |T-\operatorname{Im}\rho|\leq 1}} \frac{1}{s-\rho} + O(\log T)$$

as $T \to \infty$, where the zeros in the sum are counted according to their multiplicities.

Proof. From the definition (2.1) of the completed function ξ_F and [6, Proposition 9], we easily get

$$\begin{aligned}\frac{F'}{F}(s) = {\sum_{\rho\in Z(F)}}^{*} \frac{1}{s-\rho} - \sum_{i=1}^{N} \frac{m_i}{s-s_i} - \sum_{i=2M+\delta(\sigma_1)+1}^{N} \frac{m_i}{s-\sigma_1-\overline{s_i}} \\ - \sum_{j=1}^{r} \lambda_j \frac{\Gamma'}{\Gamma}(\lambda_j s + \mu_j) - \log Q_F,\end{aligned} \tag{2.2}$$

whenever s is not a zero or a pole of the function F. Write $s = \sigma + iT$, $T > 2$. By Stirling's formula for the digamma function $\frac{\Gamma'}{\Gamma}(s) = \log s + O(1/|s|)$, the representation (2.2) yields

$$\frac{F'}{F}(s) = {\sum_{\rho\in Z(F)}}^{*} \frac{1}{s-\rho} + O(\log|s|)$$

as $|s| \to \infty$. In particular, if we evaluate the formula above at the point $s = \sigma_0 + 2 + iT$, and subtract this, we get

$$\frac{F'}{F}(s) = {\sum_{\rho\in Z(F)}}^{*} \left(\frac{1}{s-\rho} - \frac{1}{\sigma_0 + 2 + iT - \rho}\right) + O(\log T) \tag{2.3}$$

as $T \to \infty$ uniformly in $\sigma_1 - \sigma_0 - 2 \le \operatorname{Re} s \le \sigma_0 + 2$.

On the other hand, the formula (2.2) and Stirling's formula with $s = \sigma_0 + 2 + iT$ and $T > 2$, imply

$$-\operatorname{Re}\left(\frac{F'}{F}(\sigma_0 + 2 + iT)\right) < C_1 \log T - \sum_{\rho \in Z(F)} \operatorname{Re}\left(\frac{1}{\sigma_0 + 2 + iT - \rho}\right).$$

Hence,

$$\begin{aligned}\sum_{\rho \in Z(F)} \operatorname{Re}\left(\frac{1}{\sigma_0 + 2 + iT - \rho}\right) &= \sum_{\rho \in Z(F)} \frac{\sigma_0 - \operatorname{Re}\rho + 2}{(\sigma_0 - \operatorname{Re}\rho + 2)^2 + (T - \operatorname{Im}\rho)^2} \\ &= O(\log T).\end{aligned}$$

Since, $\sigma_1 - \sigma_0 \le \operatorname{Re}\rho \le \sigma_0$, we get

$$\begin{aligned}\sum_{|T - \operatorname{Im}\rho| > 1} \frac{1}{|T - \operatorname{Im}\rho|^2} &\le \sum_{\rho \in Z(F)} \frac{1 + (2\sigma_0 + 2)^2}{(2\sigma_0 + 2)^2 + (T - \operatorname{Im}\rho)^2} \\ &\le \frac{1 + (2\sigma_0 + 2)^2}{2} \sum_{\rho \in Z(F)} \frac{\sigma_0 - \operatorname{Re}\rho + 2}{(\sigma_0 - \operatorname{Re}\rho + 2)^2 + (T - \operatorname{Im}\rho)^2} \\ &= O(\log T)\end{aligned}$$

as $T \to \infty$. This implies that

$$\sum_{|T - \operatorname{Im}\rho| > 1} \left| \frac{1}{s - \rho} - \frac{1}{\sigma_0 + 2 + iT - \rho} \right| \le (2\sigma_0 + 4) \sum_{|T - \operatorname{Im}\rho| > 1} \frac{1}{(T - \operatorname{Im}\rho)^2} = O(\log T). \tag{2.4}$$

It is thus left to estimate the part of the sum in (2.3) over ρ with $|T - \operatorname{Im}\rho| \le 1$. Since the number of such zeros is $O(\log T)$ and $|\sigma_0 + 2 + iT - \rho| \ge 2$, we obtain $\sum_{|T - \gamma| \le 1} \frac{1}{|\sigma_0 + 2 + iT - \rho|} = O(\log T)$. This, together with (2.4) and (2.3), completes the proof. □

Let $T > \max_{j=1,\ldots,N} |\operatorname{Im} s_j|$, $T \ne \operatorname{Im}\rho$, and let $N^+_{F,\sigma_0,\sigma_1}(T)$ and $N^-_{F,\sigma_0,\sigma_1}(T)$ denote the number of non-trivial zeros of the function $F \in \mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$ such that $0 \le \operatorname{Im}\rho \le T$ or $-T \le \operatorname{Im}\rho \le 0$, respectively. The approximate formulas for the numbers $N^+_{F,\sigma_0,\sigma_1}(T)$ and $N^-_{F,\sigma_0,\sigma_1}(T)$ can be obtained in the classical way by the application of the argument principle and the functional equation. Variation of the arguments of each factor in the completed function is observed. For the gamma factors, Stirling's formula is used, while estimates for the functions F are obtained using Lemma 2.1. See for example [11].

Lemma 2.2. *Assume $F \in \mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$, $T > \max_{j=1,\ldots,N} |\operatorname{Im} s_j|$, $T \ne \operatorname{Im}\rho$. Now*

$$N^{\pm}_{F,\sigma_0,\sigma_1}(T) = \frac{d_F}{2\pi} T \log T + c_F T + O(\log T), \tag{2.5}$$

where

$$c_F = \frac{\log Q_F}{\pi} - \frac{d_F}{2\pi} + \frac{1}{\pi} \sum_{j=1}^{r} \lambda_j \log \lambda_j.$$

2.3. τ-Li coefficients for the class $\mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$

In [6], the τ-Li coefficients for the class $\mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$ are defined, their existence is proved and alternative formulas are derived. Also the classical τ-Li criterion for the generalized Riemann hypothesis is proved for the class $\mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$. We will now state some of these results for the reader's convenience.

Definition 2.3. Let $\tau \in [\sigma_1, +\infty)$. For an arbitrary positive integer n, the n-th τ-Li coefficient associated to $F \in \mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$ is defined as

$$\lambda_F(n, \tau) = \sum_{\rho \in Z(F)}^{*} \left(1 - \left(\frac{\rho}{\rho - \tau}\right)^n\right),$$

where $Z(F)$ denotes the set of non-trivial zeros of F.

Theorem 2.4. *Let $F \in \mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$ and let $\tau \in [\sigma_1, +\infty)$ be an arbitrary fixed real number such that $0, \tau \notin Z(F)$. The following two statements are equivalent:*

(i) $\sigma_1 - \tau/2 \leq \operatorname{Re} \rho \leq \tau/2$ *for every* $\rho \in Z(F)$,

(ii) $\operatorname{Re} \lambda_F(n, \tau) \geq 0$ *for every positive integer* n.

It is easy to conclude that the natural interval for the values of τ is $[\sigma_1, 2\sigma_0]$, as pointed out in [6].

Theorem 2.5. *Let $F \in \mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$ and let $\tau \in [\sigma_1, 2\sigma_0]$ be an arbitrary fixed real number such that $0, \tau \notin Z(F)$. For every positive integer n, we have*

$$\lambda_F(n, \tau) = \frac{\tau}{(n-1)!} \left[\frac{d^n}{ds^n}(s^{n-1} \log \xi_F(s))\right]_{s=\tau}.$$

Theorem 2.6. *Let $F \in \mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$ and $\tau \in [\sigma_1, 2\sigma_0] \backslash \{\sigma_1 - \overline{s_i} : i = 2M + \delta(\sigma_1) + 1, \ldots, N\}$. For every positive integer n, we have*

$$\begin{aligned}\lambda_F(n, \tau) = &\sum_{\substack{i=1 \\ s_i \neq \tau}}^{N} m_i \left(1 - \left(\frac{s_i}{s_i - \tau}\right)^n\right) + \sum_{i=2M+1+\delta(\sigma_1)}^{N} m_i \left(1 - \left(\frac{\sigma_1 - \overline{s_i}}{\sigma_1 - \tau - \overline{s_i}}\right)^n\right) \\ &+ n\tau \log Q_F + \sum_{k=1}^{n} \binom{n}{k} \tau^k b_{k-1} + \sum_{k=1}^{n} \binom{n}{k} \frac{\tau^k}{(k-1)!} \sum_{j=1}^{r} \lambda_j^k \Psi^{(k-1)}(\lambda_j \tau + \mu_j),\end{aligned}$$

where $\frac{F'}{F}(s) = \sum_{\ell=-1}^{\infty} b_\ell (s - \tau)^\ell$ is the Laurent expansion at $s = \tau$. In particular, if p is the order of the pole of the function F at $s = \tau$, then $b_{-1} = -p$, and if the function $\frac{F'}{F}(s)$ does not have a pole at $s = \tau$, then $b_{-1} = 0$.

3. Asymptotic behavior of the τ-Li coefficients

In this section we derive asymptotic formulas for the two contributions to the τ-Li coefficients attached to functions from the class $\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$. From Theorem 2.6, we see that the τ-Li coefficient $\lambda_F(n,\tau)$ can be written as a sum of an archimedean and a non-archimedean part. Precisely,

$$\lambda_F(n,\tau) = S_A(n,\tau) + S_{NA}(n,\tau),$$

where

$$S_A(n,\tau) = \sum_{k=1}^{n} \binom{n}{k} \frac{\tau^k}{(k-1)!} \sum_{j=1}^{r} \lambda_j^k \Psi^{(k-1)}(\lambda_j\tau+\mu_j) + n\tau \log Q_F \tag{3.1}$$

and

$$\begin{aligned} S_{NA}(n,\tau) = {} & \sum_{k=1}^{n} \binom{n}{k} \tau^k b_{k-1} + \sum_{\substack{i=1 \\ s_i\neq\tau}}^{N} m_i \left(1 - \left(\frac{s_i}{s_i-\tau}\right)^n\right) \\ & + \sum_{i=2M+1+\delta(\sigma_1)}^{N} m_i \left(1 - \left(\frac{\sigma_1-\overline{s_i}}{\sigma_1-\tau-\overline{s_i}}\right)^n\right). \end{aligned} \tag{3.2}$$

We will next look at the asymptotic expansions of the archimedean and non-archimedean parts of the τ-Li coefficients.

3.1. Asymptotic behavior of the archimedean part of the n-th τ-Li coefficient

In the following theorem, we give the full asymptotic expansion of the archimedean part of the n-th τ-Li coefficient.

Theorem 3.1. *Let $F \in \mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$ and $\tau \in (\sigma_1, 2\sigma_0]$, then*

$$\begin{aligned} S_A(n,\tau) = {} & \left(\tau \sum_{j=1}^{r} \lambda_j\right) n\log n + \left(\sum_{j=1}^{r} \lambda_j \log(\tau\lambda_j) + \gamma \sum_{j=1}^{r}\lambda_j - \sum_{j=1}^{r}\lambda_j + \log Q_F\right) n\tau \\ & + \frac{\tau}{2}\sum_{j=1}^{r}\lambda_j + \sum_{j=1}^{r}\mu_j - \frac{r}{2} + \tau \sum_{j=1}^{r}\lambda_j \sum_{k=1}^{K} \frac{B_{2k}}{2k} n^{-2k+1} \\ & + \sum_{j=1}^{r}\sum_{t=0}^{m-1} \left(\frac{\mu_j+t}{\lambda_j\tau+\mu_j+t}\right)^n + O_K(n^{-2K}) \end{aligned} \tag{3.3}$$

as $n\to\infty$, where the numbers B_{2k} are the Bernoulli numbers and $m\in\mathbb{N}$ is such that $m+M\geq 0$ where $M=\min_{j=1,\ldots,r}\operatorname{Re}\mu_j$.

Proof. We first write the archimedean part (3.1) in terms of the Hurwitz zeta function. Depending on the values μ_j, in some cases it is convenient to do some argument shifts.

Mainly, if $M = \min_{j=1,\dots,r} \operatorname{Re}\mu_j$, then there exists $m \in \mathbb{N}$ such that $m+M \geq 0$. Repeated application of the recurrence relation for the digamma function [1, 6.4.6]

$$\psi^{(n)}(z+1) = \psi^{(n)}(z) + (-1)^n n! z^{-n-1}$$

for $n \geq 0$, implies

$$\psi^{(n)}(z) = \psi^{(n)}(z+m) - (-1)^n n!(z^{-n-1} + (z+1)^{-n-1} + \cdots + (z+m-1)^{-n-1}).$$

In addition, by [1, 6.4.10], we have

$$\psi^{(n)}(z) = (-1)^{n+1} n! \zeta(n+1, z)$$

for $z \neq 0, -1, -2, \dots$, and thus the sum in (3.1) can be written in the form

(3.4)

$$\begin{aligned}
&\sum_{k=1}^{n} \binom{n}{k} \frac{\tau^k}{(k-1)!} \sum_{j=1}^{r} \lambda_j^k \left(\Psi^{(k-1)}(\lambda_j\tau + \mu_j + m) - (-1)^{k-1}(k-1)! \sum_{t=0}^{m-1} (\lambda_j\tau + \mu_j + t)^{-k} \right) \\
&= \sum_{k=1}^{n} \binom{n}{k} \sum_{j=1}^{r} (-\tau\lambda_j)^k \zeta(k, \lambda_j\tau + \mu_j + m) + \sum_{t=0}^{m-1} \sum_{j=1}^{r} \left(1 - \frac{\lambda_j\tau}{\lambda_j\tau + \mu_j + t}\right)^n - rm \\
&= n\tau \sum_{j=1}^{r} \lambda_j \frac{\Gamma'}{\Gamma}(\lambda_j\tau + \mu_j + m) + \sum_{j=1}^{r} S_A(n,j) + \sum_{t=0}^{m-1} \sum_{j=1}^{r} \left(\frac{\mu_j + t}{\lambda_j\tau + \mu_j + t} \right)^n - rm,
\end{aligned}$$

where

$$S_A(n,j) = \sum_{k=2}^{n} \binom{n}{k} (-\tau\lambda_j)^k \zeta(k, \lambda_j\tau + \mu_j + m).$$

Main part of the proof is to approximate the expression $S_A(n,j)$. Ideas for the approximation are analogues to those used in [3]. Calculus of residues implies

$$S_A(n,j) = \frac{(-1)^n}{2\pi i} n! \int_R f_j(s)\, ds, \tag{3.5}$$

where

$$f_j(s) = \frac{\Gamma(s-n)}{\Gamma(s+1)} (\tau\lambda_j)^s \zeta(s, \lambda_j\tau + \mu_j + m),$$

and R is positively oriented rectangle with vertices at points $3/2 \pm i$ and $n + 1/2 \pm i$. Basically, the poles of the function $f_j(s)$ inside R are simple poles of the gamma function $\Gamma(s-n)$ at $s = 2, 3, \dots, n$, since the other factors of $f_j(s)$ are holomorphic in R. Calculus of residues implies

$$\frac{(-1)^n}{2\pi i} n! \int_R f_j(s)\, ds = (-1)^n n! \sum_{k=2}^{n} \operatorname*{Res}_{s=k} f_j(s),$$

which justifies (3.5), since

$$\operatorname*{Res}_{s=k} f_j(s) = \frac{1}{\Gamma(k+1)}(\lambda_j\tau)^k \zeta(k, \lambda_j\tau + \mu_j + m)\frac{(-1)^{n-k}}{(n-k)!}$$

for $k = 2, 3, \ldots, n$.

The function $f_j(s)$ is uniformly bounded on the real segment joining $n + 1/2$ and e^n, hence, the rectangle R can be deformed to the line $L : (e^n - i\infty, e^n + i\infty)$. The additional singularities of the function $f_j(s)$ are the poles at the points $s = 0$ and $s = 1$. Hence, we have

$$\mathcal{S}_A(n, j) = \frac{(-1)^n}{2\pi i} n! \int_L f_j(s)\, ds + (-1)^{n-1} n! \left(\operatorname*{Res}_{s=0} f_j(s) + \operatorname*{Res}_{s=1} f_j(s) \right),$$

and thus

$$\begin{aligned} \mathcal{S}_A(n, j) &= (-1)^{n-1} n! \left(\operatorname*{Res}_{s=0} f_j(s) + \operatorname*{Res}_{s=1} f_j(s) \right) \\ &\quad + O\left(n! \int_{-\infty}^{\infty} \left| \frac{\Gamma(e^n + it - n)}{\Gamma(e^n + it + 1)} \right| (\lambda_j\tau)^{e^n} \zeta(e^n, |\lambda_j\tau + \mu_j + m|)\, dt \right). \end{aligned} \tag{3.6}$$

We evaluate the residues in (3.6), and approximate the integral appearing there. The pole at $s = 0$ is simple and its residue can be easily calculated

$$\operatorname*{Res}_{s=0} f_j(s) = \frac{(-1)^n}{n!} \zeta(0, \lambda_j\tau + \mu_j + m) = \frac{(-1)^n}{n!} \left(\frac{1}{2} - \lambda_j\tau - \mu_j - m \right).$$

The pole at $s = 1$ is of order 2 and its residue can be found using the Laurent series representations of the factors of the function $f_j(s)$

$$\begin{aligned} \zeta(s, \lambda_j\tau + \mu_j + m) &= \frac{1}{s-1} - \frac{\Gamma'}{\Gamma}(\lambda_j\tau + \mu_j + m) + \cdots, \\ (\lambda_j\tau)^s &= \lambda_j\tau e^{(s-1)\log(\lambda_j\tau)} = \lambda_j\tau(1 + (s-1)\log\lambda_j\tau + \cdots), \\ \frac{1}{\Gamma(s+1)} &= 1 + (\gamma - 1)(s-1) + \cdots, \\ \Gamma(s-n) &= \frac{(-1)^{n-1}}{(n-1)!}\frac{1}{s-1} + \frac{(-1)^{n-1}}{(n-1)!}\frac{\Gamma'}{\Gamma}(n) + \cdots. \end{aligned}$$

This yields

$$\operatorname*{Res}_{s=1} f_j(s) = \frac{(-1)^{n-1}}{(n-1)!}\lambda_j\tau \left(\frac{\Gamma'}{\Gamma}(n) - \frac{\Gamma'}{\Gamma}(\lambda_j\tau + \mu_j + m) + \log(\lambda_j\tau) + \gamma - 1 \right).$$

Stirling's formula for the digamma function [1, 6.3.18] implies

$$\begin{aligned} \operatorname*{Res}_{s=1} f_j(s) = \frac{(-1)^{n-1}}{(n-1)!}\lambda_j\tau \Big(&\log n - \frac{1}{2n} - \frac{\Gamma'}{\Gamma}(\lambda_j\tau + \mu_j + m) + \log(\lambda_j\tau) + \gamma - 1 \\ &- \sum_{k=1}^{K} \frac{B_{2k}}{2k} n^{-2k} + O_K(n^{-1-2K}) \Big) \end{aligned}$$

as $n \to \infty$.

Let us estimate the integral appearing in (3.6). First notice that the part containing the Hurwitz zeta functions is

$$\left|(\lambda_j \tau)^{e^n} \zeta(e^n, |\lambda_j \tau + \mu_j + m|)\right| = o(1)$$

as $n \to \infty$, since $\operatorname{Re} \mu_j > -m$ for all $j = 1, \dots, r$.

The part containing the gamma functions can be modified using the reflection formula and a functional equation for the gamma function. It decays rapidly enough as $n \to \infty$, i.e.,

$$n! \int_{-\infty}^{\infty} \left| \frac{\Gamma(e^n + it - n)}{\Gamma(e^n + it + 1)} \right| dt \ll n^{-2K}$$

as $n \to \infty$, for all $K \in \mathbb{N}$, as shown in [3]. Now, (3.6) reduces to

$$\begin{aligned} S_A(n, j) = {} & \lambda_j \tau + \mu_j + m - \frac{1}{2} + \lambda_j \tau n \log n - \frac{\lambda_j \tau}{2} \\ & + n \lambda_j \tau \left(\log(\lambda_j \tau) - \frac{\Gamma'}{\Gamma}(\lambda_j \tau + \mu_j + m) + \gamma - 1 \right) \\ & - \lambda_j \tau \sum_{k=1}^{K} \frac{B_{2k}}{2k} n^{-2k+1} + O_K(n^{-2K}) \end{aligned}$$

as $n \to \infty$. Combining the expression above with the equation (3.4), and the definition of the archimedean part (3.1), implies (3.3). This completes the proof. □

3.2. Asymptotic behavior of the non-archimedean contribution to the n-th τ-Li coefficient

The bound for $S_{NA}(n, \tau)$ given in the following theorem is in terms of the incomplete τ-Li coefficient up to height T, defined by

$$\lambda_F(n, \tau, T) = \sum_{\substack{\rho \in Z(F) \\ |\operatorname{Im} \rho| \le T}} \left(1 - \left(\frac{\rho}{\rho - \tau} \right)^n \right)$$

where T is a cutoff parameter.

Theorem 3.2. *Let $F \in \mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$, $\tau \in [\sigma_1, 2\sigma_0] \setminus \{\sigma_1 - \overline{s_i} : i = 2M + \delta(\sigma_1) + 1, \dots, N\}$ and $\tau > 1/2$, then*

$$\begin{aligned} S_{NA}(n, \tau) = {} & \lambda_F(n, \tau, \sqrt{n}) - \mathcal{S}_\tau(n) \\ & + \sum_{i=2M+\delta(\sigma_1)+1}^{N} m_i \left(1 - \left(\frac{\sigma_1 - \overline{s_i}}{\sigma_1 - \tau - \overline{s_i}} \right)^n \right) + O(\sqrt{n} \log n) \end{aligned} \tag{3.7}$$

as $n \to \infty$, where

$$\mathcal{S}_\tau(n) = \sum_{j=1}^{r} \sum_{k=0}^{\lfloor -\frac{\tau\lambda_j + 2\operatorname{Re}\mu_j}{2} \rfloor} \left(\left(\frac{\mu_j + k}{\mu_j + k + \tau\lambda_j} \right)^n - 1 \right),$$

and $\lfloor x \rfloor$ denotes the integer part of a real number x.

Proof. The proof is based on the contour integration of a suitably chosen function along certain rectangle. The method is an extension of the method used in [9, 14]. To construct the function to be integrated, let

$$\begin{aligned}
k_{n,\tau}(s) &= \left(1 + \frac{\tau}{s}\right)^n - 1 = \sum_{k=1}^{n} \binom{n}{k} \left(\frac{\tau}{s}\right)^k, \\
g_{j,\tau}(s) &= \left(\frac{\tau}{s}\right)^j \frac{F'}{F}(s+\tau) \\
G_{n,\tau}(s) &= k_{n,\tau}(s) \frac{F'}{F}(s+\tau) = \sum_{k=1}^{n} \binom{n}{k} \left(\frac{\tau}{s}\right)^k \frac{F'}{F}(s+\tau) = \sum_{k=1}^{n} \binom{n}{k} g_{k,\tau}(s).
\end{aligned}$$

In order to construct a suitable rectangle to be used in the contour integration, we need to introduce some notation. We need to ensure that all the poles of the function F are in the contour, as well as all the non-trivial zeros up to the height T. Assume that $F \in \mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$ has poles of order m_i at s_i, $i = 1, \ldots, N$ and put $\eta_1 = -1 + \min_{1 \le i,j \le N}\{\operatorname{Re} s_i, \sigma_1 - \operatorname{Re} s_j\}$, $\eta_2 = 1 + \max_{1 \le i,j \le N}\{\operatorname{Re} s_i, \sigma_1 - \operatorname{Re} s_j\}$, and $T \ge \max_{1 \le i \le N}\{|\operatorname{Im} s_i|\}$. Let α be a positive real number such that $\alpha + \sigma_1 > \sigma_0$ and $[\eta_1, \eta_2] \subseteq [-\alpha, \alpha + \sigma_1]$. Non-trivial zeros ρ of the function $F \in \mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$ lie in the critical strip $\sigma_1 - \sigma_0 \le \operatorname{Re} s \le \sigma_0$, so the non-trivial zeros and the poles of the function $F \in \mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$ lie in the strip $-\alpha \le \operatorname{Re} s \le \alpha + \sigma_1$. Let us define $a = 1 + \max\{\alpha + \sigma_1, \sigma_0/\tau\}$ and choose $n \in \mathbb{N}$ such that $2\tau\sqrt{n} > a$ and $\sqrt{n} + \epsilon_n = T$ where ϵ_n is chosen such that $0 < \epsilon_n < 1$ and that the horizontal lines $\operatorname{Im} s = \pm T$ do not approach closer than $O(1/\log n)$ to any zero of $F(s)$.

Let us integrate the function $G_{n,\tau}(s)$ over the counterclockwise oriented rectangle $R(n)$ formed by the lines $\operatorname{Re} s = -\tau a$, $\operatorname{Re} s = 2\tau\sqrt{n}$, $\operatorname{Im} s = \pm T$.

Poles of the function $G_{n,\tau}(s)$ in $R(n)$ are at $s = 0$, at points that correspond to the trivial and non-trivial zeros of F and at the poles of F. The residue at $s = 0$ can be calculated using the Laurent expansion of the function $\frac{F'}{F}(s)$ at $s = \tau$ given by $\frac{F'}{F}(s) = \sum_{\ell=-1}^{\infty} b_\ell (s-\tau)^\ell$. If p is the order of the pole of the function F at $s = \tau$, then $b_{-1} = -p$, otherwise $b_{-1} = 0$. Now, we have

$$g_{j,\tau}(s) = \tau^j \sum_{\ell=-1}^{\infty} b_\ell s^{\ell-j} = \tau^j \left(\sum_{\ell=-1}^{j-1} b_\ell s^{\ell-j} + \sum_{\ell=j}^{\infty} b_\ell s^{\ell-j} \right)$$

$$= \tau^j \left(\sum_{k=1}^{j+1} b_{j-k} s^{-k} + \sum_{k=0}^{\infty} b_{j+k} s^k \right).$$

There are two possibilities, depending whether τ is a pole of the function F or not. If it is, then $b_{-1} \neq 0$, and then $g_{j,\tau}(s)$ has a pole at $s = 0$ of the order $j+1$, and it is easy to calculate that

$$\operatorname*{Res}_{s=0} g_{j,\tau}(s) = \tau^j b_{j-1}.$$

If it is not a pole of the function F, then $b_{-1} = 0$. Let $v \in \{0, 1, \dots, j-1\}$ be the smallest index such that $b_v \neq 0$. Then $g_{j,\tau}(s)$ has a pole at $s = 0$ of the order $j - v$ and it produces the same residue as in the previous case. Thus,

$$\operatorname*{Res}_{s=0} G_{n,\tau}(s) = \sum_{k=1}^{n} \binom{n}{k} \tau^k b_{k-1}. \tag{3.8}$$

It is easy to calculate that

$$\operatorname*{Res}_{s=s_i-\tau} G_{n,\tau}(s) = m_i \left(1 - \left(\frac{s_i}{s_i - \tau} \right)^n \right), \quad s_i \neq \tau, \tag{3.9}$$

$$\operatorname*{Res}_{s=\rho-\tau} G_{n,\tau}(s) = \left(\frac{\rho}{\rho - \tau} \right)^n - 1, \tag{3.10}$$

$$\operatorname*{Res}_{s=s_{j,k}-\tau} G_{n,\tau}(s) = \left(\frac{\mu_j + k}{\mu_j + k + \tau\lambda_j} \right)^n - 1, \tag{3.11}$$

where $s_{j,k} = -(\mu_j + k)/\lambda_j$ is a trivial zero of the function $F \in \mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$ ($1 \leq j \leq r$, $k \in \mathbb{N}_0$) such that $s_{j,k} - \tau$ is inside the rectangle $R(n)$. The residue theorem, (3.8), (3.9), (3.10), and (3.11) give

$$\begin{aligned}
I(n) &= \frac{1}{2\pi i} \int_{R(n)} G_{n,\tau}(s)\, ds \\
&= \operatorname*{Res}_{s=0} G_{n,\tau}(s) + \sum_{\substack{i=1 \\ s_i \neq \tau}}^{N} \operatorname*{Res}_{s=s_i-\tau} G_{n,\tau}(s) \\
&\quad + \sum_{\substack{\rho \in Z(F) \\ |\operatorname{Im}\rho| \leq T}} \operatorname*{Res}_{s=\rho-\tau} G_{n,\tau}(s) + \sum_{j=1}^{r} \sum_{\substack{k \in \mathbb{N}_0 \\ s_{j,k}-\tau \in R(n)}} \operatorname*{Res}_{s=s_{j,k}-\tau} G_{n,\tau}(s) \\
&= \sum_{k=1}^{n} \binom{n}{k} b_{k-1} \tau^k + \sum_{\substack{i=1 \\ s_i \neq \tau}}^{N} m_i \left(1 - \left(\frac{s_i}{s_i - \tau} \right)^n \right) \\
&\quad + \sum_{\substack{\rho \in Z(F) \\ |\operatorname{Im}\rho| \leq T}} \left(\left(\frac{\rho}{\rho - \tau} \right)^n - 1 \right) + \sum_{j=1}^{r} \sum_{\substack{k \in \mathbb{N}_0 \\ s_{j,k}-\tau \in R(n)}} \left(\left(\frac{\mu_j + k}{\mu_j + k + \tau\lambda_j} \right)^n - 1 \right)
\end{aligned}$$

$$= S_{NA}(n,\tau) - \lambda_F(n,\tau,T) - \sum_{i=2M+1+\delta(\sigma_1)}^{N} m_i \left(1 - \left(\frac{\sigma_1 - \overline{s_i}}{\sigma_1 - \tau - \overline{s_i}}\right)^n\right)$$
$$+ \sum_{j=1}^{r} \sum_{\substack{k\in\mathbb{N}_0 \\ s_{j,k}-\tau\in R(n)}} \left(\left(\frac{\mu_j + k}{\mu_j + k + \tau\lambda_j}\right)^n - 1\right)$$

where zeros are counted according to their multiplicities.

Since $s_{j,k} - \tau \in R(n)$, we obtain that $-(\lambda_j\tau(2\sqrt{n}+1) + \operatorname{Re}\mu_j) \le k \le (a-1)\tau\lambda_j - \operatorname{Re}\mu_j$. We choose $n \in \mathbb{N}$ suitably large (it is convenient since we are interested in the limiting process as $n \to \infty$), i.e., such that $\lambda_j\tau(2\sqrt{n}+1) + \operatorname{Re}\mu_j \ge 0$ for all $j = 1,\dots,r$. Now

$$\mathcal{S} = \sum_{j=1}^{r} \sum_{\substack{k\in\mathbb{N}_0 \\ s_{j,k}-\tau\in R(n)}} \left(\left(\frac{\mu_j + k}{\mu_j + k + \tau\lambda_j}\right)^n - 1\right)$$
$$= \sum_{j=1}^{r} \sum_{k=0}^{\lfloor (a-1)\tau\lambda_j - \operatorname{Re}\mu_j \rfloor} \left(\left(\frac{\mu_j + k}{\mu_j + k + \tau\lambda_j}\right)^n - 1\right).$$

If $(a-1)\tau\lambda_j - \operatorname{Re}\mu_j < 0$ for all $j = 1,\dots,r$, the last sum is empty.

Also, the properties

$$\left|\frac{\mu_j + k}{\mu_j + k + \tau\lambda_j}\right| < 1, \quad \tau > -\frac{2}{\lambda_j}(\operatorname{Re}\mu_j + k) \quad \text{and} \quad k > -\frac{\tau\lambda_j + 2\operatorname{Re}\mu_j}{2}$$

are equivalent. If we denote $K_{\min}(j) = \max\{0, 1 + \lfloor -(\tau\lambda_j + 2\operatorname{Re}\mu_j)/2 \rfloor\}$ and $K_{\max}(j) = \lfloor -(\tau\lambda_j + 2\operatorname{Re}\mu_j)/2 \rfloor$, we have

$$\mathcal{S} = \sum_{j=1}^{r} \sum_{k=K_{\min}(j)}^{\lfloor (a-1)\tau\lambda_j - \operatorname{Re}\mu_j \rfloor} \left(\left(\frac{\mu_j + k}{\mu_j + k + \tau\lambda_j}\right)^n - 1\right) + \sum_{j=1}^{r} \sum_{k=0}^{K_{\max}(j)} \left(\left(\frac{\mu_j + k}{\mu_j + k + \tau\lambda_j}\right)^n - 1\right)$$
$$= O(1) + \mathcal{S}_\tau(n)$$

where $\mathcal{S}_\tau(n) = \sum_{j=1}^{r} \sum_{k=0}^{K_{\max}(j)} \left(\left(\frac{\mu_j+k}{\mu_j+k+\tau\lambda_j}\right)^n - 1\right)$. Therefore,

$$\begin{aligned} I(n) &= S_{NA}(n,\tau) - \lambda_F(n,\tau,T) + \mathcal{S}_\tau(n) \\ &\quad - \sum_{i=2M+\delta(\sigma_1)+1}^{N} m_i \left(1 - \left(\frac{\sigma_1 - \overline{s_i}}{\sigma_1 - \tau - \overline{s_i}}\right)^n\right) + O(1). \end{aligned} \tag{3.12}$$

It remains to estimate the integral $I(n)$. It can be written as a sum of the four integrals over the sides of the rectangle $R(n)$. We will start from the side on the line $\operatorname{Re} s = 2\tau\sqrt{n}$, and denote integrals by $I_1(n)$, $I_2(n)$, $I_3(n)$ and $I_4(n)$, respectively.

On the line $\operatorname{Re} s = 2\tau\sqrt{n}$ integrand can be easily bounded. Namely,

$$|k_{n,\tau}(2\tau\sqrt{n}+it)| \le \sum_{k=1}^{n}\binom{n}{k}\frac{\tau^k}{(\sqrt{4\tau^2 n+t^2})^k} \le \sum_{k=1}^{n}\binom{n}{k}\left(\frac{\tau}{2\tau\sqrt{n}}\right)^k$$
$$< \left(1+\frac{1}{2\sqrt{n}}\right)^n < 2^{\sqrt{n}}$$

and

$$\left|\frac{F'}{F}(2\tau\sqrt{n}+it+\tau)\right| \le \left|\frac{F'}{F}(2\tau\sqrt{n}+\tau)\right| \le C_0 2^{-(2\tau\sqrt{n}+\tau-\sigma_0-1)},$$

since $2\tau\sqrt{n}+\tau > \sigma_0+1$, and for all $\operatorname{Re} s = \sigma > \sigma_0+1$ one has

$$\left|\frac{F'}{F}(s)\right| \le \left|\frac{F'}{F}(\sigma)\right| \le 2^{-(\sigma-\sigma_0-1)}\left|\frac{F'}{F}(\sigma_0+1)\right| \le C_0 2^{-(\sigma-\sigma_0-1)},$$

where C_0 is a constant. Therefore,

$$|k_{n,\tau}(2\tau\sqrt{n}+it)|\left|\frac{F'}{F}(2\tau\sqrt{n}+it+\tau)\right| \le 2^{\sqrt{n}}C_0 2^{-(2\tau\sqrt{n}+\tau-\sigma_0-1)}$$
$$\le C_0 2^{\sigma_0+1-\sigma_1}2^{-\sqrt{n}(2\tau-1)}.$$

The length of the vertical side located at $\operatorname{Re} s = 2\tau\sqrt{n}$, $|\operatorname{Im} s| \le T$ of the rectangle $R(n)$, is $O(\sqrt{n})$ so the assumption $\tau > 1/2$ implies the estimate $|I_1(n)| = o(1)$ as $n \to \infty$.

In the integral $I_3(n)$ over the side on the line $\operatorname{Re} s = -\tau a$, we have $|k_{n,\tau}(s)| = O(1)$ as $n \to \infty$, since $|1+\tau/s| < 1$ if and only if $a > 1/2$, but the last assumption is true by the choice of a. Additionally, $\left|\frac{F'}{F}(s+\tau)\right| = O(\log n)$ as $n \to \infty$ by Lemma 2.1. The length of the vertical segment $\operatorname{Re} s = -\tau a$, $|\operatorname{Im} s| \le T$ is $O(\sqrt{n})$ giving the bound $|I_3(n)| = O(\sqrt{n}\log n)$ as $n \to \infty$.

Integrals $I_2(n)$ and $I_4(n)$ can be estimated analogously. We will consider $I_2(n)$. Let $s = \sigma + iT$ be a point on the horizontal segment $-a\tau \le \sigma \le 2\tau\sqrt{n}$, $T = \sqrt{n}+\epsilon_n$. For $|\sigma| \le a\tau$, we have

$$\left|1+\frac{\tau}{s}\right| = \left|1+\frac{\tau}{\sigma+iT}\right| = \left|1+\frac{\tau(\sigma-iT)}{\sigma^2+T^2}\right|$$
$$= \sqrt{\left(1+\frac{\sigma\tau}{\sigma^2+T^2}\right)^2+\left(\frac{\tau^2T^2}{\sigma^2+T^2}\right)^2} \le 1+\frac{\tau(\sigma+\tau)}{\sigma^2+T^2} \le 1+\frac{\tau^2(a+1)}{n},$$

so

$$|k_{n,\tau}(s)| \le \left(1+\frac{\tau^2(a+1)}{n}\right)^n + 1 \le e^{\tau^2(a+1)}+1 = O(1), \tag{3.13}$$

and from Lemma 2.1 and the approximate number of zeros given by (2.5), we obtain

$$\left|\frac{F'}{F}(s+\tau)\right| = O((\log T)^2) = O((\log n)^2).$$

Now we step across the interval $a\tau \le \sigma \le 2\tau\sqrt{n}$ towards right, in segments of length τ, starting from $\sigma = a\tau$. At the initial point, we have $\left|\frac{F'}{F}(s+\tau)\right| = O(1)$, since we are in the region of absolute convergence. From

$$\begin{aligned}\left|\frac{k_{n,\tau}(s+\tau)+1}{k_{n,\tau}(s)+1}\right| &= \left|\frac{1+\frac{\tau}{s+\tau}}{1+\frac{\tau}{s}}\right|^n = \left|\frac{1+\frac{\tau}{\sigma+iT+\tau}}{1+\frac{\tau}{\sigma+iT}}\right|^n \\ &= \left|1-\frac{\tau^2}{(\sigma+\tau+iT)^2}\right|^n \le \left(1+\frac{\tau^2}{T^2}\right)^n \le e^{\tau^2}\end{aligned} \tag{3.14}$$

we obtain an upper bound for $\left|k_{n,\tau}(s)\frac{F'}{F}(s+\tau)\right|$ that decreases geometrically at each step, and after $O(\log n)$ steps, it becomes $O(1)$. Since the horizontal segment has length $O(\sqrt{n})$, using (3.13) and (3.14), we obtain

$$\begin{aligned}|I_2(n)| &= \left|\int_{-a\tau}^{a\tau} k_{n,\tau}(\sigma+iT)\frac{F'}{F}(\sigma+iT+\tau)\,d\sigma + \int_{a\tau}^{2\tau\sqrt{n}} k_{n,\tau}(\sigma+iT)\frac{F'}{F}(\sigma+iT+\tau)\,d\sigma\right| \\ &= O(\sqrt{n}).\end{aligned}$$

The same bound holds also for $|I_4(n)|$. Finally, combining all the bounds, we have

$$|I(n)| = O(\sqrt{n}\log n)$$

as $n \to \infty$ and from (3.12), we obtain

$$S_{NA}(n,\tau) = \lambda_F(n,\tau,T) - \mathcal{S}_\tau(n) + \sum_{i=2M+\delta(\sigma_1)+1}^{N} m_i\left(1-\left(\frac{\sigma_1-\overline{s_i}}{\sigma_1-\tau-\overline{s_i}}\right)^n\right) + O(\sqrt{n}\log n)$$

as $n \to \infty$. Additionally, the cutoff parameter in the incomplete τ-Li coefficient in the previous equation can be slightly modified without changing the error term.

The formula (2.5) implies that $N^{\pm}_{F,\sigma_0,\sigma_1}(T+1) - N^{\pm}_{F,\sigma_0,\sigma_1}(T) = O(\log T)$, since $|T-\sqrt{n}| < 1$ and therefore, there are $O(\log n)$ zeros in an interval of length one at this height. Further, for each zero $\rho = \beta + i\gamma$ with $\sqrt{n} \le |\gamma| < \sqrt{n}+1$ the contribution to the incomplete τ-Li coefficient is bounded by a constant since

$$\begin{aligned}\left|\left(\frac{\rho}{\rho-\tau}\right)^n\right| &= \left|1+\frac{\tau(\beta-\tau-i\gamma)}{(\beta-\tau)^2+\gamma^2}\right|^n = \left(\left(1+\frac{\tau(\beta-\tau)}{(\beta-\tau)^2+\gamma^2}\right)^2 + \left(\frac{\tau\gamma}{(\beta-\tau)^2+\gamma^2}\right)^2\right)^{n/2} \\ &\le \left(\left(1+\frac{\tau\sigma_0}{n}\right)^2 + \frac{\tau^2}{n}\right)^{n/2} < 2^{2\tau\sigma_0+\tau^2+\tau^2\sigma_0^2},\end{aligned}$$

so $|\lambda_F(n,\tau,\sqrt{n}) - \lambda_F(n,\tau,T)| = O(\log n)$. Consequently, the relation (3.2) can be written as (3.7). The proof is complete. □

3.3. Consequence of the $\tau/2$-Riemann hypothesis

Let us assume that the $\tau/2$-Riemann hypothesis holds for the function $F \in \mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$, i.e., assume that all the non-trivial zeros ρ of F, have the property $\sigma_1 - \tau/2 \leq \operatorname{Re}\rho \leq \tau/2$. Then $|\rho/(\rho-\tau)| \leq 1$, so

$$|\lambda_F(n, \tau, \sqrt{n})| \leq 2 \sum_{\substack{\rho \in Z(F) \\ |\operatorname{Im}\rho| \leq \sqrt{n}}} 1 = O(\sqrt{n}\log n).$$

Now, Theorems 3.1 and 3.2 imply the following corollary.

Corollary 3.3. *Let $F \in \mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$, $\tau \in [\sigma_1, 2\sigma_0] \setminus \{\sigma_1 - \overline{s_i} : i = 2M + \delta(\sigma_1) + 1, \ldots, N\}$ and $\tau > 1/2$. Assume the $\tau/2$-Riemann hypothesis for $F \in \mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$, then*

$$\begin{aligned}\lambda_F(n,\tau) = {} & \left(\tau \sum_{j=1}^{r} \lambda_j\right) n \log n + \left(\sum_{j=1}^{r} \lambda_j \log(\tau\lambda_j) + \gamma \sum_{j=1}^{r} \lambda_j - \sum_{j=1}^{r} \lambda_j + \log Q_F\right) n\tau \\ & + \sum_{j=1}^{r} \sum_{t=0}^{m-1} \left(\frac{\mu_j + t}{\lambda_j \tau + \mu_j + t}\right)^n - \sum_{j=1}^{r} \sum_{k=0}^{\lfloor -(\tau\lambda_j + 2\operatorname{Re}\mu_j)/2 \rfloor} \left(\left(\frac{\mu_j + k}{\mu_j + k + \tau\lambda_j}\right)^n - 1\right) \\ & + \sum_{i=2M+\delta(\sigma_1)+1}^{N} m_i \left(1 - \left(\frac{\sigma_1 - \overline{s_i}}{\sigma_1 - \tau - \overline{s_i}}\right)^n\right) + O(\sqrt{n}\log n)\end{aligned}$$

as $n \to \infty$ where $m \in \mathbb{N}$ is such that $m + M \geq 0$ where $M = \min_{j=1,\ldots,r} \operatorname{Re}\mu_j$.

If τ is sufficiently large, then the sums containing exponential terms disappear, and we have the following corollary:

Corollary 3.4. *Let $F \in \mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$, $\tau \in [\sigma_1, 2\sigma_0] \setminus \{\sigma_1 - \overline{s_i} : i = 2M + \delta(\sigma_1) + 1, \ldots, N\}$ and $\tau > \max_{\substack{j=1,\ldots,r \\ i=1,\ldots,N}} \{1/2, (-2\operatorname{Re}\mu_j)/\lambda_j, 2(\sigma_1 - \operatorname{Re}s_i)\}$. Assume the $\tau/2$-Riemann hypothesis for $F \in \mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$, then*

$$\begin{aligned}\lambda_F(n,\tau) = {} & \left(\tau \sum_{j=1}^{r} \lambda_j\right) n \log n + \left(\sum_{j=1}^{r} \lambda_j \log(\tau\lambda_j) + \gamma \sum_{j=1}^{r} \lambda_j - \sum_{j=1}^{r} \lambda_j + \log Q_F\right) n\tau \\ & + O(\sqrt{n}\log n)\end{aligned}$$

as $n \to \infty$.

3.4. A special case: products of shifts of the Riemann zeta function

In the first example we obtain formulas for the archimedean and non-archimedean parts of the τ-Li coefficients of a product of shifts of the Riemann zeta function and discuss its behavior for some values of τ which are of special interest. In the second example we deduce a refinement of the result under the Riemann hypothesis.

Example 3.5. Let

$$F(s) = \zeta(s-A)\zeta(s+A)$$

for an arbitrary real constant $A > 1$, where ζ denotes the Riemann zeta function. Then $F \in \mathcal{S}^{\sharp\flat}(A+1,1)$ (as shown in [6]) with $Q_F = \pi^{-1}$, $\omega = 1$, $r = 2$, $\lambda_1 = \lambda_2 = 1/2$, $\mu_1 = A/2$, $\mu_2 = -A/2$. The function F has simple poles at $s_1 = -A+1$ and $s_2 = A+1$. We consider values of τ on the set $[1, 2A+2] \setminus \{A\}$ according to Theorem 3.2.

Set $m = \lceil A/2 \rceil$, where $\lceil x \rceil$ denotes the smallest integer not less than x, and $a = A+3$. Using Theorems 3.1 and 3.2, we get

$$\begin{aligned} S_A(n,\tau) = \tau n \log n &+ \left(\log\left(\frac{\tau}{2}\right) + \gamma - 1 - \log \pi\right) n\tau + \frac{\tau}{2} - 1 \\ &+ \sum_{t=0}^{\lceil A/2 \rceil - 1} \left(\left(\frac{A+2t}{\tau + A + 2t}\right)^n + \left(\frac{-A+2t}{\tau - A + 2t}\right)^n\right) \\ &+ \tau \sum_{k=1}^{K} \frac{B_{2k}}{2k} n^{-2k+1} + O_K(n^{-2K}) \end{aligned} \tag{3.15}$$

and

$$S_{NA}(n,\tau) = \lambda_F(n,\tau,\sqrt{n}) - \mathcal{S}_\tau(n) + \sum_{i=1}^{2}\left(1 - \left(\frac{1-\overline{s_i}}{1-\tau-\overline{s_i}}\right)^n\right) + O(\sqrt{n}\log n) \tag{3.16}$$

as $n \to \infty$, where

$$\mathcal{S}_\tau(n) = \sum_{k=0}^{\lfloor -\frac{\tau/2+A}{2} \rfloor} \left(\left(\frac{A+2k}{A+2k+\tau}\right)^n - 1\right) + \sum_{k=0}^{\lfloor -\frac{\tau/2-A}{2} \rfloor} \left(\left(\frac{-A+2k}{-A+2k+\tau}\right)^n - 1\right).$$

The first sum on the right-hand side of the equation above is empty because $\tau/2 + A > 0$, so

$$\mathcal{S}_\tau(n) = \sum_{k=0}^{\lfloor A/2 - \tau/4 \rfloor} \left(\left(\frac{-A+2k}{-A+2k+\tau}\right)^n - 1\right)$$

and

$$\sum_{i=1}^{2}\left(1 - \left(\frac{1-\overline{s_i}}{1-\tau-\overline{s_i}}\right)^n\right) = -\left(\frac{A}{A-\tau}\right)^n + O(1)$$

as $n \to \infty$. Thus,

$$\begin{aligned} S_{NA}(n,\tau) = \lambda_F(n,\tau,\sqrt{n}) &- \sum_{k=0}^{\lfloor A/2 - \tau/4 \rfloor} \left(\left(\frac{-A+2k}{-A+2k+\tau}\right)^n - 1\right) \\ &- \left(\frac{A}{A-\tau}\right)^n + O(\sqrt{n}\log n) \end{aligned} \tag{3.17}$$

as $n \to \infty$.

From the well known results for the zero-free region of the Riemann zeta function, we easily derive a zero-free region for the function F. Thus, the cases with $2A < \tau < 2A+2$ and the implied strips are of special interest. With this assumption, we obtain that $\mathcal{S}_\tau(n)$ is empty, and $|A/(A-\tau)| < 1$, so

$$S_{NA}(n,\tau) = \lambda_F(n,\tau,\sqrt{n}) + O(\sqrt{n}\log n) \tag{3.18}$$

as $n \to \infty$, and sums appearing in (3.15) are $o(1)$ as $n \to \infty$, and therefore

$$\lambda_F(n,\tau) = \lambda_F(n,\tau,\sqrt{n}) + \tau n \log n + \left(\log\left(\frac{\tau}{2}\right) + \gamma - 1 - \log\pi\right) n\tau + O(\sqrt{n}\log n)$$

as $n \to \infty$.

Remark 3.6. Reasoning similarly as in Example 3.5, one may investigate the asymptotic behaviour of the τ-Li coefficients attached to various functions $G(s-A)G(s+A)$, where $G \in \mathcal{S}^{\sharp\flat}$ and $\tau \in (2A, 2A+2)$, the interval implied by the zero-free region.

Remark 3.7. In addition to those values of τ treated in Example 3.5, the approximate formulas obtained may be used to describe the asymptotic behavior of the τ-Li coefficients for all the other values of τ on the interval of specific interest. Persistence of the exponentially growing terms is expected.

Namely, in the case with $\tau = 2A$ and $\mathcal{S}_\tau(n) = (-1)^n - 1$, using the expressions (3.15) and (3.17), we get

$$\begin{aligned} S_A(n,\tau) &= 2An\log n + (\log A + \gamma - 1 - \log\pi)2An + A - 1 \\ &\quad + \sum_{t=0}^{\lceil A/2\rceil - 1}\left(\left(\frac{A+2t}{3A+2t}\right)^n + \left(\frac{-A+2t}{A+2t}\right)^n\right) \\ &\quad + 2A\sum_{k=1}^{K}\frac{B_{2k}}{2k}n^{-2k+1} + O_K(n^{-2K}) \end{aligned} \tag{3.19}$$

and

$$S_{NA}(n,\tau) = \lambda_F(n,\tau,\sqrt{n}) + 2(-1)^{n+1} + O(\sqrt{n}\log n)$$

as $n \to \infty$, so

$$\lambda_F(n,2A) = \lambda_F(n,2A,\sqrt{n}) + 2An\log n + (\log A + \gamma - 1 - \log\pi)2An + O(\sqrt{n}\log n)$$

as $n \to \infty$.

In the case with $1 \le \tau < 2A$ and $\tau \ne A$, we have $|A/(A-\tau)| > 1$, so

$$\begin{aligned} \lambda_F(n,\tau) &= \lambda_F(n,\tau,\sqrt{n}) - \left(\frac{A}{A-\tau}\right)^n + s(a,\tau) + \tau n\log n \\ &\quad + \left(\log\left(\frac{\tau}{2}\right) + \gamma - 1 - \log\pi\right) n\tau + O(\sqrt{n}\log n) \end{aligned}$$

as $n \to \infty$, where

$$s(a,\tau) = -\sum_{k=0}^{\lfloor A/2-\tau/4 \rfloor} \left(\frac{-A+2k}{-A+2k+\tau} \right)^n + \sum_{t=0}^{\lceil A/2 \rceil -1} \left(\frac{-A+2t}{\tau-A+2t} \right)^n .$$

Notice, that for some values of τ, the sums may be empty, and therefore treated as zero. Otherwise, some terms persist.

Now we consider the asymptotic behavior of the τ-Li coefficients conditionally under the Riemann hypothesis.

Example 3.8. Let

$$F(s) = \zeta(s-A)\zeta(s+A)$$

for an arbitrary real constant $A > 0$ and $\tau = 2A+1$. Assume that the Riemann hypothesis is valid, i.e., assume that all the non-trivial zeros of the function F lie on the lines $\operatorname{Re} s = 1/2 \pm A$. Under this assumption $|\rho/(\rho-\tau)| \leq 1$, so each term in $\lambda_F(n,\tau,\sqrt{n})$ contributes at most 2 in the absolute value. Since the number of zeros ρ such that $|\operatorname{Im}\rho| \leq \sqrt{n}$ is $O(\sqrt{n}\log n)$, using the previous example we obtain

$$\lambda_F(n,\tau) = \tau n \log n + \left(\log\left(\frac{\tau}{2}\right) + \gamma - 1 - \log\pi \right) n\tau + O(\sqrt{n}\log n) \quad \text{as } n \to \infty.$$

4. Numerical computations

In this section we present some numerical computations for the τ-Li coefficients for different L-functions. We concentrate on functions in the class under consideration, and in particular, on appropriate products of the Riemann zeta function.

Different approaches can be used in the numerical calculations of the τ-Li coefficients. Different definitions or arithmetic formulas for τ-Li coefficients can be used to develop codes for calculations. Calculations based on the definition of τ-Li coefficients in terms of a sum over the zeros of the corresponding L-function are done with Mathematica 9 with precise error estimates in [6,12] for some examples of functions from the class $\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$.

In the present paper, we are interested in separate calculations of the two, archimedean and non-archimedean, contributions. The definition of the τ-Li coefficients is not adequate in this case. Thus, a different approach needs to be used. We use the definition of the n-th τ-Li coefficient in terms of the n-th derivative of the completed L-function given in Proposition 2.5, together with the power series representation of the corresponding L-function. This method is analogous to the one used in [4]. This approach enables us to distinguish between different contributions.

The code is written using Arb, a C library for arbitrary-precision floating-point ball arithmetic, developed by F. Johansson [8]. The main advantage is the fact that it supports efficient high-precision computation with power series and special functions over the real and complex numbers with automatic error control.

Extensive set of data is obtained in numerical computations. In the following example, we present selected sets of data for the functions considered in Example 3.5.

Example 4.1. Let $F(s) = \zeta(s-5)\zeta(s+5)$. Let us notice that the τ-Li coefficients in this case are real, i.e., $\operatorname{Re}\lambda_F(n,\tau) = \lambda_F(n,\tau)$.

In Figures 4.1–4.6 we present values for the archimedean $S_A(n,\tau)$ and non-archimedean $S_{NA}(n,\tau)$ contributions to the τ-Li coefficients attached to the function F. Values $\tau \in \{9, 10, 11\}$ are chosen in order to illustrate the different situations discussed in Example 3.5 and Remark 3.7. These contributions are presented in Figures 4.1, 4.3 and 4.5 while the sums of these contributions, i.e., values of the τ-Li coefficients are presented in Figures 4.2, 4.4 and 4.6. All the values are calculated for n from 1 to 500 with the step equal to 1. These values are joined in order to be able to easily see the corresponding behavior.

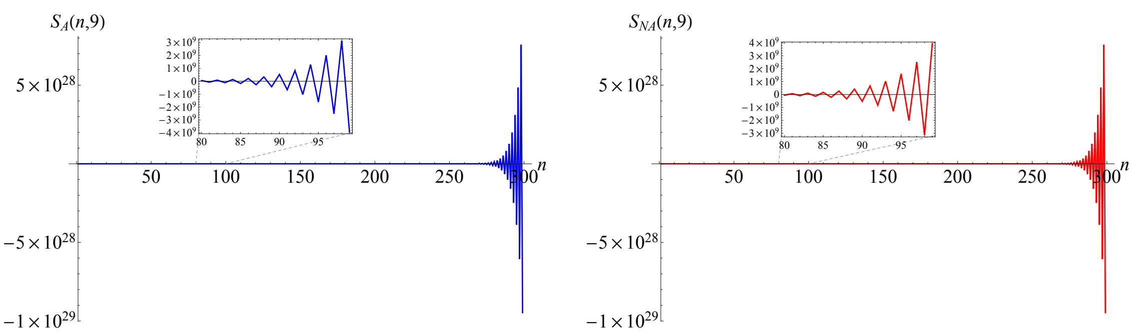

Figure 4.1: The archimedean and non-archimedean contributions to the τ-Li coefficients attached to the function F for $\tau = 9$.

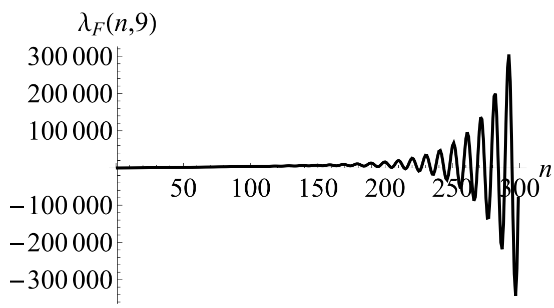

Figure 4.2: τ-Li coefficients attached to the function F for $\tau = 9$.

In the case $\tau = 9$, both contributions are oscillatory with exponentially growing amplitudes, as expected since in (3.15) and in (3.16), the sums with exponentially growing terms persist, when n changes. The obtained contributions' values produce oscillatory

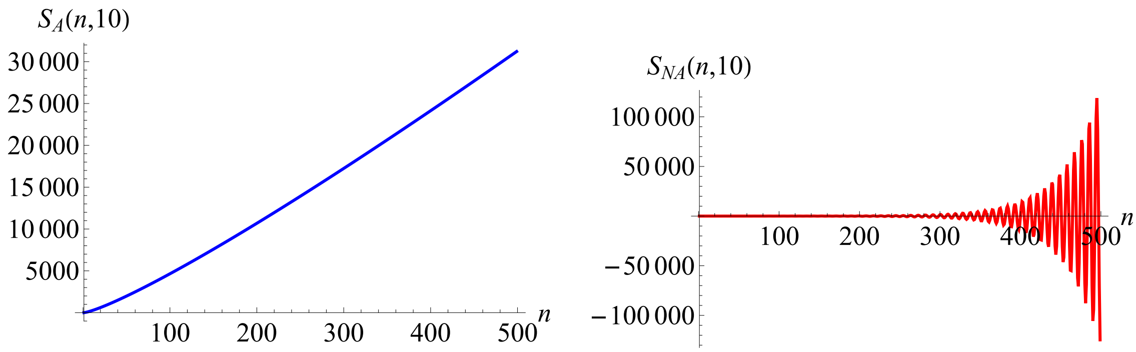

Figure 4.3: The archimedean and non-archimedean contributions to the τ-Li coefficients attached to the function F for $\tau = 10$.

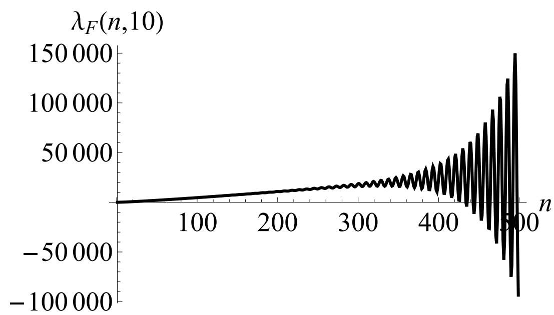

Figure 4.4: τ-Li coefficients attached to the function F for $\tau = 10$.

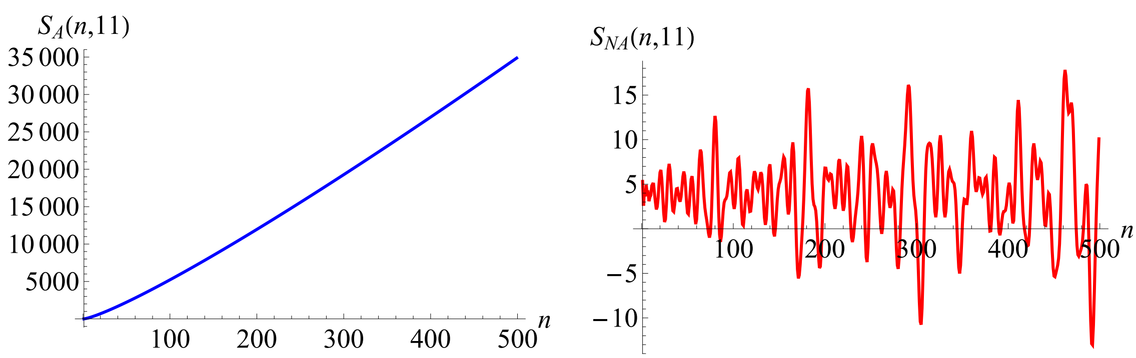

Figure 4.5: The archimedean and non-archimedean contributions to the τ-Li coefficients attached to the function F for $\tau = 11$.

τ-Li coefficients and some negative values. This is completely in agreement with the results in [6], obtained in a completely different way, for some similar functions. Namely, assuming that the Riemann hypothesis holds, all the zeros of the function $F(s)$ are located at lines $\operatorname{Re} s = a/2$, where $a \in \{-9, 11\}$, thus in the case $\tau = 9$, we have no zeros of the function $F(s)$ in the strip $\sigma_1 - \tau/2 \leq \operatorname{Re} s \leq \tau/2$ (see Theorem 2.4(i)) but all the zeros

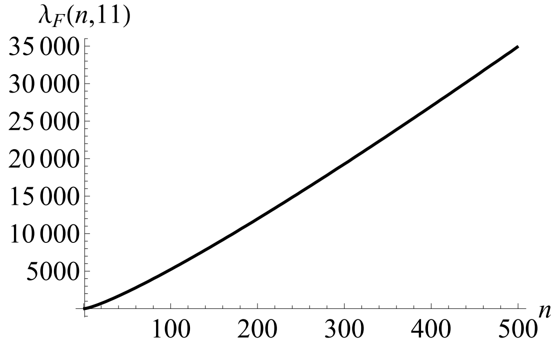

Figure 4.6: τ-Li coefficients attached to the function F for $\tau = 11$.

are outside, so negative values of the τ-Li coefficients are expected.

In the case $\tau = 10$, the archimedean part contributes nicely with the leading term of order $n \log n$ to the growth of the τ-Li coefficients, while the non-archimedean term exhibits the same behavior as in the case $\tau = 9$, oscillatory values with exponentially growing amplitudes. Graphs are in accordance with the formulas obtained in Remark 3.7, since the terms with powers of n in sums in (3.19) are less than one in the absolute value. The exponential growth of the non-archimedean terms comes from the incomplete τ-Li coefficients. As we can observe at Figure 4.4, the archimedean contribution to the τ-Li coefficients is completely dominant for n from 1 to 300, but then the impact of the non-archimedean term becomes visible, and it finally starts producing negative values of the τ-Li coefficients around $n = 430$. Appearance of the negative values of the τ-Li coefficients are expected for $\tau = 10$, having in mind Theorem 2.4 and assuming the Riemann hypothesis.

For $\tau = 11$, the archimedean contribution exhibits a completely analogous behavior as in the case $\tau = 10$. The leading term of the non-archimedean term is the incomplete τ-Li coefficient in this case, as shown in (3.18), but assuming the Riemann hypothesis it is $O(\sqrt{n} \log n)$, as shown in Example 3.8. Thus the values of the non-archimedean contribution are considerably smaller than the corresponding values of the archimedean term, as is clearly visible in Figure 4.5. There is no visible impact of the non-archimedean term to the τ-Li coefficients, which can be seen from Figure 4.6. The values of $\lambda_F(n, 11)$ grow as $11n \log n$ when n increases, which is expected from Example 3.8. There are no oscillations and no negative values.

The discussion above shows that the general behavior of the contributions to the τ-Li coefficients can be easily explained by consequences of proved theorems. In addition, in the sequel, we show that the numerical evaluations of approximations and exact values fit very well together.

In Figure 4.7 we compare the approximate values of $S_A^*(n, \tau)$ which is the archimedean contribution calculated up to a constant term (see expression (3.15)), i.e.,

$$S_A^*(n,\tau) = \tau n \log n + \left(\log\left(\frac{\tau}{2}\right) + \gamma - 1 - \log \pi\right) n\tau + \frac{\tau}{2} - 1 + \sum_{t=0}^{\lceil A/2\rceil - 1} \left(\left(\frac{A+2t}{\tau + A + 2t}\right)^n + \left(\frac{-A+2t}{\tau - A + 2t}\right)^n\right)$$

with the exact values $S_A(n,\tau)$. The results are presented for $\tau = 9$ and $\tau = 11$. Clearly, approximate values describe the behavior of the exact values with a very high precision.

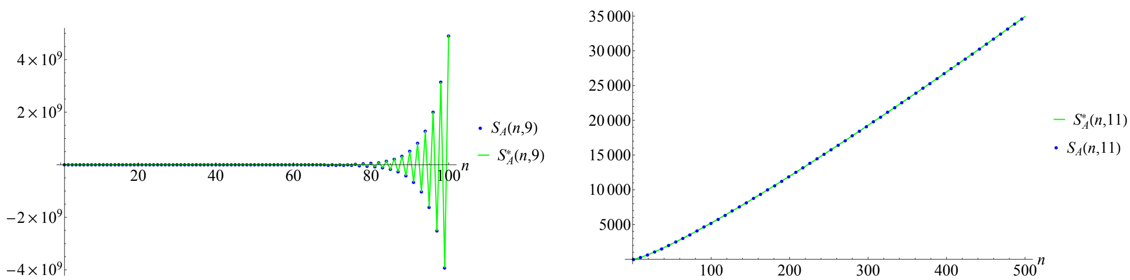

Figure 4.7: Archimedean contributions and τ-Li coefficients attached to the function F for $\tau = 9$ and $\tau = 11$ and the corresponding approximations.

In the case of the non-archemedean contribution given by (3.17), we are able to distinguish two main impacts to it: the incomplete τ-Li coefficients $\lambda_F(n,\tau,\sqrt{n})$ and the impact of the sums with exponential terms $S_{NA}^*(n,\tau)$, i.e.,

$$S_{NA}(n,\tau) = \lambda_F(n,\tau,\sqrt{n}) + S_{NA}^*(n,\tau) + O(\sqrt{n}\log n),$$

where

$$S_{NA}^*(n,\tau) = -\sum_{k=0}^{\lfloor A/2-\tau/4\rfloor} \left(\left(\frac{-A+2k}{-A+2k+\tau}\right)^n - 1\right) - \left(\frac{A}{A-\tau}\right)^n.$$

In Figures 4.8–4.10 we present these impacts for $\tau \in \{9, 10, 11\}$.

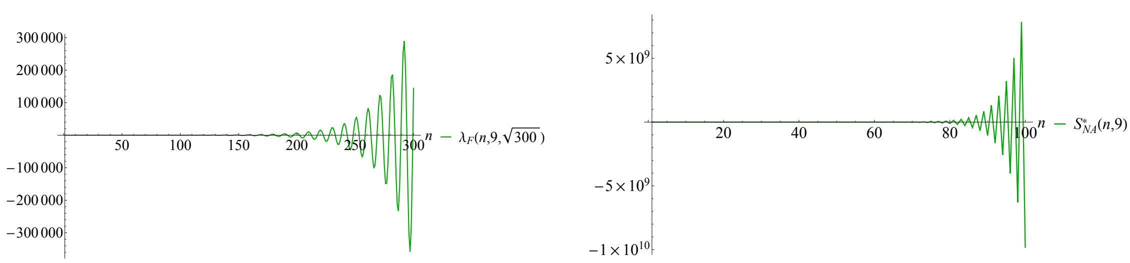

Figure 4.8: Impacts of the non-archimedean contributions to the τ-Li coefficients attached to the function F for $\tau = 9$.

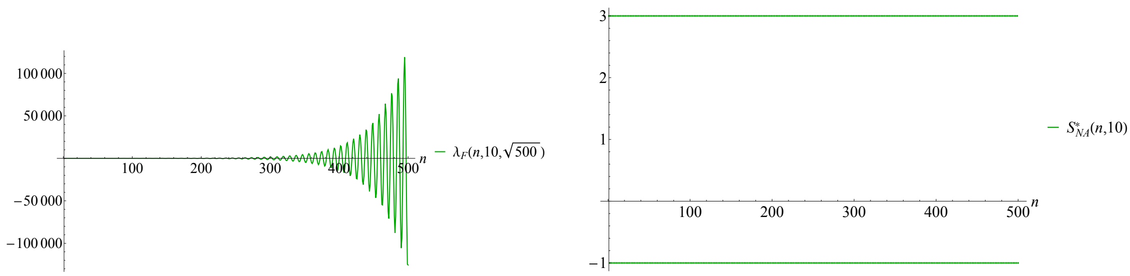

Figure 4.9: Impacts of the non-archimedean contributions to the τ-Li coefficients attached to the function F for $\tau = 10$.

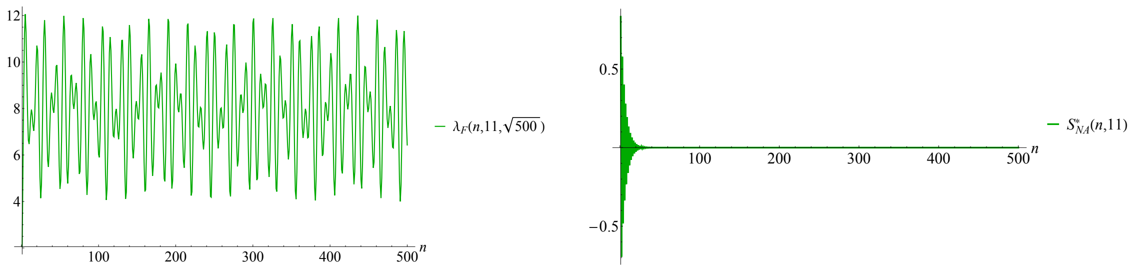

Figure 4.10: Impacts of the non-archimedean contributions to the τ-Li coefficients attached to the function F for $\tau = 11$.

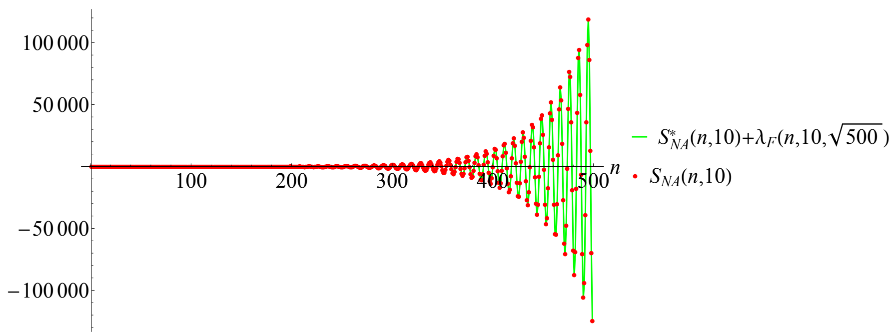

Figure 4.11: Non-archimedean contributions and τ-Li coefficients attached to the function F for $\tau = 10$ and its approximate values.

It is interesting to notice that, in the general case, it is not possible to derive a relation between these two impacts. In the case of $\tau = 9$ both impacts exhibit oscillatory behavior with exponentially growing amplitudes producing the same behavior for the non-archimedean contribution, as shown in the right panel of Figure 4.1. For $\tau = 10$, the exponential growth comes from the incomplete τ-Li coefficients, while the values of

$S_A^*(n, 10)$ are bounded. Thus, the oscillation of the τ-Li coefficients comes from the incomplete τ-Li coefficients, as mentioned earlier. For $\tau = 11$ both impacts are quite small and bounded, producing no visible effect to the τ-Li coefficients, as shown in Figure 4.6.

Figure 4.11 presents a numerical approximation of the non-archimedean contribution to the τ-Li coefficients and its values obtained from the definition in the case $\tau = 10$.

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