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# On Stronger Forms of Sensitivity in Non-autonomous Systems

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Abstract. In this paper, some stronger forms of transitivity in a non-autonomous discrete dynamical system $(X, f_{1,\infty})$ generated by a sequence $(f_n)$ of continuous self maps converging uniformly to $f$, are studied. The concepts of thick sensitivity, ergodic sensitivity and multi-sensitivity for non-autonomous discrete dynamical systems, which are all stronger forms of sensitivity, are defined and studied. It is proved that under certain conditions, if the rate of convergence at which $(f_n)$ converges to $f$ is "sufficiently fast", then various forms of sensitivity and transitivity for the non-autonomous system $(X, f_{1,\infty})$ and the autonomous system $(X, f)$ coincide. Also counter examples are given to support results.

## 1. Introduction

Dynamical system is one of the very significant and applicable branches of mathematics devoted to the study of systems governed by a consistent set of laws over time such as difference and differential equations. Beginning with the contributions of Poincaré and Lyapunov, theory of dynamical systems has seen significant developments in the recent years. This theory has gained considerable interest and has been found to have useful connections with many different areas of mathematics [8–11, 13].

Most of the natural systems in this world, whether it is the rhythm of day and night or the yearly seasons or weather patterns which vary from one year to another are subjected to time-dependent external forces and their modeling leads to a mathematical theory of non-autonomous discrete dynamical systems. An autonomous discrete dynamical system is a dynamical system which has no external input and always evolves according to the same unchanging law. The theory of non-autonomous dynamical systems helps in characterizing the behaviour of various natural phenomenons which cannot be modeled by autonomous systems. The mathematical theory of non-autonomous systems is considerably more involved than the theory of autonomous systems. Over recent years, the theory of such systems has developed into a highly active field related to, yet recognizably distinct

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from that of classical autonomous dynamical systems [2, 16, 18, 21, 24, 25]. This development was motivated by problems of applied mathematics, in particular, in life sciences where genuinely non-autonomous systems are in abundance. In general the dynamics of non-autonomous discrete dynamical system is much richer and quiet different from the dynamics of autonomous discrete dynamical systems.

In recent years, non-autonomous discrete dynamical systems have been widely studied. Let $(X, d)$ be a metric space and $f_n\colon X \to X$, $n \in \mathbb{N}$ be a continuous map. Consider the following non-autonomous discrete dynamical system (N.D.S.) $(X, f_{1,\infty})$, where

$$x_{n+1} = f_n(x_n), \quad n \in \mathbb{N}.$$

For convenience we denote $(f_n)_{n=1}^{\infty}$ by $f_{1,\infty}$. Naturally, a difference equation of the form $x_{n+1} = f_n(x_n)$ can be thought of as the discrete analogue of a non-autonomous differential equation $dx/dt = f(x,t)$. Non-autonomous discrete dynamical systems were introduced by authors in [6]. In various mathematical problems including those in the area of applied mathematics, we usually work with a sequence of maps instead of a single map.

Our main focus in this paper remains on two dynamical properties, namely, transitivity and sensitive dependence on initial conditions, or simply sensitivity [1]. Sensitive dependence on initial conditions, also known as the butterfly effect, is the main ingredient of chaos. In a dynamical system exhibiting sensitivity, a small change in the initial conditions leads to a significant change in the dynamics of the system. Sensitivity analysis has a major application in the area of population biology for studying the effect of transmission and treatment of diseases [3, 5]. Roughly speaking, a dynamical system is sensitive if given any region in the space $X$, there are always two points in that region and a unit of time say $n \in \mathbb{N}$ such that $n$-th iterations of these two points under $f_{1,\infty}$ are seperated considerably. For continuous self maps of compact metric spaces, Moothathu [19] has started a study of stronger forms of transitivity and sensitivity in terms of the largeness of subsets of $\mathbb{N}$. He has defined and studied syndetic transitivity, syndetic sensitivity, cofinite sensitivity and multi-sensitivity. Working on ideas of various types of large subsets of $\mathbb{N}$, several other stronger forms of transitivity and sensitivity have been studied for dynamical systems by different researchers [7, 15, 25]. In this paper, we define thick sensitivity, ergodic sensitivity and multi-sensitivity for non-autonomous discrete dynamical systems. In [20], the authors have studied the relations of transitivity, weak-mixing, topological mixing, sensitivity, cofinite sensitivity and various other dynamical properties of a non-autonomous discrete dynamical system $(X, f_{1,\infty})$ generated by a sequence $(f_n)$ of continuous self-maps uniformly converging to $f$ with the autonomous system $(X, f)$. For more results on non-autonomous discrete dynamical systems, one can refer to recent researches done in this direction [4, 14, 17, 22, 23].

Motivated by the work done in this area, we investigate these relations for stronger forms of transitivity and sensitivity. In Section 2, we recall some already known concepts by giving various definitions. In Section 3, we establish a necessary and sufficient condition for the non-autonomous system $(X, f_{1,\infty})$ generated by a sequence $(f_n)$ of continuous self-maps to be syndetically transitive and topologically ergodic. Further, we give examples to support our hypothesis. In Section 4, we obtain a necessary and sufficient condition for the non-autonomous system $(X, f_{1,\infty})$ generated by a sequence $(f_n)$ of continuous self-maps to be syndetically sensitive, thickly sensitive, thickly syndetic sensitive, ergodically sensitive, multi-sensitive, collective sensitive and synchronous sensitive. We also give example to establish the necessity of hypothesis.

## 2. Preliminaries

In this section we recall some well known notions.

Given a subset $A$ of a topological space $X$, $\mathrm{int}(A)$ denote the interior of $A$ in $X$. For any two open sets $U$ and $V$ of $X$, denote, $N_{f_{1,\infty}}(U,V) = \{n \in \mathbb{N} : f_1^n(U) \cap V \neq \emptyset\}$. Let $V \subset X$ be a nonempty open subset, $\mathbb{N}$ be the set of positive integers and $\delta > 0$. Denote $N_{f_{1,\infty}}(V,\delta) = \{n \in \mathbb{N} :$ there exist $x, y \in V$ satisfying $d(f_1^n(x), f_1^n(y)) > \delta\}$.

**Definition 2.1.** A set $F \subset \mathbb{N}$ is called *syndetic* if there exists a positive integer $a$ such that $\{i, i+1, \ldots, i+a\} \cap F \neq \emptyset$ for any $i \in \mathbb{N}$.

**Definition 2.2.** A *thick set* is a set of integers that contains arbitrarily long intervals. That is, given a thick set $T$, for every $p \in \mathbb{N}$, there is some $n \in \mathbb{N}$ such that $\{n, n+1, n+2, \ldots, n+p\} \subset T$.

**Definition 2.3.** A set $F \subset \mathbb{N}$ is called *thickly syndetic* if $\{n \in \mathbb{N} : n+j \in F \text{ for } 0 \leq j \leq k\}$ is syndetic for each $k \in \mathbb{N}$. Then taking $n = a$ in the definition of syndetic set, we have that every thickly syndetic subset of $\mathbb{N}$ is syndetic.

**Definition 2.4.** A system $(X, f_{1,\infty})$ is said to be *open* if for any open set $U$ in $X$, $f_n(U)$ is open for each $n \in \mathbb{N}$.

**Definition 2.5.** A system $(X, f_{1,\infty})$ is said to be *feeble open* if for any nonempty open set $U$ in $X$, $\mathrm{int}(f_n(U))$ is nonempty for each $n \in \mathbb{N}$.

**Definition 2.6.** Let $|N_{f_{1,\infty}}(U,V)|$ be the cardinal number of the set $N_{f_{1,\infty}}(U,V)$, then

$$\limsup_{n\to\infty} \frac{|N_{f_{1,\infty}}(U,V) \cap N_n|}{n}$$

is called the upper density of $N_{f_{1,\infty}}(U,V)$, where $N_n = \{0, 1, 2, \ldots, n-1\}$.

**Definition 2.7.** The system $(X, f_{1,\infty})$ is said to be *topologically transitive* if for any two nonempty open sets $U$ and $V$ in $X$, there exists a positive integer $n \in \mathbb{N}$ such that, $f_1^n(U) \cap V \neq \emptyset$. Thus, the system $(X, f_{1,\infty})$ is said to be topologically transitive if for any two nonempty open sets $U_0$ and $V_0$ of $X$, $N_{f_{1,\infty}}(U_0, V_0)$ is nonempty.

**Definition 2.8.** The system $(X, f_{1,\infty})$ is said to be *topologically mixing* if for any two nonempty open sets $U_0$ and $V_0$ in $X$, there exists a positive integer $N \in \mathbb{N}$ such that for any $n \geq N$, $U_n \cap V_0 \neq \emptyset$, where $U_{i+1} = f_i(U_i)$, $1 \leq i \leq n$, i.e., $(f_n \circ f_{n-1} \circ \cdots \circ f_1)(U_0) \cap V_0 \neq \emptyset$ for all $n \geq N$. Thus, the system $(X, f_{1,\infty})$ is said to be topologically mixing if for any two nonempty open sets $U_0$ and $V_0$ of $X$, there is a positive integer $N$ such that $N_{f_{1,\infty}}(U_0, V_0) \supset [N, \infty) \cap \mathbb{N}$.

**Definition 2.9.** The system $(X, f_{1,\infty})$ is said to be *syndetically transitive* if for any two nonempty open sets $U_0$ and $V_0$ in $X$, $N_{f_{1,\infty}}(U_0, V_0)$ is syndetic.

**Definition 2.10.** The system $(X, f_{1,\infty})$ is said to be *topologically ergodic* if for any two nonempty open sets $U_0$ and $V_0$ in $X$, $N_{f_{1,\infty}}(U_0, V_0)$ has positive upper density.

**Definition 2.11.** The system $(X, f_{1,\infty})$ is said to have *sensitive dependence on initial conditions* if there exists a constant $\delta_0 > 0$ such that for any $x_0 \in X$ and any neighbourhood $U$ of $x_0$ there exists $y_0 \in X \cap U$ and a positive integer $n$ such that $d(x_n, y_n) > \delta_0$, where $\{x_i\}_{i=0}^{\infty}$ and $\{y_i\}_{i=0}^{\infty}$ are the orbits of the system $(X, f_{1,\infty})$ starting from $x_0$ and $y_0$ respectively. The constant $\delta_0 > 0$ is called a sensitivity constant of the system $(X, f_{1,\infty})$. Here $(x_i)_{i=0}^{\infty} = \{x \in X : f_1^n(x), n \geq 1\}$ where $f_1^n(x) = f_n \circ \cdots \circ f_1(x)$. The system $(X, f_{1,\infty})$ is said to have sensitive dependence on initial conditions or is *sensitive* in $X$ if there exists a constant $\delta > 0$ such that for any nonempty open set $V$ of $X$, $N_{f_{1,\infty}}(V, \delta)$ is nonempty.

**Definition 2.12.** The system $(X, f_{1,\infty})$ is called *cofinitely sensitive* in $X$ if there exists a constant $\delta > 0$ such that for any nonempty open set $V$ of $X$, there exists $N \geq 1$ such that $[N, \infty) \cap \mathbb{N} \subset N_{f_{1,\infty}}(V, \delta)$, where $\delta$ is called a constant of cofinite sensitivity.

**Definition 2.13.** The system $(X, f_{1,\infty})$ is said to have *syndetic sensitivity* in $X$ if there exists a constant $\delta > 0$ such that for any nonempty open set $V$ of $X$, $N_{f_{1,\infty}}(V, \delta)$ is syndetic, where $\delta$ is called a constant of syndetic sensitivity.

Clearly, syndetically sensitive implies sensitive as $N_{f_{1,\infty}}(V, \delta)$ is nonempty when system is syndetically sensitive. Also, when system is cofinitely sensitive, then there exists a positive integer $N$ such that $N_{f_{1,\infty}}(V, \delta) \supset [N, \infty) \cap \mathbb{N}$, i.e., $\{N, N+1, \ldots\} \subset N_{f_{1,\infty}}(V, \delta)$. Therefore, taking $a = N$ in the definition of syndetic subset, we have that $N_\tau(V, \delta)$ is syndetic. Hence, cofinitely sensitive implies syndetically sensitive. Therefore, we have

$$\text{Cofinitely Sensitive} \Longrightarrow \text{Syndetically Sensitive} \Longrightarrow \text{Sensitive.}$$

**Definition 2.14.** The system $(X, f_{1,\infty})$ is said to be *thickly syndetic sensitive* in $X$ if there exists a constant $\delta > 0$ such that for any nonempty open set $V$ of $X$, $N_{f_{1,\infty}}(V, \delta)$ is thickly syndetic, where $\delta$ is called a constant of thickly syndetic sensitivity [17].

**Note 2.15.** Cofinitely Sensitive $\Longrightarrow$ Thickly Syndetically Sensitive $\Longrightarrow$ Syndetically Sensitive $\Longrightarrow$ Sensitive.

**Definition 2.16.** The system $(X, f_{1,\infty})$ is said to have *collective sensitivity* in $X$ if there exists some $\delta > 0$ such that for finitely many distinct points $x_1, x_2, \ldots, x_n$ of $X$ and an arbitrary $\epsilon > 0$, there exist the same number of distinct points $y_1, y_2, \ldots, y_n$ of $X$ and some positive integer $k$ satisfying the following two conditions:

(1) $d(x_i, y_i) < \epsilon$, $i = 1, 2, \ldots, n$.

(2) there exists an $i_0$ with $1 \leq i_0 \leq n$ such that $d((f_1^k(x_i)), (f_1^k(y_{i_0}))) \geq \delta$, $i = 1, 2, \ldots, n$ or $d((f_1^k(y_i)), (f_1^k(x_{i_0}))) \geq \delta$, $i = 1, 2, \ldots, n$.

Here, $\delta$ is called a constant of collective sensitivity [26].

**Definition 2.17.** The system $(X, f_{1,\infty})$ is said to have *synchronous sensitivity* in $X$ if there exists some $\delta > 0$ such that for finitely many distinct points $x_1, x_2, \ldots, x_n$ of $X$ and an arbitrary $\epsilon > 0$, there exist the same number of distinct points $y_1, y_2, \ldots, y_n$ of $X$ and some positive integer $k$ satisfying the following two conditions:

(1) $d(x_i, y_i) < \epsilon$, $i = 1, 2, \ldots, n$.

(2) $d((f_1^k(x_i)), (f_1^k(y_i))) \geq \delta$, $i = 1, 2, \ldots, n$.

Here, $\delta$ is called a constant of synchronous sensitivity [26].

Let $(X, d)$ be a compact metric space and let $C(X)$ denote the collection of continuous self-maps on $X$. For any $f, g \in C(X)$, the *Supremum metric* is defined by $D(f, g) = \sup_{x \in X} d(f(x), g(x))$. It is easy to observe that a sequence $(f_n)$ in $C(X)$ converges to $f$ in $C((X), D)$ if and only if $f_n$ converges to $f$ uniformly on $X$ and hence the topology generated by the *Supremum metric* is called the topology of uniform convergence.

In [20], authors have proved several results for topological transitivity, weak mixing, topological mixing, sensitive dependence on initial conditions and cofinite sensitivity including the following proposition and corollary for the system $(X, f_{1,\infty})$.

**Proposition 2.18.** [20] *Let $(X, f_{1,\infty})$ be an N.D.S. generated by a family $f_{1,\infty}$ and let $f$ be any continuous self map on $X$. If the family $f_{1,\infty}$ commutes with $f$ then for any $x \in X$ and any $k \in \mathbb{N}$, $d(f_1^k(x), f^k(x)) \leq \sum_{i=1}^{k} D(f_i, f)$.*

**Corollary 2.19.** [20] *Let $(X, f_{1,\infty})$ be an N.D.S. generated by a family $f_{1,\infty}$ and let $f$ be any continuous self map on $X$. If the family $f_{1,\infty}$ commutes with $f$ then for any $x \in X$ and any $k \in \mathbb{N}$, $d(f_1^{n+k}(x), f^k(f_1^n(x))) \leq \sum_{i=1}^{k} D(f_{i+1}, f)$.*

## 3. Some stronger forms of transitivity

In this section, we give a necessary and sufficient condition for the system $(X, f_{1,\infty})$ to be syndetically transitive and topologically ergodic, where $(X, f_{1,\infty})$ is an N.D.S. generated by a family $f_{1,\infty}$ of feeble open maps commuting with a continuous self map $f$. We also provide two examples to support our hypothesis.

**Theorem 3.1.** *Let $(X, f_{1,\infty})$ be an N.D.S. generated by a family $f_{1,\infty}$ of feeble open maps commuting with $f$. If $\sum_{i=1}^{\infty} D(f_i, f) < \infty$, then $(X, f)$ is syndetically transitive if and only if $(X, f_{1,\infty})$ is syndetically transitive.*

*Proof.* Let $(X, f)$ be syndetically transitive. Let $\epsilon > 0$ be given and $U = B(x, \epsilon)$ and $V = B(y, \epsilon)$ be two nonempty open sets in $X$. As $\sum_{i=1}^{\infty} D(f_i, f) < \infty$, there exists $r$ such that $\sum_{i=r}^{\infty} D(f_i, f) < \epsilon/2$. As the family $f_{1,\infty}$ consists of feeble open maps, therefore $f_1^r(U)$ has nonempty interior. Let $U' = \mathrm{int}(f_1^r(U))$ and $V' = B(y, \epsilon/2)$. Clearly, $U'$ and $V'$ are nonempty open sets in $X$. Then, since $(X, f)$ is syndetically transitive so $N_f(U', V')$ is syndetic. Let $m \in N_f(U', V')$ so there exists $u' \in U'$ such that $f^m(u') \in V'$. Now since $U' = \mathrm{int}(f_1^r(U))$ therefore there exists $u \in U$ such that $u' = f_1^r(u)$ therefore, $f^m(f_1^r(u)) \in V'$. Also by Corollary 2.19, we have

$$d(f_1^{m+r}(U), f^m(f_1^r(U))) \leq \sum_{i=1}^{m} D(f_i, f) < \epsilon/2$$

therefore by triangle inequality, we get,

$$d(y, f_1^{m+r}(U)) \leq d(y, f_1^r(U)) + d(f^m(f_1^r), f_1^{m+r}(U)) < \epsilon$$

which implies $f_1^{m+r}(U) \cap V \neq \emptyset$ hence $m + r \in N_{f_{1,\infty}}(U, V)$ and we get $N_f(U', V') + r \subseteq N_{f_{1,\infty}}(U, V)$ therefore $N_{f_{1,\infty}}(U, V)$ is syndetic. Thus, $(X, f_{1,\infty})$ is syndetically transitive.

Conversely, let $\epsilon > 0$ be given and let $B(x, \epsilon)$ and $B(y, \epsilon)$ be two nonempty open sets in $X$. As $\sum_{i=1}^{\infty} D(f_i, f) < \infty$, choose $r \in \mathbb{N}$ such that $\sum_{i=r}^{\infty} D(f_i, f) < \epsilon/2$. Further, the syndetic transitivity of the system $(X, f_{1,\infty})$ ensures that the number of times any nonempty open set $U$ visits $B(y, \epsilon)$ is infinite. Applying syndetic transitivity of $(X, f_{1,\infty})$ to open sets $U = (f_1^r)^{-1} B(x, \epsilon)$ and $V = B(y, \epsilon/2)$, we get that $N_{f_{1,\infty}}(U, V)$ is syndetic. Choose $k$ such that $f_1^{r+k}(U) \cap V \neq \emptyset$. Consequently, there exists $u \in U$ such that $d(f_1^{r+k}(u), y) < \epsilon/2$. Also by Corollary 2.19, we have

$$d(f_1^{r+k}(u), f^k(f_1^r(u))) < \sum_{i=1}^{k} D(f_{r+i}, f) < \epsilon/2$$

and therefore by triangle inequality

$$d(y, f^k(f_1^r(u))) < \epsilon.$$

As $f_1^r(u) \in B(x,\epsilon)$, we have $f^k B(x,\epsilon) \cap B(y,\epsilon) \neq \emptyset$ which implies

$$(N_{f_{1,\infty}}(U,V) - r) \subseteq N_f(B(x,\epsilon), B(y,\epsilon))$$

therefore $N_f(B(x,\epsilon), B(y,\epsilon))$ is syndetic. Hence, $(X,f)$ is syndetically transitive. □

**Theorem 3.2.** *Let $(X, f_{1,\infty})$ be an N.D.S. generated by a family $f_{1,\infty}$ of feeble open maps commuting with $f$. If $\sum_{i=1}^{\infty} D(f_i, f) < \infty$, then $(X,f)$ is topologically ergodic if and only if $(X, f_{1,\infty})$ is topologically ergodic.*

*Proof.* Let $(X,f)$ be topologically ergodic. Let $\epsilon > 0$ be given and let $U = B(x,\epsilon)$ and $V = B(y,\epsilon)$ be two nonempty open sets in $X$. As $\sum_{i=1}^{\infty} D(f_i, f) < \infty$, there exists $r$ such that $\sum_{i=r}^{\infty} D(f_n, f) < \epsilon/2$. Since the family $f_{1,\infty}$ consists of feeble open maps, therefore $f_1^r(U)$ has nonempty interior. Let $U' = \text{int}(f_1^r(U))$ and $V' = B(y, \epsilon/2)$. Clearly, $U'$ and $V'$ are nonempty open sets in $X$. Then $(X,f)$ being topologically ergodic, $N_f(U', V')$ has positive upper density. Let $m \in N_f(U', V')$ then there exists $u' \in U'$ such that $f^m(u') \in V'$. Now, since $U' = \text{int}(f_1^r(U))$ therefore there exists $u \in U$ such that $u' = f_1^r(u)$ and hence, $f^m(f_1^r(u)) \in V'$. Also, by Corollary 2.19, we have

$$d(f_1^{m+r}(U), f^m(f_1^r(U))) \leq \sum_{i=1}^{m} D(f_i, f) < \epsilon/2$$

therefore by triangle inequality, we get,

$$d(y, f_1^{m+r}(U)) \leq d(y, f_1^r(U)) + d(f^m(f_1^r), f_1^{m+r}(U)) < \epsilon$$

which implies $f_1^{m+r}(U) \cap V \neq \emptyset$. So $m + r \in N_{f_{1,\infty}}(U,V)$ and $N_f(U', V') + r \subseteq N_{f_{1,\infty}}(U,V)$ implying $N_{f_{1,\infty}}(U,V)$ also has positive upper density. Hence, $(X, f_{1,\infty})$ is topologically ergodic.

Conversely, let $\epsilon > 0$ be given and let $B(x,\epsilon)$ and $B(y,\epsilon)$ be two nonempty open sets in $X$. As $\sum_{i=1}^{\infty} D(f_i, f) < \infty$, choose $r \in \mathbb{N}$ such that $\sum_{i=r}^{\infty} D(f_i, f) < \epsilon/2$. Further, the topologically ergodicity of the system $(X, f_{1,\infty})$ ensures that the number of times any nonempty open set in $X$ visits $B(y,\epsilon)$ is infinite. Applying topologically ergodicity of $(X, f_{1,\infty})$ to open sets $U = (f_1^r)^{-1} B(x,\epsilon)$ and $V = B(y, \epsilon/2)$ we get that, $N_{f_{1,\infty}}(U,V)$ has positive upper density. Choose $k$ such that $f_1^{r+k}(U) \cap V \neq \emptyset$. Consequently, there exists $u \in U$ such that $d(f_1^{r+k}(u), y) < \epsilon/2$. Also by Corollary 2.19, we have

$$d(f_1^{r+k}(u), f^k(f_1^r(u))) < \sum_{i=1}^{k} D(f_{r+i}, f) < \epsilon/2$$

and therefore by triangle inequality

$$d(y, f^k(f_1^r(u))) < \epsilon.$$

As $f_1^r(u) \in B(x,\epsilon)$, we have $f^k(B(x,\epsilon)) \cap B(y,\epsilon) \neq \emptyset$ which implies

$$N_{f_{1,\infty}}(U,V) - r \subseteq N_f(B(x,\epsilon), B(y,\epsilon))$$

therefore $N_f(B(x,\epsilon), B(y,\epsilon))$ has positive upper density. Hence, $(X,f)$ is topologically ergodic. □

In the following example, we show that the above results fail if the condition of taking $(f_n, n \in \mathbb{N})$ to be feeble open is dropped from the hypothesis.

**Example 3.3.** Let $I$ be the unit interval $[0,1]$ and $f_1$, $f_2$ be defined by

$$f_1(x) = \begin{cases} 2x & \text{for } x \in [0,1/5], \\ 1 & \text{for } x \in [1/5,2/5], \\ x & \text{for } x \in [2/5,1], \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} 4x & \text{for } x \in [0,1/4], \\ -4x+2 & \text{for } x \in [1/4,1/2], \\ 4x-2 & \text{for } x \in [1/2,3/4], \\ -4x+4 & \text{for } x \in [3/4,1]. \end{cases}$$

Take $f_0(x) = f_1(x)$ and $f_n(x) = f_2(x)$ for all $n > 1$. Then $(f_n)_{n=0}^{\infty}$ converges uniformly to $f = f_2$ in $X$. Note that $f$ is feeble open. Clearly, $(X,f)$ is syndetically transitive and topologically ergodic being topological mixing. However, the system $(X, f_{1,\infty})$ is not even transitive because for open sets $U = (1/5, 2/5)$ and $V = (2/5, 3/4)$, $\mathbb{N}_{f_{1,\infty}}(U,V)$ is empty and hence it is not syndetically transitive or topologically ergodic. Our Theorems 3.1 and 3.2 fail here because in the family $(f_n)$, the function $f_0$ is not feeble open.

In the following example, we provide a system $(X, f_{1,\infty})$ which satisfies the hypothesis of Theorems 3.1 and 3.2.

**Example 3.4.** Let $I$ be the interval $[0,2]$ and $f_1$, $f_2$ be defined by $f_1(x) = x$ for $x \in [0,2]$ and

$$f_2(x) = \begin{cases} 4x & \text{for } x \in [0,1/4], \\ -4x+2 & \text{for } x \in [1/4,1/2], \\ 4x-2 & \text{for } x \in [1/2,3/4], \\ -4x+4 & \text{for } x \in [3/4,1]. \end{cases}$$

Let $f_0(x) = f_1(x)$ and $f_n(x) = f_2(x)$ for all $n > 1$. Note that the functions $f_1$ and $f = f_2(x)$ are feeble open. Clearly, $(X,f)$ is syndetically transitive and topologically ergodic being topological mixing. The family $f_n$ consists of feeble open maps converging uniformly to $f$. Thus, this family satisfies the hypothesis of our Theorems 3.1 and 3.2, and hence, the system $(X, f_{1,\infty})$ is also syndetically transitive and topologically ergodic.

## 4. Some stronger forms of sensitivity

In this section, we prove some results for various stronger forms of sensitivity. Some stronger forms of sensitivity like syndetic sensitivity, thickly syndetic sensitivity, collective sensitivity and synchronous sensitivity have already been defined for the non-autonomous dynamical systems. However, thick sensitivity, ergodic sensitivity and multi-sensitivity are some stronger forms of sensitivity that have been widely studied for autonomous systems [7, 12, 15, 19]. We study these notions for non-autonomous dynamical systems.

**Definition 4.1.** The system $(X, f_{1,\infty})$ is said to have *thick sensitivity* in $X$ if there exists a constant $\delta > 0$ such that for any nonempty open set $V$ of $X$, $N_{f_{1,\infty}}(V, \delta)$ is thick; $\delta$ is called a constant of thick sensitivity.

**Definition 4.2.** The system $(X, f_{1,\infty})$ is said to have *ergodic sensitivity* in $X$ if there exists a constant $\delta > 0$ such that for any nonempty open set $V$ of $X$, $N_{f_{1,\infty}}(V, \delta)$ has positive upper density; $\delta$ is called a constant of ergodic sensitivity.

**Definition 4.3.** The system $(X, f_{1,\infty})$ is said to have *multi-sensitivity* in $X$ if there exists a constant $\delta > 0$ such that for every $k \geq 1$ and any nonempty open subsets $V_1, V_2, \ldots, V_k$ of $X$, $\bigcap_{i=0}^{k} \mathbb{N}_{f_{1,\infty}}(V_i, \delta)$ is nonempty; $\delta$ is called a constant of multi-sensitivity.

*Remark* 4.4. Note that cofinite sensitivity implies each of the above defined forms of sensitivity.

In the following theorems, we give a necessary and sufficient condition for the system $(X, f_{1,\infty})$ to be syndetically sensitive, thickly sensitive, thickly syndetic sensitive, ergodically sensitive and multi-sensitive, where $(X, f_{1,\infty})$ is an N.D.S. generated by a family $f_{1,\infty}$ of feeble open maps commuting with a continuous self map $f$.

**Theorem 4.5.** *Let $(X, f_{1,\infty})$ be an N.D.S. generated by a family $f_{1,\infty}$ of feeble open maps commuting with $f$. If $\sum_{i=1}^{\infty} D(f_i, f) < \infty$, then $(X, f)$ is syndetically sensitive if and only if $(X, f_{1,\infty})$ is syndetically sensitive.*

*Proof.* Let $(X, f)$ be syndetically sensitive with constant of syndetic sensitivity $\delta > 0$. Let $\epsilon > 0$ be given and $U = B(x, \epsilon)$ be a nonempty open set in $X$. As $\sum_{i=1}^{\infty} D(f_n, f) < \infty$, by Corollary 2.19 we get, for any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $d(f_1^{n+k}(x), f^k(f_1^n(x))) < \epsilon$ for all $x \in X$, $k \in \mathbb{N}$. Choose $m \in \mathbb{N}$ such that $1/m < \delta/4$. Then, there exists $n_0$ such that $d(f_1^{n_0+k}(x), f^k(f_1^{n_0}(x))) < 1/m$ for all $x \in X$, $k \in \mathbb{N}$. As $f_n$'s are feeble open, $U' = \operatorname{int} f_1^{n_0}(U)$ is nonempty open and by syndetic sensitivity of $(X, f)$, we get $N_f(U', \delta)$ is syndetic. Let $k \in N_f(U', \delta)$ then there exist $v_1, v_2 \in f_1^{n_0}(U)$ such that $d(f^k(v_1), f^k(v_2)) > \delta$. As $v_1, v_2 \in f_1^{n_0}(U)$, there exist $v_1', v_2' \in U$such that $v_1 = f_1^{n_0}(v_1')$ and $v_2 = f_1^{n_0}(v_2')$ and

$$d(f^k(f_1^{n_0}(v_1')), f^k(f_1^{n_0}(v_2'))) > \delta.$$

Also $d(f_1^{n_0+k}(v_i'), f^k(f_1^{n_0}(v_i'))) < 1/m$ for $i = 1, 2$.

Therefore, by triangle inequality,

$$d(f_1^{n_0+k}(v_1'), f_1^{n_0+k}(v_2')) > \delta - 2/m > \delta/2.$$

Thus, we obtain $v_1'$ and $v_2' \in U$ such that

$$d(f_1^{n_0+k}(v_1'), f_1^{n_0+k}(v_2')) > \delta/2$$

which implies $n_0+k \in N_{f_{1,\infty}}(U, \delta/2)$ hence $N_f(U', \delta)+n_0 \subseteq N_{f_{1,\infty}}(U, \delta/2)$. Since $N_f(U', \delta)$ is syndetic therefore $N_{f_{1,\infty}}(U, \delta/2)$ is syndetic and hence $(X, f_{1,\infty})$ is syndetically sensitive.

Conversely, let $(X, f_{1,\infty})$ be syndetically sensitive with constant $\delta > 0$ and let $U$ be a nonempty open set in $X$. Let $m \in \mathbb{N}$ such that $1/m < \delta/4$. Then there exists $n_0 \in \mathbb{N}$ such that $d(f_1^{n_0+k}(x), f^k(f_1^{n_0}(x))) < 1/m$ for all $x \in X$, $k \in \mathbb{N}$. Since $(X, f_{1,\infty})$ is syndetically sensitive, therefore for any $n \in \mathbb{N}$, the set $N_{f_{1,\infty}}(U, \delta)$ is syndetic and hence the set $\{k : \operatorname{diam}(f_1^k(U)) > \delta\}$ is infinite. Thus, for open set $(f_1^{n_0})^{-1}(U)$, the set $N_{f_{1,\infty}}((f_1^{n_0})^{-1}(U), \delta)$ is syndetic. So there exists $k \in \mathbb{N}$ such that $k + n_0 \in N_{f_{1,\infty}}((f_1^{n_0})^{-1}(U), \delta)$ implying there exist $v_1, v_2 \in (f_1^{n_0})^{-1}(U)$ such that

$$d(f_1^{n_0+k}(v_1), f_1^{n_0+k}(v_2)) > \delta.$$

Since $v_1, v_2 \in ((f_1^{n_0})^{-1}(U))$ therefore there exist $v_1', v_2' \in U$ such that $v_1' = f_1^{n_0}(v_1)$ and $v_2' = f_1^{n_0}(v_2)$ and hence

$$d((f_{n_0+k} \circ \cdots \circ f_{n_0+1})(v_1'), (f_{n_0+k} \circ \cdots \circ f_{n_0+1})(v_2')) > \delta.$$

Also $d(f_1^{n_0+k}(v_i), f^k(f_1^{n_0}(v_i))) < 1/m$ or $d(f_1^{n_0+k}(v_i), f^k(v_i')) < 1/m$ for $i = 1, 2$.

Therefore, by triangle inequality

$$d(f^k(v_1'), f^k(v_2')) > \delta - 2/m > \delta/2.$$

Hence, we obtain $v_1', v_2' \in U$ such that $d(f^k(v_1'), f^k(v_2')) > \delta/2$ which implies $k \in N_f(U, \delta/2)$ so $(N_{f_{1,\infty}}((f_1^{n_0})^{-1}(U), \delta) - n_0) \subseteq N_f(U, \delta/2)$ and $N_{f_{1,\infty}}((f_1^{n_0})^{-1}(U), \delta)$ being syndetic implies $N_f(U, \delta/2)$ is syndetic. Therefore, $(X, f)$ is syndetically sensitive. $\square$

**Theorem 4.6.** *Let $(X, f_{1,\infty})$ be an N.D.S. generated by a family $f_{1,\infty}$ of feeble open maps commuting with $f$. If $\sum_{i=1}^{\infty} D(f_i, f) < \infty$, then $(X, f)$ is thickly sensitive if and only if $(X, f_{1,\infty})$ is thickly sensitive.*

*Proof.* Let $(X, f)$ be thickly sensitive with constant of thick sensitivity $\delta > 0$. Let $\epsilon > 0$ be given and $U = B(x, \epsilon)$ be a nonempty open set in $X$. As $\sum_{i=1}^{\infty} D(f_i, f) < \infty$, by Corollary 2.19 we obtain, for any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $d(f_1^{n+k}(x), f^k(f_1^n(x))) < \epsilon$

for all $x \in X$, $k \in \mathbb{N}$. Choose $m \in \mathbb{N}$ such that $1/m < \delta/4$. Thus, there exists $n_0$ such that $d(f_1^{n_0+k}(x), f^k(f_1^{n_0}(x))) < 1/m$ for all $x \in X$, $k \in \mathbb{N}$. As $f_n$'s are feeble open, $U' = \operatorname{int} f_1^{n_0}(U)$ is nonempty open and by thick sensitivity of $(X, f)$ we have $N_f(U', \delta)$ is thick. Let $k \in N_f(U', \delta)$ then there exist $v_1, v_2 \in f_1^{n_0}(U)$ such that $d(f^k(v_1), f^k(v_2)) > \delta$. As $v_1, v_2 \in f_1^{n_0}(U)$ so there exist $v_1', v_2' \in U$ such that $v_1 = f_1^{n_0}(v_1')$ and $v_2 = f_1^{n_0}(v_2')$ and hence

$$d(f^k(f_1^{n_0}(v_1')), f^k(f_1^{n_0}(v_2'))) > \delta.$$

Also $d(f_1^{n_0+k}(v_i'), f^k(f_1^{n_0}(v_i'))) < 1/m$ for $i = 1, 2$.

Therefore, by triangle inequality,

$$d(f_1^{n_0+k}(v_1'), f_1^{n_0+k}(v_2')) > \delta - 2/m > \delta/2.$$

Thus, we obtain $v_1', v_2' \in U$ such that

$$d(f_1^{n_0+k}(v_1'), f_1^{n_0+k}(v_2')) > \delta/2$$

which implies $n_0 + k \in N_{f_{1,\infty}}(U, \delta/2)$ so $N_f(U', \delta) + n_0 \subseteq N_{f_{1,\infty}}(U, \delta/2)$. Since $N_f(U', \delta)$ is thick therefore, $N_{f_{1,\infty}}(U, \delta/2)$ is thick and hence $(X, f_{1,\infty})$ is thickly sensitive.

Conversely, let $(X, f_{1,\infty})$ be thickly sensitive with constant $\delta > 0$ and let $U$ be a nonempty open set in $X$. Let $m \in \mathbb{N}$ such that $1/m < \delta/4$. Thus, there exists $n_0 \in \mathbb{N}$ such that $d(f_1^{n_0+k}(x), f^k(f_1^{n_0}(x))) < 1/m$ for all $x \in X$, $k \in \mathbb{N}$. Since $(X, f_{1,\infty})$ is thickly sensitive, therefore for any $n \in \mathbb{N}$, the set $N_{f_{1,\infty}}(U, \delta)$ is thick and hence the set $\{k : \operatorname{diam}(f_1^k(U)) > \delta\}$ is infinite. Thus, for open set $(f_1^{n_0})^{-1}(U)$, the set $N_{f_{1,\infty}}((f_1^{n_0})^{-1}(U), \delta)$ is thick, which implies there exists $k \in \mathbb{N}$ such that $k + n_0 \in N_{f_{1,\infty}}((f_1^{n_0})^{-1}(U), \delta)$. Thus, there exist $v_1, v_2 \in (f_1^{n_0})^{-1}(U)$ such that

$$d(f_1^{n_0+k}(v_1), f_1^{n_0+k}(v_2)) > \delta.$$

Since $v_1, v_2 \in ((f_1^{n_0})^{-1}(U))$ therefore there exist $v_1', v_2' \in U$ such that $v_1' = f_1^{n_0}(v_1)$ and $v_2' = f_1^{n_0}(v_2)$ and hence

$$d(f_{n_0+k} \circ \cdots \circ f_{n_0+1}(v_1'), f_{n_0+k} \circ \cdots \circ f_{n_0+1}(v_2')) > \delta.$$

Also $d(f_1^{n_0+k}(v_i), f^k(f_1^{n_0}(v_i))) < 1/m$ or $d(f_1^{n_0+k}(v_i), f^k(v_i')) < 1/m$ for $i = 1, 2$.

Therefore, by triangle inequality

$$d(f^k(v_1'), f^k(v_2')) > \delta - 2/m > \delta/2.$$

Hence, there exist $v_1', v_2' \in U$ such that $d(f^k(v_1'), f^k(v_2')) > \delta/2$ which implies $k \in N_f(U, \delta/2)$ so $N_{f_{1,\infty}}((f_1^{n_0})^{-1}(U), \delta) - n_0 \subseteq N_f(U, \delta/2)$. Since $N_{f_{1,\infty}}((f_1^{n_0})^{-1}(U), \delta)$ is thick therefore $N_f(U, \delta/2)$ is thick and hence $(X, f)$ is thickly sensitive. $\square$

**Theorem 4.7.** *Let $(X, f_{1,\infty})$ be an N.D.S. generated by a family $f_{1,\infty}$ of feeble open maps commuting with $f$. If $\sum_{i=1}^{\infty} D(f_i, f) < \infty$, then $(X, f)$ is thickly syndetic sensitive if and only if $(X, f_{1,\infty})$ is thickly syndetic sensitive.*

*Proof.* Let $(X, f)$ be thickly syndetic sensitive with constant of thickly syndetic sensitivity $\delta > 0$. Let $\epsilon > 0$ be given and $U = B(x, \epsilon)$ be a nonempty open set in $X$. As $\sum_{i=1}^{\infty} D(f_i, f) < \infty$, by Corollary 2.19 we obtain, for any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $d(f_1^{n+k}(x), f^k(f_1^n(x))) < \epsilon$ for all $x \in X$, $k \in \mathbb{N}$. Choose $m \in \mathbb{N}$ such that $1/m < \delta/4$. Then, there exists $n_0$ such that $d(f_1^{n_0+k}(x), f^k(f_1^{n_0}(x))) < 1/m$ for all $x \in X$, $k \in \mathbb{N}$. As $f_n$'s are feeble open, $U' = \operatorname{int} f_1^{n_0}(U)$ is nonempty open and by thickly syndetic sensitivity of $(X, f)$, we have $N_f(U', \delta)$ is thickly syndetic which implies that the set $S_{N_f}(U', \delta) = \{n \in \mathbb{N} : n + j; j \in N_f(U', \delta), 0 \leq j \leq k, k \in \mathbb{N}\}$ is syndetic. Let $k \in N_{f(U',\delta)}$ then there exist $v_1, v_2 \in f_1^{n_0}(U)$ such that $d(f^k(v_1), f^k(v_2)) > \delta$. As $v_1, v_2 \in f_1^{n_0}(U)$, implying there exist $v_1', v_2' \in U$ such that $v_1 = f_1^{n_0}(v_1')$ and $v_2 = f_1^{n_0}(v_2')$ and hence

$$d(f^k(f_1^{n_0}(v_1')), f^k(f_1^{n_0}(v_2'))) > \delta.$$

Also $d(f_1^{n_0+k}(v_i'), f^k(f_1^{n_0}(v_i'))) < 1/m$ for $i = 1, 2$.

By triangle inequality,

$$d(f_1^{n_0+k}(v_1'), f_1^{n_0+k}(v_2')) > \delta - 2/m > \delta/2.$$

Therefore, we obtain $v_1', v_2' \in U$ such that $d(f_1^{n_0+k}(v_1'), f_1^{n_0+k}(v_2')) > \delta/2$ which implies $n_0 + k \in N_{f_{1,\infty}}(U, \delta/2)$ for all $k \in N_f(U', \delta)$ and hence

$$\begin{aligned} &\{n \in \mathbb{N} : n + n_0 + j; j \in N_f(U', \delta), 0 \leq j \leq k, k \in \mathbb{N}\} \\ &\subseteq \{n \in \mathbb{N} : n + j; j \in \mathbb{N}_{f_{1,\infty}}(U, \delta/2); 0 \leq j \leq k, k \in \mathbb{N}\} \\ &= S_{N_{f_{1,\infty}}}(U, \delta/2), \end{aligned}$$

thus $S_{N_f}(U', \delta) + n_0 \subseteq S_{N_{f_{1,\infty}}}(U, \delta/2)$. Since $S_{N_f}(U', \delta)$ is syndetic therefore $S_{N_{f_{1,\infty}}}(U, \delta/2)$ is syndetic. Hence, $\mathbb{N}_{f_{1,\infty}}(U, \delta/2)$ is thickly syndetic implying $(X, f_{1,\infty})$ is thickly syndetic sensitive.

Conversely, let $(X, f_{1,\infty})$ be thickly syndetic sensitive with constant $\delta > 0$ and let $U$ be a nonempty open set in $X$. Let $m \in \mathbb{N}$ such that $1/m < \delta/4$. Then, there exists $n_0 \in \mathbb{N}$ such that $d(f_1^{n_0+k}(x), f^k(f_1^{n_0}(x))) < 1/m$ for all $x \in X$, $k \in \mathbb{N}$. Since $(X, f_{1,\infty})$ is thickly syndetic sensitive, therefore for any $n \in \mathbb{N}$, the set $N_{f_{1,\infty}}(U, \delta)$ is thickly syndetic. Hence the set $\{k : \operatorname{diam}(f_1^k(U)) > \delta\}$ is infinite implying for open set $(f_1^{n_0})^{-1}(U)$, the set $N_{f_{1,\infty}}((f_1^{n_0})^{-1}(U), \delta)$ is thickly syndetic, and hence there exists $k \in \mathbb{N}$ such that $k + n_0 \in N_{f_{1,\infty}}((f_1^{n_0})^{-1}(U), \delta)$. So, there exist $v_1, v_2 \in (f_1^{n_0})^{-1}(U)$ such that

$d(f_1^{n_0+k}(v_1), f_1^{n_0+k}(v_2)) > \delta$. Since $v_1, v_2 \in ((f_1^{n_0})^{-1}(U))$ therefore there exist $v_1', v_2' \in U$ such that $v_1' = f_1^{n_0}(v_1)$ and $v_2' = f_1^{n_0}(v_2)$ and hence

$$d(f_{n_0+k} \circ \cdots \circ f_{n_0+1}(v_1'), f_{n_0+k} \circ \cdots \circ f_{n_0+1}(v_2')) > \delta.$$

Also $d(f_1^{n_0+k}(v_i), f^k(f_1^{n_0}(v_i))) < 1/m$ or $d(f_1^{n_0+k}(v_i), f^k(v_i')) < 1/m$ for $i = 1, 2$.

By triangle inequality,

$$d(f^k(v_1'), f^k(v_2')) > \delta - 2/m > \delta/2.$$

So there exist $v_1', v_2' \in U$ such that $d(f^k(v_1'), f^k(v_2')) > \delta/2$ which implies $k \in N_f(U, \delta/2)$ and hence for every $m \in N_{f_{1,\infty}}(U, \delta), m - n_0 \in N_f(U, \delta/2)$. Thus, the set

$$\begin{aligned}
&\{n \in \mathbb{N} : n - n_0 + j; j \in \mathbb{N}_{f_{1,\infty}}(U, \delta), 0 \le j \le k, k \in \mathbb{N}\} \\
&\subseteq \{n \in \mathbb{N} : n + j; j \in \mathbb{N}_f(U, \delta/2), 0 \le j \le k, k \in \mathbb{N}\} \\
&= S_{N_f}(U, \delta/2)
\end{aligned}$$

implying $S_{N_f}(U, \delta/2)$ is syndetic and hence $N_f(U, \delta/2)$ is thickly syndetic. Therefore, $(X, f)$ is thickly syndetic sensitive □

**Theorem 4.8.** *Let $(X, f_{1,\infty})$ be an N.D.S. generated by a family $f_{1,\infty}$ of feeble open maps commuting with $f$. If $\sum_{i=1}^{\infty} D(f_i, f) < \infty$, then $(X, f)$ is ergodically sensitive if and only if $(X, f_{1,\infty})$ is ergodically sensitive.*

*Proof.* Let $(X, f)$ be ergodically sensitive with $\delta > 0$ as constant of ergodic sensitivity. Let $\epsilon > 0$ be given and $U = B(x, \epsilon)$ be a nonempty open set in $X$. As $\sum_{i=1}^{\infty} D(f_i, f) < \infty$, by Corollary 2.19 we obtain, for any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $d(f_1^{n+k}(x), f^k(f_1^n(x)))$ $< \epsilon$ for all $x \in X$, $k \in \mathbb{N}$. Choose $m \in \mathbb{N}$ such that $1/m < \delta/4$. Thus, there exists $n_0$ such that $d(f_1^{n_0+k}(x), f^k(f_1^{n_0}(x))) < 1/m$ for all $x \in X$, $k \in \mathbb{N}$. As $f_n$'s are feeble open, $U' = \operatorname{int} f_1^{n_0}(U)$ is nonempty open and by ergodic sensitivity of $(X, f)$, we have $N_f(U', \delta)$ has positive upper density. Let $k \in \mathbb{N}_f(U', \delta)$ then there exist $v_1, v_2 \in f_1^{n_0}(U)$ such that $d(f^k(v_1), f^k(v_2)) > \delta$. As $v_1, v_2 \in f_1^{n_0}(U)$, there exist $v_1', v_2' \in U$such that $v_1 = f_1^{n_0}(v_1')$ and $v_2 = f_1^{n_0}(v_2')$ and hence

$$d(f^k(f_1^{n_0}(v_1')), f^k(f_1^{n_0}(v_2'))) > \delta.$$

Also $d(f_1^n{}_{0+k}(v_i'), f^k(f_1^{n_0}(v_i'))) < 1/m$ for $i = 1, 2$.

By triangle inequality,

$$d(f_1^{n_0+k}(v_1'), f_1^{n_0+k}(v_2')) > \delta - 2/m > \delta/2.$$

Therefore, there exist $v_1', v_2' \in U$ such that

$$d(f_1^{n_0+k}(v_1'), f_1^{n_0+k}(v_2')) > \delta/2$$

which implies $n_0 + k \in N_{f_{1,\infty}}(U, \delta/2)$ and hence $N_f(U', \delta) + n_0 \subseteq N_{f_{1,\infty}}(U, \delta/2)$. Since $N_f(U', \delta)$ has positive upper density therefore $N_{f_{1,\infty}}(U, \delta/2)$ has positive upper density and hence $(X, f_{1,\infty})$ is ergodically sensitive.

Conversely, let $(X, f_{1,\infty})$ be ergodically sensitive with constant $\delta > 0$ and let $U$ be a nonempty open set in $X$. Let $m \in \mathbb{N}$ such that $1/m < \delta/4$. Then, there exists $n_0 \in \mathbb{N}$ such that $d(f_1^{n_0+k}(x), f^k(f_1^{n_0}(x))) < 1/m$ for all $x \in X$, $k \in \mathbb{N}$. Since $(X, f_{1,\infty})$ is ergodically sensitive, therefore for any $n \in \mathbb{N}$, the set $N_{f_{1,\infty}}(U, \delta)$ has positive upper density, and hence the set $\{k : \operatorname{diam}(f_1^k(U)) > \delta\}$ is infinite. Thus, for open set $(f_1^{n_0})^{-1}(U)$, the set $N_{f_{1,\infty}}((f_1^{n_0})^{-1}(U), \delta)$ has positive upper density, which implies there exists $k \in \mathbb{N}$ such that $k + n_0 \in \mathbb{N}_{f_{1,\infty}}((f_1^{n_0})^{-1}(U), \delta)$. So, there exist $v_1, v_2 \in (f_1^{n_0})^{-1}(U)$ such that

$$d(f_1^{n_0+k}(v_1), f_1^{n_0+k}(v_2)) > \delta.$$

Since $v_1, v_2 \in ((f_1^{n_0})^{-1}(U))$ therefore there exist $v_1', v_2' \in U$ such that $v_1' = f_1^{n_0}(v_1)$ and $v_2' = f_1^{n_0}(v_2)$ and hence

$$d(f_{n_0+k} \circ \cdots \circ f_{n_0+1}(v_1'), f_{n_0+k} \circ \cdots \circ f_{n_0+1}(v_2')) > \delta.$$

Also, $d(f_1^{n_0+k}(v_i), f^k(f_1^{n_0}(v_i))) < 1/m$ or $d(f_1^{n_0+k}(v_i), f^k(v_i')) < 1/m$ for $i = 1, 2$.

By triangle inequality

$$d(f^k(v_1'), f^k(v_2')) > \delta - 2/m > \delta/2.$$

Therefore, we obtain $v_1', v_2' \in U$ such that $d(f^k(v_1'), f^k(v_2')) > \delta/2$ which implies $k \in N_f(U, \delta/2)$ and hence $N_{f_{1,\infty}}((f_1^{n_0})^{-1}(U), \delta) - n_0 \subseteq N_f(U, \delta/2)$. Since $N_{f_{1,\infty}}((f_1^{n_0})^{-1}(U), \delta)$ has positive upper density therefore $N_f(U, \delta/2)$ has positive upper density and hence $(X, f)$ is ergodically sensitive. $\square$

**Theorem 4.9.** *Let $(X, f_{1,\infty})$ be an N.D.S. generated by a family $f_{1,\infty}$ of feeble open maps commuting with $f$. If $\sum_{i=1}^{\infty} D(f_i, f) < \infty$, then $(X, f)$ is multi-sensitive if and only if $(X, f_{1,\infty})$ is multi-sensitive.*

*Proof.* Let $(X, f)$ be multi-sensitive with constant $\delta > 0$. As $\sum_{i=1}^{\infty} D(f_i, f) < \infty$, by Corollary 2.19 we obtain, for any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $d(f_1^{n+k}(x), f^k(f_1^n(x))) < \epsilon$ for all $x \in X$, $k \in \mathbb{N}$. Choose $m \in \mathbb{N}$ such that $1/m < \delta/4$. Then, there exists $n_0$ such that $d(f_1^{n_0+k}(x), f^k(f_1^{n_0}(x))) < 1/m$ for all $x \in X$, $k \in \mathbb{N}$. Let $V_1, V_2, \ldots, V_k$ be nonempty open sets in $X$ for some $k \geq 1$. As $f_n$'s are feeble open, $V_i' = \operatorname{int} f_1^{n_0}(V_i)$ is nonempty open for all $1 \leq i \leq k$, and by multi-sensitivity of $(X, f)$ we have $\bigcap_{i=0}^{k} N_f(V_i', \delta)$ is nonempty. Let $m \in \bigcap_{i=0}^{k} N_f(V_i', \delta)$ then, there exist $y_i, z_i \in f_1^{n_0}(V_i)$ such that $d(f^m(y_i), f^m(z_i)) > \delta$ for all $1 \leq i \leq k$. As $y_i, z_i \in f_1^{n_0}(V_i)$, there exist $y_i', z_i' \in V_i$ such that $y_i = f_1^{n_0}(y_i')$ and $z_i = f_1^{n_0}(z_i')$ and hence

$$d(f^m(f_1^{n_0}(y_i')), f^m(f_1^{n_0}(z_i'))) > \delta.$$

Also, $d(f_1^{n_0+m}(y_i'), f^m(f_1^{n_0}(y_i'))) < 1/m$ for all $1 \leq i \leq k$.

By triangle inequality,

$$d(f_1^{n_0+m}(y_i'), f_1^{n_0+m}(z_i')) > \delta - 2/m > \delta/2.$$

Therefore, we get $y_i', z_i' \in V_i$ such that $d(f_1^{n_0+m}(y_i'), f_1^{n_0+m}(z_i')) > \delta/2$ for all $1 \leq i \leq k$ and hence $n_0 + m \in \bigcap_{i=0}^{k} N_{f_{1,\infty}}(V_i, \delta/2)$ which implies

$$\bigcap_{i=0}^{k} N_f(V_i', \delta) + n_0 \subseteq \bigcap_{i=0}^{k} N_{f_{1,\infty}}(V_i, \delta/2).$$

Since $\bigcap_{i=0}^{k} N_f(V_i', \delta)$ is nonempty therefore $\bigcap_{i=0}^{k} N_{f_{1,\infty}}(V_i, \delta/2)$ is nonempty and hence $(X, f_{1,\infty})$ is multi-sensitive.

Conversely, let $(X, f_{1,\infty})$ be multi-sensitive with constant of multi-sensitivity $\delta > 0$. Let $m \in \mathbb{N}$ such that $1/m < \delta/4$. Then, there exists $n_0 \in \mathbb{N}$ such that $d(f_1^{n_0+k}(x), f^k(f_1^{n_0}(x))) < 1/m$ for all $x \in X$, $k \in \mathbb{N}$. Let $V_1, V_2, \ldots, V_k$ be non empty open sets in $X$ for some $k \geq 1$. Now, by Archimedean Property, there exists $r \in \mathbb{N}$ such that $1/r < \epsilon$ which implies $1/n < \epsilon$ for all $n \geq r$. Now let $v_i \in V_i$ for all $1 \leq i \leq k$. By multi-sensitivity of $(X, f_{1,\infty})$, $\bigcap_{i=0}^{k} N_{f_{1,\infty}}(B(v_i, 1/n), \delta)$ is nonempty for all $n \geq r$, and therefore $\bigcap_{i=0}^{k} N_{f_{1,\infty}}(B(v_i, \epsilon), \delta)$ is nonempty and in fact infinite. Hence, $\bigcap_{i=0}^{k} N_{f_{1,\infty}}(V_i, \delta)$ is nonempty and infinite for each $V_i$, $1 \leq i \leq k$, $k \geq 1$. Since $f_n$'s are continuous, therefore $((f_1^{n_0})^{-1}(V_i))$ is open, for all $1 \leq i \leq k$ which implies $\bigcap_{i=0}^{k} N_{f_{1,\infty}}(((f_1^{n_0})^{-1}(V_i)), \delta)$ is nonempty and infinite. Hence, there exists $m \in \mathbb{N}$ such that $m + n_0 \in \bigcap_{i=0}^{k} N_{f_{1,\infty}}((f_1^{n_0})^{-1}(V_i), \delta)$ which implies there exist $y_i, z_i \in (f_1^{n_0})^{-1}(V_i)$ such that

$$d(f_1^{n_0+m}(y_i), f_1^{n_0+m}(z_i)) > \delta$$

for all $1 \leq i \leq k$. Since $y_i, z_i \in ((f_1^{n_0})^{-1}(V_i))$ therefore, there exist $y_i', z_i' \in V_i$ such that $y_i' = f_1^{n_0}(y_i)$, $z_i' = f_1^{n_0}(z_i)$ and hence

$$d((f_{n_0+m} \circ \cdots \circ f_{n_0+1})(y_i'), (f_{n_0+m} \circ \cdots \circ f_{n_0+1})(z_i')) > \delta.$$

Also $d(f_1^{n_0+m}(y_i), f^m(f_1^{n_0}(y_i))) < 1/m$ or $d(f_1^{n_0+m}(y_i), f^m(y_i')) < 1/m$ for $1 \leq i \leq k$.

By triangle inequality, $d(f^m(y_i'), f^m(z_i')) > \delta - 2/m > \delta/2$. Therefore, we get $y_i', z_i' \in V_i$ such that $d(f^m(y_i'), f^m(z_i')) > \delta/2$ for all $1 \leq i \leq k$, which implies $m \in \bigcap_{i=0}^{k} N_f(V_i, \delta/2)$. Thus,

$$\left(\bigcap_{i=0}^{k} N_{f_{1,\infty}}((f_1^{n_0})^{-1}(V_i), \delta) - n_0\right) \subseteq \bigcap_{i=0}^{k} N_f(V_i, \delta/2).$$

Since $\bigcap_{i=0}^{k} N_{f_{1,\infty}}((f_1^{n_0})^{-1}(V_i), \delta)$ is nonempty therefore $\bigcap_{i=0}^{k} N_f(V_i, \delta/2)$ is nonempty and hence $(X, f)$ is multi-sensitive. $\square$

In the following example, we show that Theorems 4.5–4.9 fail if the condition of taking $(f_n, n \in \mathbb{N})$ to be feeble open is dropped from the hypothesis.

**Example 4.10.** Let $I$ be the closed unit interval $[0,1]$ and $f_1$, $f_2$ be defined by

$$f_1(x) = \begin{cases} 2x & \text{for } x \in [0, 1/5], \\ 1 & \text{for } x \in [1/5, 2/5], \\ x & \text{for } x \in [2/5, 1], \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} 2x + 1/2 & \text{for } x \in [0, 1/4], \\ -2x + 3/2 & \text{for } x \in [1/4, 3/4], \\ 2x - 3/2 & \text{for } x \in [3/4, 1]. \end{cases}$$

Let $f_0(x) = f_1(x)$ and $f_n(x) = f_2(x)$ for all $n > 1$. Then $(f_n)_{n=0}^{\infty}$ converges uniformly to $f = f_2$ in $X$. Note that the function $f$ is feeble open. Clearly, $(X, f)$ is syndetic sensitive, thickly sensitive, thickly syndetic sensitive, multi-sensitive and ergodic sensitive being cofinitely sensitive. However, the system $(X, f_{1,\infty})$ is not even sensitive because for open set $U = (1/5, 2/5)$, $\mathbb{N}_{f_{1,\infty}}(U, \delta)$ is empty implying it is not syndetic sensitive, thickly sensitive, thickly syndetic sensitive, multi-sensitive or ergodic sensitive. Our theorems fail here because in the family $(f_n)$ the function $f_0$ is not feeble open.

In the next example, we provide a system $(X, f_{1,\infty})$ which satisfies the hypothesis of Theorems 4.5–4.9.

**Example 4.11.** Let $I$ be the interval $[0,1]$ and $f_1$, $f_2$ be defined by $f_1(x) = x$ for $x \in [0,1]$ and

$$f_2(x) = \begin{cases} 2x + 1/2 & \text{for } x \in [0, 1/4], \\ -2x + 3/2 & \text{for } x \in [1/4, 3/4], \\ 2x - 3/2 & \text{for } x \in [3/4, 1]. \end{cases}$$

Let $f_0(x) = f_1(x)$ and $f_n(x) = f_2(x)$ for all $n > 1$. Note that $f = f_2(x)$ is feeble open. Clearly, $(X, f)$ is syndetic sensitive, thickly sensitive, thickly syndetic sensitive, multi-sensitive and ergodic sensitive being cofinitely sensitive. The family $(f_n)$ consists of feeble open maps converging uniformly to $f$. Thus, this family satisfies the hypothesis of our theorem and therefore, the system $(X, f_{1,\infty})$ is also syndetic sensitive, thickly sensitive, thickly syndetic sensitive, multi-sensitive and ergodic sensitive.

In the following two theorems, we give necessary and sufficient condition for the system $(X, f_{1,\infty})$ to be collective sensitive and synchronous sensitive, where $(X, f_{1,\infty})$ is an N.D.S. generated by a family $f_{1,\infty}$ of open maps commuting with a continuous self map $f$.

**Theorem 4.12.** *Let $(X, f_{1,\infty})$ be an N.D.S. generated by a family $f_{1,\infty}$ of open maps commuting with $f$. If $\sum_{i=1}^{\infty} D(f_i, f) < \infty$, then $(X, f)$ is collectively sensitive if and only if $(X, f_{1,\infty})$ is collectively sensitive.*

*Proof.* Let $\epsilon > 0$ be given and let $x_1, x_2, \dots, x_n$ be finitely many distinct points in $X$. Let $U_i = B(x_i, \epsilon)$ for all $i = 1, 2, \dots, n$. Since $\sum_{i=1}^{\infty} D(f_i, f) < \infty$, by Corollary 2.19 we obtain, for any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $d(f_1^{n+k}(x), f^k(f_1^n(x))) < \epsilon$ for all $x \in X$, $k \in \mathbb{N}$. Let $\delta > 0$ be a constant of collective sensitivity. Choose $m \in \mathbb{N}$ such that $1/m < \delta/4$. Then, there exists $n_0$ such that $d(f_1^{n_0+k}(x), f^k(f_1^{n_0}(x))) < 1/m$ for all $x \in X$, $k \in \mathbb{N}$. Let $u_i = f_1^{n_0}(x_i) \in f_1^{n_0}(U_i)$ for all $i = 1, 2, \dots, n$. So by collective sensitivity of $(X, f)$, there exist $v_i \in X$, $1 \le i \le n$, such that $d(u_i, v_i) < \epsilon$ for all $i = 1, 2, \dots, n$ and there exists $1 \le i_0 \le n$ such that

$$d(f^k(u_i), f^k(v_{i_0})) \ge \delta \quad \text{or} \quad d(f^k(v_i), f^k(u_{i_0})) \ge \delta$$

for some $k$. Since $f_n$'s are open, therefore $f_1^{n_0}(U_i)$ is open for all $i = 1, 2, \dots, n$, and hence, $v_i \in B(u_i, \epsilon) = B(f_1^{n_0}(x_i), \epsilon) \subseteq f_1^{n_0}(U_i)$. Thus, there exists $y_i \in U_i$ such that $v_i = f_1^{n_0}(y_i)$. Since $U_i = B(x_i, \epsilon)$ therefore, $d(x_i, y_i) < \epsilon$ for all $i = 1, 2, \dots, n$ and by triangle inequality we get,

$$d(f_1^{n_0+k}(x_i), f_1^{n_0+k}(y_{i_0})) \ge \delta/2$$

for all $i = 1, 2, \dots, n$ or

$$d(f_1^{n_0+k}(y_i), f_1^{n_0+k}(x_{i_0})) \ge \delta/2$$

for all $i = 1, 2, \dots, n$. Hence, $(X, f_{1,\infty})$ is collectively sensitive.

Conversely, let $(X, f_{1,\infty})$ be collectively sensitive with constant $\delta > 0$ and let $m \in \mathbb{N}$ be such that $1/m < \delta/4$. Then, there exists $n_0 \in \mathbb{N}$ such that $d(f_1^{n_0+k}(x), f^k(f_1^{n_0}(x))) < 1/m$ for all $x \in X$, $k \in \mathbb{N}$. Let $x_1, x_2, \dots, x_n$ be finitely many distinct points in $X$. Let $U_i = B(x_i, \epsilon)$ for all $i = 1, 2, \dots, n$. For any $r \in \mathbb{N}$, the collective sensitivity of $(X, f_{1,\infty})$ ensures the existence of $k_r \in \mathbb{N}$ and $y_1^r, y_2^r, \dots, y_n^r$ such that $d(x_i, y_i^r) < 1/r$ for all $1 \le i \le n$ and $d(f_1^{k_r}(x_i), f_1^{k_r}(y_{i_0})) \ge \delta$ or $d(f_1^{k_r}(y_i), f_1^{k_r}(x_{i_0})) \ge \delta$ for some $1 \le i_0 \le n$. Therefore, for $U_i = B(x_i, \epsilon)$, there exist infinitely many $k = k_r$ such that $d(x_i, y_i^k) < \epsilon$ and $d(f_1^k(x_i), f_1^k(y_{i_0})) \ge \delta$ for some $1 \le i_0 \le n$ or $d(f_1^k(y_i), f_1^k(x_{i_0})) \ge \delta$ for some $1 \le i_0 \le n$. Let $u_i = (f_1^{n_0})^{-1}(x_i) \in (f_1^{n_0})^{-1}(U_i)$. Therefore, for open sets $(f_1^{n_0})^{-1}(U_i)$, $1 \le i \le n$, there exist $v_i \in (f_1^{n_0})^{-1}(U_i)$ and $k \in \mathbb{N}$ such that $d(u_i, v_i) < \epsilon$ and for some $1 \le i_0 \le n$, $d(f_1^{n_0+k}(u_i), f_1^{n_0+k}(v_{i_0})) \ge \delta$ for all $1 \le i \le n$ or $d(f_1^{n_0+k}(v_i), f_1^{n_0+k}(u_{i_0})) \ge \delta$ for all $1 \le i \le n$. Since $v_i \in (f_1^{n_0})^{-1}(U_i)$ there exists $y_i \in U_i$ such that $y_i = f_1^{n_0}(v_i)$, $i = 1, 2, \dots, n$. Thus

$$d((f_{n_0+k} \circ \cdots \circ f_{n_0+1})(x_i), (f_{n_0+k} \circ \cdots \circ f_{n_0+1})(y_{i_0})) \ge \delta$$

or

$$d((f_{n_0+k} \circ \cdots \circ f_{n_0+1})(y_i), (f_{n_0+k} \circ \cdots \circ f_{n_0+1})(x_{i_0})) \ge \delta$$

for all $1 \le i \le n$. Also $d(f_1^{n_0+k}(v_i),\ f^k(f_1^{n_0}(v_i))) < 1/m$ or $d(f_1^{n_0+k}(v_i), f^k(y_i)) < 1/m$. Similarly, $d(f_1^{n_0+k}(u_i), f^k(x_i)) < 1/m$. Since $y_i \in U_i = B(x_i, \epsilon)$ therefore $d(x_i, y_i) < \epsilon$ for all $i = 1, 2, \dots, n$. By triangle inequality,

$$d(f^k(x_i), f^k(y_{i_0})) \geq \delta/2 \quad \text{or} \quad d(f^k(y_i), f^k(x_{i_0})) \geq \delta/2.$$

Hence, $(X, f)$ is collectively sensitive. □

**Theorem 4.13.** *Let $(X, f_{1,\infty})$ be an N.D.S. generated by a family $f_{1,\infty}$ of open maps commuting with $f$. If $\sum_{i=1}^{\infty} D(f_n, f) < \infty$, then $(X, f)$ is synchronous sensitive if and only if $(X, f_{1,\infty})$ is synchronous sensitive.*

*Proof.* Let $\epsilon > 0$ be given and let $x_1, x_2, \dots, x_n$ be finitely many distinct points in $X$. Let $U_i = B(x_i, \epsilon)$ for all $i = 1, 2, \dots, n$. As $\sum_{i=1}^{\infty} D(f_n, f) < \infty$, by Corollary 2.19 we obtain, for any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $d(f_1^{n+k}(x), f^k(f_1^n(x))) < \epsilon$ for all $x \in X$, $k \in \mathbb{N}$. Let $\delta > 0$ be a constant of synchronous sensitivity. Choose $m \in \mathbb{N}$ such that $1/m < \delta/4$. Thus, there exists $n_0$ such that $d(f_1^{n_0+k}(x), f^k(f_1^{n_0}(x))) < 1/m$ for all $x \in X$, $k \in \mathbb{N}$. Let $u_i = f_1^{n_0}(x_i) \in f_1^{n_0}(U_i)$ for all $i = 1, 2, \dots, n$. So by synchronous sensitivity of $(X, f)$ there exist $v_i \in X$, $1 \leq i \leq n$, such that $d(u_i, v_i) < \epsilon$ for all $i = 1, 2, \dots, n$ and there exists $k \in \mathbb{N}$ such that

$$d(f^k(u_i), f^k(v_i)) \geq \delta, \quad i = 1, 2, \dots, n.$$

Since $f_n$'s are open therefore $f_1^{n_0}(U_i)$ is open for all $i = 1, 2, \dots, n$, and hence, $v_i \in B(u_i, \epsilon) = B(f_1^{n_0}(x_i), \epsilon) \subseteq f_1^{n_0}(U_i)$. Thus, there exists $y_i \in U_i$ such that $v_i = f_1^{n_0}(y_i)$ for $i = 1, 2, \dots, n$. Since $U_i = B(x_i, \epsilon)$ therefore $d(x_i, y_i) < \epsilon$ for all $i = 1, 2, \dots, n$ and by triangle inequality

$$d(f_1^{n_0+k}(x_i), f_1^{n_0+k}(y_i)) \geq \delta/2$$

for all $i = 1, 2, \dots, n$, therefore, $(X, f_{1,\infty})$ is synchronous sensitive.

Conversely, let $(X, f_{1,\infty})$ be synchronous sensitive with constant $\delta > 0$ and let $m \in \mathbb{N}$ be such that $1/m < \delta/4$. Then, there exists $n_0 \in \mathbb{N}$ such that $d(f_1^{n_0+k}(x), f^k(f_1^{n_0}(x))) < 1/m$ for all $x \in X$, $k \in \mathbb{N}$. Let $x_1, x_2, \dots, x_n$ be finitely many distinct points in $X$. Let $U_i = B(x_i, \epsilon)$ for all $i = 1, 2, \dots, n$. For any $r \in \mathbb{N}$, the synchronous sensitivity of $(X, f_{1,\infty})$ ensures the existence of $k_r \in \mathbb{N}$ and $y_1^r, y_2^r, \dots, y_n^r$ such that $d(x_i, y_i^r) < 1/n$ for all $1 \leq i \leq n$ and $d(f_1^{k_r}(x_i), f_1^{k_r}(y_i)) > \delta$. Therefore, for $U_i = B(x_i, \epsilon)$, there exist infinitely many $k = k_r$ such that $d(x_i, y_i^k) < \epsilon$ and $d(f_1^k(y_i), f_1^k(x_i)) \geq \delta$. Let $u_i = (f_1^{n_0})^{-1}(x_i) \in (f_1^{n_0})^{-1}(U_i)$. Then, for open sets $(f_1^{n_0})^{-1}(U_i)$, $1 \leq i \leq n$, there exists $v_i \in (f_1^{n_0})^{-1}(U_i)$ and $k \in \mathbb{N}$ such that $d(u_i, v_i) < \epsilon$ and $d(f_1^{n_0+k}(u_i), f_1^{n_0+k}(v_i)) \geq \delta$ for all $1 \leq i \leq n$. As $v_i \in (f_1^{n_0})^{-1}(U_i)$ there exists $y_i \in U_i$ such that $y_i = f_1^{n_0}(v_i)$, $i = 1, 2, \dots, n$. Hence

$$d((f_{n_0+k} \circ \cdots \circ f_{n_0+1})(x_i), (f_{n_0+k} \circ \cdots \circ f_{n_0+1})(y_i)) \geq \delta$$

for all $1 \leq i \leq n$. Also $d(f_1^{n_0+k}(v_i), f^k(f_1^{n_0}(v_i))) < 1/m$ or $d(f_1^{n_0+k}(v_i), f^k(y_i)) < 1/m$. Similarly, $d(f_1^{n_0+k}(u_i), f^k(x_i)) < 1/m$. Since $y_i \in U_i = B(x_i, \epsilon)$ therefore $d(x_i, y_i) < \epsilon$ for all $i = 1, 2, \ldots, n$. By triangle inequality,

$$d(f^k(x_i), f^k(y_i)) \geq \delta/2.$$

Hence, $(X, f)$ is synchronous sensitive. □

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