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Pseudo Projective Modules Which are not Quasi Projective and Quivers

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Abstract. In this paper we construct pseudo projective modules which are not quasi projective over non-commutative perfect rings. To do it we construct finite dimensional quiver algebras over the field $\mathbb{Z}_2$. The modules which are constructed will have finite length three and only three nonzero proper submodules.

1. Introduction

We consider associative rings R with identity and all modules considered are unitary left R-modules. Throughout this paper K will be any field.

Let M be a module. M is said to be *quasi projective* (*pseudo projective*) if, for any submodule X of M, any homomorphism (epimorphism) $f\colon M \to M/X$ can be lifted to an endomorphism of M. Note that pseudo projective modules are named as epi-projective in [2].

Let M be a module and (P,p) be a projective cover of M. M is called *automorphism coinvariant* if, for every automorphism $f\colon P \to P$, $f(\operatorname{Ker} p) \subseteq \operatorname{Ker} p$, equivalently, for every automorphism $f\colon P \to P$, $f(\operatorname{Ker} p) = \operatorname{Ker} p$. It is proven in [3, Theorem 2.3] that automorphism coinvariant modules and pseudo projective modules coincide over left perfect rings.

Let x be a vertex of a quiver Q. Then $S(x)$ will denote the simple representation corresponding to the vertex x. On the other hand, $P(x)$ (resp. $I(x)$) will denote the indecomposable projective (resp. injective) representation corresponding to the vertex x. Sometimes, for short, $S(x)$ is replaced by x and pictures of the form

$$\begin{matrix}1 & & 2\\ & 3 & \end{matrix},\quad \begin{matrix}1 & & 2\\ & 2 & \end{matrix},\quad \begin{matrix}1\\ 2\end{matrix},\ \begin{matrix}2\\ 2\end{matrix},\ \begin{matrix}1\\ 3\end{matrix},\ \ldots$$

denote the composition series of indecomposable modules. For more background on quivers we refer to [1,6].

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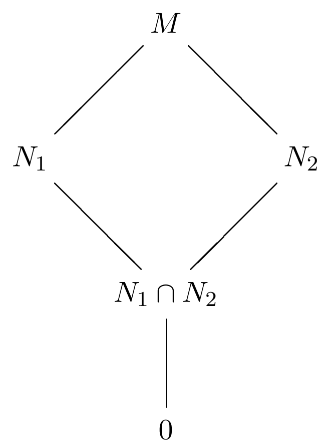
The aim of this paper is to construct pseudo projective modules which are not quasi projective. The modules that we construct by means of quivers are either injective (see Examples 2.3 and 2.4) or of injective dimension one and infinity (see Theorem 2.8). Moreover they have finite length three and only three nonzero proper submodules.

This paper is organized as follows. In Section 1 we recall some definitions and conventions. In Section 2 we collect all the results.

2. Results

The following lemma is the dual of Lemma 2 in [5]. We are giving its proof for sake of completeness.

Lemma 2.1. (see also [2, Exercises 4.45(8)]) *Let M be an R-module whose lattice of submodules is*



with $M/N_1 \ncong M/N_2$. The following facts hold:

(i) *M is not quasi projective.*

(ii) *M is pseudo projective if and only if* $\mathrm{End}_R(M/N_i) \cong \mathbb{Z}_2$ *for $i = 1, 2$.*

Proof. (i) We first note that the two submodules of length 2 of M are uniserial. On the other hand, the unique factor module of M of length 2 is the direct sum of two nonisomorphic simple modules. Hence every nonzero endomorphism f of M which is not an automorphism has the property that $f(M) = N_1 \cap N_2$.

Let $\pi\colon M \to M/(N_1 \cap N_2)$ be the natural epimorphism and let $g\colon M \to M/(N_1 \cap N_2)$ be the homomorphism defined by $g(n_1 + n_2) = n_1 + N_1 \cap N_2$ for all $n_1 \in N_1$ and $n_2 \in N_2$. Since $g(M) = N_1/(N_1 \cap N_2)$, we conclude that there is no $f \in \mathrm{End}_R(M)$ such that $\pi f = g$.

(ii) Assume M is pseudo projective. Suppose that f_1, f_2 are two nonzero endomorphisms of M/N_1 with $f_1 \neq f_2$. Since M/N_1 is simple, f_1 and f_2 are isomorphisms. Let $n_2 \in N_2$. Then there exist $n_2', n_2'' \in N_2$ such that $f_1(n_2 + N_1) = n_2' + N_1$ and

$f_2(n_2+N_1) = n_2''+N_1$. With n_2, n_2', n_2'' as above and $n_1 \in N_1$ let $g_1, g_2\colon M \to M/(N_1\cap N_2)$ be the homomorphisms defined by the formula

$$g_1(n_1+n_2) = n_1+n_2'+N_1\cap N_2, \quad g_2(n_1+n_2) = n_1+n_2''+N_1\cap N_2.$$

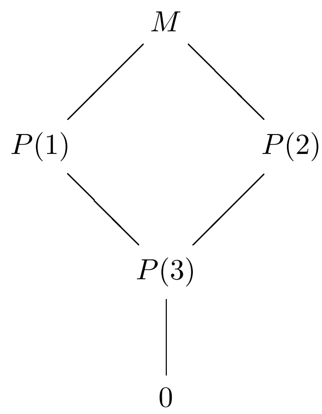
Then $N_1/(N_1\cap N_2) \subsetneqq g_i(M)$ and so g_i is an epimorphism for every $i = 1, 2$. Let $\pi\colon M \to M/(N_1\cap N_2)$ be the natural epimorphism and let h_1, h_2 be two endomorphisms of M such that $\pi h_1 = g_1$ and $\pi h_2 = g_2$. Set $h = h_1 - h_2$. Since $g_1 \neq g_2$, we have $h \neq 0$ and $h(M) \neq N_1 \cap N_2$. Fix $m = n_1 + n_2 \in M$ with $n_1 \in N_1$ and $n_2 \in N_2$. Then there exist $n_2', n_2'' \in N_2$ such that $h(m) + N_1 \cap N_2 = \pi h(m) = g_1(m) - g_2(m) = n_2' - n_2'' + N_1 \cap N_2$. It follows that $h(M) \subseteq N_2$. This implies that $h(M) = N_2$, but this contradicts the first remark on the endomorphisms of M. Consequently, $\mathrm{End}_R(M/N_1) \cong \mathbb{Z}_2$. Similarly, $\mathrm{End}_R(M/N_2) \cong \mathbb{Z}_2$.

Assume now $\mathrm{End}_R(M/N_i) \cong \mathbb{Z}_2$ for $i = 1, 2$. Let $f\colon M \to M/N_1$ be an epimorphism. Since N_1 and N_2 are the unique submodules of M of length 2 and $M/N_1 \ncong M/N_2$ we have $\mathrm{Ker}\, f = N_1$. This remark and the hypothesis $\mathrm{End}_R(M/N_1) \cong \mathbb{Z}_2$ imply that f is the natural epimorphism $\pi\colon M \to M/N_1$. Therefore $f = \pi$ lifts to the identity of M. The same is true for any epimorphism $f\colon M \to M/N_2$. Now assume $f\colon M \to M/(N_1 \cap N_2)$ is an epimorphism. Since $N_1 \cap N_2$ is the unique simple submodule of M and $M/\mathrm{Ker}\, f \cong M/(N_1 \cap N_2)$, it follows that $\mathrm{Ker}\, f = N_1 \cap N_2$. On the other hand, we have $\mathrm{End}_R(M/(N_1 \cap N_2)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. This implies that f is the natural epimorphism $\pi\colon M \to M/(N_1 \cap N_2)$. Thus also in this case f lifts to the identity of M, and the proof of (ii) is completed. □

Remark 2.2. Let R be a ring and let a be an element of the center of R. Assume that N is an R-module which has two elements x and y such that $N = Rx$, $y = ax$ and $S = Ry$ is the unique nonzero proper submodule of N. Let $f\colon N \to N$ be the R-homomorphism defined by $f(n) = an$, for any $n \in N$. Then we have that $f(rx) = arx = rax = ry \in S$ for any $r \in R$. Consequently, $f(N) = S$. Since N and S are not isomorphic, $\mathrm{Ker}\, f = S$. Hence $N/S \cong S$. Now let R, M, N_1 and N_2 be as in Lemma 2.1. Assume that R is commutative. Then N_1 and N_2 satisfy the hypotheses on N above and clearly $N_1 \cap N_2$ is the unique nonzero proper submodule of N_1 and N_2. Therefore we have $N_1/(N_1 \cap N_2) \cong N_1 \cap N_2$ and $N_2/(N_1 \cap N_2) \cong N_1 \cap N_2$. This is a contradiction since $M/N_1 \ncong M/N_2$. Therefore R cannot be commutative in Lemma 2.1.

Example 2.3. There is a hereditary K-algebra R and an R-module M with projective cover (P, p) such that M satisfies the hypotheses of Lemma 2.1, $\mathrm{End}_R(M) \cong K$ and $\mathrm{Aut}\, P \cong K^* \times K^*$.

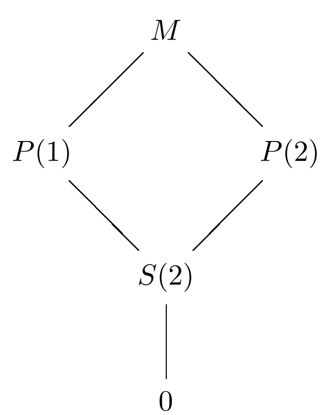
Construction. Let R be the K-algebra given by the quiver $1 \longrightarrow 3 \longleftarrow 2$ and let M be the R-module $I(3) = {}^{1}{}_{3}{}^{2}$. Note that $P(1) = \frac{1}{3}$, $P(2) = \frac{2}{3}$ and $P(3) = 3$. Then the lattice of submodules of M is



and we have $I(3)/P(1) \ncong I(3)/P(2)$. (Also $\mathrm{End}_R(I(3)/P(1)) \cong \mathrm{End}_R(I(3)/P(2)) \cong K$ because $I(3)/P(1)$ and $I(3)/P(2)$ are one dimensional vector spaces.) Moreover we clearly have $\mathrm{End}_R(I(3)) \cong K$ and $P \cong P(1) \oplus P(2)$. On the other hand, we have $\mathrm{End}_R(P) \cong K \oplus K$ and so $\mathrm{Aut}\, P \cong K^* \times K^*$.

Example 2.4. There is a non hereditary K-algebra R and an R-module M with projective cover (P, p) such that M satisfies the hypotheses of Lemma 2.1 and $\mathrm{End}_R(M) \cong K[x]/(x^2)$. Moreover, if $K = \mathbb{Z}_2$, then $\mathrm{Aut}\, P$ is $C_2 \times C_2$ and for any $f \in \mathrm{Aut}\, P$ we have $f(v) = v$ for all $v \in \mathrm{Ker}\, p$.

Construction. Let R be the K-algebra given by the quiver $1 \xrightarrow{a} 2 \circlearrowleft b$ with relations $ba = b^2 = 0$ and let $M = I(2) = {}^{1}{}_{2}{}^{2}$. Note that $P(1) = \substack{1\\2}$, $P(2) = \substack{2\\2}$ and $S(2) = 2$. Also in this case the lattice of submodules of M is of the form



with $I(2)/P(1) \ncong I(2)/P(2)$ and $\mathrm{End}_R(I(2)) \cong K[x]/(x^2)$. (Also $\mathrm{End}_R(I(2)/P(1)) \cong \mathrm{End}_R(I(2)/P(2)) \cong K$ because $I(2)/P(1)$ and $I(2)/P(2)$ are one dimensional vector spaces.) Then $P \cong P(1) \oplus P(2)$ is of the form $V_1 \xrightarrow{a} V_2 \circlearrowleft b$ and there exists a basis $\{v_1, v_2, v_3, v_4\}$ of P such that $V_1 = \langle v_1 \rangle$, $V_2 = \langle v_2, v_3, v_4 \rangle$, $\mathrm{Ker}\, p = \langle v_2 - v_4 \rangle$, $av_1 = v_2$

and $bv_3 = v_4$. Hence $bv_2 = bv_4 = 0$. Assume now $K = \mathbb{Z}_2$, and take any $f \in \operatorname{Aut} P$. Then $f(v_1) = v_1$ and $f(v_3)$ is one of the vectors v_3, $v_2 + v_3$, $v_3 + v_4$, $v_2 + v_3 + v_4$. Since $f(v_2) = f(av_1) = v_2$ and $f(v_4) = f(bv_3) = v_4$, we have $f(v) = v$ for all $v \in \langle v_1, v_2, v_4 \rangle$. This means that f is the identity on $\operatorname{Ker} p = \langle v_2 - v_4 \rangle$. Since P is generated, as a left R-module, by v_1 and v_3 and $f^2(v_3) = v_3$, it follows that f^2 is the identity map. Consequently $\operatorname{Aut} P \cong C_2 \times C_2$.

Theorem 2.5. *There exist pseudo projective modules which are not quasi projective over a non-commutative perfect ring.*

Proof. This follows from Lemma 2.1 and Examples 2.3 and 2.4 when the field K is the field $\mathbb{Z}_2$. Note that in Examples 2.3 and 2.4, the R-modules M are also automorphism coinvariant and R is a perfect ring. □

Remark 2.6. We should point out that, under the assumption that $K = \mathbb{Z}_2$, the module M in Example 2.3 is isomorphic to the module over a matrix algebra considered in [4, Example 5.1], the dual of [4, Example 3.1]. On the other hand, the module M in Example 2.4 is the dual of the module considered in [4, Example 3.2] defined over a matrix algebra.

Remark 2.7. Any module M satisfying the hypotheses of Lemma 2.1 has finite length three and at least two non-isomorphic composition factors. Moreover the modules $I(3) = {}^{1}{}_{3}{}^{2}$ in Example 2.3 and $I(2) = {}^{1}{}_{2}{}^{2}$ in Example 2.4 are the last terms of two Auslander-Reiten sequences, involving all their submodules, of the form

$$0 \longrightarrow 3 \longrightarrow \begin{array}{c}1\\3\end{array} \oplus \begin{array}{c}2\\3\end{array} \longrightarrow \begin{array}{ccc}1 & & 2\\ & 3 & \end{array} \longrightarrow 0$$

and

$$0 \longrightarrow 2 \longrightarrow \begin{array}{c}1\\2\end{array} \oplus \begin{array}{c}2\\2\end{array} \longrightarrow \begin{array}{ccc}1 & & 2\\ & 2 & \end{array} \longrightarrow 0.$$

The middle term is always the projective cover of $I(3)$ and $I(2)$, respectively.

Theorem 2.8. *There exist non injective modules satisfying the hypotheses of Lemma* 2.1 *and their injective dimensions may be either one or infinite.*

Proof. Let R be the hereditary K-algebra given by the following quiver:

$$\begin{array}{ccc}1 & 2 & 4\\ \searrow & \downarrow & \swarrow\\ & 3 & \end{array}$$

Then $\begin{smallmatrix}1&&2\\&3&\end{smallmatrix}$ satisfies the hypotheses of Lemma 2.1, but it is not a direct summand of the R-module $I(3) = \begin{smallmatrix}1&2&4\\&3&\end{smallmatrix}$. Consequently the injective dimension of $\begin{smallmatrix}1&&2\\&3&\end{smallmatrix}$ is equal to one. Similarly, let R be the K-algebra given by the quiver

$$1 \xrightarrow{a} 2 \circlearrowleft b$$

with relations $ba = b^3 = 0$. Then $\begin{smallmatrix}1&&2\\&2&\end{smallmatrix}$ satisfies the hypotheses of Lemma 2.1, but it is not a direct summand of the R-module $I(2) = \begin{smallmatrix}&&2\\1&&2\\&2&\end{smallmatrix}$. It is easy to check that the injective dimension of $\begin{smallmatrix}1&&2\\&2&\end{smallmatrix}$ is infinite. □

Remark 2.9. Let $\begin{smallmatrix}1&&2\\&3&\end{smallmatrix}$ be the module used in the proof of Theorem 2.8. Then the embedding $\begin{smallmatrix}1\\3\end{smallmatrix} \hookrightarrow \begin{smallmatrix}1&&2\\&3&\end{smallmatrix}$ has a factorization of the form $\begin{smallmatrix}1\\3\end{smallmatrix} \hookrightarrow \begin{smallmatrix}1&&2&&4\\&3&&3&\end{smallmatrix} \twoheadrightarrow \begin{smallmatrix}1&&2\\&3&\end{smallmatrix}$. Consequently, $\begin{smallmatrix}1\\3\end{smallmatrix} \hookrightarrow \begin{smallmatrix}1&&2\\&3&\end{smallmatrix}$ is a reducible map.

Theorem 2.10. *There exist a K-algebra R and R-modules M, N_1, N_2 as in Lemma* 2.1 *with the following properties:*

(a) *N_1 and N_2 are projective and $N_i \hookrightarrow M$ and $N_1 \cap N_2 \hookrightarrow N_i$ are irreducible, for $i = 1, 2$.*

(b) *$N_1 \cap N_2$ is projective and* $\mathrm{Hom}_R(M, N_1 \cap N_2) = 0$.

(c) *$N_1 \cap N_2$ has infinite projective dimension and* $\mathrm{Hom}_R(M, N_1 \cap N_2) \cong K$.

(d) *M admits a projective cover (P, p) such that $P \cong N_1 \oplus N_2$ and* $\mathrm{Ker}\, p \cong N_1 \cap N_2$.

Proof. Let M be the R-module $\begin{smallmatrix}1&&2\\&3&\end{smallmatrix}$ of Example 2.3. As already observed, there is an Auslander-Reiten sequence of the form

$$0 \longrightarrow 3 \longrightarrow \begin{smallmatrix}1\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \longrightarrow \begin{smallmatrix}1&&2\\&3&\end{smallmatrix} \longrightarrow 0.$$

Consequently (a), (b) and (d) hold. Next let M be the R-module $\begin{smallmatrix}1&&2\\&2&\end{smallmatrix}$ of Example 2.4. In this case we already know that there is an Auslander-Reiten sequence of the form

$$0 \longrightarrow 2 \longrightarrow \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}1&&2\\&2&\end{smallmatrix} \longrightarrow 0.$$

Since we clearly have $\mathrm{pdim}(2) = \infty$, we conclude that (a), (c) and (d) hold. □

Example 2.11. There exist M, N_1, N_2 as in the hypotheses of Lemma 2.1 with the following properties:

(a) N_1 is projective and N_2 is not projective.

(b) $N_2 \hookrightarrow M$ and $N_1 \cap N_2 \hookrightarrow N_1$ are irreducible maps.

(c) $N_1 \hookrightarrow M$ and $N_1 \cap N_2 \hookrightarrow N_2$ are reducible maps.

Construction. Let R be the hereditary K-algebra given by the following quiver:

$$1 \longrightarrow 3 \longleftarrow 2 \longrightarrow 4.$$

Assume $M = I(3) = \begin{smallmatrix}1&&2\\&3&\end{smallmatrix}$, $N_1 = P(1) = \begin{smallmatrix}1\\3\end{smallmatrix}$ and $N_2 = \begin{smallmatrix}2\\3\end{smallmatrix}$. Then there exist Auslander-Reiten sequences of the form

$$0 \longrightarrow 3 \longrightarrow \begin{smallmatrix}1\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\end{smallmatrix} \longrightarrow \begin{smallmatrix}1&&2&\\&3&&4\end{smallmatrix} \longrightarrow 0$$

and

$$0 \longrightarrow \begin{smallmatrix}&2&\\3&&4\end{smallmatrix} \longrightarrow \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}1&&2&\\&3&&4\end{smallmatrix} \longrightarrow \begin{smallmatrix}1&&2\\&3&\end{smallmatrix} \longrightarrow 0.$$

Hence M, N_1, N_2 satisfy the hypotheses of Lemma 2.1 and (a) and (b) clearly hold. On the other hand the embeddings $\begin{smallmatrix}1\\3\end{smallmatrix} \hookrightarrow \begin{smallmatrix}1&&2\\&3&\end{smallmatrix}$ and $3 \hookrightarrow \begin{smallmatrix}2\\3\end{smallmatrix}$ have a factorization of the form $\begin{smallmatrix}1\\3\end{smallmatrix} \hookrightarrow \begin{smallmatrix}1&&2&\\&3&&4\end{smallmatrix} \twoheadrightarrow \begin{smallmatrix}1&&2\\&3&\end{smallmatrix}$ and $3 \hookrightarrow \begin{smallmatrix}&2&\\3&&4\end{smallmatrix} \twoheadrightarrow \begin{smallmatrix}2\\3\end{smallmatrix}$, respectively. Therefore (c) holds.

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