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Local Well-posedness for Semilinear Heat Equations on H type Groups

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Abstract. In this paper, we discuss the local existence and uniqueness for the Cauchy problem of semi heat equations with an initial data in the space L^q on H type group $\mathbb{H}_p^d$, which has the dimension p of the center, like the argument on the Euclidean space given by F. B. Weissler. That is, the Cauchy problem

$$\begin{cases} \left(\partial_t - \Delta_{\mathbb{H}_p^d}\right) u(g,t) = |u|^{r-1}u, & g \in \mathbb{H}_p^d,\ t > 0, \\ u(g,0) = u_0(g) \in L^q(\mathbb{H}_p^d) \end{cases}$$

has a unique solution if $q > N(r-1)/2$ ($q = N(r-1)/2$) and $q \geq r$ ($q > r$), where $r > 1$ and $N = 2d + 2p$ is the homogeneous dimension of $\mathbb{H}_p^d$.

1. Introduction

For $d = 1, 2, \ldots$, let $\mathbb{H}_p^d$ ($= \mathbb{R}^{2d+p}$) be an H type group (the group of Heisenberg type) with the dimension $p \geq 1$ of center and $\Delta_{\mathbb{H}_p^d}$ be the sublaplacian on $\mathbb{H}_p^d$. H type groups were first introduced by A. Kaplan [6]. In this paper we consider the Cauchy problem of the form

$$\begin{cases} \left(\partial_t - \Delta_{\mathbb{H}_p^d}\right) u(g,t) = |u|^{r-1}u, & g \in \mathbb{H}_p^d,\ t > 0, \\ u(g,0) = u_0(g) \in L^q(\mathbb{H}_p^d). \end{cases} \tag{1.1}$$

On the Euclidean space, there exist enormous investigations of local well-posed for the semi-linear heat equations. In [11], F. B. Weissler gave an existence and nonexistence theorem for local solutions of the Cauchy problem for the semi-linear heat equation with an initial value in $L^q(\mathbb{R}^n)$

$$\begin{cases} (\partial_t - \Delta)\, u(x,t) = |u|^{r-1}u, & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = u_0(x) \in L^q(\mathbb{R}^n). \end{cases} \tag{1.2}$$

After [11], the argument of this direction has been deepened by many mathematicians (for example, [1, 3, 4, 9, 12] and so on).

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In this paper, our goal is to obtain the results of the Cauchy problem (1.1) like those of the Cauchy problem (1.2), following [1,11]. Roughly speaking, our results are as follows: if $q \geq N(r-1)/2$, $N = 2d+2p$ is the homogeneous dimension of $\mathbb{H}_p^d$, then the problem (1.1) is locally well-posedness whenever initial functions $u_0(g) \in L^q(\mathbb{H}_p^d)$. As is standard method, we consider (1.1) via the corresponding integral equation

$$u(t) = e^{t\Delta_{\mathbb{H}_p^d}} u_0 + \int_0^t e^{(t-\sigma)\Delta_{\mathbb{H}_p^d}} (|u(\sigma)|^{r-1} u(\sigma))\, d\sigma.$$

The statements of our results are as follows:

Theorem 1.1. *Let $q > N(r-1)/2$ and $q \geq 1$, $N \geq 4$. Then for any $u_0 \in L^q(\mathbb{H}_p^d)$, there exists a positive T and a solution $u \in C([0,T]; L^q(\mathbb{H}_p^d))$ of (1.1). Moreover there exists a positive constant C, independent of t, such that*

$$\|u(t) - v(t)\|_{L^q(\mathbb{H}_p^d)} \leq C \|u_0 - v_0\|_{L^q(\mathbb{H}_p^d)}$$

for almost all $t \in [0,T]$.

Theorem 1.2. *Let $q = N(r-1)/2$ and $q > 1$, $N \geq 4$. Then for any $u_0 \in L^q(\mathbb{H}_p^d)$, there exists a positive T and a solution $u \in C([0,T]; L^q(\mathbb{H}_p^d))$ of (1.1). Moreover there exists a positive constant C, independent of t and T, such that*

$$\|u(t) - v(t)\|_{L^q(\mathbb{H}_p^d)} \leq C \|u_0 - v_0\|_{L^q(\mathbb{H}_p^d)}$$

for all $t \in [0,T]$.

Uniqueness holds in that class:

Theorem 1.3. *Assume that $q > N(r-1)/2$ (resp. $q = N(r-1)/2$) and $q \geq r$ (resp. $q > r$), $N \geq 4$. Then uniqueness for the solution*

$$u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-\sigma)\Delta} |u(\sigma)|^{r-1} u(\sigma)\, d\sigma$$

holds in the class $C([0,T]; L^q(\mathbb{H}_p^d))$.

If $N = 4$ and $1 < r < 2$ in the assumption of Theorems 1.1 and 1.3, then by these theorems, we see that for any $u_0 \in L^2(\mathbb{H}_1^1)$, $\mathbb{H}_1^1$ $(= \mathbb{R}^3)$ is called Heisenberg group, the solution u of the Cauchy problem

$$\begin{cases} \left(\partial_t - \Delta_{\mathbb{H}_1^1}\right) u(g,t) = |u|^{r-1} u, & g \in \mathbb{H}_1^1,\ t > 0, \\ u(g,0) = u_0(g) \end{cases}$$

is locally wellposed. That is, the solution u of the Cauchy problem

$$\begin{cases} \left(\partial_t - \partial_x^2 - \partial_y^2 - \frac{1}{4}(x^2+y^2)\partial_s^2 - (y\partial_x - x\partial_y)\partial_s\right) u(x,y,s,t) = |u|^{r-1}u, \\ u(x,y,s,0) = u_0(x,y,s) \end{cases}$$

for $(x,y,s) \in \mathbb{H}_1^1$ and $t > 0$ is locally wellposed.

The outline of this paper is as follows. In Section 2, we recall the definition of H type groups. In Section 3, the needed lemmas are given. For example, L^α–L^β estimate, a singular Gronwall lemma and so on. In Sections 4 and 5, we prove local existence theorems and continuous dependence of the cases $q > N(r-1)/2$ (Theorem 1.1) and $q = N(r-1)/2$ (Theorem 1.2), respectively. Finally, in Section 6, we show a uniqueness theorem (Theorem 1.3).

2. H type group

Let $\mathcal{G}$ be a two step nilpotent Lie algebra endowed with an inner product $\langle\,\cdot\,,\cdot\,\rangle$ and we denote by $\mathfrak{z}$ its center. Then $\mathcal{G}$ is said to be of H type if $\mathcal{G}$ satisfies the following two conditions:

1. $[\mathfrak{z}^\perp, \mathfrak{z}^\perp] = \mathfrak{z}$.
2. For any $S \in \mathfrak{z}$, we define the mapping J_S from $\mathfrak{z}^\perp$ to $\mathfrak{z}^\perp$ by $\langle J_S u, w\rangle = \langle S, [u,w]\rangle$, $u, w \in \mathfrak{z}^\perp$. If $|S| = 1$, J_S is an orthogonal mapping.

Let G be a connected and simply connected Lie group. Then G is said to be a group of H type if its Lie algebra $\mathcal{G}$ is of H type. Let $\mathfrak{z}^*$ be the dual of $\mathfrak{z}$. For a given $a\,(\neq 0) \in \mathfrak{z}^*$, a skew symmetric mapping $B(a)$ on $\mathfrak{z}^\perp$ is defined by

$$B(a)(u,w) := a([u,w]), \quad u, w \in \mathfrak{z}^\perp.$$

We denote by z_a an element of $\mathfrak{z}$ determined by

$$B(a)(u,w) = a([u,w]) = \langle J_{z_a} u, w\rangle, \quad u, w \in \mathfrak{z}^\perp.$$

Since $B(a)$ is non-degenerate and a symplectic form, we can see that the dimension of $\mathfrak{z}^\perp = 2d$. For a given $a\,(\neq 0) \in \mathfrak{z}^*$, we can choose an orthonormal basis of $\mathfrak{z}^\perp$

$$\{E_1(a), E_2(a), \dots, E_d(a), \overline{E}_1(a), \overline{E}_2(a), \dots, \overline{E}_d(a)\}$$

such that

$$B(a)E_i(a) = |z_a| J_{z_a/|z_a|} E_i(a) = \varepsilon_i |z_a| \overline{E}_i(a)$$

and

$$B(a)\overline{E}_i(a) = -\varepsilon_i |z_a| E_i(a),$$

where $\varepsilon_i = \pm 1$. Set $p = \dim \mathfrak{z}$. Then we can denote the elements of $\mathcal{G}$ by

$$(z,s) = (x,y,s) = \sum_{i=1}^{d} (x_i E_i + y_i \overline{E}_i) + \sum_{j=1}^{p} s_j \widetilde{E}_j,$$

where $\{\widetilde{E}_1, \ldots, \widetilde{E}_p\}$ is an orthonormal basis such that $a(\widetilde{E}_1) = |a|$, $a(\widetilde{E}_j) = 0$, $(j = 2, 3, \ldots, p)$. We identify the H type Lie algebra $\mathcal{G}$ with the H type Lie group G. Then the group law on H type group has the form

$$(z,s) \circ (z', s') = \left(z + z', s + s' + \frac{1}{2}[z, z'] \right), \tag{2.1}$$

where $[z, z']_j = \langle z, U^j z' \rangle$, $j = 1, 2, \ldots, p$ and U^j satisfies the following conditions:

1. U^j is a $2d \times 2d$ skew symmetric and orthogonal matrix,
2. for any $i, j \in \{1, 2, \ldots, p\}$, $i \neq j$, $U^i U^j + U^j U^i = 0$.

Remark 2.1. H type groups G must satisfy $p + 1 \leq 2d$ (see [7]).

Remark 2.2. If the matrix U^j is skew symmetric (linearly independent), G is called Carnot group.

By the definition, the unit element of H type groups is $(0,0)$ and the inverse element is $(-z, -s)$. Moreover the left invariant vector fields are given by

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} \sum_{k=1}^{p} \left(\sum_{l=1}^{2d} z_l U^k_{l,j} \right) \frac{\partial}{\partial s_k}, \quad Y_j = \frac{\partial}{\partial y_j} + \frac{1}{2} \sum_{k=1}^{p} \left(\sum_{l=1}^{2d} z_l U^k_{l,j+d} \right) \frac{\partial}{\partial s_k},$$

where, $z_l = x_l$, $z_{l+d} = y_l$ $(l = 1, 2, \ldots, d)$ and $U^k_{i,j}$, $U^k_{i,j+d}$ are the (i,j) and $(i, j+d)$ components of the matrix U^k, respectively. We denote by $\mathbb{H}^d_p$ $(= \mathbb{R}^{2d+p})$ H type groups G to emphasize the dimension p of the center.

Example 2.3 (1-dimensional Heisenberg group $\mathbb{H} = \mathbb{H}^1_1$)**.** Let U^1 be a (2×2) skew symmetric matrix defined by

$$U^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

By (2.1), the group law of $\mathbb{H}$ is given by

$$(z,s) \circ (z', s') = \left(z + z', s + s' + \frac{1}{2}(yx' - xy') \right),$$

where, $z = (x, y), z' = (x', y') \in \mathfrak{z}^{\perp}$, $s \in \mathfrak{z}$. Moreover the left invariant vector fields are given by

$$X = \frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial s}, \quad Y = \frac{\partial}{\partial y} - \frac{1}{2}x\frac{\partial}{\partial s}.$$

Example 2.4 (2-dimensional Heisenberg group $\mathbb{H}_1^2$)**.** Let U^1 be a (4×4) skew symmetric matrix defined by

$$U^1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

By (2.1), the group law of $\mathbb{H}^2$ is given by

$$(z, s) \circ (z', s') = \left(z + z', s + s' + \frac{1}{2}(-x_1 y_1' - x_2 y_2' + y_1 x_1' + y_2 x_2') \right),$$

where, $z = (x_1, x_2, y_1, y_2), z' = (x_1', x_2', y_1', y_2') \in \mathfrak{z}^{\perp}$, $s \in \mathfrak{z}$. Moreover the left invariant vector fields are given by

$$X_1 = \frac{\partial}{\partial x_1} + \frac{1}{2}y_1\frac{\partial}{\partial s}, \quad X_2 = \frac{\partial}{\partial x_2} + \frac{1}{2}y_2\frac{\partial}{\partial s}, \quad Y_1 = \frac{\partial}{\partial y_1} - \frac{1}{2}x_1\frac{\partial}{\partial s}, \quad Y_2 = \frac{\partial}{\partial y_2} - \frac{1}{2}x_2\frac{\partial}{\partial s}.$$

Example 2.5 ($\mathbb{H}_2^2$ case)**.** Let U^1 and U^2 be (4×4) skew symmetric matrices defined by

$$U^1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad U^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

By (2.1), the group law of $\mathbb{H}_2^2$ is given by

$$(z, s) \circ (z', s') = \begin{pmatrix} z + z' \\ s_1 + s_1' + \frac{1}{2}(-x_1 x_2' + x_2 x_1' - y_1 y_2' + y_2 y_1') \\ s_2 + s_2' + \frac{1}{2}(x_1 y_1' - x_2 y_2' - x_1' y_1 + y_2 x_2') \end{pmatrix},$$

where, $z = (x_1, x_2, y_1, y_2), z' = (x_1', x_2', y_1', y_2') \in \mathfrak{z}^{\perp}$, $s = (s_1, s_2), s' = (s_1', s_2') \in \mathfrak{z}$. Moreover the left invariant vector fields are given by

$$X_1 = \frac{\partial}{\partial x_1} + \frac{1}{2}\left(x_2 \frac{\partial}{\partial s_1} - y_1 \frac{\partial}{\partial s_2} \right), \qquad X_2 = \frac{\partial}{\partial x_2} + \frac{1}{2}\left(-x_1 \frac{\partial}{\partial s_1} + y_2 \frac{\partial}{\partial s_2} \right),$$

$$Y_1 = \frac{\partial}{\partial y_1} - \frac{1}{2}\left(y_2 \frac{\partial}{\partial s_1} + x_1 \frac{\partial}{\partial s_2} \right), \qquad Y_2 = \frac{\partial}{\partial y_2} - \frac{1}{2}\left(-y_1 \frac{\partial}{\partial s_1} - x_2 \frac{\partial}{\partial s_2} \right).$$

Let $\mathcal{B}_0 = (X_1, \ldots, X_{2d})$ be an orthonomal basis of $\mathfrak{z}$ and $\mathcal{F}_0 = (T_1, \ldots, T_p)$ be an orthonomal basis $\mathfrak{z}$. By using these basis, we identify $\mathfrak{z}^\perp$ with $\mathbb{R}^{2d}$ and $\mathfrak{z}$ with $\mathbb{R}^p$, respectively. The sublaplacian of $\mathbb{H}_p^d$ is denoted by $\Delta_{\mathbb{H}_p^d} = \sum_{i=1}^{2d} X_i^2$. This essentially self adjoint positive operator does not depend on the choice of $\mathcal{B}_0$. Thanks to Hörmander's result, the sublaplacian $\Delta_{\mathbb{H}_p^d}$ is subelliptic. H type groups $\mathbb{H}_p^d$ have a Haar measure. This does not depend on the choice of $\mathcal{B}_0$ and $\mathcal{F}_0$ (see [2]). Let the homogeneous dimension $N = \dim \mathfrak{z}^\perp + 2 \dim \mathfrak{z} = 2d + 2p$.

3. Technical lemmas

We summarize the some lemmas to show our assertion. D. S. Jerison and A. Sánchez-Calle gave the estimate of the heat kernel associated to $\Delta_{\mathbb{H}_p^d}$ as follows:

Lemma 3.1. [5] *Let $h_t(g)$ be the heat kernel associated to $\Delta_{\mathbb{H}_p^d}$. Then there exist positive constants C_1 and $C_{I,l}$ depending Δ such that*

$$\left|\partial_t^l X_I h_t(g)\right| \leq C_{I,l} t^{-l-\frac{|I|}{2}-\frac{N}{2}} e^{-\frac{c_1 \rho(g)^2}{t}},$$

where $I = (i_1, \ldots, i_m)$ with $|I| = m$ and $X_I = X_{i_1} X_{i_2} \cdots X_{i_m}$. Moreover ρ is the Caratheodory distance on H type group.

By Young's inequality and Lemma 3.1, we have the following L^α–L^β estimate.

Lemma 3.2 (L^α–L^β estimate)**.** *Assume $N = 2d + 2p$, $1 \leq \alpha < \beta \leq \infty$ and $\frac{1}{\gamma} = \frac{1}{\alpha} - \frac{1}{\beta}$. Then there exists a positive constant C such that*

$$\left\|e^{t\Delta_{\mathbb{H}_p^d}} \varphi\right\|_{L^\beta(\mathbb{H}_p^d)} \leq C t^{-\frac{N}{2\gamma}} \|\varphi\|_{L^\alpha(\mathbb{H}_p^d)}, \quad t > 0,$$

for any $\varphi \in L^\alpha(\mathbb{H}_p^d)$.

Proof. Let $\varphi \in \mathcal{S}(\mathbb{H}_p^d)$. By Young's inequality and Lemma 3.1, we have

$$\left\|e^{t\Delta_{\mathbb{H}_p^d}} \varphi\right\|_{L^\infty(\mathbb{H}_p^d)} \leq C t^{-\frac{N}{2}} \|\varphi\|_{L^1(\mathbb{H}_p^d)}.$$

On the other hand, by L^2 boundedness of the semigroup $e^{t\Delta}$, we obtain

$$\left\|e^{t\Delta_{\mathbb{H}_p^d}} \varphi\right\|_{L^2(\mathbb{H}_p^d)} \leq C \|\varphi\|_{L^2(\mathbb{H}_p^d)}.$$

By Riesz-Thorin interpolation theorem, we have

$$\left\|e^{t\Delta_{\mathbb{H}_p^d}} \varphi\right\|_{L^\beta(\mathbb{H}_p^d)} \leq C t^{-\frac{N}{2}(\frac{1}{\alpha}-\frac{1}{\beta})} \|\varphi\|_{L^\alpha(\mathbb{H}_p^d)}.$$

Since the space $\mathcal{S}(\mathbb{H}_p^d)$ is dense subset of $L^\alpha(\mathbb{H}_p^d)$, this estimate holds for $\varphi \in L^\alpha(\mathbb{H}_p^d)$. □

We use the following singular Gronwall lemma to show the continuous dependence.

Lemma 3.3. [1] *Let $T > 0$, $A \geq 0$, $0 \leq \alpha, \beta \leq 1$ and let f be a nonnegative function with $f \in L^p(0,T)$ for some $p > 1$ such that $p' \max\{\alpha, \beta\} < 1$. Consider a nonnegative function $\varphi \in L^\infty(0,T)$ such that*

$$\varphi(t) \leq At^{-\alpha} + \int_0^t (t-\sigma)^{-\beta} f(\sigma)\varphi(\sigma)\, d\sigma$$

for almost all $t \in [0,T]$. Then there exists a positive constant C, depending only on T, α, β, p and $\|f\|_{L^p}$, such that

$$\varphi(t) \leq Ct^{-\alpha}$$

for almost all $t \in [0,T]$.

Similarly as Theorem A2 in [1], we have the following lemma.

Lemma 3.4. *Let $N \geq 4$, $T > 0$ and $a \in C([0,T]; L^{N/2}(\mathbb{H}_p^d))$. If $u \in L^\infty((0,T); L^q(\mathbb{H}_p^d))$ with $q > N/(N-2)$ satisfies*

$$u(t) = \int_0^t e^{(t-\sigma)\Delta_{\mathbb{H}_p^d}} a(\sigma)u(\sigma)\, d\sigma$$

for any $t \in [0,T]$, then $u(t) \equiv 0$.

As for the proof, we refer to [1].

4. Proof of Theorem 1.1

Let $T > 0$ and $Y_T = L^\infty((0,T); L^q(\mathbb{H}_p^d)) \cap L^\infty((0,T); L^{qr}(\mathbb{H}_p^d))$ with a norm

$$\|u\|_{Y_T} = \max\left\{ \sup_{0<t<T} \|u(t)\|_{L^q(\mathbb{H}_p^d)},\ \sup_{0<t<T} t^\lambda \|u(t)\|_{L^{qr}(\mathbb{H}_p^d)} \right\}, \quad \lambda = \frac{N(r-1)}{2qr} < 1$$

and

$$B_{M+1} = \{u \mid \|u\|_{Y_T} \leq M+1\}$$

as a subset of Y_T, where $M = \max\{M_1, M_2\}$ such that

$$\|u_0\|_{Y_T} \leq M_1 \quad \text{and} \quad \left\| e^{t\Delta_{\mathbb{H}_p^d}} u_0 \right\|_{Y_T} \ (\leq C\|u_0\|_{Y_T}) \leq M_2.$$

M depends on $\|u_0\|_{Y_T}$, independent of t. Moreover the mapping Φ from B_{M+1} to Y_T is defined by

$$\Phi[u](t) = e^{t\Delta_{\mathbb{H}_p^d}} u_0 + \int_0^t e^{(t-\sigma)\Delta_{\mathbb{H}_p^d}} (|u(\sigma)|^{r-1} u(\sigma))\, d\sigma.$$

At first, we show that Φ is the mapping from B_{M+1} into B_{M+1}. By Lemma 3.2 (L^q–L^m estimate) and $q > N(r-1)/2$, we have that for any $u \in B_{M+1}$,

$$\begin{aligned}
&\left\| \int_0^t e^{(t-\sigma)\Delta_{\mathbb{H}_p^d}} (|u(\sigma)|^{r-1} u(\sigma))\, d\sigma \right\|_{L^m(\mathbb{H}_p^d)} \\
&\le C \int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{m})} \|u(\sigma)\|_{L^{qr}(\mathbb{H}_p^d)}^r \, d\sigma \\
&\le C(M+1)^r \int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{m})} \sigma^{-r\lambda}\, d\sigma \\
&= C(M+1)^r t^{1-r\lambda-\frac{N}{2}(\frac{1}{q}-\frac{1}{m})} \int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{m})} \sigma^{-r\lambda}\, d\sigma
\end{aligned}$$

for $m \ge q$. Since

$$\int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{m})} \sigma^{-r\lambda}\, d\sigma < \infty \quad \text{and} \quad 1 - r\lambda > 0,$$

we see that

$$t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{m})} \left\| \int_0^t e^{(t-\sigma)\Delta_{\mathbb{H}_p^d}} (|u(\sigma)|^{r-1} u(\sigma))\, d\sigma \right\|_{L^m(\mathbb{H}_p^d)} \le C(M+1)^r T^{1-r\lambda}. \tag{4.1}$$

If we take $m = q$ or $m = qr$ in (4.1), then we have

$$\left\| \int_0^t e^{(t-\sigma)\Delta_{\mathbb{H}_p^d}} (|u(\sigma)|^{r-1} u(\sigma))\, d\sigma \right\|_{L^q(\mathbb{H}_p^d)} \le C_1 (M+1)^r T^{1-r\lambda}$$

or

$$t^{\lambda} \left\| \int_0^t e^{(t-\sigma)\Delta_{\mathbb{H}_p^d}} (|u(\sigma)|^{r-1} u(\sigma))\, d\sigma \right\|_{L^{qr}(\mathbb{H}_p^d)} \le C_2 (M+1)^r T^{1-r\lambda}.$$

Hence we obtain

$$\|\Phi[u]\|_{Y_T} \le M + \max\{C_1, C_2\}(M+1)^r T^{1-r\lambda}.$$

For a sufficiently small $T > 0$, we have

$$\max\{C_1, C_2\}(M+1)^r T^{1-r\lambda} \le 1.$$

Therefore we see that Φ is the mapping from B_{M+1} into B_{M+1}.

Next we proceed to proving that the mapping Φ from B_{M+1} to Y_T is the contraction mapping. By the inequality

$$\left\| |u|^{r-1}u - |v|^{r-1}v \right\|_{L^q(\mathbb{H}_p^d)} \le r \left(\|u\|_{L^{qr}(\mathbb{H}_p^d)}^{r-1} + \|v\|_{L^{qr}(\mathbb{H}_p^d)}^{r-1} \right) \|u - v\|_{L^{qr}(\mathbb{H}_p^d)} \tag{4.2}$$

and Lemma 3.2 (L^q–L^m estimate), for $u_1, u_2 \in B_{M+1}$, we obtain that

$$\|\Phi[u_1](t) - \Phi[u_2](t)\|_{L^m(\mathbb{H}_p^d)} \le C_3 (M+1)^{r-1} t^{1-r\lambda-\frac{N}{2}(\frac{1}{q}-\frac{1}{m})} \|u_1 - u_2\|_{Y_T} \tag{4.3}$$

for a constant $C_3 > 0$. If we take $m = q$ or $m = qr$ in (4.3), then we have that

$$\|\Phi[u_1](t) - \Phi[u_2](t)\|_{L^q(\mathbb{H}_p^d)} \leq C_3(M+1)^{r-1}T^{1-r\lambda}\|u_1 - u_2\|_{Y_T}$$

or

$$t^\lambda\|\Phi[u_1](t) - \Phi[u_2](t)\|_{L^{qr}(\mathbb{H}_p^d)} \leq C_3(M+1)^{r-1}T^{1-r\lambda}\|u_1 - u_2\|_{Y_T}.$$

Hence we obtain

$$\|\Phi[u_1](t) - \Phi[u_2](t)\|_{Y_T} \leq C_4(M+1)^{r-1}T^{1-r\lambda}\|u_1 - u_2\|_{Y_T}$$

for a constant $C_4 > 0$. Since $1 - r\lambda > 0$, we have for a sufficient small $T > 0$,

$$C_4(M+1)^{r-1}T^{1-r\lambda} \leq \frac{1}{2}.$$

Therefore we see that the mapping Φ is the contraction mapping for a sufficiently small T. By Banach fixed point theorem, there exists a unique fixed point u of the mapping Φ in B_{M+1}.

Similarly as [1], we see that $u \in C([0,T]; L^q(\mathbb{H}_p^d))$. Indeed by $u \in B_{M+1}$ and $r\lambda < 1$, $|u|^{r-1}u \in L^1((0,T); L^q(\mathbb{H}_p^d))$. This implies $u \in C([0,T]; L^q(\mathbb{H}_p^d))$.

Finally we show the continuous depending on the initial value. Let $u(t)$ and $v(t)$ be solutions of the Cauchy problem (1.1) with $u(0) = u_0$ and $v(0) = v_0$, respectively. By the inequality (4.2) and Lemma 3.2 (L^q–L^{qr} estimate), we obtain

$$\begin{aligned}
&\|u(t) - v(t)\|_{L^{qr}(\mathbb{H}_p^d)} \\
&\leq \left\|e^{t\Delta_{\mathbb{H}_p^d}}(u_0 - v_0)\right\|_{L^{qr}(\mathbb{H}_p^d)} + \int_0^t \left\|e^{(t-\sigma)\Delta_{\mathbb{H}_p^d}}(|u(\sigma)|^{r-1}u(\sigma) - |v(\sigma)|^{r-1}v(\sigma))\right\|_{L^{qr}(\mathbb{H}_p^d)} d\sigma \\
&\leq C_5 t^{-\lambda}\|u_0 - v_0\|_{L^q(\mathbb{H}_p^d)} + C_6(M+1)^r \int_0^t (t-\sigma)^{-\lambda}\sigma^{(r-1)\lambda}\|u(\sigma) - v(\sigma)\|_{L^{qr}(\mathbb{H}_p^d)} d\sigma
\end{aligned}$$

for positive constants C_5 and C_6. By Lemma 3.3 (Gronwall Lemma), we have

$$\|u(t) - v(t)\|_{L^{qr}(\mathbb{H}_p^d)} \leq C_7 t^{-\lambda}\|u_0 - v_0\|_{L^q(\mathbb{H}_p^d)}, \quad \text{a.a. } t \in [0,T].$$

Hence we obtain

$$t^\lambda\|u(t) - v(t)\|_{L^{qr}(\mathbb{H}_p^d)} \leq C_7\|u_0 - v_0\|_{L^q(\mathbb{H}_p^d)}, \quad \text{a.a. } t \in [0,T], \tag{4.4}$$

where a constant $C_7 > 0$ depends on T, q, r and N. By the inequality (4.2) and (4.4), we have

$$\begin{aligned}
&\|u(t) - v(t)\|_{L^q(\mathbb{H}_p^d)} \\
&\leq \left\|e^{t\Delta_{\mathbb{H}_p^d}}(u_0 - v_0)\right\|_{L^q(\mathbb{H}_p^d)} + \int_0^t \left\|e^{(t-\sigma)\Delta_{\mathbb{H}_p^d}}(|u(\sigma)|^{r-1}u(\sigma) - |v(\sigma)|^{r-1}v(\sigma))\right\|_{L^q(\mathbb{H}_p^d)} d\sigma \\
&\leq \|u_0 - v_0\|_{L^q(\mathbb{H}_p^d)} + C_8(M+1)^{r-1} \sup_{0\leq t\leq T} t^\lambda\|u(t) - v(t)\|_{L^{qr}(\mathbb{H}_p^d)} \\
&\leq C_9\|u_0 - v_0\|_{L^q(\mathbb{H}_p^d)}
\end{aligned}$$

for positive constants C_8 and C_9. Therefore we obtain

$$\|u(t) - v(t)\|_{L^q(\mathbb{H}_p^d)} \le C_9 \|u_0 - v_0\|_{L^q(\mathbb{H}_p^d)}, \quad \text{a.a. } t \in [0,T].$$

This completes the proof of Theorem 1.1.

Remark 4.1. We can show $u \in C^1((0,T]; L^q(\mathbb{H}_p^d))$. Let $\nu(t) = u(t+\varepsilon)$ on the interval $(0, T-\varepsilon]$ for any $\varepsilon > 0$. Then we have

$$\nu(t) = e^{t\Delta_{\mathbb{H}_p^d}} u(\varepsilon) + \int_0^t e^{(t-\sigma)\Delta} |\nu(\sigma)|^{r-1} \nu(\sigma)\, d\sigma.$$

By the inequality (4.2), we can see $\sigma \mapsto |\nu(\sigma)|^{r-1}\nu(\sigma)$ is Hölder continuous from $[\varepsilon, T-\varepsilon]$ to $L^q(\mathbb{H}_p^d)$. If $\nu_1(t) = u(t+2\varepsilon)$, then $\sigma \mapsto |\nu_1(\sigma)|^{r-1}\nu_1(\sigma)$ is Hölder continuous from $[0, T-2\varepsilon]$ to $L^q(\mathbb{H}_p^d)$. By Theorem 1.27 in [8], ν_1 is continuously differentiable for $t > 0$ and satisfies $\nu_1'(t) = \Delta_{\mathbb{H}_p^d}\nu_1 + |\nu_1(t)|^{r-1}\nu_1(t)$. Since $\varepsilon > 0$ is arbitrary, $\nu \in C^1((0,T]; L^q(\mathbb{H}_p^d))$.

5. Proof of Theorem 1.2

Fix any θ such that $q < \theta < qr$, $\theta \ge r$ and set

$$\widetilde{E}_T = L^\infty((0,T); L^q(\mathbb{H}_p^d)) \cap \{u \in L^\infty_{\mathrm{loc}}((0,T); L^\theta(\mathbb{H}_p^d)), t^\alpha u \in L^\infty((0,T); L^\theta(\mathbb{H}_p^d))\}$$

and

$$E_T = L^\infty((0,T); L^q(\mathbb{H}_p^d)) \cap \{u \in L^\infty_{\mathrm{loc}}((0,T); L^\theta(\mathbb{H}_p^d)), t^\alpha u \in C_0([0,T]; L^\theta(\mathbb{H}_p^d))\},$$

where $\alpha = \frac{N}{2}\left(\frac{1}{q} - \frac{1}{\theta}\right) < 1$ and C_0 means the set of functions which vanish at $t = 0$ following [1]. Fix $M = \max\{M_1, M_2\}$ such that

$$\|u_0\|_{L^q(\mathbb{H}_p^d)} \le M_1 \quad \text{and} \quad \left\| e^{t\Delta_{\mathbb{H}_p^d}} u_0 \right\|_{L^q(\mathbb{H}_p^d)} \le M_2.$$

M is independent of t. For some $\delta > 0$ to be chosen later, let

$$\widetilde{K}_T = \{u \in \widetilde{E}_T \mid \|u(t)\|_{L^q(\mathbb{H}_p^d)} \le M+1 \text{ and } t^\alpha \|u(t)\|_{L^\theta(\mathbb{H}_p^d)} \le \delta\}$$

for $t \in (0,T)$ and

$$K_T = \widetilde{K}_T \cap E_T$$

with a norm

$$\|u\|_{\widetilde{K}_T} = \sup_{0<t<T} t^\alpha \|u(t)\|_{L^\theta(\mathbb{H}_p^d)}.$$

Let the mapping Φ be defined in Section 4 and

$$c = \frac{N}{2}\left(\frac{r}{\theta} - \frac{1}{q}\right).$$

For $u \in \widetilde{K}_T$, by Lemma 3.2 ($L^{\theta/r}$–L^q estimate), we have

$$\begin{aligned}
\|\Phi[u](t)\|_{L^q(\mathbb{H}_p^d)} &\le \left\| e^{t\Delta_{\mathbb{H}_p^d}} u_0 \right\|_{L^q(\mathbb{H}_p^d)} + \int_0^t \left\| e^{(t-\sigma)\Delta_{\mathbb{H}_p^d}} (|u(\sigma)|^{r-1} u(\sigma)) \right\|_{L^q(\mathbb{H}_p^d)} d\sigma \\
&\le C\|u_0\|_{L^q(\mathbb{H}_p^d)} + \int_0^t (t-\sigma)^{-c} \|u(\sigma)\|_{L^\theta(\mathbb{H}_p^d)}^r \, d\sigma \\
&\le C\|u_0\|_{L^q(\mathbb{H}_p^d)} + \left(\sup_{0<t<T} t^\alpha \|u(t)\|_{L^\theta(\mathbb{H}_p^d)} \right)^r \int_0^t (t-\sigma)^{-c} \sigma^{-r\alpha} \, d\sigma \\
&\le C\|u_0\|_{L^q(\mathbb{H}_p^d)} + C_1 \delta^r,
\end{aligned}$$

where C_1 depends only on N, q, θ and r. Therefore we obtain

$$\|\Phi[u](t)\|_{L^q(\mathbb{H}_p^d)} \le M + 1$$

provided

$$C_1 \delta^r \le 1. \tag{5.1}$$

On the other hand, by Lemma 3.2 ($L^{\theta/r}$–L^θ estimate), we have

$$\begin{aligned}
& t^\alpha \|\Phi[u](t)\|_{L^\theta(\mathbb{H}_p^d)} \\
&\le \sup_{0<t<T} t^\alpha \left\| e^{t\Delta_{\mathbb{H}_p^d}} u_0 \right\|_{L^\theta(\mathbb{H}_p^d)} + t^\alpha \int_0^t (t-\sigma)^{-\frac{N(r-1)}{2\theta}} \|u(\sigma)\|_{L^\theta(\mathbb{H}_p^d)}^r \, d\sigma \\
&\le \sup_{0<t<T} t^\alpha \left\| e^{t\Delta_{\mathbb{H}_p^d}} u_0 \right\|_{L^\theta(\mathbb{H}_p^d)} \\
&\quad + \left(\sup_{0<t<T} t^\alpha \|u(t)\|_{L^\theta(\mathbb{H}_p^d)} \right)^r t^\alpha \int_0^t (t-\sigma)^{-\frac{N(r-1)}{2\theta}} \sigma^{-r\alpha} \, d\sigma \\
&\le \sup_{0<t<T} t^\alpha \left\| e^{t\Delta_{\mathbb{H}_p^d}} u_0 \right\|_{L^\theta(\mathbb{H}_p^d)} + C_2 \delta^r,
\end{aligned} \tag{5.2}$$

where C_2 also depends only on N, q, θ and r. Therefore we obtain

$$\sup_{0<t<T} t^\alpha \|\Phi[u](t)\|_{L^\theta(\mathbb{H}_p^d)} \le \sup_{0<t<T} t^\alpha \left\| e^{t\Delta_{\mathbb{H}_p^d}} u_0 \right\|_{L^\theta(\mathbb{H}_p^d)} + \frac{\delta}{2}$$

provided

$$C_2 \delta^{r-1} \le \frac{1}{2}. \tag{5.3}$$

Similarly, we have for $u, v \in \widetilde{K}_T$,

$$\begin{aligned}
\sup_{0<t<T} t^\alpha \|\Phi[u](t) - \Phi[v](t)\|_{L^\theta(\mathbb{H}_p^d)} &\le C_3 \delta^{r-1} \sup_{0<t<T} t^\alpha \|u(t) - v(t)\|_{L^\theta(\mathbb{H}_p^d)} \\
&\le \frac{1}{2} \sup_{0<t<T} t^\alpha \|u(t) - v(t)\|_{L^\theta(\mathbb{H}_p^d)}
\end{aligned} \tag{5.4}$$

provided

$$C_3\delta^{r-1} \le \frac{1}{2}, \tag{5.5}$$

where C_3 depends only on N, q, θ and r. Therefore the mapping $\Phi\colon \widetilde{K}_T \to \widetilde{E}_T$ follows from the above estimates. We fix any $\delta > 0$ sufficiently small such that (5.1), (5.3) and (5.5) are satisfied. Furthermore, we fix $T > 0$ so that

$$\sup_{0<t<T} t^{\alpha} \left\| e^{t\Delta_{\mathbb{H}_p^d}} u_0 \right\|_{L^{\theta}(\mathbb{H}_p^d)} \le \frac{\delta}{2}. \tag{5.6}$$

T can be chosen independent of δ by the same reason as Lemma 8 in [1]. By (5.2), (5.4) and (5.6), we see that $\Phi\colon \widetilde{K}_T \to \widetilde{K}_T$ is a strict contraction. Hence Φ has a unique fixed point in $\widetilde{K}_T$.

Next we show that this fixed point belongs to K_T. It is sufficient to show that $\Phi\colon K_T \to K_T$. For this purpose, we check that $\Phi[u] \in C((0,T];L^{\theta}(\mathbb{H}_p^d))$ and $\lim_{t\to 0} t^{\alpha}\Phi[u](t) = 0$ in $L^{\theta}(\mathbb{H}_p^d)$. Similarly as the proof of Lemma 2.1 in [10], we see that $\Phi[u] \in C((0,T];L^{\theta}(\mathbb{H}_p^d))$. On the other hand, by Lemma 3.2 ($L^{\theta/r}$–L^{θ} estimate), we have that

$$\begin{aligned}
&t^{\alpha}\|\Phi[u](t)\|_{L^{\theta}(\mathbb{H}_p^d)} \\
&\le t^{\alpha}\left\| e^{t\Delta_{\mathbb{H}_p^d}} u_0 \right\|_{L^{\theta}(\mathbb{H}_p^d)} + \left(\sup_{0<t<T} t^{\alpha}\|u(t)\|_{L^{\theta}(\mathbb{H}_p^d)}\right)^r t^{\alpha}\int_0^t (t-\sigma)^{-\frac{N(r-1)}{2\theta}}\sigma^{-r\alpha}\,d\sigma \\
&\le t^{\alpha}\left\| e^{t\Delta_{\mathbb{H}_p^d}} u_0 \right\|_{L^{\theta}(\mathbb{H}_p^d)} + \left(\sup_{0<t<T} t^{\alpha}\|u(t)\|_{L^{\theta}(\mathbb{H}_p^d)}\right)^r \int_0^1 (1-\sigma)^{-\frac{N(r-1)}{2\theta}}\sigma^{-r\alpha}\,d\sigma \\
&\to 0
\end{aligned}$$

as $t \to 0$, since $u \in E_T$ and $r\alpha + N(r-1)/(2\theta) = \alpha + 1$. Let any interval $[a,b] \subset (0,T)$ and $t \in [a,b]$. Then we obtain

$$\begin{aligned}
&\sup_{a\le t\le b}\|\Phi[u](t)\|_{L^{\theta}(\mathbb{H}_p^d)} \\
&\le \left\| e^{t\Delta_{\mathbb{H}_p^d}} u_0 \right\|_{L^{\theta}(\mathbb{H}_p^d)} + \left(\sup_{a\le t\le b}\|u(t)\|_{L^{\theta}(\mathbb{H}_p^d)}\right)^r \int_0^t (t-\sigma)^{-\frac{N(r-1)}{2\theta}}\sigma^{-r\alpha}\,d\sigma \\
&\le C\|u_0\|_{L^{\theta}(\mathbb{H}_p^d)} + a^{-\alpha}\left(\sup_{a\le t\le b}\|u(t)\|_{L^{\theta}(\mathbb{H}_p^d)}\right)^r \int_0^1 (1-\sigma)^{-\frac{N(r-1)}{2\theta}}\sigma^{-r\alpha}\,d\sigma.
\end{aligned}$$

Hence we also see that $\Phi[u] \in L^{\infty}_{\mathrm{loc}}((0,T);L^{\theta}(\mathbb{H}_p^d))$. Next we show that $u \in C([0,T],L^q(\mathbb{H}_p^d))$. It is sufficient to show that

$$\lim_{t\to 0}\left\|\int_0^t e^{(t-\sigma)\Delta_{\mathbb{H}_p^d}}|u(\sigma)|^{r-1}u(\sigma)\right\|_{L^q(\mathbb{H}_p^d)} = 0.$$

Indeed, by Lemma 3.2 ($L^{\theta/r}$–L^q estimate), we have

$$\left\| \int_0^t e^{(t-\sigma)\Delta_{\mathbb{H}_p^d}} |u(\sigma)|^{r-1} u(\sigma) \right\|_{L^q(\mathbb{H}_p^d)} \le \int_0^t (t-\sigma)^{-a} \|u(\sigma)\|_{L^\theta}^r \, d\sigma$$
$$\le C_4 \left(\sup_{0<t<T} t^\alpha \|u(t)\|_{L^\theta(\mathbb{H}_p^d)} \right)^r \to 0$$

as $t \to 0$, since $u \in E_T$.

Finally we show the continuous depending on the initial value. Let $u(t)$ and $v(t)$ be solutions of the Cauchy problem (1.1) with $u(0) = u_0$ and $v(0) = v_0$, respectively. Similarly as the argument of (5.2), we have

$$\sup_{0\le t\le T} t^\alpha \|u(t) - v(t)\|_{L^\theta(\mathbb{H}_p^d)}$$
$$\le \sup_{0\le t\le T} t^\alpha \left\| e^{t\Delta_{\mathbb{H}_p^d}} (u_0 - v_0) \right\|_{L^\theta(\mathbb{H}_p^d)} + \frac{1}{2} \sup_{0\le t\le T} t^\alpha \|u(t) - v(t)\|_{L^\theta(\mathbb{H}_p^d)}.$$

By this, we obtain

$$\sup_{0\le t\le T} t^\alpha \|u(t) - v(t)\|_{L^\theta(\mathbb{H}_p^d)} \le 2\|u_0 - v_0\|_{L^q(\mathbb{H}_p^d)}. \tag{5.7}$$

On the other hand, by (4.2), (5.7) and Lemma 3.2 ($L^{\theta/r}$–L^q estimate), we have

$$\|u(t) - v(t)\|_{L^q(\mathbb{H}_p^d)}$$
$$\le \left\| e^{t\Delta_{\mathbb{H}_p^d}} (u_0 - v_0) \right\|_{L^q(\mathbb{H}_p^d)} + \int_0^t \left\| e^{(t-\sigma)\Delta_{\mathbb{H}_p^d}} (|u(\sigma)|^{r-1} u(\sigma) - |v(\sigma)|^{r-1} v(\sigma)) \right\|_{L^q(\mathbb{H}_p^d)} d\sigma$$
$$\le \|u_0 - v_0\|_{L^q(\mathbb{H}_p^d)} + C_5 \int_0^t (t-\sigma)^{-c} \left\| |u(\sigma)|^{r-1} u(\sigma) - |v(\sigma)|^{r-1} v(\sigma) \right\|_{L^{\theta/r}(\mathbb{H}_p^d)} d\sigma$$
$$\le \|u_0 - v_0\|_{L^q(\mathbb{H}_p^d)} + C_6 \int_0^t (t-\sigma)^{-c} (\|u(\sigma)\|_{L^\theta(\mathbb{H}_p^d)}^{r-1} + \|v(\sigma)\|_{L^\theta(\mathbb{H}_p^d)}^{r-1}) \|u(\sigma) - v(\sigma)\|_{L^\theta(\mathbb{H}_p^d)} d\sigma$$
$$\le \|u_0 - v_0\|_{L^q(\mathbb{H}_p^d)} + C_7 \delta^{r-1} \sup_{0\le t\le T} t^\alpha \|u(t) - v(t)\|_{L^\theta(\mathbb{H}_p^d)}$$
$$\le C_8 \|u_0 - v_0\|_{L^q(\mathbb{H}_p^d)}$$

for positive constants C_5, C_6, C_7 and C_8. C_8 is independent of T. Therefore we obtain

$$\|u(t) - v(t)\|_{L^q(\mathbb{H}_p^d)} \le C_8 \|u_0 - v_0\|_{L^q(\mathbb{H}_p^d)}, \quad t \in [0, T].$$

This completes the proof of Theorem 1.2.

6. Proof of Theorem 1.3

We consider separately two cases: Case (i): $q > N(r-1)/2$ and $q \ge r$, Case (ii): $q = N(r-1)/2$ and $q > r$.

Case (i): Let $u, v \in C([0,T]; L^q(\mathbb{H}_p^d))$ be two solutions. Then we have

$$u(t) - v(t) = \int_0^t e^{(t-\sigma)\Delta}(|u(\sigma)|^{r-1}u(\sigma) - |v(\sigma)|^{r-1}v(\sigma))\, d\sigma.$$

By Lemma 3.2 ($L^{q/r}$–L^q estimate) and by the inequality

$$\left\| |u|^{r-1}u - |v|^{r-1}v \right\|_{L^{\frac{q}{r}}(\mathbb{H}_p^d)} \leq r \left(\|u\|^{r-1}_{L^q(\mathbb{H}_p^d)} + \|v\|^{r-1}_{L^q(\mathbb{H}_p^d)} \right) \|u - v\|_{L^q(\mathbb{H}_p^d)},$$

we obtain that

$$\begin{aligned} \|u(t) - v(t)\|_{L^q(\mathbb{H}_p^d)} &\leq C \int_0^t (t-\sigma)^{-\theta} \left\| |u|^{r-1}u - |v|^{r-1}v \right\|_{L^{\frac{q}{r}}(\mathbb{H}_p^d)} d\sigma \\ &\leq C' \int_0^t (t-\sigma)^{-\theta} \left(\|u\|^{r-1}_{L^q(\mathbb{H}_p^d)} + \|v\|^{r-1}_{L^q(\mathbb{H}_p^d)} \right) \|u - v\|_{L^q(\mathbb{H}_p^d)}\, d\sigma, \end{aligned} \tag{6.1}$$

where $\theta = N(r-1)/(2q) < 1$ and positive constants C, C' are independent of t.

Let $M = \sup_{0 \leq t \leq T} \left(\|u\|_{L^q(\mathbb{H}_p^d)} + \|v\|_{L^q(\mathbb{H}_p^d)} \right)$ and $\psi(t) = \sup_{0 \leq \sigma \leq t} \|u(t) - v(t)\|_{L^q(\mathbb{H}_p^d)}$ for $t \in [0,T]$. By the estimate (6.1), we deduce that

$$\psi(t) \leq CM^{r-1} \frac{T^{1-\theta}}{1-\theta} \psi(t). \tag{6.2}$$

Let T' be sufficiently small such that $0 < T' < T$ and let $t \in [0,T']$. Then by (6.2), we can see $\psi(t) = 0$. Finitely repeating the same argument, we can see that $\psi(t) = 0$ for $t \in [0,T]$.

Cases (ii): Let $q = N(r-1)/2 > r$ and $N \geq 4$. Let u, v be two solutions and let $w = u - v$. We put

$$a(g,t) = \begin{cases} \frac{|u|^{r-1}u - |v|^{r-1}v}{u-v} & \text{if } u \neq v, \\ r|u|^{r-1} & \text{if } u = v \end{cases}$$

so that

$$w(t) = \int_0^t e^{(t-\sigma)\Delta} a(\sigma) w(\sigma)\, d\sigma.$$

By the same argument as in [1], we can see that $a \in C([0,T] : L^{N/2}(\mathbb{H}_p^d))$. By Lemma 3.4, we see that $w \equiv 0$. Note that $q > N/(N-2)$.

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