Inverses and Determinants of Toeplitz-Hessenberg Matrices

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Abstract. The inverses of Toeplitz-Hessenberg matrices are investigated. It is known that each inverse of such a matrix is a sum of a lower triangular matrix L and a matrix R of rank 1. The formulas of L and x, y such that $xy^T = R$ are derived. Using this result we propose an algorithm for inverting Toeplitz-Hessenberg matrices. Moreover, from the expression of the inverse a formula for the determinant is deduced.

1. Introduction

The Hessenberg (or lower Hessenberg) matrices are the matrices $H = [h_{ij}]$ satisfying condition $h_{ij} = 0$ for $j - i > 1$. More general, the matrix is said to be k-Hessenberg if and only if $H_{ij} = 0$ for $j - i > k$.

Asplund [3] was probably the first who discovered that a nonsingular matrix is strictly 1-Hessenberg if and only if its inverse is a sum of a matrix of rank one and a lower triangular matrix with zeros in the main diagonal. Analogous theorem is true for k-Hessenberg matrices. It is interesting that even more can be said: the right upper block of the inverse of k-Hessenberg matrix can be written as a product of $n \times k$ and $k \times n$ matrices [10]. Moreover, it turned out that also inverses of $\mathbb{N} \times \mathbb{N}$ Hessenberg matrices (on the condition they exist) can also be written as such sums [2]. This property of inverses was discussed most intensively for the tridiagonal matrices [4–6]. Clearly, such representation of inverses can result in proposing algorithms for finding its entries [12].

In this paper we would like to continue the discussion on this subject and assuming that $H^{-1} = L - \frac{1}{\gamma} xy^T$ (where L is lower triangular and x, y are vectors) present the formulas for L, x and y. Clearly, when the entries of H are arbitrarily chosen it is very complicated. Therefore, we limit ourselves to the class of Toeplitz-Hessenberg matrices,

i.e., matrices of the form

(1.1) $H = H_n = \begin{bmatrix} a_0 & a_{-1} & & & & \\ a_1 & a_0 & a_{-1} & & & \\ a_2 & a_1 & a_0 & a_{-1} & & \\ \vdots & & & & \ddots & \\ a_{n-2} & a_{n-3} & \cdots\cdots\cdots & & a_0 & a_{-1} \\ a_{n-1} & a_{n-2} & \cdots\cdots\cdots & & a_1 & a_0 \end{bmatrix}_{n\times n}$.

We will assume that H is strictly Hessenberg, i.e., $a_{-1} \neq 0$. The first result we are going to prove in this paper is the following.

Theorem 1.1. *Let H be an invertible matrix of form (1.1), where $a_{-1} \neq 0$. Then H is invertible if and only if*

$$\gamma = a_{n-2} - a_{n-1} - \sum_{i=2}^{n-1} a_{n-i-1}\left(\sum_{j=1}^{i-1} a_{j-1}b_{i-j-1}\right) \neq 0$$

and in this case

$$H^{-1} = \begin{bmatrix} 0 & 0 \\ \widetilde{A} & 0 \end{bmatrix} - \frac{1}{\gamma}x_H y_H^T,$$

where

$$\widetilde{A}_{ij} = \begin{cases} b_{i-j} & \text{if } i \geq j, \\ 0 & \text{otherwise,} \end{cases} \quad \text{with } b_0 = 1,\ b_k = -\frac{1}{a_{-1}}\sum_{r=0}^{k-1} a_r b_{k-r-1} \text{ for } k \geq 1,$$

$$(x_H)_i = \begin{cases} 1 & \text{if } i = 1, \\ -\sum_{j=1}^{i-1} a_{j-1}b_{i-j-1} & \text{if } 1 < i \leq n, \end{cases}$$

$$(y_H)_i = (x_H)_{n+1-i} = \begin{cases} -\sum_{j=i}^{n-1} a_{n-j-1}b_{j-1} & \text{if } 1 \leq i < n, \\ 1 & \text{if } i = n. \end{cases}$$

Based on this theorem we propose an algorithm for inverting Toeplitz-Hessenberg matrices. Moreover, we will present one more conclusion following from Theorem 1.1. It is natural that in some formulas and algorithms for finding inverses one may sometimes use submatrices and subminors. For instance in [1,7] there are formulas for the entries of H^{-1} that involve the minors of H. In Section 3 we will present some other formula for the inverse of a Toeplitz-Hessenberg matrix and comparing it with Theorem 1.1 we will obtain some formula for $\det(H_n)$.

2. Inverses

Before we start, let's introduce some notation. By H_n or H we will mean a matrix of form (1.1). The symbol $\mathcal{M}_{n\times m}(F)$ will stand for the set of $n \times m$ matrices over a (commutative) field F, and we will abbreviate $\mathcal{M}_{n\times n}(F)$ to $\mathcal{M}_n(F)$. I_n will denote the $n \times n$ identity matrix, whereas $0_{n\times m}$ the $n \times m$ zero matrix. Moreover, let E_{km} be a matrix with 1 in the position (k,m) and 0 in every other position (in the case of using this symbol the dimension of E_{km} will follow from the context). We will also abbreviate $(x_H)_i$ $((y_H)_i$ respectively) to $x_{H,i}$.

To prove Theorem 1.1 we are going to use the result from [11] where more precise expressions for L, x and y are presented. Namely, we need the following theorem.

Theorem 2.1. [11, Theorem 3.1] *Let H be a strict k-Hessenberg matrix with the block decomposition*

$$H = \begin{bmatrix} B & A \\ D & C \end{bmatrix}, \quad A \in \mathcal{M}_{n-k}(F), \ B \in \mathcal{M}_{(n-k)\times k}(F), \ C \in \mathcal{M}_{k\times(n-k)}(F), \ D \in \mathcal{M}_k(F).$$

Then H is invertible if and only if $CA^{-1}B - D$ is invertible and if H is invertible we have

$$(2.1) \qquad H^{-1} = \begin{bmatrix} 0 & 0 \\ A^{-1} & 0 \end{bmatrix} - \begin{bmatrix} I_k \\ -A^{-1}B \end{bmatrix} (CA^{-1}B - D)^{-1} \begin{bmatrix} -CA^{-1} & I_k \end{bmatrix}.$$

For $k = 1$ the blocks A, B, C, D are

$$(2.2) \qquad A = A_n = \begin{bmatrix} a_{-1} & & & & & \\ a_0 & a_{-1} & & & & \\ a_1 & a_0 & a_{-1} & & & \\ \vdots & & \ddots & \ddots & & \\ a_{n-4} & a_{n-5} & \cdots & a_0 & a_{-1} & \\ a_{n-3} & a_{n-4} & \cdots & a_1 & a_0 & a_{-1} \end{bmatrix}_{(n-1)\times(n-1)},$$

$$B = B_n = [a_0 a_1 \ldots a_{n-2}]^T, \quad C = C_n = [a_{n-2} a_{n-3} \ldots a_0], \quad D = D_n = [a_{n-1}].$$

Note that $(B^T)_i = C_{n-i-1}$. This fact is going to be useful later on. Formula (2.1) can be simplified to

$$H^{-1} = \begin{bmatrix} 0 & 0 \\ A^{-1} & 0 \end{bmatrix} - \frac{1}{CA^{-1}B - D} \begin{bmatrix} 1 \\ -A^{-1}B \end{bmatrix} \begin{bmatrix} -CA^{-1} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ A^{-1} & 0 \end{bmatrix} - \frac{1}{CA^{-1}B - D} \begin{bmatrix} -CA^{-1} & 1 \\ -A^{-1}BCA^{-1} & -A^{-1}B \end{bmatrix}.$$

First let's focus on determining A^{-1}. For any $1 \leq k \leq n-1$, let $C_{n-1}^{(k)}(A)$ denote the matrix $I_{n-1} + \sum_{i=k+1}^{n-1} \frac{a_{i-k-1}}{a_{-1}} E_{ik}$. Additionally, let $C_{n-1}^{(0)}(A)$ denote I_{n-1}. We can observe the following.

Remark 2.2. Let A be given as in (2.2). Then $A = a_{-1} \prod_{k=1}^{n-2} C_{n-1}^{(k)}(A)$.

The decomposition from Remark 2.2 can be used to find the inverse of A. First we find the inverses of $C_{n-1}^{(k)}(A)$. Namely, one can check that they are given in

Remark 2.3. Let $n, k \in \mathbb{N}$, $n \geq 3$, $\alpha \in F \setminus \{0\}$, and let A be given as in (2.2). Then

$$[C_{n-1}^{(k)}(A)]^{-1} = I_{n-1} - \sum_{r=k+1}^{n-1} a_{i-k-1} E_{ik}.$$

From Remarks 2.2 and 2.3 we get now the form of A^{-1}.

Lemma 2.4. *Let A be given as in (2.2) with $n > 3$. Then*

$$(2.3) \qquad A^{-1} = \frac{1}{a_{-1}} \sum_{k=0}^{n-2} \sum_{r=1}^{n-1-k} b_k E_{r+k,r},$$

where $b_0 = 1$ and $b_k = -\frac{1}{a_{-1}} \sum_{r=0}^{k-1} a_r b_{k-r-1}$ for $k \geq 1$.

Proof. Since A is a triangular Toeplitz matrix, so is A^{-1}. We prove (2.3) inductively on n. We have

$$\begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & -\frac{a_0}{a_{-1}} & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -\frac{a_0}{a_{-1}} & 1 & \\ -\frac{a_1}{a_{-1}} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -b_1 & 1 & \\ b_2 & b_1 & 1 \end{bmatrix}$$

with

$$b_1 = -\frac{a_0}{a_{-1}}, \quad b_2 = \left(\frac{a_0}{a_{-1}}\right)^2 - \frac{a_1}{a_{-1}} = -\frac{1}{a_{-1}}(a_0 b_1 + a_1 b_0).$$

The first step of induction holds.

Consider now $n > 4$. We have

$$\begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & b_1 & 1 & & \\ \vdots & \vdots & & \ddots & \\ 0 & b_{n-3} & \cdots & b_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & \\ -\frac{a_0}{a_{-1}} & 1 & & & \\ -\frac{a_1}{a_{-1}} & & 1 & & \\ \vdots & & & \ddots & \\ -\frac{a_{n-3}}{a_{-1}} & & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ b_1 & 1 & & & \\ b_2 & b_1 & 1 & & \\ \vdots & \vdots & & \ddots & \\ b_{n-2} & b_{n-3} & \cdots & b_1 & 1 \end{bmatrix},$$

$$b_{n-2} = -\frac{a_0}{a_{-1}} b_{n-3} - \frac{a_1}{a_{-1}} b_{n-4} - \cdots - \frac{a_{n-4}}{a_{-1}} b_1 - \frac{a_{n-3}}{a_{-1}} b_0. \qquad \square$$

As it was mentioned before, from Theorem 1.1 it follows from the fact that H^{-1} can be represented as $L - \frac{1}{\gamma}xy^T$. Clearly, by Theorem 2.1, in our case x, y are of the forms:

$$x_H = \begin{bmatrix} 1 \\ -A^{-1}B \end{bmatrix}, \quad y_H = \begin{bmatrix} -CA^{-1} & 1 \end{bmatrix}.$$

Thus, we can give the following proof.

Proof of Theorem 1.1. Let $2 \le i \le n$. Then

$$x_{H,i} = (-A^{-1}B)_{i-1} = -\sum_{j=1}^{i-1}(A^{-1})_{i-1,j} \cdot B_j = -\sum_{j=1}^{i-1}b_{i-1-j}a_{j-1}.$$

To find $(CA^{-1})_i$ for $1 \le i \le n-1$ we can either perform the same calculations for $A^{-1}B$ or make use of the following two facts:

(1) $B_i^T = C_{n+1-i}$,

(2) the i-th entry of the k-th column of A^{-1} is equal to the $(n-1-i)$-th entry of $(n-1-i)$-th row of A^{-1}.

From the first of these two ways we get the formula for y_i that appears in Theorem 1.1. From the second one we obtain the equality $y_i = x_{n+1-i}$.

We finish the proof by determining $CA^{-1}B - D$:

$$CA^{-1}B - D = \sum_{i=2}^{n}C_i(A^{-1}B)_i - D$$

$$= a_{n-2} - \sum_{i=1}^{n-1}\left[a_{n-i-1} \cdot \left(\sum_{j=1}^{i-1} a_{j-1}b_{i-j-1} \right) \right] - a_{n-1}. \qquad \square$$

From Theorem 1.1 we derive the following algorithm for calculating H^{-1}.

Algorithm 2.5. (1) *let* $H = [0]_{n \times n}$;

(2) *for* $k = 2, 3, \ldots, n$ *put* $H'_{kk} = \frac{1}{a_{-1}}$;

(3) *put* $b_0 = 1$,

for $k = 1, 2, \ldots, n-2$

$b_k = 0$,

for $r = 0, 1, \ldots, k-1$

$\quad b_k = b_k + a_r b_{k-r-1}$,

$\quad b_k = -\frac{1}{a_{-1}}b_k$,

for $r = 0, 1, \ldots, k-1$

$\quad H'_{k+r+2,k+1} = b_{r+1}$,

(4) $x_1 = 1$,

for $i = 2, 3, \ldots, n-1$

$\quad x_i = 0$,

$$for \ j = 1, 2, \ldots, i - 1$$

$$x_i = x_i - a_{j-1}b_{i-j-1},$$

(5) $\gamma = 0$,

$$for \ i = 2, 3, \ldots, n - 1$$

$$\gamma = \gamma + a_{n-i-1}x_i,$$

$$\gamma = a_{n-2} - a_{n-1} - \gamma,$$

(6) $for \ i = 1, 2, \ldots, n$

$$for \ j = 1, 2, \ldots, n$$

$$H'_{ij} = H'_{ij} - \tfrac{1}{\gamma}x_i x_{n+1-j}.$$

3. Determinants of Toeplitz-Hessenberg matrices

It is known that the determinant of H_n can be evaluated using the Trudi formula (see [8,9]). Namely,

$$(3.1) \quad \det H_n = \sum_{k_1+2k_2+\cdots+nk_n=n} \binom{k_1 + k_2 + \cdots + k_n}{k_1, k_2, \ldots, k_n}(-a_{-1})^{n-k_1-\cdots-k_n}a_0^{k_1}a_1^{k_2}\cdots a_{n-1}^{k_n}.$$

In the present section we will obtain some other formula for $\det(H_n)$. As it contains some multiple sums, in comparison to (3.1) it may seem more complicated. However, its advantage is that it does not involve the partitions of n. Let's copy from [1] a formula for the entries of $(H^{-1})_{ij}$ that the authors have concluded from the Cayley formula and the Sylvester theorem on determinants:

$$(3.2) \qquad (H^{-1})_{ij} = (-1)^{i+j}\left(\prod_{k=0}^{i-j-1} h_{i-k,i-k+1}\right) \cdot \frac{\det H_{1,2,\ldots,j-1}^{1,2,\ldots,j-1} \cdot \det H_{i+1,i+2,\ldots,n}^{i+1,i+2,\ldots,n}}{\det H},$$

where $H_{l_1,\ldots,l_m}^{k_1,\ldots,k_m}$ denotes the submatrix of H consisting of the entries standing in the intersections of the rows $k_1, \ldots, k_m$ and columns $l_1, \ldots, l_m$.

Using (3.2) we will prove the following theorem.

Theorem 3.1. *Let H_n be defined as in (1.1) with $a_{-1} \neq 0$. Then*

$$\det(H_n) = (-a_{-1})^{n-1} \cdot \left\{b_{n-1} - \frac{\left(\sum_{j=1}^{n-2} a_{j-1}b_{n-j-2}\right)^2}{a_{n-2} - a_{n-1} - \sum_{i=2}^{n-1} a_{n-i-1}\left(\sum_{j=1}^{i-1} a_{j-1}b_{i-j-1}\right)}\right\}^{-1}.$$

Proof. Since we are dealing with Toeplitz-Hessenberg matrices, the submatrices of H_n consisting of some k consecutive rows and columns are simply the matrices H_k. Moreover, since all the entries $h_{i-k,i-k+1}$ lie in the first superdiagonal, in this case we have $h_{i-k,i-k+1} = a_{-1}$ for all i, k. Thus (3.2) can be transformed into

$$(3.3) \qquad\qquad (H_n^{-1})_{ij} = \frac{(-1)^{i+j}a_{-1}^{i-j}\det(H_{j-1})\det(H_{n-i})}{\det(H_n)}.$$

Substituting $i = n$, $j = 1$ we obtain the equality: $\det(H_n) = \frac{(-a_{-1})^{n-1}}{(H_n^{-1})_{n1}}$. Now it is a natural idea to use Theorem 1.1. This leads to

$$\det(H_n) = \frac{(-a_{-1})^{n-1}}{(H_n^{-1})_{n1}} = \frac{(-a_{-1})^{n-1}}{b_{n-2} - \frac{1}{\gamma} x_n y_1} = \frac{(-a_{-1})^{n-1}}{b_{n-2} - \frac{x_n^2}{\gamma}}$$

$$= (-a_{-1})^{n-1} \cdot \left\{ b_{n-2} - \frac{\left(\sum_{j=1}^{n-2} a_{j-1} b_{n-j-2} \right)^2}{a_{n-2} - a_{n-1} - \sum_{i=2}^{n-1} a_{n-i-1} \left(\sum_{j=1}^{i-1} a_{j-1} b_{i-j-1} \right)} \right\}^{-1} . \square$$

Obviously, (3.3) used for various i, j can lead to some combinatorial identities. One example is presented below.

Corollary 3.2. *If $a_0, a_1, \ldots, a_{n-1}$ are arbitrary numbers, then*

$$\sum_{k_1 + 2k_2 + \cdots + nk_n = n} \binom{k_1 + \cdots + k_n}{k_1, \ldots, k_n} (-1)^{n - k_1 - \cdots - k_n} a_0^{k_1} a_1^{k_2} \cdots a_{n-1}^{k_n}$$

$$= (-a_{-1})^{n-1} \cdot \left\{ b_{n-2} - \frac{\left(\sum_{j=1}^{n-2} a_{j-1} b_{n-j-2} \right)^2}{a_{n-2} - a_{n-1} - \sum_{i=2}^{n-1} a_{n-i-1} \left(\sum_{j=1}^{i-1} a_{j-1} b_{i-j-1} \right)} \right\}^{-1} .$$

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