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Global Well-posedness of Weak Solutions to the Time-dependent Ginzburg-Landau Model for Superconductivity

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Abstract. We prove the global existence and uniqueness of weak solutions to the time dependent Ginzburg-Landau system in superconductivity with Coulomb gauge.

1. Introduction

We consider the existence and uniqueness problem for the 3D Ginzburg-Landau model in superconductivity:

$$\eta\partial_t\psi + i\eta k\phi\psi + \left(\frac{i}{k}\nabla + A\right)^2\psi + (|\psi|^2 - g)\psi = 0, \tag{1.1}$$

$$\partial_t A + \nabla\phi + \operatorname{curl}^2 A + \operatorname{Re}\left\{\left(\frac{i}{k}\nabla\psi + \psi A\right)\overline{\psi}\right\} = \operatorname{curl} H \tag{1.2}$$

in $Q_T := (0,T)\times\Omega$, with boundary and initial conditions

$$\nabla\psi\cdot\nu = 0, \quad A\cdot\nu = 0, \quad \operatorname{curl} A\times\nu = H\times\nu \quad \text{on } (0,T)\times\partial\Omega, \tag{1.3}$$

$$(\psi, A)(\,\cdot\,,0) = (\psi_0, A_0)(\cdot) \qquad \text{in } \Omega. \tag{1.4}$$

Here, the unknowns ψ, A, and ϕ are $\mathbb{C}$-valued, $\mathbb{R}^2$-valued, and $\mathbb{R}$-valued functions, respectively, and they stand for the order parameter, the magnetic potential, and the electric potential, respectively. Two positive constants η and κ are Ginzburg-Landau constants, g is a positive function that depends on the material as well as on the temperature and other variables, H is the applied magnetic field, and $i := \sqrt{-1}$. $\overline{\psi}$ denotes the complex conjugate of ψ, $\operatorname{Re}\psi := (\psi + \overline{\psi})/2$ is the real part of ψ, and $|\psi|^2 := \psi\overline{\psi}$ is the density of superconductivity carriers. T is any given positive constant. Ω is a simply connected and bounded domain with smooth boundary $\partial\Omega$ and ν is the outward unit normal to $\partial\Omega$.

It is well-known that the Ginzburg-Landau equations are gauge invariant, namely, if (ψ, A, ϕ) is a solution of (1.1)–(1.2), then $(\psi e^{ik\chi}, A + \nabla\chi, \phi - \partial_t\chi)$ is also a solution for any real-valued smooth function χ. Accordingly, in order to obtain the well-posedness of the problem, we need to impose some gauge condition. From physical point of view, one may usually think of four types of the gauge condition:

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(1) Coulomb gauge: $\operatorname{div} A = 0$ in Ω and $\int_\Omega \phi\, dx = 0$.

(2) Lorentz gauge: $\phi = -\operatorname{div} A$ in Ω.

(3) Lorenz gauge: $\partial_t \phi = -\operatorname{div} A$ in Ω.

(4) Temporal gauge (Weyl gauge): $\phi = 0$ in Ω.

For the initial data $(\psi_0, A_0) \in W_0 := \{(\psi_0, A_0) \mid \psi_0 \in L^\infty \cap H^1, A_0 \in H^1\}$, Chen et al. [4, 5], Du [6], Fan and Ozawa [9], and Tang [12] proved the existence and uniqueness of global strong solutions to (1.1)–(1.4) in the case of the Coulomb, Lorenz and Lorentz as well as temporal gauges.

For the initial data $\psi_0, A_0 \in L^2$, under the Coulomb or Lorentz gauge, Tang and Wang (2-D) [13], Fan and Jiang (3-D) [8] proved the global existence of weak solutions. Fan and Ozawa (2-D) [10] and Fan, Gao and Guo (3-D) [7] proved the global existence and uniqueness of weak solutions for $\psi_0, A_0 \in L^d$ with $d = 2, 3$.

Here we point out that all the above results [4–10, 12, 13] require $g = 1$ and H is smooth.

We will assume that

$$g := g(x,t) \in L^p(0,T; L^q(\Omega)) \quad \text{with } \frac{2}{p} + \frac{3}{q} = 2,\ 1 \le p < \infty \text{ and } \frac{3}{2} < q \le \infty, \tag{1.5}$$

$$H := H(x,t) \in L^2(0,T; L^3(\Omega)) \cap L^{3/2}(0,T; L^3(\partial\Omega)). \tag{1.6}$$

The aim of this paper is to study the well-posedness of the problem (1.1)–(1.4) under the conditions (1.5) and (1.6), we will prove

Theorem 1.1. *Let $\psi_0, A_0 \in L^3$ and* (1.5) *and* (1.6) *hold true. Then there exists a unique weak solution (ψ, A) of* (1.1)–(1.4) *with the choice of Coulomb gauge, such that*

$$\psi, A \in W := L^\infty(0,T; L^3) \cap L^2(0,T; H^1) \cap L^5(\Omega \times (0,T)), \tag{1.7}$$

$$\partial_t \psi, \partial_t A \in W' := \text{the dual of } W \tag{1.8}$$

for any $T > 0$.

In our proofs, we will use the following lemmas.

Lemma 1.2. [1, 11] *Let Ω be a smooth and bounded open set in $\mathbb{R}^3$. Then there exists $C > 0$ such that*

$$\|f\|_{L^p(\partial\Omega)} \le C \|f\|_{L^p(\Omega)}^{1-1/p} \|f\|_{W^{1,p}(\Omega)}^{1/p} \tag{1.9}$$

for any $1 < p < \infty$ and $f\colon \Omega \to \mathbb{R}^3$ in $W^{1,p}(\Omega)$.

Lemma 1.3. [2] *Let Ω be a regular bounded domain in $\mathbb{R}^3$, let $f\colon \Omega \to \mathbb{R}^3$ be a smooth enough vector field, and let $1 < p < \infty$. Then, the following identity holds true:*

$$-\int_\Omega \Delta f \cdot f|f|^{p-2}\,dx = \int_\Omega |f|^{p-2}|\nabla f|^2\,dx + \frac{4(p-2)}{p^2}\int_\Omega |\nabla |f|^{p/2}|^2\,dx - \int_{\partial\Omega} |f|^{p-2}(\nu\cdot\nabla)f\cdot f\,dS. \tag{1.10}$$

Lemma 1.4. [3,8] *Let $\psi, A \in W$ and $\left|\frac{i}{k}\nabla\psi + \psi A\right| |\psi|^{1/2} \in L^2(Q_T)$, then $\nabla\phi \in L^{5/3}(Q_T) \cap L^2(0,T;L^{3/2})$ satisfies*

$$-\Delta\phi = \operatorname{div}\operatorname{Re}\left\{\left(\frac{i}{k}\nabla\psi + \psi A\right)\overline{\psi}\right\} \qquad \text{in } Q_T, \tag{1.11}$$

$$\nabla\phi\cdot\nu = 0 \qquad \text{on } (0,T)\times\partial\Omega. \tag{1.12}$$

2. Proof of Theorem 1.1

By the results proved in [7, 8], one can prove a similar well-posedness result of strong solutions. We take $\psi_{0n} \in H^1 \cap L^\infty$, $A_{0n} \in H^1$, $g_n \in H^2(\Omega\times(0,T))$ and $H_n \in H^2(\Omega\times(0,T))$ such that

$$\|\psi_{0n} - \psi_0\|_{L^3} \to 0, \quad \|A_{0n} - A_0\|_{L^3} \to 0,$$
$$\|g_n - g\|_{L^p(0,T;L^q(\Omega))} \to 0, \quad \|H_n - H\|_{L^2(0,T;L^3(\Omega))\cap L^{3/2}(0,T;L^3(\partial\Omega))} \to 0$$

as $n \to \infty$. Thus we have a unique strong solution ψ_n, A_n with the data $(\psi_{0n}, A_{0n}, g_n, H_n)$. We want to establish a priori estimates (1.7) and (1.8) uniformly with respect to n. Then by the standard compactness argument, we can get $\psi_n \to \psi$ and $A_n \to A$ as $n \to \infty$ (see Section 3 below), thus we conclude that the existence of weak solutions and a prior estimates (1.7) and (1.8). Now we drop the subscript "n" of ψ_n and A_n and do as follows.

Multiplying (1.1) by $\overline{\psi}$, integrating by parts, and then taking the real part, we see that

$$\begin{aligned}
&\frac{\eta}{2}\frac{d}{dt}\int |\psi|^2\,dx + \int \left|\frac{i}{k}\nabla\psi + \psi A\right|^2 dx + \int |\psi|^4\,dx \\
&= \int g|\psi|^2\,dx \le \|g\|_{L^q}\|\psi\|^2_{L^{2q/(q-1)}} \\
&\le C\|g\|_{L^q}\|\psi\|^{2-3/q}_{L^2}\|\nabla|\psi|\|^{3/q}_{L^2} + C\|g\|_{L^q}\|\psi\|^2_{L^2} \\
&\le \frac{1}{2}\left\|\frac{1}{k}\nabla|\psi|\right\|^2_{L^2} + C\|g\|^p_{L^q}\|\psi\|^2_{L^2} + C\|\psi\|^2_{L^2} \\
&\le \frac{1}{2}\left\|\frac{i}{k}\nabla\psi + \psi A\right\|^2_{L^2} + C\|g\|^p_{L^q}\|\psi\|^2_{L^2} + C\|\psi\|^2_{L^2},
\end{aligned}$$

which gives

$$\|\psi\|_{L^\infty(0,T;L^2)} + \left\|\frac{i}{k}\nabla\psi + \psi A\right\|_{L^2(0,T;L^2)} \le C. \tag{2.1}$$

Here we have used the Gagliardo-Nirenberg inequality

$$\|\psi\|_{L^{2q/(q-1)}} \le C\|\psi\|_{L^2}^{1-3/(2q)}\|\nabla|\psi|\|_{L^2}^{3/(2q)} + C\|\psi\|_{L^2}, \tag{2.2}$$

and the diamagnetic inequality

$$\left|\frac{1}{k}\nabla|\psi|\right| \le \left|\frac{i}{k}\nabla\psi + \psi A\right|. \tag{2.3}$$

Similarly, multiplying (1.1) by $|\psi|\overline{\psi}$, integrating by parts, and then taking the real part, and using (1.10), (2.2) and (2.3), we have

$$\begin{aligned}
&\frac{1}{3}\frac{d}{dt}\int |\psi|^3\,dx + \int \left|\frac{i}{k}\nabla\psi + \psi A\right|^2 |\psi|\,dx + \int |\psi|^5\,dx \\
&\le \int g|\psi|^3\,dx \le \|g\|_{L^q}\||\psi|^{3/2}\|_{L^{2q/(q-1)}}^2 = \|g\|_{L^q}\|w\|_{L^{2q/(q-1)}}^2 \quad (w := |\psi|^{3/2}) \\
&\le \|g\|_{L^q}\|w\|_{L^2}^{2-3/q}\|\nabla w\|_{L^2}^{3/q} + C\|g\|_{L^q}\|w\|_{L^2}^2 \\
&\le \frac{1}{2}\left\|\frac{1}{k}\nabla w\right\|_{L^2}^2 + C(\|g\|_{L^q}^p + 1)\|w\|_{L^2}^2 \\
&\le \frac{1}{2}\int \left|\frac{i}{k}\nabla\psi + \psi A\right|^2 |\psi|\,dx + C(\|g\|_{L^q}^p + 1)\|\psi\|_{L^3}^3,
\end{aligned}$$

which leads to

$$\sup_{0\le t\le T}\int |\psi|^3\,dx + \int_0^T\int \left|\frac{i}{k}\nabla\psi + \psi A\right|^2 |\psi|\,dxdt + \int_0^T\int |\psi|^5\,dxdt \le C. \tag{2.4}$$

Testing (1.2) by A and using (2.4), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |A|^2\,dx + \int |\operatorname{curl} A|^2\,dx - \int H\operatorname{curl} A\,dx \\
&\le \int \left|\frac{i}{k}\nabla\psi + \psi A\right| |\psi||A|\,dx \\
&= \int \left|\frac{i}{k}\nabla\psi + \psi A\right| \cdot |\psi|^{1/2} \cdot |\psi|^{1/2} \cdot |A|\,dx \\
&\le \left\|\left|\frac{i}{k}\nabla\psi + \psi A\right| |\psi|^{1/2}\right\|_{L^2} \||\psi|^{1/2}\|_{L^6}\|A\|_{L^3} \\
&\le C\left\|\left|\frac{i}{k}\nabla\psi + \psi A\right| |\psi|^{1/2}\right\|_{L^2} (\|A\|_{L^2} + \|\operatorname{curl} A\|_{L^2}) \\
&\le \frac{1}{2}\|\operatorname{curl} A\|_{L^2}^2 + C\left\|\left|\frac{i}{k}\nabla\psi + \psi A\right| |\psi|^{1/2}\right\|_{L^2}^2 + C\|A\|_{L^2}^2,
\end{aligned}$$

which implies

$$\|A\|_{L^\infty(0,T;L^2)} + \|A\|_{L^2(0,T;H^1)} \le C.$$

Since

$$\int_0^T \int |\psi A|^2 \, dxdt \le \|\psi\|_{L^3}^2 \int_0^T \|A\|_{L^6}^2 \, dt \le C,$$

it follows from (2.1) that

$$\|\psi\|_{L^2(0,T;H^1)} \le C.$$

Testing (1.2) by $|A|A$ and letting $u := |A|^{3/2}$, using (1.3), (1.9), (1.10), (1.11), (1.12), (2.4), and the vector identities

$$(\nu \cdot \nabla)A \cdot A = (A \cdot \nabla)A \cdot \nu + (\operatorname{curl} A \times \nu)A,$$

and

$$(A \cdot \nabla)A \cdot \nu = -(A \cdot \nabla)\nu \cdot A, \quad (A \cdot \nu = 0 \text{ on } (0,T) \times \partial\Omega),$$

we arrive that

$$\begin{aligned}
&\frac{d}{dt}\int u^2 \, dx + C_0 \int |\nabla u|^2 \, dx + C_0 \int |A||\nabla A|^2 \, dx \\
&\le \int \left|\frac{i}{k}\nabla\psi + \psi A\right| |\psi| u^{4/3} \, dx + \int |\nabla\phi| u^{4/3} \, dx + C\int_{\partial\Omega} u^2 \, dS \\
&\quad + C\int_{\partial\Omega} |H \times \nu| u^{4/3} \, dS + C\int_\Omega |H||A|^{1/2}|\nabla u| \, dx + C\int_\Omega |H||A||\nabla A| \, dx \\
&\le C\left\|\left|\frac{i}{k}\nabla\psi + \psi A\right| |\psi|^{1/2}\right\|_{L^2} \||\psi|^{1/2}\|_{L^6}\|u^{4/3}\|_{L^3} + \|\nabla\phi\|_{L^{3/2}}\|u^{4/3}\|_{L^3} \\
&\quad + C\|u\|_{L^2(\Omega)}\|u\|_{H^1(\Omega)} + C\|H\|_{L^3(\partial\Omega)}\|u^{4/3}\|_{L^{3/2}(\partial\Omega)} \\
&\quad + C\|H\|_{L^3(\Omega)}\|A\|_{L^3}^{1/2}\|\nabla u\|_{L^2} + C\int_\Omega |H||A||\nabla A| \, dx \\
&\le C\left\|\left|\frac{i}{k}\nabla\psi + \psi A\right| |\psi|^{1/2}\right\|_{L^2} \||\psi|^{1/2}\|_{L^6}\|u^{4/3}\|_{L^3} + C\|u\|_{L^2(\Omega)}\|u\|_{H^1(\Omega)} \\
&\quad + C\|H\|_{L^3(\partial\Omega)}\|u\|_{L^2(\partial\Omega)}^{4/3} + C\|H\|_{L^3(\Omega)}\|A\|_{L^3}^{1/2}\|\nabla u\|_{L^2} + C\int_\Omega |H||A||\nabla A| \, dx \\
&\le C\left\|\left|\frac{i}{k}\nabla\psi + \psi A\right| |\psi|^{1/2}\right\|_{L^2} \|u\|_{L^4}^{4/3} + C\|u\|_{L^2}\|u\|_{H^1} \\
&\quad + C\|H\|_{L^3(\partial\Omega)}(\|u\|_{L^2}\|u\|_{H^1})^{2/3} + C\|H\|_{L^3(\Omega)}\|A\|_{L^3}^{1/2}(\|\nabla u\|_{L^2} + \||A|^{1/2}\nabla A\|_{L^2}) \\
&\le C\left\|\left|\frac{i}{k}\nabla\psi + \psi A\right| |\psi|^{1/2}\right\|_{L^2} (\|u\|_{L^2}^{1/4}\|\nabla u\|_{L^2}^{3/4} + \|u\|_{L^2})^{4/3} + C\|u\|_{L^2}\|u\|_{H^1} \\
&\quad + C\|H\|_{H^3(\partial\Omega)}(\|u\|_{L^2}\|u\|_{H^1})^{2/3} + C\|H\|_{L^3(\Omega)}\|A\|_{L^3}^{1/2}(\|\nabla u\|_{L^2} + \||A|^{1/2}\nabla A\|_{L^2}) \\
&\le \frac{C_0}{2}\|\nabla u\|_{L^2}^2 + \frac{C_0}{2}\||A|^{1/2}\nabla A\|_{L^2}^2 + C\left\|\left|\frac{i}{k}\nabla\psi + \psi A\right| |\psi|^{1/2}\right\|_{L^2}^2 \|u\|_{L^2}^{2/3}
\end{aligned}$$

$$+ C\left\|\left|\frac{i}{k}\nabla\psi + \psi A\right||\psi|^{1/2}\right\|_{L^2}\|u\|_{L^2}^{4/3} + C\|u\|_{L^2}^2 + C\|H\|_{L^3(\partial\Omega)}\|u\|_{L^2}^{4/3}$$
$$+ C\|H\|_{L^3(\partial\Omega)}^{3/2}\|u\|_{L^2} + C\|H\|_{L^3(\Omega)}^2\|A\|_{L^3},$$

which implies

$$\|A\|_{L^\infty(0,T;L^3)} + \|A\|_{L^5(Q_T)} + \|\nabla u\|_{L^2(Q_T)} \le C.$$

Here, we have used

$$\|\nabla\phi\|_{L^{3/2}} \le C\left\|\left(\frac{i}{k}\nabla\psi + \psi A\right)\overline{\psi}\right\|_{L^{3/2}} \le C\left\|\left|\frac{i}{k}\nabla\psi + \psi A\right||\psi|^{1/2}\right\|_{L^2}\||\psi|^{1/2}\|_{L^6}$$

and the Gagliardo-Nirenberg inequality

$$\|u\|_{L^4} \le C\|u\|_{L^2}^{1/4}\|\nabla u\|_{L^2}^{3/4} + C\|u\|_{L^2}.$$

The proof of uniqueness follows from [7] and the a priori estimates (1.7) and thus we omit the details here.

This completes the proof.

3. Appendix

In this section, we will give the precise definition of weak solutions and explain more details of the proof of the existence of weak solutions.

Definition 3.1 (Weak solutions)**.** (ψ, A, ϕ) is called a weak solution to the problem (1.1)–(1.4) in $\Omega \times (0,T)$ under the Coulomb gauge if

$$\psi, A \in W := L^\infty(0,T;L^3) \cap L^2(0,T;H^1) \cap L^5(0,T;L^5), \quad \nabla\phi \in L^{5/3}(Q_T) \cap L^2(0,T;L^{3/2}),$$

and

$$\int_0^T\!\!\int \left[-\eta\psi w_t + i\eta k\phi\psi w + \left(\frac{i}{k}\nabla\psi + \psi A\right)\left(\frac{i}{k}\nabla w + Aw\right) + (|\psi|^2 - g)\psi w\right] dxdt = 0$$

for any $w \in C_0^\infty(\Omega \times [0,T])$, $w(\,\cdot\,,0) = w(\,\cdot\,,T) = 0$, and

$$\int_0^T\!\!\int \left[-AB_t + \nabla\phi B + (\operatorname{curl} A - H)\operatorname{curl} B + \operatorname{Re}\left(\frac{i}{k}\nabla\psi + \psi A\right)\overline{\psi}\cdot B\right] dxdt = 0$$

for any $B \in C_0^\infty(\Omega \times [0,T])$, $B(\,\cdot\,,0) = B(\,\cdot\,,T) = 0$.

We have an approximate solution (ψ_n, A_n, ϕ_n) satisfying

$$\|\psi_n\|_W + \|A_n\|_W \le C,$$
$$\|\partial_t\psi_n\|_{W'} + \|\partial_t A_n\|_{W'} \le C,$$
$$\|\nabla\phi_n\|_{L^{5/3}(Q_T)} + \|\nabla\phi_n\|_{L^2(0,T;L^{3/2})} \le C.$$

By standard compactness principle (e.g., Lions-Aubin lemma) we have

$$\phi_n \rightharpoonup \phi \quad \text{weakly in } L^2(0,T;L^3),$$
$$\psi_n \to \psi \quad \text{strongly in } L^2(0,T;L^{3/2}),$$

which gives

$$\phi_n\psi_n \to \phi\psi \quad \text{in the sense of distributions.}$$

On the other hand, we have

$$\nabla\psi_n \rightharpoonup \nabla\psi \quad \text{weakly in } L^2(0,T;L^2),$$
$$A_n \to A \quad \text{strongly in } L^2(0,T;L^2),$$

which implies

$$\nabla\psi_n \cdot A_n \to \nabla\psi \cdot A \quad \text{in the sense of distributions.}$$

It is easy to verify that

$$\psi_n \to \psi \quad \text{strongly in } L^p(0,T;L^p), \quad 1 < p < 5,$$
$$A_n \to A \quad \text{strongly in } L^p(0,T;L^p), \quad 1 < p < 5,$$

which implies

$$|\psi_n|^2\psi_n \to |\psi|^2\psi \quad \text{srtongly in } L^1(0,T;L^1),$$
$$\psi_n|A_n|^2 \to \psi|A|^2 \quad \text{srtongly in } L^1(0,T;L^1),$$
$$|\psi_n|^2 A_n \to |\psi|^2 A \quad \text{srtongly in } L^1(0,T;L^1).$$

Now it is easy to completes the proof.

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