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Asymptotic Stability of the Viscoelastic Equation with Variable Coefficients and the Balakrishnan-Taylor Damping

Tae Gab Ha

Abstract. In this paper, we consider the viscoelastic equation with variable coefficients and Balakrishnan-Taylor damping and source terms. This work is devoted to prove, under suitable conditions on the initial data, the asymptotic stability without imposing any restrictive growth assumption on the damping term and weakening of the usual assumptions on the relaxation function.

1. Introduction

In this paper, we are concerned with the uniform energy decay rates of solutions for the viscoelastic equation:

$$(1.1)\qquad \begin{cases} u'' - M(t)Lu + \int_0^t h(t-s)Lu(s)\,ds + g(u') = |u|^\rho u & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \Gamma \times (0,\infty), \\ u(x,0) = u_0, \quad u'(x,0) = u_1, \end{cases}$$

where $Lu = \operatorname{div}(A\nabla u) = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)$ and $M(t) = \xi_1 + \xi_2 \int_\Omega A\nabla u \nabla u\,dx + \xi_3 \int_\Omega A\nabla u\nabla u'\,dx$, where ξ_1, ξ_2, ξ_3 are positive constants. Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with boundary Γ. $'$ denotes the derivative with respect to time t.

When $A = I$ with the Balakrishnan-Taylor damping ($\xi_3 \neq 0$), the model was initially proposed by Balakrishnan and Taylor in [1] and Bass and Zes [2]. The original motivation for studying this model seemed to solve the spillover problem, namely, to design a feedback control function that involves only finitely many modes in order to achieve a high performance of the closed-loop systems, such as a robust and exponential stabilization of the system when there might be some uncertainty in values of the parameters. So far, there are some stability results for the problem having the Balakrishnan-Taylor damping (see [15, 16, 20, 21]). For instance, Tatar and Zaraï [16] proved an exponential decay result

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of the energy provided that the kernel decays exponentially. Recently, Ha [8] studied the uniform decay rates of the energy without imposing any restrictive growth assumption on the damping term and weakening the usual assumptions on the relaxation function applying the method in [5, 7].

When A is a general matrix without the Balakrishnan-Taylor damping ($\xi_2 = \xi_3 = 0$), such a problem is called a wave equation with variable coefficients in principle. These equations arise in mathematical modeling of inhomogeneous media in solid mechanics, electromagnetic, fluid flows through porous media, and other areas of physics and engineering. For the variable coefficients problem, the main tool is the Riemannian geometry method, which was introduced by Yao [19] and has been widely used in the literature (see [6, 12, 13, 17] and a list of references therein). For instance, Wu [18] proved the uniform decay of the energy without any geometrical conditions on the shape of the dissipative portion of the boundary, whereas [10] studied the general decay rates of the energy without imposing any restrictive growth near zero assumption on the damping term having the Balakrishnan-Taylor damping. However, above mentioned references did not consider the relaxation function h. On the other hand, the viscoelastic type problems with variable coefficients and source term are very few results (cf. [4, 9]). For example, Boukhatem and Benabderrahmane [3] studied the uniform decay rate of the energy to the viscoelastic wave equation with variable coefficients and acoustic boundary conditions without damping term. But there is none, to our knowledge, for the viscoelastic problem with variable coefficients and the Balakrishnan-Taylor damping.

Motivated by previous works, the goal of this paper is to study the asymptotic stability of the viscoelastic equation with variable coefficients and Balakrishnan-Taylor damping and source terms by applying the method developed in [7]. This paper is organized as follows: In Section 2, we recall the hypotheses to prove our main result and introduce the existence and energy decay rate theorem. In Section 3, we prove the uniform decay rates of the energy without imposing any restrictive growth near zero assumption on the damping term and weakening of the usual assumptions on the relaxation function.

2. Preliminaries

We begin this section introducing some notations and our main results. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $n \geq 1$, with smooth boundary Γ. Throughout this paper we define the Hilbert space $\mathcal{H} = \{u \in H^1(\Omega) : Lu \in L^2(\Omega)\}$ with the norm $\|u\|_{\mathcal{H}} = \left(\|u\|^2_{H^1(\Omega)} + \|Lu\|_2^2\right)^{1/2}$ and $H^1_0(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma\}$. Moreover, $L^p(\Omega)$-norm is denoted by $\|\cdot\|_p$ and $\langle u, v\rangle = \int_\Omega u(x)v(x)\,dx$.

(H$_1$) Hypotheses on ξ_1, ξ_2, ξ_3, ρ. Let $\xi_i > 0$, $i = 1, 2, 3$, and let ρ be a constant satisfying the following condition:

$$0 < \rho < \frac{2}{n-2} \quad \text{if } n \geq 3 \quad \text{and} \quad \rho > 0 \quad \text{if } n = 1, 2.$$

(H$_2$) Hypotheses on A. The matrix $A = \big(a_{ij}(x)\big)$, where $a_{ij} \in C^1(\overline{\Omega})$, is symmetric and there exists a positive constant a_0 such that for all $x \in \overline{\Omega}$ and $\omega = (\omega_1, \ldots, \omega_n)$ we have

$$\sum_{i,j=1}^{n} a_{ij}(x)\omega_j\omega_i \geq a_0|\omega|^2. \tag{2.1}$$

(H$_3$) Hypotheses on g. Let $g\colon \mathbb{R} \to \mathbb{R}$ be a nondecreasing C^1 function such that $g(0) = 0$ and suppose that there exists a strictly increasing and odd function β of C^1 class on $[-1, 1]$ such that

$$\begin{aligned} |\beta(s)| \leq |g(s)| \leq |\beta^{-1}(s)| \quad &\text{if } |s| \leq 1, \\ c_1|s| \leq |g(s)| \leq c_2|s| \quad &\text{if } |s| > 1, \end{aligned} \tag{2.2}$$

where β^{-1} denotes the inverse function of β and c_1, c_2 are positive constants.

(H$_4$) Hypotheses on h. Let $h\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a bounded C^1 function satisfying

$$\xi_1 - \int_0^\infty h(s)\,ds = \ell > 0. \tag{2.3}$$

Moreover, we assume that $h'(t) < 0$ for all $t \geq 0$.

By using the hypothesis (H$_2$), we verify that the bilinear form $a(\,\cdot\,,\cdot\,)\colon H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ defined by

$$a(u(t), v(t)) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x)\frac{\partial u(t)}{\partial x_j}\frac{\partial v(t)}{\partial x_i}\,dx = \int_\Omega A\nabla u(t)\nabla v(t)\,dx$$

is symmetric and continuous. On the other hand, from (2.1) for $\omega = \nabla u$, we get

$$a(u(t), u(t)) \geq a_0 \int_\Omega \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^2 dx = a_0\|\nabla u(t)\|_2^2, \tag{2.4}$$

which implies that $a(\,\cdot\,,\cdot\,)$ is coercive.

Now, we state the local existence theorem which can be complete arguing as [3,19,20].

Theorem 2.1. *Suppose that* (H$_1$)–(H$_4$) *hold. Then given* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, *there exist* $T > 0$ *and a unique solution* u *of the problem* (1.1) *such that*

$$u \in C(0, T; H_0^1(\Omega)) \cap C^1(0, T; L^2(\Omega)).$$

In order to study the global existence and the decay of a local solution for problem (1.1) given by Theorem 2.1, we will find a stable region. First of all, we define the energy associated to problem (1.1) by

$$\begin{aligned} E(t) &= \frac{1}{2}\|u'(t)\|_2^2 + \frac{1}{2}\left(\xi_1 + \frac{\xi_2}{2}a(u(t),u(t)) - \int_0^t h(t-s)\,ds\right)a(u(t),u(t)) \\ &\quad + \frac{1}{2}(h \diamond u)(t) - \frac{1}{\rho+2}\|u(t)\|_{\rho+2}^{\rho+2}, \end{aligned} \tag{2.5}$$

where

$$(h \diamond u)(t) = \int_0^t h(t-s)a(u(t)-u(s),u(t)-u(s))\,ds.$$

Then

$$\begin{aligned} E'(t) &= -\xi_3\left(\frac{1}{2}\frac{d}{dt}a(u(t),u(t))\right)^2 + \frac{1}{2}(h' \diamond u)(t) \\ &\quad - \frac{1}{2}h(t)a(u(t),u(t)) - \langle g(u'(t)),u'(t)\rangle \\ &\le 0. \end{aligned} \tag{2.6}$$

We also define the following functional in order to obtain the potential well:

$$\begin{aligned} J(t) &= \frac{1}{2}\left(\xi_1 + \frac{\xi_2}{2}a(u(t),u(t)) - \int_0^t h(t-s)\,ds\right)a(u(t),u(t)) \\ &\quad + \frac{1}{2}(h \diamond u)(t) - \frac{1}{\rho+2}\|u(t)\|_{\rho+2}^{\rho+2} \end{aligned} \tag{2.7}$$

and

$$I(t) = \left(\xi_1 - \int_0^t h(t-s)\,ds\right)a(u(t),u(t)) + (h \diamond u)(t) - \|u(t)\|_{\rho+2}^{\rho+2}.$$

Lemma 2.2. *Suppose that* (H_4) *holds and* $(u_0,u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *such that*

$$\zeta := \frac{C_{\rho+2}^{\rho+2}}{a_0\ell}\left(\frac{2(\rho+2)}{a_0\rho\ell}E(0)\right)^{\rho/2} < 1 \quad \text{and} \quad I(0) > 0, \tag{2.8}$$

where $C_{\rho+2}$ *is an imbedding constant of* $H_0^1(\Omega) \hookrightarrow L^{\rho+2}(\Omega)$. *Then* $I(t) > 0$ *for all* $t \ge 0$.

Proof. From the continuity of $u(t)$ and since $I(0) > 0$, it follows that

$$I(\widetilde{t}) \ge 0$$

for all $\widetilde{t}$ belonging to some neighborhood of $t = 0$. Let denote by $[0, t_{\max}]$ the maximal interval where the above inequality hold. Then from the above inequality, we deduce that

$$\begin{aligned} J(\widetilde{t}) &= \frac{\rho}{2(\rho+2)}\left(\left(\xi_1 - \int_0^{\widetilde{t}} h(\widetilde{t}-s)\,ds\right)a(u(\widetilde{t}),u(\widetilde{t})) + (h \diamond u)(\widetilde{t})\right) \\ &\quad + \frac{\xi_2}{4}a^2(u(\widetilde{t}),u(\widetilde{t})) + \frac{1}{\rho+2}I(\widetilde{t}) \\ &\ge \frac{\rho}{2(\rho+2)}\left(\left(\xi_1 - \int_0^{\widetilde{t}} h(\widetilde{t}-s)\,ds\right)a(u(\widetilde{t}),u(\widetilde{t})) + (h \diamond u)(\widetilde{t})\right). \end{aligned} \tag{2.9}$$

By using (2.3), (2.4), (2.5), (2.7) and (2.9) we have

$$\begin{aligned}\ell\|\nabla u(\widetilde{t})\|_2^2 &\le \frac{1}{a_0}\left(\xi_1 - \int_0^{\widetilde{t}} h(\widetilde{t}-s)\,ds\right) a(u(\widetilde{t}),u(\widetilde{t})) \le \frac{2(\rho+2)}{a_0\rho} J(\widetilde{t}) \\ &\le \frac{2(\rho+2)}{a_0\rho} E(\widetilde{t}) \le \frac{2(\rho+2)}{a_0\rho} E(0)\end{aligned} \tag{2.10}$$

for all $\widetilde{t} \in [0, t_{\max}]$. Hence, by (2.3), (2.8), (2.10) and the imbedding $H_0^1(\Omega) \hookrightarrow L^{\rho+2}(\Omega)$ we get

$$\begin{aligned}\|u(\widetilde{t})\|_{\rho+2}^{\rho+2} &\le C_{\rho+2}^{\rho+2}\|\nabla u(\widetilde{t})\|_2^{\rho+2} \le \frac{C_{\rho+2}^{\rho+2}}{\ell}\|\nabla u(\widetilde{t})\|_2^{\rho}\ell\|\nabla u(\widetilde{t})\|_2^2 \\ &\le a_0\zeta\ell\|\nabla u(\widetilde{t})\|_2^2 < \left(\xi_1 - \int_0^{\widetilde{t}} h(\widetilde{t}-s)\,ds\right) a(u(\widetilde{t}),u(\widetilde{t})),\end{aligned} \tag{2.11}$$

which implies that

$$I(\widetilde{t}) > 0 \quad \text{for all } \widetilde{t} \in [0, t_{\max}].$$

By repeating this procedure and using the fact that

$$\lim_{\widetilde{t}\to t_{\max}} \frac{C_{\rho+2}^{\rho+2}}{a_0\ell}\left(\frac{2(\rho+2)}{a_0\rho\ell}E(\widetilde{t})\right)^{\rho/2} \le \zeta < 1,$$

$t_{\max}$ is extended to for all t. □

Theorem 2.3. *Let $u(t)$ be the solution of* (1.1). *If $(u_0,u_1) \in H_0^1(\Omega)\times L^2(\Omega)$ satisfies* (2.8)*, then the solution $u(t)$ is global.*

Proof. It suffices to show that $\|u'(t)\|_2^2 + a(u(t),u(t))$ is bounded independent of t. By virtue of (2.9) and Lemma 2.2, we get

$$\begin{aligned}J(t) &= \frac{\rho}{2(\rho+2)}\left(\left(\xi_1 - \int_0^t h(t-s)\,ds\right)a(u(t),u(t)) + (h\diamond u)(t)\right) \\ &\quad + \frac{\xi_2}{4}a^2(u(t),u(t)) + \frac{1}{\rho+2}I(t) \\ &> \frac{\rho}{2(\rho+2)}\left(\left(\xi_1 - \int_0^t h(t-s)\,ds\right)a(u(t),u(t)) + (h\diamond u)(t)\right).\end{aligned}$$

From the fact $(h\diamond u)(t) > 0$, for all $t \ge 0$ and above inequality, we obtain

$$\ell a(u(t),u(t)) \le \left(\xi_1 - \int_0^t h(t-s)\,ds\right)a(u(t),u(t)) < \frac{2(\rho+2)}{\rho}J(t).$$

Hence,

$$\frac{1}{2}\|u'(t)\|_2^2 + \frac{\rho\ell}{2(\rho+2)}a(u(t),u(t)) < \frac{1}{2}\|u'(t)\|_2^2 + J(t) = E(t) < E(0).$$

Therefore, there exists a positive constant C depending only on ρ and ℓ such that

$$\|u'(t)\|_2^2 + a(u(t),u(t)) \le CE(0).$$ □

Now we are in the position to state the energy decay rates result.

Theorem 2.4. *Suppose that* (H_1)–(H_4) *and* (2.8) *hold. Then we have following cases about decay rates:*

Case 1. *β is linear.*

- *If $-h'/h \geq m$ for all t, where m is some positive constant. Then we have*

$$E(t) \leq CE(0)e^{-\omega t},$$

where C and ω are some positive constants.

- *If $-h'/h$ decays to zero at infinity, then we have*

$$E(t) \leq CE(0)h(t)^{\omega}.$$

Case 2. *β has polynomial growth near zero. $\beta(s) = s^{\gamma}$ for some $\gamma > 1$.*

- *If $-h'/h \geq m$ for all t, where m is some positive constant, then*

$$E(t) \leq \frac{CE(0)}{(1+t)^{2/(\gamma-1)}}.$$

- *If $-h'/h$ decays to zero at infinity, then*

$$E(t) \leq \frac{CE(0)}{\big(-\ln h(t)\big)^{2/(\gamma-1)}}.$$

Case 3. *β does not necessarily have polynomial growth near zero. Assume that the function $G(s) = \beta(s)/s$ is nondecreasing on $(0,1)$ and $G(0) = 0$.*

- *If $-h'/h \geq m$ for all t, where m is some positive constant, then*

$$E(t) \leq CE(0)\left(\beta^{-1}\left(\frac{1}{t}\right)\right)^{2}.$$

- *If $-h'/h$ decays to zero at infinity, then there exists a nondecreasing concave function $\phi \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\phi(t) \to \infty$ as $t \to \infty$ and such that the energy satisfies the following decay rate*

$$E(t) \leq \frac{CE(0)}{\phi(t)}.$$

In particular, if the inequality

$$\frac{d}{dt}\left[h^{-1}\left(\frac{h(0)}{e^{t}}\right)\right] \geq \frac{1}{t\beta(1/t)} \quad \text{for all } t \geq 1$$

is satisfied, then the energy decays as

$$E(t) \leq \frac{CE(0)}{(-\ln h(t))^{2}}.$$

3. Asymptotic stability

In this section we prove the uniform decay rates of equation (1.1). In the following section, the symbol C indicates positive constants, which may be different.

Let us multiply equation (1.1) by $E(t)\phi'(t)u$, $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a concave nondecreasing function of class C^2, such that $\phi(t) \to +\infty$ as $t \to +\infty$, and then integrate the obtained result over $\Omega \times [S,T]$. Then we have

$$
\begin{aligned}
0 &= \int_S^T E(t)\phi'(t) \\
&\quad \times \int_\Omega u(t) \left(u''(t) - M(t)Lu(t) + \int_0^t h(t-s)Lu(s)\,ds + g(u'(t)) - |u(t)|^\rho u(t) \right) dxdt \\
&= \int_S^T E(t)\phi'(t) \int_\Omega u(t)u''(t)\,dxdt - \int_S^T E(t)\phi'(t)M(t) \int_\Omega u(t)Lu(t)\,dxdt \\
&\quad + \int_S^T E(t)\phi'(t) \int_\Omega u(t) \int_0^t h(t-s)Lu(s)\,dsdxdt \\
&\quad + \int_S^T E(t)\phi'(t) \int_\Omega u(t)g(u'(t))\,dxdt - \int_S^T E(t)\phi'(t) \int_\Omega |u(t)|^{\rho+2}\,dxdt.
\end{aligned} \tag{3.1}
$$

We note that

$$
\begin{aligned}
\int_S^T E(t)\phi'(t) \int_\Omega u(t)u''(t)\,dxdt &= \big[E(t)\phi'(t)\langle u(t), u'(t)\rangle\big]_S^T \\
&\quad - \int_S^T (E'(t)\phi'(t) + E(t)\phi''(t)) \int_\Omega u(t)u'(t)\,dxdt \\
&\quad - \int_S^T E(t)\phi'(t)\|u'(t)\|_2^2\,dt,
\end{aligned}
$$

$$
\begin{aligned}
&- \int_S^T E(t)\phi'(t)M(t) \int_\Omega u(t)Lu(t)\,dxdt \\
&= \xi_1 \int_S^T E(t)\phi'(t)a(u(t),u(t))\,dt + \xi_2 \int_S^T E(t)\phi'(t)a^2(u(t),u(t))\,dt \\
&\quad + \frac{\xi_3}{4} \big[E(t)\phi'(t)a^2(u(t),u(t))\big]_S^T - \frac{\xi_3}{4} \int_S^T (E'(t)\phi'(t) + E(t)\phi''(t))a^2(u(t),u(t))\,dt
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_S^T E(t)\phi'(t) \int_\Omega u(t) \int_0^t h(t-s)Lu(s)\,dsdxdt \\
&= - \int_S^T E(t)\phi'(t) \int_0^t h(s)\,ds\, a(u(t),u(t))\,dt \\
&\quad - \int_S^T E(t)\phi'(t) \int_0^t h(t-s)a(u(t),u(s)-u(t))\,dsdt.
\end{aligned}
$$

By replacing above identities in (3.1) and having in mind the definition of the energy associated to problem (1.1), it follows that

$$
\begin{aligned}
&2\int_S^T E^2(t)\phi'(t)\,dt\\
&= 2\int_S^T E(t)\phi'(t)\|u'(t)\|_2^2\,dt + \int_S^T E(t)\phi'(t)(h\diamond u)(t)\,dt\\
&\quad - \frac{\xi_2}{2}\int_S^T E(t)\phi'(t)a^2(u(t),u(t))\,dt\\
&\quad - \left[E(t)\phi'(t)\left(\langle u(t),u'(t)\rangle + \frac{\xi_3}{4}a^2(u(t),u(t))\right)\right]_S^T\\
&\quad + \int_S^T (E'(t)\phi'(t)+E(t)\phi''(t))\left(\langle u(t),u'(t)\rangle + \frac{\xi_3}{4}a^2(u(t),u(t))\right)dt\\
&\quad + \int_S^T E(t)\phi'(t)\int_0^t h(t-s)a(u(t),u(s)-u(t))\,dsdt\\
&\quad - \int_S^T E(t)\phi'(t)\int_\Omega u(t)g(u'(t))\,dxdt + \frac{\rho}{\rho+2}\int_S^T E(t)\phi'(t)\|u(t)\|_{\rho+2}^{\rho+2}\,dt\\
&:= 2\int_S^T E(t)\phi'(t)\|u'(t)\|_2^2\,dt + \int_S^T E(t)\phi'(t)(h\diamond u)(t)\,dt\\
&\quad - \frac{\xi_2}{2}\int_S^T E(t)\phi'(t)a^2(u(t),u(t))\,dt + I_1+I_2+I_3+I_4+I_5.
\end{aligned}
\tag{3.2}
$$

Now we are going to estimate terms on the right-hand side of (3.2).

Estimate for $I_1 := -\left[E(t)\phi'(t)\left(\langle u(t),u'(t)\rangle + \frac{\xi_3}{4}a^2(u(t),u(t))\right)\right]_S^T$. By using Young's and Poincaré's inequalities and (2.10), we obtain

$$|\langle u(t),u'(t)\rangle| \le CE(t) \tag{3.3}$$

and

$$a^2(u(t),u(t)) \le \left(\frac{2(\rho+2)}{\rho\ell}\right)^2 E(0)E(t). \tag{3.4}$$

Since $E(t)$ is nonincreasing and $\phi(t)$ is nondecreasing, we have

$$I_1 \le -C[E(t)\phi'(t)E(t)]_S^T \le CE^2(S).$$

Estimate for $I_2 := \int_S^T (E'(t)\phi'(t)+E(t)\phi''(t))\left(\langle u(t),u'(t)\rangle + \frac{\xi_3}{4}a^2(u(t),u(t))\right)dt$. From (3.3) and (3.4), we have

$$
\begin{aligned}
|I_2| &\le C\int_S^T |E'(t)\phi'(t)+E(t)\phi''(t)|E(t)\,dt\\
&\le C\int_S^T -E'(t)E(t)\,dt + CE^2(S)\int_S^T -\phi''(t)\,dt \le CE^2(S).
\end{aligned}
$$

Estimate for $I_3 := \int_S^T E(t)\phi'(t)\int_0^t h(t-s)a(u(t),u(s)-u(t))\,dsdt$. From Young's inequality and (2.6), we obtain

$$\begin{aligned}|I_3| &\le \frac{1}{2}\int_S^T E(t)\phi'(t)\int_0^t h(t-s)a(u(t)-u(s),u(t)-u(s))\,dsdt\\&\quad+\frac{1}{2}\int_S^T E(t)\phi'(t)\int_0^t h(s)\,ds\,a(u(t),u(t))\,dt\\&\le \frac{1}{2}\int_S^T E(t)\phi'(t)(h\diamond u)(t)\,dt + C\int_S^T E(t)\phi'(t)h(t)a(u(t),u(t))\,dt\\&\le \frac{1}{2}\int_S^T E(t)\phi'(t)(h\diamond u)(t)\,dt + CE^2(S).\end{aligned}$$

Estimate for $I_4 := -\int_S^T E(t)\phi'(t)\int_\Omega u(t)g(u'(t))\,dxdt$. By Young's and Poincaré's inequalities and (2.10), we have

$$\begin{aligned}|I_4| &\le \frac{a_0\rho\ell\epsilon}{2(\rho+2)C_P}\int_S^T E(t)\phi'(t)\|u(t)\|_2^2\,dt + C(\epsilon)\int_S^T E(t)\phi'(t)\int_\Omega |g(u'(t)|^2\,dxdt\\&\le \frac{a_0\rho\ell\epsilon}{2(\rho+2)}\int_S^T E(t)\phi'(t)\|\nabla u(t)\|_2^2\,dt + C(\epsilon)\int_S^T E(t)\phi'(t)\int_\Omega |g(u'(t)|^2\,dxdt\\&\le \epsilon\int_S^T E^2(t)\phi'(t)\,dt + C(\epsilon)\int_S^T E(t)\phi'(t)\int_\Omega |g(u'(t)|^2\,dxdt,\end{aligned}$$

where C_P is a Poincaré constant.

Estimate for $I_5 := \frac{\rho}{\rho+2}\int_S^T E(t)\phi'(t)\|u(t)\|_{\rho+2}^{\rho+2}\,dt$. By (2.10) and (2.11), it follows that

$$I_5 \le \frac{\rho a_0\zeta\ell}{\rho+2}\int_S^T E(t)\phi'(t)\|\nabla u(t)\|_2^2\,dt \le 2\zeta\int_S^T E(t)^2\phi'(t)\,dt.$$

By replacing all estimates $I_1,\ldots,I_5$ in (3.2), and taking ϵ sufficiently small, we get that

$$\begin{aligned}\int_S^T E^2(t)\phi'(t)\,dt &\le CE^2(S) + C\int_S^T E(t)\phi'(t)\|u'(t)\|_2^2\,dt\\&\quad+ C\int_S^T E(t)\phi'(t)\int_\Omega |g(u'(t)|^2\,dxdt\\&\quad+ C\int_S^T E(t)\phi'(t)(h\diamond u)(t)\,dt.\end{aligned}\tag{3.5}$$

Now we are going to estimate the most important term: $I_6 := \int_S^T E(t)\phi'(t)\|u'(t)\|_2^2\,dt$, $I_7 := \int_S^T E(t)\phi'(t)\int_\Omega |g(u'(t))|^2\,dxdt$ and $I_8 := \int_S^T E(t)\phi'(t)(h\diamond u)(t)\,dt$.

Terms of I_6, I_7 and I_8 can be estimated by the same arguments as Section 4 in [9]. For the convenience of readers, we will adapt the procedure. First of all, we present two technical lemma which will play an essential role when establishing the asymptotic behavior.

Lemma 3.1. [14] *Let $E\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a nonincreasing function and $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ a strictly increasing function of class C^1 such that*

$$\phi(0) = 0 \quad \text{and} \quad \phi(t) \to +\infty \quad \text{as } t \to +\infty.$$

Assume that there exists $\sigma > 0$ and $\omega > 0$ such that

$$\int_S^{+\infty} E^{1+\sigma}(t)\phi'(t)\,dt \le \frac{1}{\omega}E^{\sigma}(0)E(S)$$

for all $S \ge 0$. Then E has the following decay property:

(i) *if $\sigma = 0$, then $E(t) \le E(0)e^{1-\omega\phi(t)}$ for all $t \ge 0$;*

(ii) *if $\sigma > 0$, then $E(t) \le E(0)\left(\frac{1+\sigma}{1+\omega\sigma\phi(t)}\right)^{1/\sigma}$ for all $t \ge 0$.*

Lemma 3.2. [14] *Let E and ϕ be satisfied the condition of Lemma 3.1. Assume that there exists $\sigma > 0$, $\sigma' \ge 0$ and $C > 0$ such that*

$$\int_S^{+\infty} E^{1+\sigma}(t)\phi'(t)\,dt \le CE^{1+\sigma}(S) + \frac{C}{(1+\phi(S))^{\sigma'}}E^{\sigma}(0)E(S), \quad 0 \le S < +\infty.$$

Then, there exists $C > 0$ such that

$$E(t) \le E(0)\frac{C}{(1+\phi(t))^{(1+\sigma')/\sigma}} \quad \text{for all } t > 0.$$

3.1. Energy decay rate when β is linear and $-h'/h \ge m$ for all t, where m is some positive constant

Since β is linear, $|g(s)| \le C|s|$ for all $s \in \mathbb{R}$. Then, we can easily check that

$$I_6 + I_7 \le C\int_S^T E(t)\phi'(t)\int_\Omega u'(t)g(u'(t))\,dxdt \le C\int_S^T E(t)(-E'(t))\,dt \le CE^2(S). \tag{3.6}$$

Now, we assume that there is $\phi(t)$ concave nondecreasing such that $\phi'(t) \le -h'(t)/h(t)$ and $\phi(t) \to \infty$ as $t \to \infty$. Then we have

$$\phi'(t)h(t-s) \le \phi'(t-s)h(t-s) \le -h'(t-s). \tag{3.7}$$

Hence,

$$\begin{aligned} I_8 &\le -\int_S^T E(t)\int_0^t h'(t-s)a(u(t)-u(s),u(t)-u(s))\,dsdt \\ &\le 2\int_S^T E(t)(-E'(t))\,dt \le E^2(S). \end{aligned} \tag{3.8}$$

By replacing (3.6) and (3.8) in (3.5), we have

$$\int_S^T E^2(t)\phi'(t)\,dt \le CE^2(S),$$

which implies by Lemma 3.1 that

$$E(t) \le E(0)e^{1-\phi(t)/(2C)}. \tag{3.9}$$

It remains to estimate the growth of ϕ. However, it is very simple in this case. Indeed, let us set $\phi(t) := kt$, where k is for some positive constant, then $\phi(t)$ satisfies all the required properties and we obtain that the energy decays exponentially to zero.

3.2. Energy decay rate when β is linear and $-h'/h$ decays to zero at infinity

In order to estimate I_8, we have only to find $\phi(t)$ satisfying $\phi'(t) = -h'(t)/h(t)$. Indeed, let us set $\phi(t) = \ln(h(0)/h(t))$. Then $\phi(t)$ satisfies all the required properties and $\phi'(t) = -h'(t)/h(t)$, i.e., $\phi'(t)$ satisfies (3.7). Hence,

$$I_8 \le E^2(S).$$

Thus, by using the same argument as Subsection 3.1 we also obtain (3.9). Furthermore, we replace $\phi(t) = \ln(h(0)/h(t))$ in (3.9), then we get

$$E(t) \le CE(0)(h(t))^{1/(2C)}.$$

3.3. Energy decay rate when β has polynomial growth near zero and $-h'/h \ge m$ for all t, where m is some positive constant

Assume that $\beta(s) = s^\gamma$ for some $\gamma > 1$. In this particular case, it is interesting to estimate using the method developed by [11]. It is necessary to use the multiplier $E^{(\gamma-1)/2}(t)\phi'(t)u$ in place of $E(t)\phi'(t)u$. Then similar computations lead to

$$\begin{aligned}\int_S^T E^{(\gamma+1)/2}(t)\phi'(t)\,dt \le CE^{(\gamma+1)/2}(S) + C\int_S^T E^{(\gamma-1)/2}(t)\phi'(t)\|u'(t)\|_2^2\,dt\\ + C\int_S^T E^{(\gamma-1)/2}(t)\phi'(t)\int_\Omega |g(u'(t))|^2\,dxdt\\ + C\int_S^T E^{(\gamma-1)/2}(t)\phi'(t)(h\diamond u)(t)\,dt.\end{aligned} \tag{3.10}$$

By hypotheses on g, we have

$$\begin{aligned}\int_{|u'|\le 1}|u'|^2\,dx &\le \int_{|u'|\le 1}(u'g(u'))^{2/(\gamma+1)}\,dx\\ &\le C\left(\int_{|u'|\le 1}u'g(u')\,dx\right)^{2/(\gamma+1)} \le \left(-E'(t)\right)^{2/(\gamma+1)}\end{aligned}$$

and

$$\int_{|u'|>1} |u'|^2\,dx \le C \int_{|u'|>1} u'g(u')\,dx \le -CE'(t).$$

Hence

$$\begin{aligned}
&\int_S^T E^{(\gamma-1)/2}(t)\phi'(t)\int_\Omega |u'|^2\,dxdt \\
&= \int_S^T E^{(\gamma-1)/2}(t)\phi'(t)\int_{|u'|\le 1}|u'|^2\,dxdt + \int_S^T E^{(\gamma-1)/2}(t)\phi'(t)\int_{|u'|>1}|u'|^2\,dxdt \\
&\le C\int_S^T E^{(\gamma-1)/2}(t)\phi'(t)\left(-E'(t)\right)^{2/(\gamma+1)}\,dt + C\int_S^T E^{(\gamma-1)/2}(t)\left(-E'(t)\right)\,dt \\
&\le \epsilon\int_S^T E^{(\gamma+1)/2}(t)\phi'(t)\,dt + C(\epsilon)E(S) + CE^{(\gamma+1)/2}(S).
\end{aligned} \tag{3.11}$$

Similarly,

$$\begin{aligned}
&\int_S^T E^{(\gamma-1)/2}(t)\phi'(t)\int_\Omega g(u')^2\,dxdt \\
&\le \epsilon\int_S^T E^{(\gamma+1)/2}(t)\phi'(t)\,dt + C(\epsilon)E(S) + E^{(\gamma+1)/2}(S).
\end{aligned} \tag{3.12}$$

Moreover, by using the same argument as (3.8), we have

$$\int_S^T E^{(\gamma-1)/2}(t)\phi'(t)(h\diamond u)(t)\,dt \le CE^{(\gamma+1)/2}(S). \tag{3.13}$$

By replacing (3.11), (3.12) and (3.13) in (3.10) and choosing ϵ sufficiently small, then we get

$$\int_S^T E^{(\gamma+1)/2}(t)\phi'(t)\,dt \le CE(S),$$

which implies by Lemma 3.1 and choosing $\phi(t) = kt$ that

$$E(t) \le \frac{CE(0)}{(1+t)^{2/(\gamma-1)}}.$$

3.4. Energy decay rate when β has polynomial growth near zero and $-h'/h$ decays to zero at infinity

By using the same argument as Subsection 3.2, if we set $\phi(t) = \ln(h(0)/h(t))$, then we get (3.13). Therefore, by using the same argument as Subsection 3.3 and replacing $\phi(t) = \ln(h(0)/h(t))$, then we obtain

$$E(t) \le \frac{CE(0)}{(-\ln h(t))^{2/(\gamma-1)}}.$$

3.5. Energy decay rate when β does not necessarily have polynomial growth near zero and $-h'/h \geq m$ for all t, where m is some positive constant

Assume that the function $G(s) = \beta(s)/s$ is nondecreasing on $(0,1)$ and $G(0) = 0$. Let $\phi(t)$ be the concave function such that its inverse is defined by

$$\phi^{-1}(t) = 1 + \int_1^t \frac{1}{G(1/s)}\, ds$$

for all $t \geq 1$. Then $\phi(t)$ satisfies all the required properties and can be easily extended on $[0,1)$ such that it remains concave nondecreasing (see [14]). Moreover,

$$\begin{aligned} \int_S^\infty \phi'(t)(G^{-1}(\phi'(t)))^2\, dt &= \int_{\phi(S)}^\infty (G^{-1}(\phi'(\phi^{-1}(s))))^2\, ds \\ &= \int_{\phi(S)}^\infty \left(G^{-1}\left(\frac{1}{(\phi^{-1})'(s)}\right)\right)^2 ds = \int_{\phi(S)}^\infty \frac{1}{s^2}\, ds \\ &= \frac{1}{\phi(S)} \leq \beta^{-1}\left(\frac{1}{S}\right). \end{aligned} \tag{3.14}$$

Assume that $\phi'(t) \leq -m_1 \frac{h'(t)}{h(t)}$ for some positive constant m_1. Then we have

$$I_8 \leq m_1 E^2(S). \tag{3.15}$$

Next we will estimate I_6 and I_7.

Estimate for $I_6 = \int_S^T E(t)\phi'(t)\|u'(t)\|_2^2\, dt$. For every $t \geq 1$ let us define

$$\begin{aligned} \Omega_1 &= \{x \in \Omega : |u'(t)| \leq f(t)\}, \\ \Omega_2 &= \{x \in \Omega : f(t) < |u'(t)| \leq f(1)\}, \\ \Omega_3 &= \{x \in \Omega : |u'(t)| > f(1)\}, \end{aligned}$$

where each Ω_i depends on t and $f(t) = G^{-1}(\phi'(t))$ such that f is a decreasing positive function which satisfies $f(t) \to 0$ as $t \to +\infty$.

Now, let us consider these three cases.

Case 1. Part on Ω_3.

First, we claim that $f(1) > 0$. Indeed, we assume that $f(1) = 0$. Then $G^{-1}(\phi'(1)) = f(1) = 0$, i.e., $\phi'(1) = G(0) = 0$. It implies that $\phi'(t) \leq \phi'(1)$ for all $t \geq 1$, consequently, $\phi'(t) = 0$ for all $t \geq 1$. This contradicts the fact that ϕ is strictly increasing. Thus, $f(1) > 0$.

If $f(1) > 1$, then from (2.2), $|g(u'(t))| \geq c_1|u'(t)|$.

If $f(1) \leq 1$, we note that the function $F\colon s \mapsto g(s)/s$ is positive and continuous on $[-1,-f(1)] \cup [f(1),1]$ which implies that there exists a positive constant c_3 satisfying

$g(s)/s \geq c_3$ for $|s| \in [f(1), 1]$, i.e., $|g(u'(t))| \geq c_3|u'(t)|$. Hence, for $c_4 = \min\{c_1, c_3\}$, $|u'(t)| \leq \frac{1}{c_4}|g(u'(t))|$. Then we have

$$\begin{aligned}\int_S^T E(t)\phi'(t)\int_{\Omega_3}|u'(t)|^2\,dxdt &\leq \frac{1}{c_4}E(S)\int_S^T \phi'(t)\int_{\Omega_3}|u'(t)||g(u'(t))|\,dxdt \\ &\leq \frac{\phi'(S)}{c_4}E(S)\int_S^T\int_\Omega u'(t)g(u'(t))\,dxdt \\ &\leq \frac{\phi'(S)}{c_4}E^2(S).\end{aligned} \tag{3.16}$$

Case 2. Part on Ω_2.

Since G is nondecreasing on $(0,1)$, then we have

$$\phi'(t)|u'(t)|^2 = G(f(t))|u'(t)|^2 \leq G(|u'(t)|)|u'(t)|^2 = u'(t)\beta(u'(t)) \leq u'(t)g(u'(t))$$

for all $x \in \Omega_2$, consequently, we obtain

$$\int_S^T E(t)\phi'(t)\int_{\Omega_2}|u'(t)|^2\,dxdt \leq E(S)\int_S^T\int_\Omega u'(t)g(u'(t))\,dxdt \leq E^2(S). \tag{3.17}$$

Case 3. Part on Ω_1.

By the definition of the boundary of this paper, we have

$$\begin{aligned}\int_S^T E(t)\phi'(t)\int_{\Omega_1}|u'(t)|^2\,dxdt &\leq E(S)\int_S^T \phi'(t)\int_{\Omega_1} f^2(t)\,dxdt \\ &\leq \operatorname{meas}(\Omega)E(S)\int_S^T \phi'(t)(G^{-1}(\phi'(t)))^2\,dt.\end{aligned} \tag{3.18}$$

From (3.16)–(3.18), we get

$$I_6 \leq CE^2(S) + CE(S)\int_S^T \phi'(t)(G^{-1}(\phi'(t)))^2\,dt. \tag{3.19}$$

Estimate for $I_7 = \int_S^T E(t)\phi'(t)\int_\Omega |g(u'(t))|^2\,dxdt$. For every $t \geq 1$ let us define

$$\begin{aligned}\Omega_4 &= \{x \in \Omega : \beta^{-1}(|u'(t)|) \leq f(t)\}, \\ \Omega_5 &= \{x \in \Omega : f(t) < \beta^{-1}(|u'(t)|) \leq f(1)\}, \\ \Omega_6 &= \{x \in \Omega : \beta^{-1}(|u'(t)|) > f(1)\}.\end{aligned}$$

By using the similar arguments as the estimate for I_6, we obtain the same estimate

$$I_7 \leq CE^2(S) + CE(S)\int_S^T \phi'(t)(G^{-1}(\phi'(t)))^2\,dt. \tag{3.20}$$

By replacing (3.15), (3.19) and (3.20) in (3.5) and using Lemma 3.2 and (3.14), we obtain

$$E(t) \leq CE(0)\left(\beta^{-1}\left(\frac{1}{t}\right)\right)^2.$$

3.6. Energy decay rate when β does not necessarily have polynomial growth near zero and $-h'/h$ decays to zero at infinity

In order to prove this subsection, we introduce the useful lemma.

Lemma 3.3. [5] *Given $f_1, f_2\colon \mathbb{R}_+ \to \mathbb{R}_+$ two continuous functions such that $f_1(t) \to 0$ as $t \to \infty$ and $\int_0^\infty f_1 = +\infty$, f_2 is nondecreasing on a neighborhood of* 0 *and $f_2(0) = 0$. Then there always exists $f_3\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$f_3 \le f_1, \quad \int_0^\infty f_3 = +\infty, \quad \int_0^\infty f_3(t) f_2(f_3(t))\, dt < +\infty. \tag{3.21}$$

Moreover, if f_1 is nonincreasing, then f_3 can also be chosen nonincreasing.

Now, we define $f_1 := -h'(t)/h(t)$ and choose $f_2 := G^{-1}(\cdot)^2$. Then $\int_0^t f_1 = \ln(h(0)/h(t)) \to \infty$ as $t \to \infty$ and f_1 is nonincreasing. Also, by the definition of G, f_2 is nondecreasing on a neighborhood of 0 and $f_2(0) = 0$. Hence, by using Lemma 3.3 there exists $\phi' := f_3$ that is nonincreasing and that satisfies (3.21). So we get that

$$\phi'(t) h(t-s) \le \phi'(t-s) h(t-s) \le -h'(t-s).$$

Hence,

$$I_8 \le E^2(S). \tag{3.22}$$

And by using the same arguments as Subsection 3.5 and (3.21), we obtain

$$I_6 \le CE^2(S) + CE(S) \int_S^T \phi'(t)(G^{-1}(\phi'(t)))^2\, dt \le CE^2(S) + CE(S) \tag{3.23}$$

and

$$I_7 \le CE^2(S) + CE(S) \int_S^T \phi'(t)(G^{-1}(\phi'(t)))^2\, dt \le CE^2(S) + CE(S). \tag{3.24}$$

By replacing (3.22), (3.23) and (3.24) in (3.5) and using Lemma 3.1, we obtain

$$E(t) \le \frac{CE(0)}{\phi(t)}.$$

Next, we assume that

$$\frac{d}{dt}\left[h^{-1}\left(\frac{h(0)}{e^t}\right)\right] \ge \frac{1}{t\beta(1/t)} \quad \text{for all } t \ge 1. \tag{3.25}$$

Let us set $\phi(t) = \ln(h(0)/h(t))$, then $\phi(t)$ satisfies all the required properties and $\phi'(t) = -h'(t)/h(t)$. So we get that

$$\phi'(t) h(t-s) \le \phi'(t-s) h(t-s) \le -h'(t-s).$$

Hence,

$$I_8 \le E^2(S). \tag{3.26}$$

On the other hand, by the same argument as (3.19) and (3.20), we have

$$I_6,\, I_7 \le CE^2(S) + CE(S)\int_S^T \phi'(t)(G^{-1}(\phi'(t)))^2\,dt. \tag{3.27}$$

By replacing (3.26) and (3.27) in (3.5), we obtain

$$\begin{aligned}
\int_S^T E^2(t)\phi'(t)\,dt &\le CE^2(S) + CE(S)\int_S^\infty \phi'(t)(G^{-1}(\phi'(t)))^2\,dt \\
&\le CE^2(S) + CE(S)\int_{\phi(S)}^\infty (G^{-1}(\phi'(\phi^{-1}(s))))^2\,ds \\
&\le CE^2(S) + CE(S)\int_{\phi(S)}^\infty \left(G^{-1}\left(\frac{1}{(\phi^{-1})'(s)}\right)\right)^2\,ds.
\end{aligned} \tag{3.28}$$

We now solve the $\phi^{-1}(t)$. Since $\phi(t) = \ln(h(0)/h(t))$, we can obtain by simple computation that $\phi^{-1}(t) = h^{-1}(h(0)/e^t)$. We note that since $h^{-1}(t)$ is decreasing positive function, $\phi^{-1}(t)$ is increasing positive function. By using the assumption (3.25) and the fact G^{-1} is increasing, we deduce that

$$\begin{aligned}
\int_{\phi(S)}^\infty \left(G^{-1}\left(\frac{1}{(\phi^{-1})'(s)}\right)\right)^2\,ds &\le \int_{\phi(S)}^\infty \left(G^{-1}\left(G\left(\frac{1}{t}\right)\right)\right)^2\,ds \\
&= \int_{\phi(S)}^\infty \frac{1}{s^2}\,ds = \frac{1}{\phi(S)}.
\end{aligned} \tag{3.29}$$

By replacing (3.29) in (3.28), we have

$$\int_S^T E^2(t)\phi'(t)\,dt \le CE^2(S) + \frac{CE(S)}{\phi(S)},$$

which implies by Lemma 3.2 and definition of $\phi(t)$ that

$$E(t) \le \frac{CE(0)}{(-\ln h(t))^2}.$$

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Tae Gab Ha
Department of Mathematics, and Institute of Pure and Applied Mathematics, Chonbuk National University, Jeonju 54896, Republic of Korea
E-mail address: `tgha@jbnu.ac.kr`