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Carleson Measures and Trace Theorem for β-harmonic Functions

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Abstract. General harmonic extension has no uniqueness and harmonic functions may have different non-tangential boundary values in different convergence sense. In this paper, we establish first β-harmonic functions in ultra-distribution frame. Further, we consider the characterization between Carleson measure space and boundary distribution space. For β-harmonic functions with boundary distributions, there exists no maximum value principle. We apply Meyer wavelets to introduce basic harmonic functions and basic observers. We apply Meyer wavelets and vaguelette knowledge to prove the uniqueness of β-harmonic extension and prove also that β-harmonic function converges to boundary distribution in the relative norm sense.

1. Introduction

A classic harmonic function in $\mathbb{R}^{n+1}_+$ is a function satisfying the following equation:

$$\left(\partial_t^2 + \sum_{i=1,\dots,n} \partial_{x_i}^2\right) f(t,x) = 0 \quad \text{in } \mathbb{R}^{n+1}_+. \tag{1.1}$$

In this paper, we extend the classic harmonic functions to β-harmonic flow functions with boundary distributions. Denote $(-\Delta)^\beta$ the β-order Laplace operator defined by the Fourier transform:

$$\widehat{(-\Delta)^\beta u}(\xi) = |\xi|^{2\beta} \widehat{u}(\xi) \quad \text{in } \mathbb{R}^n.$$

β-harmonic function in $\mathbb{R}^{n+1}_+$ is defined as the flow distribution in $\mathbb{R}^n$ of parameter t. The exact sense of the following equation (1.2) will be discussed in Section 2.

Definition 1.1. Given $\beta > 0$. A function $f(t,x)$ on $\mathbb{R}^{n+1}_+$ is called to be a β-harmonic function, if

$$\partial_t^2 f(t,x) - (-\Delta)^\beta f(t,x) = 0 \quad \text{in } \mathbb{R}^{n+1}_+. \tag{1.2}$$

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The study of β-harmonic functions plays an important role in the PDE problem. For example, the following generalized Navier-Stokes equations:

$$\begin{cases} \partial_t + (-\Delta)^\beta u + u \cdot \nabla u - \nabla p = 0 & \text{in } \mathbb{R}^{1+n}_+, \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}^{1+n}_+, \\ u|_{t=0} = u_0 & \text{in } \mathbb{R}^n. \end{cases} \tag{1.3}$$

Picard's iterative process is to find out a mild solution near the 2β-harmonic function $u^{(0)} = e^{-t(-\Delta)^\beta} u_0$ where u_0 is a distribution. Denote

$$B(u,u)(t,x) \equiv \int_0^t e^{(t-s)\Delta} \left\{ \sum_l \partial_{x_l}(u_l u) - \sum_l \sum_{l'} (-\Delta)^{-1} \partial_{x_l} \partial_{x_{l'}} \nabla(u_l u_{l'}) \right\} ds.$$

For all $j = 0, 1, 2, \ldots$, denote

$$u^{(j+1)}(t,x) = u^{(0)}(t,x) - B(u^{(j)}, u^{(j)})(t,x).$$

For small initial value u_0, $B(u^{(j)}, u^{(j)})(t,x)$ is an error term. Picard's contraction principle tells that $u^{(j)}$ converges to a unique solution of the above Navier-Stokes equations (1.3). See Cannone [2] and Koch-Tataru [8] for $\beta = 2$; see Li-Yang [10, 11] and Lin-Yang [14] for $\beta > 1$. Hence, it is helpful for the non-linear problem to understand better the properties of β-harmonic functions generated from distributions.

β-harmonic function $f(t,x)$ satisfying (1.1) is just a ultra-distribution. In this paper, we study the relation between distributions $f(x)$ on $\mathbb{R}^n$ and the β-harmonic functions $f(t,x)$ on $\mathbb{R}^{n+1}_+$. That is to say, we consider the following Cauchy problem of the equation (1.2):

$$\begin{cases} \partial_t^2 f(t,x) - (-\Delta)^\beta f(t,x) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ f(0,x) = f(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{1.4}$$

But we know, even for $\beta = 1$, the above problem is an ill-posed problem. A distribution $f(x)$ on $\mathbb{R}^n$ can be extend to different β-harmonic functions $f(t,x)$ on $\mathbb{R}^{n+1}_+$. Further, a β-harmonic function may have different non-tangential boundary value for different convergence sense. In Section 2, we establish β-harmonic functions in ultra-distribution frame. We provide a strict sense of boundary distribution and β-harmonic function. See Theorems 2.8 and 2.10.

Let $C_n = \Gamma((n+1)/2)/\pi^{(n+1)/2}$ and let

$$\begin{aligned} P(x) &= \frac{C_n}{(1+|x|^2)^{(n+1)/2}}, \\ P_t(x) &= t^{-n} P\left(\frac{x}{t}\right) = \frac{C_n t}{(t^2+|x|^2)^{(n+1)/2}}, \\ \widehat{P}_t(\xi) &= e^{-t|\xi|}. \end{aligned}$$

The Poisson integral of f is defined by

$$f(t,x) = P_t f(x) =: \int_{\mathbb{R}^n} P_t(x-y) f(y)\, dy.$$

For $f(x)$ satisfies the following condition

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{1+|x|^{n+1}}\, dx < \infty,$$

its Poisson integration $P_t f(x)$ corresponding to a classic harmonic function. For $\beta = 1$ and $f(t,x)$ satisfies the maximum value principle, the harmonic extension $f(t,x)$ of function $f(x)$ becomes Poisson extension. Triebel [19] considered the Poisson characterization of Besov spaces where Poisson extension has uniqueness. Alvarez-Guzmán-Partida-Pérez-Esteva [1] extend harmonic extension to distributions.

If we replace harmonic extension to Poisson type extension, the equation (1.4) becomes the following equation (1.5). For $\beta > 0$, we consider the Cauchy problem of the following heat equations:

$$\begin{cases} \{\partial_t + (-\Delta)^{\beta/2}\} u(t,x) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ u(0,x) = f(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{1.5}$$

Poisson type extension is unique. See Proposition 2.5. Harmonic extension has no uniqueness. See Theorem 2.8. Harmonic functions have different non-tangential boundary values in different convergence sense. See Proposition 2.9.

But in Section 3, we consider a one-to-one relation between Carleson measure and the q-mean oscillation functions. β-harmonic functions with boundary distributions have no maximum value principle. We apply Meyer wavelets to establish theorem about the uniqueness of the generalized β-harmonic functions extension and to established that the β-harmonic function converges to boundary distribution in the relative norm sense. See Theorems 3.6 and 3.7. From Sections 4 to 7, we present some results on Meyer wavelets, vaguelets and Poisson type extension. In last section, we apply these results to prove these two theorems.

In fact, in Section 4, we present some preliminaries on Meyer wavelets and wavelet characterization of q-mean oscillation space $M^{\alpha,q}$. Meyer introduced the conception of vaguelettes in [15]. In Section 5, we use Meyer wavelets to study two kinds of vaguelettes relative to β-harmonic functions. By applying Meyer wavelets, the Poisson type extension is changed to the sum of basic β-harmonic functions. The computation about boundary distribution is changed to the inner product of β-harmonic function and the basic observers. The basic β-harmonic functions and the basic observers are all vagulettes on the parameter t. The derivative of basic β-harmonic functions and the integration of basic observers are all located at $t2^{j\beta}$.

In Sections 6 and 7, we apply vaguelette knowledge to prove the properties of Poisson type extension of q-mean oscillation spaces and boundary distribution of Carleson measures. In Section 8, we prove Theorems 3.6 and 3.7 which are independent to Poisson type extension.

Some notations:

- $\mathsf{U} \lesssim \mathsf{V}$ represents that there is a constant $C > 0$ such that $\mathsf{U} \leq C\mathsf{V}$.
- For convenience, the positive constants C may change from one line to another and usually depend on the dimension n, α, β and other fixed parameters.
- The Schwartz class of rapidly decreasing functions and its dual will be denoted by $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$, respectively. Let

  $$\mathcal{S}_0(\mathbb{R}^n) = \left\{ f \in \mathcal{S}(\mathbb{R}^n), \int x^\gamma f(x)\, dx = 0, \forall\, \gamma \in \mathbb{N}^n \right\}$$

  and denote its dual by $\mathcal{S}_0'(\mathbb{R}^n)$.
- Denote $f(x) \in \mathcal{S}_{\text{strip}}(\mathbb{R}^n)$, if $f(x) \in \mathcal{S}(\mathbb{R}^n)$ and there exists $0 < C_1 < C_2$ such that

  $$\operatorname{Supp} \widehat{f}(\xi) \subset \{\xi \in \mathbb{R}^n, C_1 \leq |\xi| \leq C_2\}.$$
- For $\beta > 0$ and $t > 0$, denote $\mathcal{S}^t_{\beta,0}(\mathbb{R}^n) = e^{-t(-\Delta)^{\beta/2}} \mathcal{S}_0(\mathbb{R}^n)$ and

  $$\mathcal{S}'_{\beta,t}(\mathbb{R}^n) = \left(\mathcal{S}^t_{\beta,0}(\mathbb{R}^n)\right)'.$$
- $\widehat{f}$ denotes the Fourier transform of f.

2. β-harmonic functions in ultra-distribution frame

The meaning of the Cauchy problem of the equation (1.2) is not clear. We will give a clear meaning of the Cauchy problem (1.4) in this section. First, we present two kinds of harmonic flows relative to harmonic extension. Then we consider harmonic extension. In the end of this section, we consider the meaning of boundary value.

2.1. Poisson type flow

Poisson integrals are applied to characterize a series of function spaces. See [4, 17, 19]. Denote $L^1_{\text{loc}}(\mathbb{R}^n)$ the locally integrable function spaces on $\mathbb{R}^n$. When one studies the harmonic function spaces with Poisson kernel, one assumes often the local integrability condition

$$f \in L^1_{\text{loc}}(\mathbb{R}^n).$$

See Essen-Janson-Peng-Xiao [4], Fabes-Neri [6] and Wu-Xie [21]. Local integrable property may cause the incompleteness of function space. In fact, $\forall\, 1/2 < p < 1$, Hardy spaces $H^p(\mathbb{R}^n)$ are completed quasi-norm spaces, but $H^p(\mathbb{R}) \cap L^1_{\text{loc}}(\mathbb{R})$ are not completed spaces.

Example 2.1. Let $\varphi \in \mathcal{S}(\mathbb{R})$ and $\int \varphi(x)\, dx = 1$. We have

$$\frac{1}{t}\left\{\phi\left(\frac{x}{t}\right) - \phi\left(\frac{x-1}{t}\right)\right\} \in H^p(\mathbb{R}) \cap L^1_{\text{loc}}(\mathbb{R}).$$

The limitation of the above Cauchy sequence is not a locally integrable function. In fact,

$$\lim_{t\to 0}\frac{1}{t}\left\{\phi\left(\frac{x}{t}\right) - \phi\left(\frac{x-1}{t}\right)\right\} = \delta(x) - \delta(x-1) \notin L^1_{\text{loc}}(\mathbb{R}).$$

Further,

$$H\{\delta(x) - \delta(x-1)\} = \frac{1}{\pi}\left\{\frac{1}{x} - \frac{1}{x-1}\right\}.$$

For $1/2 < p < 1$, we have

$$\|\delta(x) - \delta(x-1)\|_{L^p(\mathbb{R})} = 0.$$
$$0 < \left\|\frac{1}{\pi}\left\{\frac{1}{x} - \frac{1}{x-1}\right\}\right\|_{L^p} < \infty.$$

Hence for $1/2 < p < 1$,

$$0 < \|\delta(x) - \delta(x-1)\|_{H^p} < \infty.$$

To avoid the local integrability condition, Stein-Weiss [18] used the Poisson semi-group operators and bounded distributions to consider harmonic functions in Hardy spaces $H^p(\mathbb{R}^n)$

$$\widehat{P_t f}(\xi) =: e^{-t|\xi|}\widehat{f}(\xi). \tag{2.1}$$

If $f(x)$ is a bounded distribution in $\mathcal{S}'(\mathbb{R}^n)$, then $P_t f$ is a distribution in $\mathcal{S}'(\mathbb{R}^n)$ and one can define the non-tangential maximum function.

But $e^{-t|\xi|}$ is not smooth at zero. The above equation (2.1) can not extend a distribution $f(x)$ in $\mathcal{S}'(\mathbb{R}^n)$ to the Poisson flow $f(t, x)$ in $\mathcal{S}'(\mathbb{R}^n)$. In fact, we have

Example 2.2. Let $f(x) = x^2 \in \mathcal{S}'(\mathbb{R})$. Then $\forall\, t > 0$, we have

(i) $P_t f(x)$ can not be defined as a distribution in $\mathcal{S}'(\mathbb{R})$.

(ii) $P_t f(x)$ is the zero element in $\mathcal{S}'_0(\mathbb{R})$.

In fact, we know $f(x) = x^2 \in \mathcal{S}'(\mathbb{R})$. Note that,

$$\frac{y^2}{t^2+(x-y)^2} = \frac{(x-y)^2}{t^2+(x-y)^2} - 2x\frac{x-y}{t^2+(x-y)^2} + x^2\frac{1}{t^2+(x-y)^2}.$$

For all $g(x) \in \mathcal{S}(\mathbb{R})$, formally,

$$\begin{aligned}\int P_t f(x) g(x)\,dx &= \int y^2 P_t g(y)\,dy = \iint \frac{ty^2}{t^2+(x-y)^2} g(x)\,dxdy \\ &= \int g(x)\,dx \int \frac{ty^2}{\pi(t^2+y^2)}\,dy - 2\int xg(x)\,dx \int \frac{ty}{\pi(t^2+y^2)}\,dy \\ &\quad + \int x^2 g(x)\,dx \int \frac{t}{\pi(t^2+y^2)}\,dy.\end{aligned}$$

(i) If we take $g(x) \in \mathcal{S}(\mathbb{R})$ such that $\int g(x)\,dx = 1$ and $\int xg(x)\,dx = \int x^2 g(x)\,dx = 0$. By a direct calculation, for all $t > 0$,

$$\int P_t f(x) g(x)\,dx = \int \frac{ty^2}{\pi(t^2+y^2)}\,dy = \infty.$$

For all $t > 0$, we know $P_t f(x) \notin \mathcal{S}'(\mathbb{R})$.

(ii) If $g(x) \in \mathcal{S}_0(\mathbb{R})$, then $\int g(x)\,dx = \int xg(x)\,dx = \int x^2 g(x)\,dx = 0$. Hence $\forall\, t > 0$,

$$\int P_t f(x) g(x)\,dx = 0.$$

That is to say, $\forall\, t > 0$, $P_t f(x)$ is the zero element in $\mathcal{S}'_0(\mathbb{R})$.

We clarify first the sense of Poisson type semigroup operator. Denote P_t^β a Poisson type semigroup operator defined as follows:

$$f(t,x) = P_t^\beta f(x) =: e^{-t(-\Delta)^{\beta/2}} f(x) \equiv (2\pi)^{-n} \int e^{-t|\xi|^\beta} \widehat{f}(\xi) e^{ix\xi}\,d\xi. \tag{2.2}$$

For $\beta \in 2\mathbb{N}$, we have

Theorem 2.3. (i) *$\forall\, f(x) \in \mathcal{S}'(\mathbb{R}^n)$ and $\forall\, t > 0$, we have $P_t^\beta f(x) \in \mathcal{S}'(\mathbb{R}^n)$.*

(ii) *$\forall\, f(x) \in \mathcal{S}'_0(\mathbb{R}^n)$ and $\forall\, t > 0$, we have $P_t^\beta f(x) \in \mathcal{S}'_0(\mathbb{R}^n)$.*

(iii) *Further, $f(x) \in \mathcal{S}'(\mathbb{R}^n)$, $\forall\, t > 0$, $x \in \mathbb{R}^n$, $\forall\, \gamma_1 \in \mathbb{N}$, $\gamma_2 \in \mathbb{N}^n$, there exists a positive real number $C_{t,x}^{\gamma_1,\gamma_2}$ such that*

$$|\partial_t^{\gamma_1} \partial_x^{\gamma_2} f(t,x)| < C_{t,x}^{\gamma_1,\gamma_2}. \tag{2.3}$$

Proof. (i) For $\beta \in 2\mathbb{N}$ and $g \in \mathcal{S}(\mathbb{R}^n)$, we know $e^{-t|\xi|^\beta} \widehat{g}(\xi) \in \mathcal{S}(\mathbb{R}^n)$. For all $f(x) \in \mathcal{S}'(\mathbb{R}^n)$ and any parameter t,

$$|\langle P_t^\beta f(x), g(x) \rangle| < C_t.$$

Hence we have $P_t^\beta f(x) \in \mathcal{S}'(\mathbb{R}^n)$.

(ii) is similar obtained as (i).

(iii) $\forall\, t > 0$, $x \in \mathbb{R}^n$, $\forall\, \gamma_1 \in \mathbb{N}$, $\gamma_2 \in \mathbb{N}^n$, we have

$$\partial_t^{\gamma_1}\partial_x^{\gamma_2} f(t,x) = (2\pi)^{-n} \int e^{-t|\xi|^\beta}(-|\xi|^\beta)^{\gamma_1}(i\xi)^{\gamma_2}\widehat{f}(\xi)e^{ix\xi}\, d\xi.$$

Further, for $\widehat{f} \in \mathcal{S}'(\mathbb{R}^n)$, there exists $N, M \in \mathbb{N}$ and $\gamma \in \mathbb{N}^n$, there exist $a_{\gamma,N}$ and $f_\gamma(\xi) \in L^\infty(\mathbb{R}^n)$ such that

$$\widehat{f}(\xi) = (1 + |\xi|^2)^N \sum_{|\gamma|\le M} a_{\gamma,N}\partial_\xi^\gamma f_\gamma(\xi).$$

Hence there exist polynomial functions $a_{\gamma,N}(t,x)$ and $P_{\gamma,N}(\xi)$ such that

$$\partial_t^{\gamma_1}\partial_x^{\gamma_2} f(t,x) = \sum_{|\gamma|\le M} \int e^{-t|\xi|^\beta} a_{\gamma,N}(t,x)P_{\gamma,N}(\xi)f_\gamma(\xi)e^{ix\xi}\, d\xi.$$

So we get the estimate (2.3). □

For $\beta > 0$ and $\beta \notin 2\mathbb{N}$, the equation (2.2) can not extend a distribution in $\mathcal{S}'(\mathbb{R}^n)$ to a distribution in $\mathcal{S}'(\mathbb{R}^n)$. But we have

Theorem 2.4. *Given $f(x) \in \mathcal{S}'(\mathbb{R}^n)$ or $f(x) \in \mathcal{S}_0'(\mathbb{R}^n)$. For all $t > 0$, we have $P_t^\beta f(x) \in \mathcal{S}_0'(\mathbb{R}^n)$.*

Proof. Since $\mathcal{S}'(\mathbb{R}^n) \subset \mathcal{S}_0'(\mathbb{R}^n)$, it is sufficient to consider the case where $f(x) \in \mathcal{S}_0'(\mathbb{R}^n)$. For $\beta > 0$, $g \in \mathcal{S}_0(\mathbb{R}^n)$, we know $e^{-t|\xi|^\beta}\widehat{g}(\xi) \in \mathcal{S}_0(\mathbb{R}^n)$. That is to say, $\forall\, f(x) \in \mathcal{S}_0'(\mathbb{R}^n)$ and for any parameter t,

$$|\langle P_t^\beta f(x), g(x)\rangle| = |\langle f(x), P_t^\beta g(x)\rangle| < C_t.$$

Hence we have $P_t^\beta f(x) \in \mathcal{S}_0'(\mathbb{R}^n)$. □

For $\beta > 0$, we know

Proposition 2.5. *For $f(x) \in \mathcal{S}_0'(\mathbb{R}^n)$, $P_t^\beta f(x)$ is the unique solution of* (1.5) *and it is a β-harmonic function.*

Proof. Applying Fourier transform to the first equation of (1.5),

$$(\partial_t + |\xi|^\beta)\widehat{u}(t,\xi) = 0.$$

We get formally the equation (2.2). Hence $P_t^\beta f(x)$ is the unique solution of the equation (1.5). □

2.2. Ultra-distribution flow H_t^β and Harmonic extension

For $\beta > 0$, denote H_t^β the operator defined as follows:

$$H_t^\beta f(x) =: e^{t(-\Delta)^{\beta/2}} f(x) \equiv (2\pi)^{-n} \int e^{t|\xi|^\beta} \widehat{f}(\xi) e^{ix\xi}\, d\xi.$$

H_t^β can map good function to good function. It is easy to see

Theorem 2.6. *For all $\beta > 0$, $t > 0$ and $f \in \mathcal{S}_{\text{strip}}(\mathbb{R}^n)$, we have $H_t^\beta f(x) \in \mathcal{S}_{\text{strip}}(\mathbb{R}^n)$.*

But H_t^β maps only distribution to ultra-distribution:

Theorem 2.7. *For $\beta > 0$, $\forall\, f(x) \in \mathcal{S}'_0(\mathbb{R}^n)$ and $\forall\, t > 0$, we have*

$$H_t^\beta f(x) \in \mathcal{S}'_{\beta,t}(\mathbb{R}^n).$$

Now we consider the harmonic extension. The Cauchy problem of the equation (1.4) is an ill-posed problem. For all $C_1 \in \mathbb{C}$ and $C_2 = 1 - C_1$, denote

$$P_t^{\beta,C_1} f(x) = C_1 H_t^\beta f(x) + C_2 P_t^\beta f(x).$$

For all $C_1 \in \mathbb{C}$, $P_t^{\beta,C_1} f(x)$ is a β-harmonic function with initial value $f(x)$.

Theorem 2.8. *For $f(x) \in \mathcal{S}'_0(\mathbb{R}^n)$, we have*

(i) *The distribution $f(t,x) = P_t^{\beta,C_1} f(x) \in \mathcal{S}'_{\beta,t}(\mathbb{R}^n)$, $\forall\, t > 0$.*

(ii) *For all $C_1 \in \mathbb{C}$, $P_t^{\beta,C_1} f(x)$ satisfies the equation (1.4).*

(iii) *All the β-harmonic extension functions must be the form $P_t^{\beta,C_1} f(x)$.*

Proof. The conclusion of $f(t,x) = P_t^{\beta,C_1} f(x) \in \mathcal{S}'_{\beta,t}(\mathbb{R}^n)$ is the direct corollary of the theorems at the begin of this subsection.

Further, applying Fourier transform to the equation (1.2), we have

$$\partial_t^2 \widehat{f}(t,\xi) - |\xi|^{2\beta} \widehat{f}(t,\xi) = 0.$$

The formal solution of the above equation is

$$\widehat{f}(t,\xi) = \left\{ C_1 e^{t|\xi|^\beta} + C_2 e^{-t|\xi|^\beta} \right\} \widehat{f}(\xi),$$

where $C_1, C_2 \in \mathbb{C}$.

Considering the initial value condition, we must take $C_2 = 1 - C_1$. Then the function $f(t,x) = P_t^{\beta,C_1} f(x)$ satisfies the equation (1.4). □

We can not use maximum value principle to get the uniqueness of the β-harmonic extension for β-harmonic function with boundary distribution. We will present how to use Meyer wavelets to get the uniqueness of the β-harmonic extension after Section 3.

2.3. Integral boundary value

For harmonic function, different convergence sense may produce different non-tangential boundary value. In fact, Stein-Weiss [18] considered Hardy spaces H^p. We know that $P_t(x) - P_t(x-1) \in H^p(\mathbb{R}^2_+)$, $\forall\, 1/2 < p < 1$. We consider the almost everywhere convergence, L^p norm convergence, H^p norm convergence and convergence in distribution sense of this flow function. We have

Proposition 2.9. *(i) $\lim_{t\to 0}\{P_t(x) - P_t(x-1)\} = 0$, $\forall\, x \neq 0, 1$.*

(ii) $\lim_{t\to 0} \|P_t(x) - P_t(x-1)\|_{L^p(\mathbb{R})} = 0$, $\forall\, 0 < p < 1$.

(iii) $\lim_{t\to 0}\langle P_t(x) - P_t(x-1), \phi(x)\rangle = \phi(0) - \phi(1)$, $\forall\, \phi(x) \in \mathcal{S}(\mathbb{R})$.

(iv) $\lim_{t\to 0} \|P_t(x) - P_t(x-1) - \{\delta(x) - \delta(x-1)\}\|_{H^p(\mathbb{R})} = 0$, $\forall\, 1/2 < p < 1$.

Proof. The proof of (i), (ii) and (iii) is direct. To prove (iv), we use the following characterization of $H^p(\mathbb{R})$ $(0 < p \leq 1)$:

$$f \in H^p(\mathbb{R}) \quad \Longleftrightarrow \quad f \in L^p(\mathbb{R}) \text{ and } Hf \in L^p(\mathbb{R}).$$

It is easy to see that the following equation is true:

$$\lim_{t\to 0} \|P_t(x) - P_t(x-1) - \{\delta(x) - \delta(x-1)\}\|_{L^p(\mathbb{R})} = 0, \quad \forall\, 0 < p < 1. \tag{2.4}$$

Further

$$H\{P_t(x) - P_t(x-1)\} = \frac{1}{\pi}\left\{\frac{x}{x^2+t^2} - \frac{x-1}{(x-1)^2+t^2}\right\}$$

and

$$H\{\delta(x) - \delta(x-1)\} = \frac{1}{\pi}\left\{\frac{1}{x} - \frac{1}{x-1}\right\}.$$

Hence $\forall\, 1/2 < p < 1$, we have

$$\lim_{t\to 0} \|H\{P_t(x) - P_t(x-1)\} - H\{\delta(x) - \delta(x-1)\}\|_{L^p(\mathbb{R})} = 0. \tag{2.5}$$

By (2.4) and (2.5), we get the conclusion of (iv). $\square$

The classic harmonic functions may have different non-tangential boundary distribution in different convergence sense. **Hence we introduce integral boundary value of the β-harmonic functions**. See also [7, 24]. Let $\widetilde{\phi} \in \mathcal{S}(\mathbb{R}^n)$ be a radial real valued function such that

$$\operatorname{Supp} \widehat{\widetilde{\phi}} \subset \left\{\xi : \frac{1}{2} \leq |\xi| \leq 2\right\} \quad \text{and} \quad \int_0^\infty \widehat{\widetilde{\phi}}(t^{1/\beta}) e^{-t}\, \frac{dt}{t} \neq 0.$$

Denote

$$\phi = \left\{ \int_0^\infty \widehat{\widetilde{\phi}}(t^{1/\beta}) e^{-t} \frac{dt}{t} \right\}^{-1} \widetilde{\phi}.$$

Write $\phi_t(x) = t^{-n/\beta}\phi(t^{-1/\beta}x)$ with $\widehat{\phi}_t(\xi) = \widehat{\phi}(t^{1/\beta}\xi)$. For a β-harmonic function $f(t,x)$, its boundary distribution is denoted as follows

$$b_f(x) = \int_{\mathbb{R}^{n+1}_+} f(t,y)\phi_t(x-y) \frac{dt}{t} dy.$$

The following boundary distribution theorem shows that the integral boundary value $b_f(x)$ plays the same role as the classic trace:

Theorem 2.10. *For β-harmonic function $f(t,x)$, if the boundary distribution b_f belong to $\mathcal{S}'(\mathbb{R}^n)$, then the boundary distribution of β-harmonic function $P^\beta_t b_f(x)$ is still b_f.*

Proof. By Fourier transformation, we have

$$\widehat{g}(\xi) = \int_0^\infty \widehat{\phi}(t^{1/\beta}) e^{-t} \frac{dt}{t} \widehat{g}(\xi) = \int_0^\infty \widehat{\phi}(t^{1/\beta}\xi) e^{-t|\xi|^\beta} \widehat{g}(\xi) \frac{dt}{t}.$$

If $g \in \mathscr{S}(\mathbb{R}^n)$, then

$$g(x) = \int_{\mathbb{R}^{n+1}_+} P^\beta_t g(y)\phi_t(x-y) \frac{dt}{t} dy. \tag{2.6}$$

Further,

$$\left\langle \int_{\mathbb{R}^{n+1}_+} P^\beta_t b_f(x-y)\phi_t(y) \frac{dt}{t} dy, g(x) \right\rangle = \left\langle b_f, \int_{\mathbb{R}^{n+1}_+} P^\beta_t g(x-y)\phi_t(y) \frac{dt}{t} dy \right\rangle.$$

Applying the equation (2.6), we get the conclusion. □

3. Carleson measures and q-mean oscillation functions

Carleson measures play an important role in the classic harmonic function theory. We extend here these meaures to the β-harmonic functions. For $\beta > 0$, the relative Carleson box based on a cube I is defined by

$$S_\beta(I) = I \times (0, \ell(I)^\beta] = \{(t,x) \in \mathbb{R}^{n+1}_+ : x \in I, t \in (0, \ell(I)^\beta]\}.$$

Definition 3.1. (i) A positive measure μ is called to be a Carleson measure in $\mathbb{R}^{n+1}_+$ if

$$\sup_I \frac{\mu(S_\beta(I))}{|I|} < \infty,$$

where $\sup_I$ indicates the supremum take over all cubes in $\mathbb{R}^n$.

(ii) A Carleson measure μ is called to be a local compact Carleson measure in $\mathbb{R}^{n+1}_+$ if

$$\lim_{|I|\to 0} \frac{\mu(S_\beta(I))}{|I|} = 0.$$

For $m \in \mathbb{N}$, denote $\nabla^m = (\partial^m_{x_1}, \ldots, \partial^m_{x_n})$.

Definition 3.2. Given $m \in \mathbb{N}$, $|\alpha| < m$.

(i) If $1 \leq q < \infty$, we define Carleson spaces $C^{\alpha,q}_m(\mathbb{R}^{n+1}_+)$ as the space of all β-harmonic functions such that

$$\|f\|_{C^{\alpha,q}_m} = \sup_I \{(f, C^{\alpha,q}_m)(I)\}^{1/q} < +\infty,$$

where

$$(f, C^{\alpha,q}_m)(I) = |I|^{-1} \int_{S_\beta(I)} |\nabla^m f(t,x)|^q t^{q(m-\alpha)/\beta - 1}\, dxdt.$$

Further, $f \in C^{\alpha,q}_{m,0}(\mathbb{R}^{n+1}_+)$, if $f \in C^{\alpha,q}_m(\mathbb{R}^{n+1}_+)$ satisfying that

$$\lim_{|I|\to 0} \{(f, C^{\alpha,q}_m)(I)\}^{1/q} = 0.$$

(ii) We define β-harmonic function $f(t,x) \in C^{\alpha,\infty}_m$, if

$$\sup_{t>0} \sup_{x\in\mathbb{R}^n} t^{(m-\alpha)/\beta} \sum_{i=1}^n |\partial^m_{x_i} f(t,x)| < \infty.$$

Remark 3.3. For $\beta = 1$, $\alpha = 0$ and $q = 2$, $C^{0,2}_m(\mathbb{R}^{n+1}_+)$ becomes the space $\mathrm{HMO}(\mathbb{R}^{n+1}_+)$ introduced by Fabes-Johnson-Neri [5].

The above generalized harmonic functions spaces correspond to the following bounded q-mean oscillation space $M^{\alpha,q}(\mathbb{R}^n)$ whose functions need not to be locally integrable. The space $M^{\alpha,q}(\mathbb{R}^n)$ is the dual space of the end point Triebel-Lizorkin spaces $\dot{F}_1^{-\alpha,q/(q-1)}(\mathbb{R}^n)$. See Lin-Lin-Yang [13] and Triebel [19]. Let $\varphi \in C^\infty_0(B(0,2n))$ and $\varphi(x) = 1$ for $x \in B(0,\sqrt{n})$. Let $Q(x_0,r)$ be the cube centered at x_0 with edge parallel to the coordinate axis and with side length r. For simplicity, sometimes, we denote $Q = Q(r)$ the cube $Q(x_0,r)$ and let $\varphi_Q(x) = \varphi((x-x_Q)/r)$. Given $m \in \mathbb{N}$, $|\alpha| < m$, $1 \leq q \leq \infty$. For arbitrary function f, let $S^{m,q,f}$ be the class of the polynomial functions $P_{Q,f} = \sum_{|\gamma|\leq m} a_\gamma x^\gamma$ with degree less than m.

Definition 3.4. Given $m \in \mathbb{N}$, $|\alpha| < m$.

(i) If $1 \leq q < \infty$, we say that f belongs to the fractional q-mean oscillation space $M^{\alpha,q}(\mathbb{R}^n)$, provided

$$\sup_Q |Q|^{-1/q} \inf_{P_{Q,f} \in S^{m,q,f}} \|\varphi_Q(f - P_{Q,f})\|_{\dot{B}_q^{\alpha,q}} < \infty,$$

where the superum is taken over all cubes Q. f belongs to the local compact fractional q-mean oscillation space $M_0^{\alpha,q}(\mathbb{R}^n)$, if $f \in M^{\alpha,q}(\mathbb{R}^n)$ and

$$\lim_{|Q| \to 0} |Q|^{-1/q} \inf_{P_{Q,f} \in S^{m,q,f}} \|\varphi_Q(f - P_{Q,f})\|_{\dot{B}_q^{\alpha,q}} = 0.$$

(ii) $M^{\alpha,\infty}(\mathbb{R}^n)$ is the fractional Bloch space $\dot{B}_\infty^{\alpha,\infty}(\mathbb{R}^n)$.

Remark 3.5. (i) The definition $M^{\alpha,q}(\mathbb{R}^n)$ ($m \in \mathbb{N}$, $|\alpha| < m$, $1 \leq q \leq \infty$) has no relation with φ_Q and $P_{Q,f}$. Because their wavelet characterization spaces have no relation with these quantities. See Theorem 4.2.

(ii) $M^{-1,2} = \mathrm{BMO}^{-1}$ is the famous space in Koch-Tataru [8].

(iii) Local compact property is equivalent to that the norm of the high frequency party of a function is small. Hence the local compact property is satisfied for all the Besov spaces $\dot{B}_p^{\alpha,q}$ and Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}$ where $\alpha \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$.

For classic harmonic function, one consider often the locally integrable function spaces. For example: Fefferman-Stein consider the Hardy space $H^1 = \dot{F}_1^{0,2}$. Fabes-Johnson-Neri [5] characterized the spaces $\mathrm{HMO}(\mathbb{R}_+^{n+1})$ with trace in $\mathrm{BMO}(\mathbb{R}^n)$. Essen-Janson-Peng-Xiao [4] considered Q spaces. Further, Sjögren [17] considered the symmetric spaces by Poisson integrals. Triebel [19] considered the Poisson characterization of Besov spaces. Alvarez-Guzmán-Partida-Pérez-Esteva [1] extend harmonic extension to distributions. Here, we establish the trace theorem on the basis of β-harmonic extension. We prove the uniqueness of the generalized β-harmonic functions extension and the convergence sense on basis of the function norm for boundary distributions.

By Theorem 2.8, harmonic extension has no uniqueness. In classic harmonic analysis, we apply maximum value principle to get the uniqueness of harmonic extension. But β-harmonic functions with boundary distributions do not have maximum value principle. In this paper, we apply Meyer wavelets to get still the uniqueness of harmonic extension and establish a one-to-one relation between β-harmonic function in Careleson measure space $C_m^{\alpha,q}$ and integral boundary value in q-mean oscillation space $M^{\alpha,q}$. More precisely, we have

Theorem 3.6. *$m \in \mathbb{N}$, $|\alpha| < m$, $1 \leq q \leq \infty$.*

(i) *If* $f \in M^{\alpha,q}$, *then* f *can extend uniquely to a* β*-harmonic function in* $C_m^{\alpha,q}$.

(ii) *If* $f(t,x) \in C_m^{\alpha,q}$, *then the relative boundary distribution* b_f *must belong to* $M^{\alpha,q}$.

Further, for local compact spaces, we have, the integral boundary value is just the non-tangential boundary value. That is to say, $f(t,x)$ converges to $f(x)$ in the relative norm sense:

Theorem 3.7. $m \in \mathbb{N}$, $|\alpha| < m$, $1 \le q < \infty$.

(i) *If* $f \in M_0^{\alpha,q}$, *then* f *can extend to a unique* β*-harmonic function in* $C_{m,0}^{\alpha,q}$.

(ii) *If* $f(t,x) \in C_{m,0}^{\alpha,q}$, *then the relative boundary distribution* b_f *belongs to* $M_0^{\alpha,q}$. *Further,*

$$\lim_{t\to 0} \|f(t,x) - b_f\|_{M^{\alpha,q}} = 0. \tag{3.1}$$

We will prove Theorems 3.6 and 3.7 in the last section. We use Meyer wavelets and relative vaguelette knowledge to consider the β-harmonic extension and trace theorem in precise meaning. In Section 4, we present some preliminaries on Meyer wavelets and characterization of certain function spaces. In Section 5, we consider some properties on basic harmonic functions and basic observers. In Sections 6 and 7, we consider Carleson measures. To simplify the notations, we consider only the case $m = 1$. For $m > 1$, it is sufficient consider more derivatives for basic harmonic functions and more integration for basic observers.

Remark 3.8. Given $m, m' \in \mathbb{N}$, $m > |\alpha|$, $m' > |\alpha|$ and $1 \le q \le \infty$. From the above theorem, we know that $C_m^{\alpha,q} = C_{m'}^{\alpha,q}$. That is to say, different derivatives produce the same space. If $\alpha = 0$, $q = 2$ and $m > 0$, then $C_m^{\alpha,q} = \mathrm{BMO}$. That is to say, BMO space has different characterization.

Remark 3.9. (i) For $\beta = 1$, Triebel [19] considered the Poisson characterization of Besov spaces. Essen-Janson-Peng-Xiao [4] considered the Poisson characterization of Q spaces. It is known that the Poisson extension has uniqueness. We consider β-harmonic extension for arbitrary positive β. It is known that the β-harmonic extension has no uniqueness. So even for $\beta = 1$ and Besov spaces or Q spaces, we need new skills.

(ii) For a distribution f, normally, it can extend to many β-harmonic functions. The uniqueness in the above two theorems is obtained by using wavelets and the restriction of the norm of β-harmonic functions in $C_m^{\alpha,q}$. See the proof of the main theorems in Section 8.

(iii) In the above section, we have seen that a classic harmonic function may have different boundary value for different convergence sense. See Proposition 2.9. But in Theorems 3.6 and 3.7, we have proved the existence of boundary distributions in the equation (3.1) under the relative given function norm sense.

Remark 3.10. (i) $M^{\alpha,q}(\mathbb{R}^n)$ can be seen as the generalizations of the following function spaces:

$$\begin{aligned}
M^{0,2}(\mathbb{R}^n) &= \mathrm{BMO}(\mathbb{R}^n), && \text{bounded mean oscillation space;}\\
M^{-1,2}(\mathbb{R}^n) &= \mathrm{BMO}^{-1}(\mathbb{R}^n), && \text{Koch and Tataru's space;}\\
M^{0,\infty}(\mathbb{R}^n) &= B^{0,\infty}_{\infty}(\mathbb{R}^n), && \text{Bloch space;}\\
M^{\alpha,\infty}(\mathbb{R}^n) &= C^{\alpha}(\mathbb{R}^n)\ (0<\alpha<1), && \text{Hölder spaces.}
\end{aligned}$$

See [15, Section 6.10].

(ii) More generalized function spaces have been considered extensively in real analysis. See Cui-Yang [3], Li-Yang [10,11], Liang et al. [12], Lin-Yang [14], Yang-Yuan [25], Yuan-Sickel-Yang [26] and the reference therein. If we replace the $\dot{B}^{\alpha,q}_q$ norm to $\dot{B}^{\alpha,q}_p$ norm or $\dot{F}^{\alpha,q}_p$ norm in Definition 3.4, and we make the respective modification for Carleson measures in Definition 3.2, our method can be applied to these general function spaces. To simplify the notations, we restrict ourselves only to q-mean oscillation spaces.

4. Preliminaries on wavelets

4.1. Meyer wavelets

We present some preliminaries on Meyer wavelets Φ^ϵ and refer the reader to Meyer [15], Wojtaszczyk [20] and Yang [23] for further information. Let

$$\begin{aligned}
E_n &= \{0,1\}^n \setminus \{0\},\\
F_n &= \{(\epsilon,k) : \epsilon \in E_n, k \in \mathbb{Z}^n\},\\
\Lambda_n &= \{(\epsilon,j,k) : \epsilon \in E_n, j \in \mathbb{Z}, k \in \mathbb{Z}^n\}.
\end{aligned}$$

We will use the real-valued Meyer wavelets. Let Ψ^0 be an even function in $C_0^\infty([-4\pi/3, 4\pi/3])$ with

$$0 \le \Psi^0(\xi) \le 1 \quad \text{and} \quad \Psi^0(\xi) = 1 \quad \text{for } |\xi| \le \frac{2\pi}{3}.$$

From now on, let

$$\Omega(\xi) = \sqrt{(\Psi^0(\xi/2))^2 - (\Psi^0(\xi))^2}.$$

Then $\Omega(\xi)$ is an even function in $C_0^\infty([-8\pi/3, 8\pi/3])$. Clearly,

$$\begin{aligned}
\Omega(\xi) &= 0 && \text{for } |\xi| \le \frac{2\pi}{3},\\
\Omega^2(\xi) + \Omega^2(2\xi) = 1 &= \Omega^2(\xi) + \Omega^2(2\pi - \xi) && \text{for } \xi \in \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right].
\end{aligned}$$

Let $\Psi^1(\xi) = \Omega(\xi)e^{-i\xi/2}$. For any $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \{0,1\}^n$, define $\Phi^\epsilon(x)$ by $\widehat{\Phi}^\epsilon(\xi) = \prod_{i=1}^n \Psi^{\epsilon_i}(\xi_i)$. For $(\epsilon, j, k) \in \Lambda_n$, let

$$\Phi^\epsilon_{j,k}(x) = 2^{jn/2}\Phi^\epsilon(2^j x - k).$$

The set $\left\{\Phi^\epsilon_{j,k} : (\epsilon, j, k) \in \Lambda_n\right\}$ forms a wavelet basis. For any $\epsilon \in \{0,1\}^n$, $k \in \mathbb{Z}^n$ and a function f on $\mathbb{R}^n$, we write $f^\epsilon_{j,k} = \left\langle f, \Phi^\epsilon_{j,k}\right\rangle$. The following result is well-known.

Theorem 4.1. *The Meyer wavelets $\left\{\Phi^\epsilon_{j,k}\right\}_{(\epsilon,j,k)\in\Lambda_n}$ form an orthogonal basis of $L^2(\mathbb{R}^n)$. Consequently, for any $f \in L^2(\mathbb{R}^n)$, the following wavelet decomposition holds in the L^2 convergence sense:*

$$f = \sum_{(\epsilon,j,k)\in\Lambda_n} f^\epsilon_{j,k}\Phi^\epsilon_{j,k}.$$

4.2. Wavelet characterization of q-mean oscillation spaces

Meyer [15] gave the wavelet characterizations of Besov spaces. See Sections 6 and 10 of [15]. By this result, we could get the following result. We omit the proof and refer the reader to [9, Lemma 2.2] and [23, Theorem 5.4] for the details.

For $f = \sum_{(\epsilon,j,k)\in\Lambda_n} f^\epsilon_{j,k}\Phi^\epsilon_{j,k}$, we have

Theorem 4.2. *Given $m \in \mathbb{N}$, $|\alpha| < m$.*

(i) *Let $1 \le q < \infty$. A function $f \in M^{\alpha,q}(\mathbb{R}^n)$ if*

$$\left[\sup_Q |Q|^{-1} \sum_{Q_{j,k}\subset Q} 2^{qj(\alpha+n/2-n/q)} \left|f^\epsilon_{j,k}\right|^q\right]^{1/q} < \infty.$$

(ii) *A function $f \in M^{\alpha,\infty}(\mathbb{R}^n)$ if*

$$\sup_{(\epsilon,j,k)} 2^{nj/2+j\alpha}\left|f^\epsilon_{j,k}\right| < \infty.$$

Applying Theorem 4.2, we get

Corollary 4.3. *Given $m \in \mathbb{N}$, $|\alpha| < m$ and $1 \le q < \infty$.*

$$M^{\alpha,q}(\mathbb{R}^n) \subsetneq M^{\alpha,\infty}(\mathbb{R}^n).$$

5. Some properties on vaguelettes

In this section, we consider two kinds of vaguelettes derived from β-harmonic functions and Meyer wavelets. It allows us to use the method of discretization to consider β-harmonic functions. Hence, we do not need the skills of Calderón-Zygmund operators like the corresponding author of this paper did in [22]. In Section 5, Chapter 8 of [15], Meyer introduced the conception of vaguelettes in $\mathbb{R}^n$ which are introduced to study the L^2 boundedness. See also [16]. Vaguelettes there are defined as Hölder smooth functions decreasing at certain speed and with a certain zero vanishing moments. To make the vagulettes to adapt our situation, we assume that vaguelettes are sufficient smooth functions decreasing enough fast and with enough zero vanishing moments. That is to say,

Definition 5.1. The functions $f_{j,k}(x)$ ($j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$) on $\mathbb{R}^n$ are called to be vaguelettes, if there exists $m \in \mathbb{N}$ and a sufficient big $N > n$ such that

$$|\partial_x^\gamma f_{j,k}(x)| \leq C 2^{(n/2+|\gamma|)j}(1+|2^j x-k|)^{-N}, \quad \forall\, |\gamma| \leq m,$$
$$\int x^\gamma f_{j,k}(x)\, dx = 0, \quad \forall\, |\gamma| \leq m-1.$$

We generalize this conception to two kinds of vaguelettes in $\mathbb{R}^{n+1}_+$ with parameter t. For wavelettes $\Phi^\epsilon_{j,k}(x)$, we consider $P^\beta_t \Phi^\epsilon_{j,k}(x)$ and $\int \phi_t(x-y)\Phi^\epsilon_{j,k}(y)\, dy$.

5.1. Two kinds of vaguelettes

By wavelette theory, all the β-harmonic functions can be seen as the linear combination of the basic β-harmonic functions $P^\beta_t \Phi^\epsilon_{j,k}(x)$.

Lemma 5.2. *There exists small positive real number c such that*

$$\left|P^\beta_t \Phi^\epsilon_{j,k}(x)\right| \leq C 2^{nj/2}(1+|2^j x-k|)^{-N}, \quad \forall\, t2^{j\beta} \leq 1,$$
$$\left|P^\beta_t \Phi^\epsilon_{j,k}(x)\right| \leq C 2^{nj/2}(t2^{j\beta})^N e^{-ct2^{j\beta}}(1+|2^j x-k|)^{-N}, \quad \forall\, t2^{j\beta} > 1,$$
$$\int P^\beta_t \Phi^\epsilon_{j,k}(x)\, dx = 0.$$

Proof. By Fourier transform, we have

$$\begin{aligned} P^\beta_t \Phi^\epsilon_{j,k}(x) &= (2\pi)^{-n} 2^{-nj/2} \int e^{-t|\xi|^\beta} \widehat{\Phi^\epsilon}(2^{-j}\xi) e^{i(x-2^{-j}k)\xi}\, d\xi \\ &= (2\pi)^{-n} 2^{nj/2} \int e^{-t2^{j\beta}|\xi|^\beta} \widehat{\Phi^\epsilon}(\xi) e^{i(2^j x-k)\xi}\, d\xi. \end{aligned}$$

Hence, we get

$$\begin{aligned} \left|P^\beta_t \Phi^\epsilon_{j,k}(x)\right| &\leq (2\pi)^{-n} 2^{nj/2} \int \left| e^{-t2^{j\beta}|\xi|^\beta} \widehat{\Phi^\epsilon}(\xi) e^{i(2^j x-k)\xi} \right| d\xi \\ &\leq C 2^{nj/2} e^{-ct2^{j\beta}}. \end{aligned}$$

If $|2^j x - k| > 1$, we assume that the first biggest component of $2^j x - k$ is the v-th component. By integration by parts for the v-th component N times, we get

$$\begin{aligned} P_t^\beta \Phi^\epsilon_{j,k}(x) &= i^{-N}(2^j x_v - k_v)^{-N}(2\pi)^{-n}2^{j+nj/2}\int e^{-t2^{j\beta}|\xi|^\beta}\widehat{\Phi^\epsilon}(\xi)\left\{\partial^N_{\xi_v} e^{i(2^j x-k)\xi}\right\} d\xi \\ &= i^{-N}(-1)^N(2^j x_v - k_v)^{-N}(2\pi)^{-n}2^{j+nj/2}\int \partial^N_{\xi_v}\left\{e^{-t2^{j\beta}|\xi|^\beta}\widehat{\Phi^\epsilon}(\xi)\right\} e^{i(2^j x-k)\xi}\, d\xi. \end{aligned}$$

Hence, we get the proof of the conclusion of this lemma. □

Further, we consider some vaguelette relative to boundary distribution. Let $\Phi^\epsilon_{j,k}(t,x) = \int \phi_t(x-y)\Phi^\epsilon_{j,k}(y)\,dy$. Then the observed quantity

$$\int_{\mathbb{R}^{n+1}_+} f(t,y)\Phi^\epsilon_{j,k}(t,y)\,\frac{dt}{t}dy$$

of $f(t,y)$ equals to the wavelet coefficient $\langle b_f(x), \Phi^\epsilon_{j,k}\rangle$. In fact,

$$\langle b_f(x), \Phi^\epsilon_{j,k}\rangle = \left\langle \int_{\mathbb{R}^{n+1}_+} f(t,y)\phi_t(x-y)\,\frac{dt}{t}dy, \Phi^\epsilon_{j,k}\right\rangle = \int_{\mathbb{R}^{n+1}_+} f(t,y)\Phi^\epsilon_{j,k}(t,y)\,\frac{dt}{t}dy.$$

The basic oscillators $\Phi^\epsilon_{j,k}(t,x)$ play a role of observers. They can pull back a β-harmonic functions to its boundary distribution. Further,

Lemma 5.3. *There exists $0 < C_1 < C_2$ such that*

(i) $\Phi^\epsilon_{j,k}(t,x) = 0$ *for* $t2^{j\beta} \geq C_2$ *or* $t2^{j\beta} \leq C_1$.

(ii) *If* $C_1 \leq t2^{j\beta} \leq C_2$, *then*

$$\left|\Phi^\epsilon_{j,k}(t,x)\right| \leq C_N 2^{nj/2}(1+|2^j x-k|)^{-N}, \quad \forall\, N > n.$$

(iii) $\int \Phi^\epsilon_{j,k}(t,x)\,dx = 0$.

Proof. The Fourier transform of $\Phi^\epsilon_{j,k}(t,x)$ is the following function:

$$\begin{aligned} \int \Phi^\epsilon_{j,k}(t,x)e^{-ix\xi}\,dx &= \iint \phi_t(x-y)\Phi^\epsilon_{j,k}(y)e^{-ix\xi}\,dxdy \\ &= \int \phi_t(x)e^{-ix\xi}\,dx \int \Phi^\epsilon_{j,k}(y)e^{-iy\xi}\,dy \\ &= \widehat{\phi}(t^{1/\beta}\xi)\widehat{\Phi^\epsilon}(2^{-j}\xi)e^{-i2^{-j}k\xi}. \end{aligned}$$

By Meyer wavelettes properties, (i) and (iii) are true.

By inverse Fourier transform, we have

$$\begin{aligned} \int \phi_t(x-y)\Phi^\epsilon_{j,k}(y)\,dy &= (2\pi)^{-n}2^{-nj/2}\int \widehat{\phi}(t^{1/\beta}\xi)\widehat{\Phi^\epsilon}(2^{-j}\xi)e^{i(x-2^{-j}k)\xi}\,d\xi \\ &= (2\pi)^{-n}2^{nj/2}\int \widehat{\phi}(2^j t^{1/\beta}\xi)\widehat{\Phi^\epsilon}(\xi)e^{i(2^j x-k)\xi}\,d\xi. \end{aligned}$$

By Meyer wavelette properties, we get (ii). □

Remark 5.4. In this paper, we use $P_t^\beta \Phi_{j,k}^\epsilon(x)$ and $\Phi_{j,k}^\epsilon(t,x)$ to consider the Poisson type extension and relative boundary distribution theorem.

(i) $P_t^\beta \Phi_{j,k}^\epsilon(x)$ is a vagulette which is not located at $t = 2^{-j\beta}$, so we consider its m order derivatives.

(ii) $\Phi_{j,k}^\epsilon(t,x)$ can not be adapted to the norm of Carleson space $C_m^{\alpha,q}$, we need its m order integration.

(iii) In the next subsection, we consider the derivatives for basic β-harmonic functions and the integrations for the basic observers. To simplify the notations, we consider only the case $m = 1$. For $m > 1$, we need only consider m order derivatives for basic β-harmonic functions and m order integrations for the basic observers.

5.2. Basic β-harmonic functions

For $u = 1, \ldots, n$, we consider $P_{j,k}^{\beta,u,\epsilon}(t,x) = \partial_{x_u} P_t^\beta \Phi_{j,k}^\epsilon(x)$, the derivative of the basic β-harmonic functions $P_t^\beta \Phi_{j,k}^\epsilon(x)$.

Lemma 5.5. (i) *If $t2^{j\beta} \leq 1$, then*

$$\left| P_{j,k}^{\beta,u,\epsilon}(t,x) \right| \leq C_N 2^{j+nj/2} (1 + |2^j x - k|)^{-N}, \quad \forall N > n.$$

(ii) *If $t2^{j\beta} > 1$, then there exists small positive real number c such that*

$$\left| P_{j,k}^{\beta,u,\epsilon}(t,x) \right| \leq C_N 2^{j+nj/2} (t2^{j\beta})^N e^{-ct2^{j\beta}} (1 + |2^j x - k|)^{-N}, \quad \forall N > n.$$

Proof. By Fourier transform, we have

$$\begin{aligned} P_{j,k}^{\beta,u,\epsilon}(t,x) &= \partial_{x_u} P_t^\beta \Phi_{j,k}^\epsilon(x) \\ &= i(2\pi)^{-n} 2^{-nj/2} \int \xi_u e^{-t|\xi|^\beta} \widehat{\Phi^\epsilon}(2^{-j}\xi) e^{i(x-2^{-j}k)\xi} \, d\xi \\ &= i(2\pi)^{-n} 2^{j+nj/2} \int \xi_u e^{-t2^{j\beta}|\xi|^\beta} \widehat{\Phi^\epsilon}(\xi) e^{i(2^j x - k)\xi} \, d\xi. \end{aligned}$$

Hence, we get

$$\begin{aligned} \left| \partial_{x_u} P_t^\beta \Phi_{j,k}^\epsilon(x) \right| &\leq (2\pi)^{-n} 2^{j+nj/2} \int \left| \xi_u e^{-t2^{j\beta}|\xi|^\beta} \widehat{\Phi^\epsilon}(\xi) e^{i(2^j x - k)\xi} \right| \, d\xi \\ &\leq C 2^{j+nj/2} e^{-ct2^{j\beta}}. \end{aligned}$$

If $|2^j x - k| > 1$, we assume that the first biggest component of $2^j x - k$ is the v-th component. By integration by parts for the v-th component N times, we get

$$\begin{aligned}
&\partial_{x_u} P_t^\beta \Phi_{j,k}^\epsilon(x) \\
&= i^{1-N}(2^j x_v - k_v)^{-N}(2\pi)^{-n}2^{j+nj/2}\int \xi_u e^{-t2^{j\beta}|\xi|^\beta}\widehat{\Phi^\epsilon}(\xi)\left\{\partial_{\xi_v}^N e^{i(2^j x-k)\xi}\right\} d\xi \\
&= i^{1-N}(-1)^N(2^j x_v - k_v)^{-N}(2\pi)^{-n}2^{j+nj/2}\int \partial_{\xi_v}^N\left\{\xi_u e^{-t2^{j\beta}|\xi|^\beta}\widehat{\Phi^\epsilon}(\xi)\right\} e^{i(2^j x-k)\xi}\, d\xi.
\end{aligned}$$

Hence, we get the proof of the conclusion of this lemma. □

We consider then the integration of the basic β-harmonic functions. For $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in E_n$, denote by i_ϵ the smallest index such that $\epsilon_{i_\epsilon} = 1$. Let $\partial_\epsilon = \partial_{x_{i_\epsilon}}$ and $I_\epsilon\Phi^\epsilon(x) = I_{i_\epsilon}\Phi^\epsilon(x)$. For $I_\epsilon P_{j,k}^{\beta,\epsilon}(t,x) = I_\epsilon P_t^\beta \Phi_{j,k}^\epsilon(x)$, by similar way in the above lemma, we have

Lemma 5.6. (i) *If $t2^{j\beta} \le 1$, then*

$$\left|I_\epsilon P_{j,k}^{\beta,\epsilon}(t,x)\right| \le C_N 2^{-j+nj/2}(1+|2^j x - k|)^{-N}, \quad \forall\, N > n.$$

(ii) *If $t2^{j\beta} > 1$, then there exists a small positive real number c such that*

$$\left|I_\epsilon P_{j,k}^{\beta,\epsilon}(t,x)\right| \le C_N 2^{-j+nj/2}(t2^{j\beta})^N e^{-ct2^{j\beta}}(1+|2^j x - k|)^{-N}, \quad \forall\, N > n.$$

5.3. Basic observers

We consider some properties on the integration of the basic oscillators $\Phi_{j,k}^\epsilon(t,x) = \int \phi_t(x-y)\Phi_{j,k}^\epsilon(y)\,dy$. For $i = 1, \ldots, n$ and any function f, define

$$I_i f(x) = \int_{-\infty}^{x_i} f(x_1, \ldots, x_{-1+i_\epsilon}, y, x_{1+i_\epsilon}, \ldots, x_n)\,dy.$$

Let $B_{j,k}^\epsilon(t,x) = I_\epsilon \Phi_{j,k}^\epsilon(t,x)$ and

$$f_{j,k}^\epsilon = \left\langle b_f(x), \Phi_{j,k}^\epsilon\right\rangle = -\int_{\mathbb{R}_+^{n+1}} \partial_\epsilon f(t,y) B_{j,k}^\epsilon(t,y)\,\frac{dt}{t}dy. \tag{5.1}$$

Lemma 5.7. *There exists $0 < C_1 < C_2$ such that*

(i) *$B_{j,k}^\epsilon(t,x) = 0$ for $t2^{j\beta} \ge C_2$ or $t2^{j\beta} \le C_1$.*

(ii) *If $C_1 \le t2^{j\beta} \le C_2$, then*

$$\left|B_{j,k}^\epsilon(t,x)\right| \le C_N 2^{nj/2-j}(1+|2^j x - k|)^{-N}, \quad \forall\, N > n.$$

Proof. By a direct computation, we have

$$\int I_\epsilon \Phi^\epsilon(x) e^{-ix\xi}\, dx = (i\xi_{i_\epsilon})^{-1} \int \Phi^\epsilon(x) e^{-ix\xi}\, dx.$$

Hence, for $B^\epsilon_{j,k}(t,x) = I_\epsilon \Phi^\epsilon_{j,k}(t,x)$, we have

$$\int B^\epsilon_{j,k}(t,x) e^{-ix\xi}\, dx = (i\xi_{i_\epsilon})^{-1} \widehat{\phi}(t^{1/\beta}\xi) \widehat{\Phi^\epsilon}(2^{-j}\xi) e^{-i2^{-j}k\xi}.$$

By inverse Fourier transformation, we get

$$\begin{aligned} B^\epsilon_{j,k}(t,x) &= (2\pi)^{-n} 2^{-nj/2} \int \widehat{\phi}(t^{1/\beta}\xi)(i\xi_{i_\epsilon})^{-1} \widehat{\Phi^\epsilon}(2^{-j}\xi) e^{i(x-2^{-j}k)\xi}\, d\xi \\ &= (2\pi)^{-n} 2^{nj/2-j} \int \widehat{\phi}(2^j t^{1/\beta}\xi)(i\xi_{i_\epsilon})^{-1} \widehat{\Phi^\epsilon}(\xi) e^{i(2^j x-k)\xi}\, d\xi, \end{aligned}$$

which implies the desired conclusion of the above lemma. □

6. Fractional Bloch spaces and β-harmonic functions

The boundary value of a harmonic function in $C^{\alpha,\infty}(\mathbb{R}^{n+1}_+)$ may not be locally integrable. We can characterize the boundary distribution in such spaces with fractional Bloch spaces $\dot{B}^{\alpha,\infty}_\infty(\mathbb{R}^n)$. For Poisson extension, we use the properties of basic β-harmonic functions. For trace result, we use the properties of the basic observers.

Theorem 6.1. *Let $|\alpha| < m$.*

(i) *For any $f \in M^{\alpha,\infty}(\mathbb{R}^n)$, we have*

$$f(t,x) =: P^\beta_t f(x) \in C^{\alpha,\infty}_m(\mathbb{R}^{n+1}_+).$$

(ii) *For any $f(t,x) \in C^{\alpha,\infty}_m(\mathbb{R}^n)$, there exists a function $f \in M^{\alpha,\infty}(\mathbb{R}^n)$ such that*

$$f(t,x) = P^\beta_t f(x).$$

Proof. For $m > 1$, we need only to consider more derivatives and more integrations. To simplify the proof, we consider only the case $m = 1$.

(i) Note that

$$\partial_{x_u} P^\beta_t f = \sum_{\epsilon,j,k} f^\epsilon_{j,k} \partial_{x_u} P^\beta_t \Phi^\epsilon_{j,k}(x).$$

Applying Theorem 4.2 and Lemma 5.5, we have

$$\begin{aligned}\left|\partial_{x_u} P_t^\beta f\right| &\le C \sum_{t2^{j\beta}\le 1,k} 2^{j(1-\alpha)}(1+|2^j x-k|)^{-N}\\ &\quad + C \sum_{t2^{j\beta}>1,k} 2^{j(1-\alpha)}(t2^{j\beta})^N e^{-ct2^{j\beta}}(1+|2^j x-k|)^{-N}\\ &\le C \sum_{t2^{j\beta}\le 1} 2^{j(1-\alpha)} + C \sum_{t2^{j\beta}>1} 2^{j(1-\alpha)}(t2^{j\beta})^N e^{-ct2^{j\beta}}\\ &\le C t^{(\alpha-1)/\beta}.\end{aligned}$$

(ii) Applying Lemma 5.7 and the equation (5.1), we get

$$\begin{aligned}\left|f^\epsilon_{j,k}\right| &\le C \int_{C_1\le t2^{j\beta}\le C_2}\int_{\mathbb{R}^n} t^{(\alpha-1)/\beta}\left|B^\epsilon_{j,k}(t,x)\right| dx\frac{dt}{t}\\ &\le C \int_{C_1\le t2^{j\beta}\le C_2}\int_{\mathbb{R}^n} t^{(\alpha-1)/\beta}2^{nj/2-j}(1+|2^j x-k|)^{-N} dx\frac{dt}{t}\\ &\le C \int_{C_1\le t2^{j\beta}\le C_2} t^{(\alpha-1)/\beta}2^{-nj/2-j}\,\frac{dt}{t}\\ &\le C2^{-nj/2-j\alpha}.\end{aligned}$$

□

7. β-harmonic function and oscillation spaces

In Section 7.1, by Theorem 7.1, we extend the functions in $M^{\alpha,q}(\mathbb{R}^n)$ to β-harmonic functions on $\mathbb{R}^{n+1}_+$. The essential ideas are to decompose functions with Meyer wavelets, then apply the properties of basic β-harmonic functions to prove the relative results. In Section 7.2, by Theorems 2.10 and 7.2, we apply the basic observers to pull back the β-harmonic functions in $C^{\alpha,q}_m(\mathbb{R}^{n+1}_+)$ into their relative boundary distribution in $M^{\alpha,q}(\mathbb{R}^n)$.

7.1. β-Poisson extension

In this subsection, we extend the functions in $M^{\alpha,q}(\mathbb{R}^n)$ to β-harmonic functions in Carleson spaces. In fact,

Theorem 7.1. *Let $1\le q<\infty$ and $|\alpha|<m$. For any $f\in M^{\alpha,q}(\mathbb{R}^n)$, we have*

$$f(t,x) =: P_t^\beta f(x) \in C^{\alpha,q}_m(\mathbb{R}^{n+1}_+).$$

Proof. For $m>1$, we need only to consider m order derivatives. To simplify the notations, we consider only the case $m=1$. For $i=1,\dots,n$, denote

$$C_{I,i} = |I|^{-1}\int_{S_\beta(I)} |\partial_{x_i} f(t,x)|^q t^{q(1-\alpha)/\beta-1}\,dxdt.$$

By wavelet characterization, we write $\partial_{x_i} f(t,x)$ to the sum of basic harmonic functions

$$\partial_{x_i} f(t,x) = \sum_{\epsilon,j,k} f^{\epsilon}_{j,k} P^{\beta,i,\epsilon}_{j,k}(t,x).$$

We decompose it into two terms

$$\begin{aligned} I_i(t,x) &= \sum_{\epsilon,j,k,2^{nj}|I|<1} f^{\epsilon}_{j,k} P^{\beta,i,\epsilon}_{j,k}(t,x), \\ II_i(t,x) &= \sum_{\epsilon,j,k,2^{nj}|I|\geq 1} f^{\epsilon}_{j,k} P^{\beta,i,\epsilon}_{j,k}(t,x). \end{aligned}$$

For $i = 1, \ldots, n$, denote

$$\begin{aligned} C^1_{I,i} &= |I|^{-1} \int_{S_\beta(I)} |I_i(t,x)|^q t^{q(1-\alpha)/\beta - 1}\, dxdt, \\ C^2_{I,i} &= |I|^{-1} \int_{S_\beta(I)} |II_i(t,x)|^q t^{q(1-\alpha)/\beta - 1}\, dxdt. \end{aligned}$$

For $\|f\|_{M^{\alpha,q}} \leq 1$, we only need to prove that

$$\sup_I C^s_{I,i} \lesssim C, \quad s = 1,2,\ i = 1,2,\ldots,n.$$

On Carleson box $S_\beta(I)$, $I_i(t,x)$ can be estimated by constant depending on I. In fact, by Corollary 4.3, $|f^{\epsilon}_{j,k}| \leq C 2^{-j\alpha - nj/2}$. For all $(t,x) \in S_\beta(I)$, we have

$$\begin{aligned} |I_i(t,x)| &\lesssim \sum_{\epsilon,j,k,2^{nj}|I|<1} 2^{-j\alpha - nj/2} \left| P^{\beta,i,\epsilon}_{j,k}(t,x) \right| \\ &\lesssim \sum_{\epsilon,j,k,2^{nj}|I|<1} 2^{j(1-\alpha)} (1 + |2^j x - k|)^{-N} \\ &\lesssim C|I|^{(\alpha-1)/n}. \end{aligned}$$

Hence, we have

$$C^1_{I,i} \lesssim |I|^{-1} \int_{S_\beta(I)} |I|^{q(\alpha-1)/n} t^{q(1-\alpha)/\beta - 1}\, dxdt \leq C.$$

For arbitrary small positive δ, we have

$$\begin{aligned} &|II_i(t,x)|^q \\ &\lesssim \sum_{\epsilon,j,k,2^{nj}|I|\geq 1} 2^{(q-1)(j+nj/2-j\delta)} \left| f^{\epsilon}_{j,k} \right|^q \left| P^{\beta,i,\epsilon}_{j,k}(t,x) \right| \left(\sum_{\epsilon,j,k,2^{nj}|I|\geq 1} 2^{-j-nj/2+j\delta} \left| P^{\beta,i,\epsilon}_{j,k}(t,x) \right| \right)^{q-1} \\ &\lesssim t^{-(q-1)\delta/\beta} \sum_{\epsilon,j,k,2^{nj}|I|\geq 1} 2^{(q-1)(j+nj/2-j\delta)} \left| f^{\epsilon}_{j,k} \right|^q \left| P^{\beta,i,\epsilon}_{j,k}(t,x) \right|. \end{aligned}$$

Hence, we get

$$\begin{aligned}
& C^2_{I,i} \\
&\lesssim |I|^{-1}\int_{S_\beta(I)}\sum_{\epsilon,j,k,2^{nj}|I|\geq 1} 2^{(q-1)(j+nj/2-j\delta)}\left|f^\epsilon_{j,k}\right|^q\left|P^{\beta,i,\epsilon}_{j,k}(t,x)\right| t^{q(1-\alpha)/\beta-1-(q-1)\delta/\beta}\,dxdt \\
&= |I|^{-1}\int_{I\times[0,2^{-j\beta}]}\sum_{\epsilon,j,k,2^{nj}|I|\geq 1} 2^{(q-1)(j+nj/2-j\delta)}\left|f^\epsilon_{j,k}\right|^q\left|P^{\beta,i,\epsilon}_{j,k}(t,x)\right| t^{q(1-\alpha)/\beta-1-(q-1)\delta/\beta}\,dxdt \\
&\quad + |I|^{-1}\int_{I\times[2^{-j\beta},l(I)^\beta]}\sum_{\epsilon,j,k,2^{nj}|I|\geq 1} 2^{(q-1)(j+nj/2-j\delta)}\left|f^\epsilon_{j,k}\right|^q\left|P^{\beta,i,\epsilon}_{j,k}(t,x)\right| t^{q(1-\alpha)/\beta-1-(q-1)\delta/\beta}\,dxdt.
\end{aligned}$$

There exists 2^n different dyadic cube I_i such that

(i) $\operatorname{dist}(I_i,I_j)=0$, $\forall\, i,j=1,\ldots,2^n$;

(ii) $2^{-n}|I|\leq|I_i|=|I_j|<|I|$, $\forall\, i,j=1,\ldots,2^n$;

(iii) $I\subset\bigcup_{i=1,\ldots,2^n} I_i$.

For all $l\in\mathbb{Z}^n$, denote $S_{I,l}=\{(j,k):2^{nj}|I|\geq 1, I_{j,k}\subset l|I_1|^{1/n}+I_1\}$. Denote

$$\begin{aligned}
C^{2,1,l}_{I,i} &= |I|^{-1}\int_{I\times[0,2^{-j\beta}]}\sum_{\epsilon,(j,k)\in S_{I,l}} 2^{(q-1)(j+nj/2-j\delta)} \\
&\qquad\times\left|f^\epsilon_{j,k}\right|^q\left|P^{\beta,i,\epsilon}_{j,k}(t,x)\right| t^{q(1-\alpha)/\beta-1-(q-1)\delta/\beta}\,dxdt, \\
C^{2,2,l}_{I,i} &= |I|^{-1}\int_{I\times[2^{-j\beta},l(I)^\beta]}\sum_{\epsilon,(j,k)\in S_{I,l}} 2^{(q-1)(j+nj/2-j\delta)} \\
&\qquad\times\left|f^\epsilon_{j,k}\right|^q\left|P^{\beta,i,\epsilon}_{j,k}(t,x)\right| t^{q(1-\alpha)/\beta-1-(q-1)\delta/\beta}\,dxdt.
\end{aligned}$$

By this way,

$$C^2_{I,i}\lesssim\sum_{l\in\mathbb{Z}^n}C^{2,1,l}_{I,i}+\sum_{l\in\mathbb{Z}^n}C^{2,2,l}_{I,i}.$$

We apply (i) of Lemma 5.5 to the estimate of $C^{2,1,l}_{I,i}$. For $|l|\leq 4^n$, we have

$$\begin{aligned}
C^{2,1,l}_{I,i} &\leq C|I|^{-1}\int_{[0,2^{-j\beta}]}\sum_{\epsilon,(j,k)\in S_{I,l}} 2^{q(j+n/2)-nj-(q-1)j\delta}\left|f^\epsilon_{j,k}\right|^q t^{q(1-\alpha)/\beta-1-\delta/\beta}\,dxdt \\
&\leq C\sup_Q|Q|^{-1}\sum_{Q_{j,k}\subset Q}2^{qj(\alpha+n/2-n/q)}\left|f^\epsilon_{j,k}\right|^q.
\end{aligned}$$

For $|l|>4^n$, we have

$$\begin{aligned}
C^{2,1,l}_{I,i} &\leq C|I|^{-1}\int_{[0,2^{-j\beta}]}\sum_{\epsilon,(j,k)\in S_{I,l}} 2^{q(j+n/2)-nj-(q-1)j\delta}\left|f^\epsilon_{j,k}\right|^q(2^j|I|^{1/n}|l|)^{n-N}t^{q(1-\alpha)/\beta-1-\delta/\beta}\,dxdt \\
&\leq C\sup_Q|Q|^{-1}\sum_{Q_{j,k}\subset Q}2^{qj(\alpha+n/2-n/q)}\left|f^\epsilon_{j,k}\right|^q|l|^{n-N}.
\end{aligned}$$

We apply (ii) of Lemma 5.5 to the estimate of $C^{2,2,l}_{I,i}$. For $|l|\leq 4^n$,

$$\begin{aligned}
&C^{2,2,l}_{I,i}\\
&\leq C|I|^{-1}\int_{[2^{-j\beta},l(I)^\beta]}\sum_{\epsilon,(j,k)\in S_{I,l}}2^{q(j+n/2)-nj-(q-1)j\delta}(t2^{j\beta})^N e^{-ct2^{j\beta}}\left|f^\epsilon_{j,k}\right|^q t^{q(1-\alpha)/\beta-1-\delta/\beta}\,dxdt\\
&\leq C\sup_Q|Q|^{-1}\sum_{Q_{j,k}\subset Q}2^{qj(\alpha+n/2-n/q)}\left|f^\epsilon_{j,k}\right|^q.
\end{aligned}$$

For $|l|>4^n$, we have

$$\begin{aligned}
C^{2,2,l}_{I,i}&\leq C|I|^{-1}\int_{[2^{-j\beta},l(I)^\beta]}\sum_{\epsilon,(j,k)\in S_{I,l}}2^{q(j+n/2)-nj-(q-1)j\delta}(t2^{j\beta})^N e^{-ct2^{j\beta}}\left|f^\epsilon_{j,k}\right|^q\\
&\quad\times(2^j|I|^{1/n}|l|)^{n-N}t^{q(1-\alpha)/\beta-1-\delta/\beta}\,dxdt\\
&\leq C\sup_Q|Q|^{-1}\sum_{Q_{j,k}\subset Q}2^{qj(\alpha+n/2-n/q)}\left|f^\epsilon_{j,k}\right|^q|l|^{n-N}.
\end{aligned}$$

Since $\sup_Q|Q|^{-1}\sum_{Q_{j,k}\subset Q}2^{qj(\alpha+n/2-n/q)}\left|f^\epsilon_{j,k}\right|^q\leq C$, we get

$$C^2_{I,i}\leq C\sum_{l\in\mathbb{Z}^n}(1+|l|)^{n-N}+C\sum_{l\in\mathbb{Z}^n}(1+|l|)^{n-N}\leq C. \qquad \square$$

7.2. Boundary distribution

The boundary value of a β-harmonic function in $C^{\alpha,q}(\mathbb{R}^{n+1}_+)$ may not be locally integrable. But we have

Theorem 7.2. *Let $1\leq q<\infty$ and $|\alpha|<m$. For any $f(t,x)\in C^{\alpha,q}(\mathbb{R}^n)$, there exists a function $f\in M^{\alpha,q}(\mathbb{R}^n)$ such that*

$$f(t,x)=P^\beta_t f(x).$$

Proof. For $m>1$, we need only to consider m order integrations. To simplify the notations, we consider only the case $m=1$. For simplicity, for any ϵ, let

$$(f,W^{\alpha,q})_\epsilon(I)=:|I|^{-1}\sum_{(j,k):I_{j,k}\subset I}2^{qj(\alpha+n/2)-nj}\left|f^\epsilon_{j,k}\right|^q.$$

We write

$$(f,W^{\alpha,q})(I)=|I|^{-1}\sum_{(\varepsilon,j,k)\in\Lambda_n:I_{j,k}\subset I}2^{qj(\alpha+n/2)-nj}\left|f^\epsilon_{j,k}\right|^q\equiv\sum_\epsilon(f,W^{\alpha,q})_\epsilon(I).$$

Because the contribution of β-harmonic function $f(t,x)$ to the vaguelette $\Phi^{\epsilon}_{j,k}(t,x)$ is $f^{\epsilon}_{j,k} = -\langle \partial_\epsilon f(t,x), B^{\epsilon}_{j,k}(t,x)\rangle$, we have

$$\begin{aligned}
&(f, W^{\alpha,q})_\epsilon(I)\\
&= |I|^{-1} \sum_{I_{j,k}\subset I} 2^{qj(\alpha+n/2)-nj} \left|f^{\epsilon}_{j,k}\right|^q\\
&= |I|^{-1} \sum_{I_{j,k}\subset I} 2^{qj(\alpha+n/2)-nj} \left|\int_0^\infty\int_{\mathbb{R}^n} \partial_\epsilon f(t,y) B^{\epsilon}_{j,k}(t,y)\, dy\frac{dt}{t}\right|^q\\
&\leq |I|^{-1} \sum_{I_{j,k}\subset I} 2^{qj(\alpha+n/2)-nj} \int_0^\infty\int_{\mathbb{R}^n} |\partial_\epsilon f(t,y)|^q \left|B^{\epsilon}_{j,k}(t,y)\right|\, dy\frac{dt}{t^q}\\
&\quad\times\left(\int_0^\infty\int_{\mathbb{R}^n} \left|B^{\epsilon}_{j,k}(t,y)\right|\, dydt\right)^{q-1}\\
&\lesssim |I|^{-1} \sum_{I_{j,k}\subset I} 2^{qj(\alpha+n/2)-nj-(q-1)(n/2+1+\beta)j} \int_0^\infty\int_{\mathbb{R}^n} |\partial_\epsilon f(t,y)|^q \left|B^{\epsilon}_{j,k}(t,y)\right|\, dy\frac{dt}{t^q}.
\end{aligned}$$

Denote $I = I_{j_0,k_0}$ and $\forall\, l\in\mathbb{Z}^n$, denote $I_l = 2^{-j_0}l + I_{j_0,k_0} = 2^{-j_0}l + I$. Hence

$$\begin{aligned}
(f, W^{\alpha,q})_\epsilon(I) &\lesssim \sum_{l\in\mathbb{Z}^n} |I|^{-1} \sum_{I_{j,k}\subset I} 2^{qj(\alpha+n/2)-nj-(q-1)(n/2+1+\beta)j}\\
&\quad\times\int_0^\infty\int_{I_l} |\partial_\epsilon f(t,y)|^q \left|B^{\epsilon}_{j,k}(t,y)\right|\, dy\frac{dt}{t^q}\\
&\equiv \sum_{l\in\mathbb{Z}^n} C_{\epsilon,I,l}.
\end{aligned}$$

For $|l| < 2^{n+1}$, we have

$$\begin{aligned}
C_{\epsilon,I,l} &\lesssim |I|^{-1} \sum_{2^{nj}|I|\geq 1} 2^{qj(\alpha-1)-(q-1)\beta j} \int_{C_1 2^{-j\beta}}^{C_2 2^{-j\beta}}\int_{I_l} |\partial_\epsilon f(t,y)|^q\, dy\frac{dt}{t^q}\\
&\lesssim |I|^{-1} \sum_{2^{nj}|I|\geq 1} \int_{C_1 2^{-j\beta}}^{C_2 2^{-j\beta}}\int_{I_l} |\partial_\epsilon f(t,y)|^q t^{q(1-\alpha)/\beta-1}\, dydt.
\end{aligned}$$

For $|l| \geq 2^{n+1}$, we have

$$\begin{aligned}
C_{\epsilon,I,l} &\lesssim |I|^{-1} \sum_{2^{nj}|I|\geq 1} 2^{qj(\alpha-1)-(q-1)\beta j}(2^{j-j_0}|l|)^{-N} \int_{C_1 2^{-j\beta}}^{C_2 2^{-j\beta}}\int_{I_l} |\partial_\epsilon f(t,y)|^q\, dy\frac{dt}{t^q}\\
&\lesssim |I|^{-1} \sum_{2^{nj}|I|\geq 1} \int_{C_1 2^{-j\beta}}^{C_2 2^{-j\beta}}\int_{I_l} |\partial_\epsilon f(t,y)|^q t^{q(1-\alpha)/\beta-1}\, dydt|l|^{-N}.
\end{aligned}$$

Since $|I|^{-1}\sum_{2^{nj}|I|\geq 1}\int_{C_1 2^{-j\beta}}^{C_2 2^{-j\beta}}\int_{I_l} |\partial_\epsilon f(t,y)|^q t^{q(1-\alpha)/\beta-1}\, dydt \leq C$, we get

$$(f, W^{\alpha,q})_\epsilon(I) \leq C\sum_{l\in\mathbb{Z}^n}(1+|l|)^{-N} \leq C. \qquad \square$$

8. Proof of Theorems 3.6 and 3.7

In this section, we apply the above results to prove Theorems 3.6 and 3.7 in Section 3.

8.1. Proof of Theorem 3.6

We prove first two preliminaries theorems.

Theorem 8.1. *Given $m \in \mathbb{N}$, $|\alpha| < m$, $1 \le q \le \infty$. If $f(x,t) \in C^{\alpha,q}_m$, then $\forall\, (\epsilon, j, k) \in \Lambda_n$,*

$$\left| \int_0^\infty \left\langle f(t,x), P^\beta_t \Phi^\epsilon_{j,k}(x) \right\rangle \, dt \right| < C^\epsilon_{j,k}.$$

Proof. We consider only $m = 1$. By integration by parts, we get

$$\int_0^\infty \left\langle f(t,x), P^\beta_t \Phi^\epsilon_{j,k}(x) \right\rangle \, dt = - \int_0^\infty \left\langle \partial_\epsilon f(t,x), I_\epsilon P^\beta_t \Phi^\epsilon_{j,k}(x) \right\rangle \, dt.$$

By applying Lemma 5.6 and the property $f(x,t) \in C^{\alpha,q}_m$, we get the conclusion. $\square$

Theorem 8.2. *Given $m \in \mathbb{N}$, $|\alpha| < m$, $1 \le q \le \infty$. If $0 \neq f(x) \in M^{\alpha,q}$, then there exists $(\epsilon, j, k) \in \Lambda_n$ such that*

$$\left| \int_0^\infty \left\langle H^\beta_t f(x), P^\beta_t \Phi^\epsilon_{j,k}(x) \right\rangle \, dt \right| = \infty.$$

Proof. By Fourier transform and inverse Fourier transform, we have

$$\begin{aligned}
&\left| \int_0^\infty \left\langle H^\beta_t f(x), P^\beta_t \Phi^\epsilon_{j,k}(x) \right\rangle \, dt \right| \\
&= C 2^{-nj/2} \int_0^\infty dt \left\{ \int_{\mathbb{R}^n} e^{t|\xi|^\beta} \widehat{f}(\xi) e^{-t|\xi|^\beta} \widehat{\Phi^\epsilon}(2^{-j}\xi) e^{i2^{-j}k\xi} \, d\xi \right\} \\
&= C 2^{-nj/2} \int_0^\infty dt \left\{ \int_{\mathbb{R}^n} \widehat{f}(\xi) \widehat{\Phi^\epsilon}(2^{-j}\xi) e^{i2^{-j}k\xi} \, d\xi \right\} \\
&= C \int_0^\infty dt \left\{ \left\langle f(x), \Phi^\epsilon_{j,k}(x) \right\rangle \right\} \\
&= C \left| f^\epsilon_{j,k} \right| \int_0^\infty dt.
\end{aligned}$$

That is to say, if $f^\epsilon_{j,k} \neq 0$, then

$$\left| \int_0^\infty \langle H^\beta_t f(x), P^\beta_t \Phi^\epsilon_{j,k}(x) \rangle \, dt \right| = C \left| f^\epsilon_{j,k} \right| \int_0^\infty dt = \infty. \qquad \square$$

Proof of Theorem 3.6. By Theorems 6.1, 7.1 and 7.2 in Sections 6 and 7, we need only prove the uniqueness of β-harmonic functions. In fact, if $f \in M^{\alpha,q}$, then

$$P^\beta_t f \in C^{\alpha,q}_m \subset \mathcal{S}'_0(\mathbb{R}^n).$$

Further, $H^\beta_t f \in \mathcal{S}'_{\beta,t}(\mathbb{R}^n)$. By Theorem 8.2, $H^\beta_t f \notin \mathcal{S}'_0(\mathbb{R}^n)$. Hence the unique β-harmonic function in the relative Carleson measure space is $P^{\beta,0}_t f = P^\beta_t f$. This completes the proof. $\square$

8.2. Proof of Theorem 3.7

First we prove a preliminary theorem.

Theorem 8.3. *Given $m \in \mathbb{N}$, $|\alpha| < m$, $1 \leq q < \infty$. If $f \in M_0^{\alpha,q}$, then*

$$\lim_{t \to 0} \left\| f(x) - P_t^\beta f(x) \right\|_{M^{\alpha,q}} = 0.$$

Proof. If $f \in M_0^{\alpha,q}$, then for all $0 < \delta < \|f\|_{M^{\alpha,q}}$, there exists $j_0 \in \mathbb{N}$ such that

$$\|f_{1,\delta}(x)\|_{M^{\alpha,q}} < \delta \quad \text{and} \quad \|f_{2,\delta}(x)\|_{M^{\alpha,q}} < \delta + \|f\|_{M^{\alpha,q}},$$

where $f_{1,\delta}(x) = \sum_{\epsilon, j \geq j_0, k} f^\epsilon_{j,k} \Phi^\epsilon_{j,k}(x)$ and $f_{2,\delta}(x) = \sum_{\epsilon, j < j_0, k} f^\epsilon_{j,k} \Phi^\epsilon_{j,k}(x)$. It is easy to see that

$$\left\| f_{1,\delta}(x) - P_t^\beta f_{1,\delta}(x) \right\|_{M^{\alpha,q}} < 2\delta.$$

Let

$$a^{\epsilon,\epsilon'}_{j,k,j',k'} = \left\langle (e^{-t(-\Delta)^{\beta/2}} - 1)\Phi^\epsilon_{j,k}(x), \Phi^{\epsilon'}_{j',k'}(x) \right\rangle.$$

By Fourier transform, we have

$$\begin{aligned} a^{\epsilon,\epsilon'}_{j,k,j',k'} &= 2^{-\frac{n}{2}(j+j')} \int (e^{-t|\xi|^\beta} - 1)\widehat{\Phi^\epsilon}(2^{-j}\xi)\widehat{\Phi^{\epsilon'}}(2^{-j'}\xi) e^{-i(2^{-j}k - 2^{-j'}k')\xi}\, d\xi \\ &= 2^{\frac{n}{2}(j-j')} \int (e^{-t2^{j\beta}|\xi|^\beta} - 1)\widehat{\Phi^\epsilon}(\xi)\widehat{\Phi^{\epsilon'}}(2^{j-j'}\xi) e^{-i(k - 2^{j-j'}k')\xi}\, d\xi. \end{aligned}$$

The support of the Fourier transform of Meyer wavelets is contained in a ring, then

$$\forall\, |j - j'| \geq 2, \quad a^{\epsilon,\epsilon'}_{j,k,j',k'} = 0.$$

Further, for $|\xi| \sim 1$, $\alpha, u \in \mathbb{N}$, $v \in \mathbb{R}$, we have $\partial^\alpha_{\xi_i}(\xi^u |\xi|^v) \leq C$. If $|j - j'| \leq 1$, we have

$$\left| a^{\epsilon,\epsilon'}_{j,k,j',k'} \right| \leq Ct2^{j\beta}(1 + |k - 2^{j-j'}k'|)^{-n-N}.$$

By the orthogonality of Meyer wavelets, we have

$$f_{2,\delta}(x) - P_t^\beta f_{2,\delta}(x) = \sum_{\epsilon,\epsilon',k,k',j<j_0,|j-j'|\leq 1} f^\epsilon_{j,k} a^{\epsilon,\epsilon'}_{j,k,j',k'} \Phi^{\epsilon'}_{j',k'}(x).$$

For a dyadic cube Q with side length 2^{-j_0}, denote $\widetilde{Q}$ the dyadic cube which contains Q with double side length. We decompose $\mathbb{R}^n$ into the union $2^{1-j_0}l + \widetilde{Q}$, $l \in \mathbb{Z}^n$. If $q = 1$, then

$$\begin{aligned}
&|Q|^{-1}\sum_{\epsilon',Q_{j',k'}\subset Q}2^{j'(\alpha-n/2)}\left|\sum_{\epsilon,j<j_0,|j-j'|\le 1,k}f^{\epsilon}_{j,k}a^{\epsilon,\epsilon'}_{j,k,j',k'}\right|\\
&\le |Q|^{-1}\sum_{\epsilon',Q_{j',k'}\subset Q}2^{j'(\alpha-n/2)}\sum_{l\in\mathbb{Z}^n}\sum_{\epsilon,j<j_0,|j-j'|\le 1,Q_{j,k}\subset 2^{1-j_0}l+\widetilde{Q}}\left|f^{\epsilon}_{j,k}\right|\left|a^{\epsilon,\epsilon'}_{j,k,j',k'}\right|\\
&\le \sum_{l\in\mathbb{Z}^n}|\widetilde{Q}|^{-1}\left\{\sup_{Q_{j,k}\subset 2^{1-j_0}l+\widetilde{Q}}\sum_{\epsilon',Q_{j',k'}\subset Q,j'\le 1+j_0,|j-j'|\le 1}\left|a^{\epsilon,\epsilon'}_{j,k,j',k'}\right|\right\}\sum_{\epsilon,Q_{j,k}\subset 2^{1-j_0}l+\widetilde{Q}}2^{j(\alpha-n/2)}\left|f^{\epsilon}_{j,k}\right|\\
&\le t2^{\beta j_0}\sum_{l\in\mathbb{Z}^n}(1+|l|)^{-n-1}|\widetilde{Q}|^{-1}\sum_{\epsilon,Q_{j,k}\subset 2^{1-j_0}l+\widetilde{Q}}2^{j(\alpha-n/2)}\left|f^{\epsilon}_{j,k}\right|.
\end{aligned}$$

If $1<q<\infty$, we choose a sufficient small positive real number δ and denote $\tau=(q-1)(n+\delta)/q$.

If $Q_{j',k'}\subset Q$ and $|l|\le 4^n$, then

$$\left|\sum_{Q_{j,k}\subset 2^{1-j_0}l+\widetilde{Q}}f^{\epsilon}_{j,k}a^{\epsilon,\epsilon'}_{j,k,j',k'}\right|^q\le\sum_{Q_{j,k}\subset 2^{1-j_0}l+\widetilde{Q}}\left|f^{\epsilon}_{j,k}\right|^q.$$

If $Q_{j',k'}\subset Q$ and $|l|>4^n$, then

$$\left|\sum_{Q_{j,k}\subset 2^{1-j_0}l+\widetilde{Q}}f^{\epsilon}_{j,k}a^{\epsilon,\epsilon'}_{j,k,j',k'}\right|^q\le\sum_{Q_{j,k}\subset 2^{1-j_0}l+\widetilde{Q}}\left|f^{\epsilon}_{j,k}\right|^q$$

and

$$\begin{aligned}
&|Q|^{-1}\sum_{\epsilon',Q_{j',k'}\subset Q}2^{qj'(\alpha+n/2-n/q)}\left|\sum_{\epsilon,j<j_0,|j-j'|\le 1,k}f^{\epsilon}_{j,k}a^{\epsilon,\epsilon'}_{j,k,j',k'}\right|^q\\
&\le |Q|^{-1}\sum_{\epsilon',Q_{j',k'}\subset Q}2^{qj'(\alpha+n/2-n/q)}\sum_{\epsilon,j<j_0,|j-j'|\le 1,k}\left|f^{\epsilon}_{j,k}\right|^q\left|a^{\epsilon,\epsilon'}_{j,k,j',k'}\right|\\
&\le |Q|^{-1}\sum_{\epsilon',Q_{j',k'}\subset Q}2^{qj'(\alpha+n/2-n/q)}\sum_{l\in\mathbb{Z}^n}\sum_{\epsilon,j<j_0,|j-j'|\le 1,Q_{j,k}\subset 2^{1-j_0}l+\widetilde{Q}}\left|f^{\epsilon}_{j,k}\right|^q\left|a^{\epsilon,\epsilon'}_{j,k,j',k'}\right|\\
&\le \sum_{l\in\mathbb{Z}^n}|Q|^{-1}\left\{\sup_{Q_{j,k}\subset 2^{1-j_0}l+\widetilde{Q}}\sum_{\epsilon',j'<1+j_0,Q_{j',k'}\subset Q}\left|a^{\epsilon,\epsilon'}_{j,k,j',k'}\right|\right\}\sum_{\epsilon,Q_{j,k}\subset 2^{1-j_0}l+\widetilde{Q}}2^{qj(\alpha+n/2-n/q)}\left|f^{\epsilon}_{j,k}\right|^q\\
&\le t2^{\beta j_0}\sum_{l\in\mathbb{Z}^n}(1+|l|)^{-n-1}|\widetilde{Q}|^{-1}\sum_{\epsilon,Q_{j,k}\subset 2^{1-j_0}l+\widetilde{Q}}2^{qj(\alpha+n/2-n/q)}\left|f^{\epsilon}_{j,k}\right|^q.
\end{aligned}$$

We choose t sufficient small such that $t2^{\beta j_0}<\delta$, we get the conclusion. □

Proof of Theorem 3.7. (i) By Theorem 3.6, the unique β-harmonic function in $C^{\alpha,q}_m$ is $P^{\beta}_t f$. Now we only need to prove

$$P^{\beta}_t f\in C^{\alpha,q}_{m,0}. \tag{8.1}$$

For all $0 < \delta < \|f\|_{M_0^{\alpha,q}}$, there exists j_0 such that $f_{j_0} = \sum_{(\epsilon, j\geq j_0, k)} f^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}(x)$ and $\|f_{j_0}\|_{M_0^{\alpha,q}} < \delta$. Hence $\left\|P^{\beta}_t f_{j_0}\right\|_{C^{\alpha,q}_m} < C\delta$.

To prove the conclusion of (8.1), we need to consider

$$f - f_{j_0} = \sum_{(\epsilon, j<j_0, k)} f^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}(x).$$

For $j < j_0$ and $\gamma \in \mathbb{N}^n$ and $|\gamma| = m$, denote

$$g_{j,\gamma}(t,x) = \sum_{(\epsilon,k)} f^{\epsilon}_{j,k}\partial^{\gamma}P^{\beta}_t\Phi^{\epsilon}_{j,k}(x).$$

Since $f(x) \in M^{\alpha,q} \subset M^{\alpha,\infty}$, we have

$$|g_{j,\gamma}| \leq C2^{(\alpha-m)j}.$$

Since $\alpha < m$, hence

$$\left|\partial^{\gamma}P^{\beta}_t(f - f_{j_0})\right| \leq C2^{(\alpha-m)j_0}.$$

That is to say,

$$\begin{aligned}
&|I|^{-1}\int_{S_\beta(I)}\left|\nabla^m P^{\beta}_t(f-f_{j_0})\right|^q t^{q(m-\alpha)/\beta-1}\,dxdt\\
&\leq |I|^{-1}\int_{S_\beta(I)} 2^{q(\alpha-m)j_0}t^{q(m-\alpha)/\beta-1}\,dxdt\\
&\leq 2^{q(\alpha-m)j_0}l(I)^{q(m-\alpha)} = \{2^{j_0}l(I)\}^{q(m-\alpha)}.
\end{aligned}$$

We choose $2^{j_0}l(I)$ sufficient small, we get the conclusion.

(ii) By Theorems 7.1 and 7.2, we get $b_f \in M^{\alpha,q}$. Now we prove that $b_f \in M^{\alpha,q}_0$. Similar to the proof of Theorem 7.2, we have

$$\begin{aligned}
&(f, W^{\alpha,q})_{\epsilon}(I)\\
&\lesssim |I|^{-1}\sum_{I_{j,k}\subset I} 2^{qj(\alpha+n/2)-nj-(q-1)(n/2+1+\beta)j}\int_0^{\infty}\int_{\mathbb{R}^n}|\partial_\epsilon f(t,y)|^q\left|B^{\epsilon}_{j,k}(t,y)\right|\,dy\frac{dt}{t^q}.
\end{aligned}$$

Denote $I = I_{j_0,k_0}$ and $\forall\, l \in \mathbb{Z}^n$, denote $I_l = 2^{-j_0}l + I_{j_0,k_0} = 2^{-j_0}l + I$. Hence

$$\begin{aligned}
(f, W^{\alpha,q})_{\epsilon}(I) &\lesssim \sum_{l\in\mathbb{Z}^n}|I|^{-1}\sum_{I_{j,k}\subset I} 2^{qj(\alpha+n/2)-nj-(q-1)(n/2+1+\beta)j}\\
&\quad\times\int_0^{\infty}\int_{I_l}|\partial_\epsilon f(t,y)|^q\left|B^{\epsilon}_{j,k}(t,y)\right|\,dy\frac{dt}{t^q}\\
&\equiv \sum_{l\in\mathbb{Z}^n} C_{\epsilon,I,l}.
\end{aligned}$$

For $|l| < 2^{n+1}$, we have

$$
\begin{aligned}
C_{\epsilon,I,l} &\lesssim |I|^{-1} \sum_{2^{nj}|I|\geq 1} 2^{qj(\alpha-1)-(q-1)\beta j} \int_{C_1 2^{-j\beta}}^{C_2 2^{-j\beta}} \int_{I_l} |\partial_\epsilon f(t,y)|^q \, dy \frac{dt}{t^q} \\
&\lesssim |I|^{-1} \sum_{2^{nj}|I|\geq 1} \int_{C_1 2^{-j\beta}}^{C_2 2^{-j\beta}} \int_{I_l} |\partial_\epsilon f(t,y)|^q t^{q(1-\alpha)/\beta - 1} \, dy dt.
\end{aligned}
$$

For $|l| \geq 2^{n+1}$, we have

$$
\begin{aligned}
C_{\epsilon,I,l} &\lesssim |I|^{-1} \sum_{2^{nj}|I|\geq 1} 2^{qj(\alpha-1)-(q-1)\beta j} (2^{j-j_0}|l|)^{-N} \int_{C_1 2^{-j\beta}}^{C_2 2^{-j\beta}} \int_{I_l} |\partial_\epsilon f(t,y)|^q \, dy \frac{dt}{t^q} \\
&\lesssim |I|^{-1} \sum_{2^{nj}|I|\geq 1} \int_{C_1 2^{-j\beta}}^{C_2 2^{-j\beta}} \int_{I_l} |\partial_\epsilon f(t,y)|^q t^{q(1-\alpha)/\beta - 1} \, dy dt |l|^{-N}.
\end{aligned}
$$

Since $f(t,x) \in C^{\alpha,q}_{m,0}$, then for any $0 < \delta < \|f\|_{C^{\alpha,q}_m}$, there exists I_δ such that, for any $|I| \leq I_\delta$, we have $|I|^{-1} \sum_{2^{nj}|I|\geq 1} \int_{C_1 2^{-j\beta}}^{C_2 2^{-j\beta}} \int_{I_l} |\partial_\epsilon f(t,y)|^q t^{q(1-\alpha)/\beta - 1} \, dy dt \leq C\delta$, we get

$$
(f, W^{\alpha,q})_\epsilon(I) \leq C \sum_{l\in\mathbb{Z}^n} (1+|l|)^{-N} \leq C\delta.
$$

By Theorem 8.3, we get

$$
\lim_{t\to 0} \|b_f - f(t,x)\|_{M^{\alpha,q}} = 0. \qquad \square
$$

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