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Invasion Entire Solutions for a Three Species Competition-diffusion System

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Abstract. The purpose of this paper is to study a three species competition model with diffusion. It is well known that there exists a family of traveling wave solutions connecting two equilibria $(0,1,1)$ and $(1,0,0)$. In this paper, we first establish the exact asymptotic behavior of the traveling wave profiles at $\pm\infty$. Then, by constructing a pair of explicit upper and lower solutions via the combination of traveling wave solutions, we derive the existence of some new entire solutions which behave as two traveling fronts moving towards each other from both sides of x-axis. Such entire solution provides another invasion way of the stronger species to the weak ones.

1. Introduction

This paper is concerned with the following three species Lotka-Volterra competition reaction-diffusion system (c.f. [3]):

$$
\begin{aligned}
&\frac{\partial v_1(x,t)}{\partial t} = d_1 \frac{\partial^2 v_1(x,t)}{\partial x^2} + r_1 v_1(x,t)[1 - v_1(x,t) - a_{11} v_2(x,t) - a_{12} v_3(x,t)], \\
&\frac{\partial v_2(x,t)}{\partial t} = d_2 \frac{\partial^2 v_2(x,t)}{\partial x^2} + r_2 v_2(x,t)[1 - v_2(x,t) - a_{21} v_1(x,t)], \\
&\frac{\partial v_3(x,t)}{\partial t} = d_3 \frac{\partial^2 v_3(x,t)}{\partial x^2} + r_3 v_3(x,t)[1 - v_3(x,t) - a_{31} v_1(x,t)],
\end{aligned} \tag{1.1}
$$

where $x, t \in \mathbb{R}$, $v_1(x,t)$, $v_2(x,t)$ and $v_3(x,t)$ denote the population densities of the three different species, $a_{11} > 0$, $a_{12} > 0$, $a_{21} > 0$ and $a_{31} > 0$ are interaction coefficients respectively, $r_i > 0$ $(i = 1,2,3)$ stands for the relative intrinsic growth rate of the species i. From the view of the intra-specific competitions, the system (1.1) formulates the relation that the species v_1 competes with v_2 and v_3 respectively, while there is no competition between species v_2 and v_3.

It is obvious that $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and $(0,1,1)$ are equilibria of (1.1). Moreover, it is easy to check that the equilibrium $(1,0,0)$ is stable and $(0,1,1)$ is unstable under the following assumption

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(H) $a_{21}, a_{31} > 1$, $a_{11} + a_{12} < 1$.

This assumption implies that the species v_1 is stronger than v_2 and v_3, and hence the species v_1 shall invade v_2 and v_3 and eventually v_2 and v_3 will be extinct. Therefore, an interesting problem is to know how the stronger species invades the weaker ones. It is no doubt that the traveling wave solutions connecting $(0, 1, 1)$ and $(1, 0, 0)$ can provide an invasion way of v_1 to v_2 and v_3. Under the assumption (H), Guo et al. [3] given some conditions on the parameters of the competition system such that the minimal wave speed $c_{\min}$ of traveling wave fronts connecting $(0, 1, 1)$ and $(1, 0, 0)$ equals to $c^* := 2\sqrt{d_1 r_1 (1 - a_{11} - a_{12})} > 0$. This result is called the linear determinacy (c.f. [3, 9]).

It is natural to ask, in addition to the traveling wave solutions, whether there exists another way of v_1 invades v_2 and v_3. In this paper, we give an affirmative answer. More precisely, we shall construct some new entire solution of (1.1) which behave as two traveling fronts moving towards each other from both sides of x-axis (see Theorem 3.1). Such entire solution provides another invasion way of the stronger species to the weak ones.

We end the introduction with the following remarks. First, since the work of Hamel and Nadirashvili [5], there are many results devoted to the entire solutions to scalar evolution equations, see e.g. [2, 6, 10, 12, 14, 16]. Morita-Tachibana [8] first extended the results of scalar equations to a two-component competition-diffusion system. Wang and Lv [17] and Wu and Wang [20] considered the entire solutions for a L-V competition system with spatial-temporal delay and general reaction-diffusion system, respectively. For other related results on entire solutions of two component systems, we refer to [4, 11, 18, 19].

Secondly, we remark that for a system enjoying the comparison principle, one can obtain the desired solution by constructing appropriate upper and lower solutions (c.f. [2, 4–6, 12, 13, 15, 17, 19, 21]). Since (1.1) can be transformed to an equivalent cooperative system, we shall prove the existence of entire solution by constructing a pair of explicit upper and lower solutions. The construction of the sub- and super-solution is based on the exact asymptotic behavior of traveling wave fronts. However, the Ikehara's theorem which is always used in scalar equations can not be applied to obtain the asymptotic behavior of traveling wave fronts. In this paper, we shall establish the exact asymptotic behavior of the traveling wave fronts by applying the asymptotic theory (c.f. [12, 17]).

Thirdly, it should be mentioned that our results can be applied to the following Lotka-Volterra competition-cooperation model (c.f. [7])

$$\begin{aligned} u_t &= d_1 u_{xx} + u(1 - u - a_1 w), \\ v_t &= d_2 v_{xx} + r v(1 - a_2 u - v), \\ w_t &= d_3 w_{xx} + b(v - w), \end{aligned} \tag{1.2}$$

where $u(x, t)$, $v(x, t)$ and $w(x, t)$ represent the population densities of three different species, respectively, $a_1 > 0$ and $a_2 > 0$ are interaction coefficients, $r > 0$ ($b > 0$) stands for the relative intrinsic growth rate of the species v and (w, respectively). Hou and Li [7] obtained the existence, asymptotic and uniqueness of traveling wave solutions of the model (1.2). However, there has been no results on the entire solutions of system (1.2).

The rest of this paper is planned as follows. In Section 2, we establish the asymptotic behavior of the traveling wave fronts at $\pm\infty$. In Section 3, by constructing a pair of appropriate super- and sub-solutions, we prove the existence of entire solutions.

2. Asymptotic behavior of traveling wave front

In this section, we establish the asymptotic behavior of the traveling wave profiles at $\pm\infty$.

By letting $u_1 = v_1$, $u_2 = 1 - v_2$ and $u_3 = 1 - v_3$, (1.1) becomes the following equivalent system:

$$\begin{aligned} \frac{\partial u_1(x,t)}{\partial t} &= d_1 \frac{\partial^2 u_1(x,t)}{\partial x^2} + r_1 u_1[1 - a_{11} - a_{12} - u_1 + a_{11}u_2 + a_{12}u_3], \\ \frac{\partial u_2(x,t)}{\partial t} &= d_2 \frac{\partial^2 u_1(x,t)}{\partial x^2} + r_2(1 - u_2)[a_{21}u_1 - u_2], \\ \frac{\partial u_3(x,t)}{\partial t} &= d_3 \frac{\partial^2 u_1(x,t)}{\partial x^2} + r_3(1 - u_3)[a_{31}u_1 - u_3]. \end{aligned} \tag{2.1}$$

It is clear that the equalibria $(0,1,1)$ and $(1,0,0)$ become $(0,0,0)$ and $(1,1,1)$, respectively, and (2.1) is cooperative on $[\mathbf{0}, \mathbf{K}]$, where $\mathbf{K} = (1,1,1)$.

Throughout this paper, a solution $(u_1(x,t), u_2(x,t), u_3(x,t))$ of (2.1) is called a traveling wave solution connecting $(0,0,0)$ and $(1,1,1)$ with speed c and profile $(\varphi_1, \varphi_2, \varphi_3)$ if $(u_1(x,t), u_2(x,t), u_3(x,t)) = (\varphi_1(\xi), \varphi_2(\xi), \varphi_3(\xi))$, $\xi = x + ct$, such that

$$\begin{aligned} &c\varphi_1'(\xi) = d_1\varphi_1''(\xi) + r_1\varphi_1(\xi)[1 - a_{11} - a_{12} - \varphi_1(\xi) + a_{11}\varphi_2(\xi) + a_{12}\varphi_3(\xi)], \\ &c\varphi_2'(\xi) = d_2\varphi_2''(\xi) + r_2(1 - \varphi_2(\xi))[a_{21}\varphi_1(\xi) - \varphi_2(\xi)], \\ &c\varphi_3'(\xi) = d_3\varphi_3''(\xi) + r_3(1 - \varphi_3(\xi))[a_{31}\varphi_1(\xi) - \varphi_3(\xi)], \\ &\varphi_1' > 0,\ \varphi_2' > 0,\ \varphi_3' > 0 \end{aligned} \tag{2.2}$$

with

$$\lim_{\xi\to-\infty}(\varphi_1(\xi), \varphi_2(\xi), \varphi_3(\xi)) = (0,0,0), \quad \lim_{\xi\to+\infty}(\varphi_1(\xi), \varphi_2(\xi), \varphi_3(\xi)) = (1,1,1). \tag{2.3}$$

In the sequel, we always assume that $\Psi = (\varphi_1, \varphi_2, \varphi_3)$ is a solution of problem (2.2)–(2.3) with positive speed $c \geq c^* = 2\sqrt{d_1 r_1(1 - a_{11} - a_{12})}$. By differentiating the differential equations (2.2) with respect to ξ, and denote $(\varphi_1', \varphi_2', \varphi_3')$ by (ψ_1, ψ_2, ψ_3). Then we obtain the following system:

(2.4)
$$\begin{aligned}
c\psi_1' &= d_1\psi_1'' + r_1\{\psi_1[1 - a_{11} - a_{12} - \varphi_1 + a_{11}\varphi_2 + a_{12}\varphi_3] + \varphi_1[-\psi_1 + a_{11}\psi_2 + a_{12}\psi_3]\},\\
c\psi_2' &= d_2\psi_2'' + r_2\{-\psi_2[a_{21}\varphi_1 - \varphi_2] + (1-\varphi_2)[a_{21}\psi_1 - \psi_2]\},\\
c\psi_3' &= d_3\psi_3'' + r_3\{-\psi_3[a_{31}\varphi_1 - \varphi_3] + (1-\varphi_3)[a_{31}\psi_1 - \psi_3]\}.
\end{aligned}$$

To obtain the asymptotic behavior of traveling waves, we consider the following two cases:

(I) $\xi \to \infty$: The limiting system of (2.4) as $\xi \to \infty$ has the following form:

$$\begin{aligned}
c\psi_{1+}' &= d_1\psi_{1+}'' - r_1\psi_{1+} + r_1a_{11}\psi_{2+} + r_1a_{12}\psi_{3+},\\
c\psi_{2+}' &= d_2\psi_{2+}'' - r_2\psi_{2+}(a_{21} - 1),\\
c\psi_{3+}' &= d_3\psi_{3+}'' - r_3\psi_{3+}(a_{31} - 1).
\end{aligned} \tag{2.5}$$

Let $\psi_{1+}' = \psi_{12+}$, $\psi_{2+}' = \psi_{22+}$ and $\psi_{3+}' = \psi_{32+}$. Then system (2.5) can be transformed into the following form

$$X' = P_1X, \quad X = (\psi_{1+}, \psi_{12+}, \psi_{2+}, \psi_{22+}, \psi_{3+}, \psi_{32+})^{\mathrm{T}}, \tag{2.6}$$

where

$$P_1 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0\\
r_1/d_1 & c/d_1 & -r_1a_{11}/d_1 & 0 & -r_1a_{12}/d_1 & 0\\
0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & r_2(a_{21}-1)/d_2 & c/d_2 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & r_3(a_{31}-1)/d_3 & c/d_3
\end{pmatrix}.$$

By a direct calculation, the eigenvalues of the matrix P_1 are $\Lambda_1 := \Lambda_1(c), \ldots, \Lambda_6 := \Lambda_6(c)$ and the corresponding eigenvectors are $h_1^+ := h_1^+(c), \ldots, h_6^+ := h_6^+(c)$, where

$$\begin{aligned}
\Lambda_1 &= \frac{c - \sqrt{c^2 + 4r_1d_1}}{2}, & \Lambda_2 &= \frac{c + \sqrt{c^2 + 4r_1d_1}}{2},\\
\Lambda_3 &= \frac{c - \sqrt{c^2 - 4r_2d_2(1 - a_{21})}}{2}, & \Lambda_4 &= \frac{c + \sqrt{c^2 - 4r_2d_2(1 - a_{21})}}{2},\\
\Lambda_5 &= \frac{c - \sqrt{c^2 - 4r_3d_3(1 - a_{31})}}{2}, & \Lambda_6 &= \frac{c + \sqrt{c^2 - 4r_3d_3(1 - a_{31})}}{2},
\end{aligned}$$

and

$$h_i^+ = \begin{pmatrix} 1 \\ \Lambda_i \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad h_j^+ = \begin{pmatrix} a_j \\ \Lambda_j a_j \\ 1 \\ \Lambda_j \\ 0 \\ 0 \end{pmatrix}, \quad h_k^+ = \begin{pmatrix} \overline{a}_k \\ \Lambda_k \overline{a}_k \\ 0 \\ 0 \\ 1 \\ \Lambda_k \end{pmatrix},$$

$$a_j = -\frac{r_1 a_{11}}{d_2 \Lambda_j^2 - c\Lambda_j - r_1}, \quad \overline{a}_k = -\frac{r_1 a_{12}}{d_3 \Lambda_k^2 - c\Lambda_k - r_1}, \quad i = 1,2,\ j = 3,4,\ k = 5,6.$$

For the sake of convenience, throughout this paper, it is always assumed that $r_1 d_1$, $r_2 d_2(a_{21} - 1)$, and $r_3 d_3(a_{31} - 1)$ differ from each other. Then the general solution of system (2.6) has the following expression:

$$(\psi_{1+}, \psi_{12+}, \psi_{2+}, \psi_{22+}, \psi_{3+}, \psi_{32+})^{\mathrm{T}} = \sum_{p=1}^{6} B_p h_p^+ e^{\Lambda_p \xi}, \tag{2.7}$$

where B_p denotes arbitrary constant. Since $X \to 0$ as $\xi \to \infty$, one has $B_2 = B_4 = B_6 = 0$. Hence, any solution of (2.7) which can converge to zeros as $\xi \to \infty$ is represented as

$$X(\xi) = B_1 h_1^+ e^{\Lambda_1 \xi} + B_3 h_3^+ e^{\Lambda_3 \xi} + B_5 h_5^+ e^{\Lambda_5 \xi}.$$

It then follows from the stable manifold theorem that

$$\begin{aligned} 1 - \varphi_1(\xi) &= \alpha e^{\Lambda_1 \xi} + \beta a_3 e^{\Lambda_3 \xi} + \gamma \overline{a}_5 e^{\Lambda_5 \xi} + \text{h.o.t.}, \\ 1 - \varphi_2(\xi) &= \beta e^{\Lambda_3 \xi} + \text{h.o.t.}, \\ 1 - \varphi_3(\xi) &= \gamma e^{\Lambda_5 \xi} + \text{h.o.t.}, \end{aligned}$$

where $\alpha \geq 0$ and $\beta, \gamma > 0$.

(II) $\xi \to -\infty$: In this case, the limiting system of (2.4) is

$$\begin{aligned} & d_1 \psi_{1-}'' - c\psi_{1-}' + r_1 \psi_{1-}(1 - a_{11} - a_{12}) = 0, \\ & d_2 \psi_{2-}'' - c\psi_{2-}' + r_2 a_{21} \psi_{1-} - r_2 \psi_{2-} = 0, \\ & d_3 \psi_{3-}'' - c\psi_{3-}' + r_3 a_{31} \psi_{1-} - r_3 \psi_{3-} = 0. \end{aligned} \tag{2.8}$$

By taking $\psi_{1-}' = \psi_{12-}$, $\psi_{2-}' = \psi_{22-}$ and $\psi_{3-}' = \psi_{32-}$, system (2.8) can be expressed as the following first-order ordinary differential system:

$$X' = P_2 X, \quad X = (\psi_{1-}, \psi_{12-}, \psi_{2-}, \psi_{22-}, \psi_{3-}, \psi_{32-})^{\mathrm{T}}, \tag{2.9}$$

where

$$P_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -r_1(1-a_{11}-a_{12})/d_1 & c/d_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -r_2a_{21}/d_2 & 0 & r_2/d_2 & c/d_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -r_3a_{31}/d_3 & 0 & 0 & 0 & r_3/d_3 & c/d_3 \end{pmatrix}.$$

Direct computation shows that the eigenvalues of the matrix P_2 are $\lambda_1 := \lambda_1(c), \ldots, \lambda_6 := \lambda_6(c)$ and the corresponding eigenvectors are $h_1^- := h_1^-(c), \ldots, h_6^- := h_6^-(c)$, respectively, where

$$\lambda_1 = \frac{c - \sqrt{c^2 - 4r_1d_1(1-a_{11}-a_{12})}}{2}, \qquad \lambda_2 = \frac{c + \sqrt{c^2 - 4r_1d_1(1-a_{11}-a_{12})}}{2},$$
$$\lambda_3 = \frac{c + \sqrt{c^2 + 4r_2d_2}}{2}, \qquad \lambda_4 = \frac{c - \sqrt{c^2 + 4r_2d_2}}{2},$$
$$\lambda_5 = \frac{c + \sqrt{c^2 + 4r_3d_3}}{2}, \qquad \lambda_6 = \frac{c - \sqrt{c^2 + 4r_3d_3}}{2},$$

and

$$h_i^- = \begin{pmatrix} 1 \\ \lambda_i \\ s_{1i} \\ \lambda_i s_{1i} \\ s_{2i} \\ \lambda_i s_{2i} \end{pmatrix}, \quad h_j^- = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \lambda_j \\ 0 \\ 0 \end{pmatrix}, \quad h_k^- = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ \lambda_k \end{pmatrix},$$
$$s_{1i} = -\frac{r_2a_{21}}{d_2\lambda_i^2 - c\lambda_i - r_2}, \quad s_{2i} = -\frac{r_3a_{31}}{d_3\lambda_i^2 - c\lambda_i - r_3}, \quad i = 1, 2,\ j = 3, 4,\ k = 5, 6.$$

Throughout this paper, for convenience to discuss, it is always assumed that $r_1d_1(1 - a_{11} - a_{12})$, r_2d_2, and r_2d_3 differ from each other. If $c = c^* = 2\sqrt{r_1d_1(1-a_{11}-a_{12})}$, then $\lambda_1 = \lambda_2$ and the matrix P_1 possesses a generalized eigenvector as follows

$$h^* = (1, 0, k_1((3\lambda_1d_2-2c)\lambda_1-r_2), k_1\lambda_1^2(2\lambda_1d_2-c), k_2((3\lambda_1d_3-2c)\lambda_1-r_2), k_2\lambda_1^2(2\lambda_1d_3-c))^{\mathrm{T}},$$

where

$$k_1 = \frac{r_2a_{21}d_2}{r_2(2\lambda_1d_2-c)^2 - [(\lambda_1d_2-c)^2 + r_2d_2](d_2\lambda_1^2 + r_2)}$$

and

$$k_2 = \frac{r_3a_{31}d_3}{r_3(2\lambda_1d_3-c)^2 - [(\lambda_1d_3-c)^2 + r_3d_3](d_3\lambda_1^2 + r_3)}.$$

It can be easily seen that $\lambda_1, \lambda_2 > 0$ and $\min\{\lambda_3, \lambda_5\} > \lambda_2 \geq \lambda_1$ under the assumption (H).

Based on the basic theory related with the first order ordinary differential system, we can easily find the general solution of system (2.9) as follows

$$(\psi_{1-}, \psi_{12-}, \psi_{2-}, \psi_{22-}, \psi_{3-}, \psi_{32-})^{\mathrm{T}} = \sum_{l=1}^{6} A_l h_l^- e^{\lambda_l \xi}, \tag{2.10}$$

where A_l denotes arbitrary constant. Since $X \to 0$ as $\xi \to -\infty$, we arrive at the conclusion that $A_4 = A_6 = 0$.

Thus, if $\lambda_1 \neq \lambda_2$, then every solution of (2.10) which converges to $(0, \ldots, 0)$ as $\xi \to -\infty$ can be denoted by

$$X(\xi) = A_1 h_1^- e^{\lambda_1 \xi} + A_2 h_2^- e^{\lambda_2 \xi} + A_3 h_3^- e^{\lambda_3 \xi} + A_5 h_5^- e^{\lambda_5 \xi}.$$

According to the unstable manifold theorem, we can obtain the asymptotic behaviors of $\varphi_1(\xi)$, $\varphi_2(\xi)$ and $\varphi_3(\xi)$ as follows

$$\begin{aligned} \varphi_1(\xi) &= \alpha e^{\lambda_1 \xi} + \beta e^{\lambda_2 \xi} + \text{h.o.t.}, \\ \varphi_2(\xi) &= \alpha s_{11} e^{\lambda_1 \xi} + \beta s_{12} e^{\lambda_2 \xi} + \gamma e^{\lambda_3 \xi} + \text{h.o.t.}, \\ \varphi_3(\xi) &= \alpha s_{21} e^{\lambda_1 \xi} + \beta s_{22} e^{\lambda_2 \xi} + \eta e^{\lambda_5 \xi} + \text{h.o.t.}, \end{aligned} \tag{2.11}$$

where h.o.t. denotes the higher order term and $\alpha, \beta, \gamma, \eta \geq 0$. Based on the same analysis as in [13], we obtain $(\alpha, \beta) \neq (0, 0)$.

If $\lambda_1 = \lambda_2$, then every solution of (2.10) which converges to $(0, \ldots, 0)$ as $\xi \to -\infty$ can be expressed by

$$X(\xi) = (C_1 h_1^- + C_2 h^* \xi) e^{\lambda_1 \xi} + C_3 h_3^- e^{\lambda_3 \xi} + C_5 h_5^- e^{\lambda_5 \xi}.$$

Thanks to the unstable manifold theorem, the following asymptotic behaviors can be obtained:

$$\begin{aligned} \varphi_1(\xi) &= \alpha e^{\lambda_1 \xi} + \beta \xi e^{\lambda_1 \xi} + \text{h.o.t.}, \\ \varphi_2(\xi) &= \alpha s_{11} e^{\lambda_1 \xi} + k_1 [(3\lambda_1 d_2 - 2c)\lambda_1 - r_2)] \beta \xi e^{\lambda_1 \xi} + \gamma e^{\lambda_3 \xi} + \text{h.o.t.}, \\ \varphi_3(\xi) &= \alpha s_{21} e^{\lambda_1 \xi} + k_2 [(3\lambda_1 d_3 - 2c)\lambda_1 - r_3)] \beta \xi e^{\lambda_1 \xi} + \eta e^{\lambda_5 \xi} + \text{h.o.t.}, \end{aligned} \tag{2.12}$$

where $(\alpha, \beta) \neq (0, 0)$ and $\gamma, \eta \geq 0$.

From (2.7), (2.11) and (2.12), we have the following result. It is obvious that $\lambda_1(c) \leq \lambda_2(c)$, $\lambda_1'(c) < 0$, $\lambda_j'(c) > 0$ and $\Lambda_k'(c) > 0$, $j = 2, 3, 5$, $k = 1, 3, 5$.

Theorem 2.1. *Assume that the condition* (H) *holds. Let* $(\varphi_1, \varphi_2, \varphi_3)$ *be a traveling wave front of* (2.2) *with speed* $c \geq c^*$. *Then the following asymptotic properties hold:*

(i) *As* $\xi \to -\infty$,

$$\begin{pmatrix} \varphi_1(\xi) \\ \varphi_2(\xi) \\ \varphi_3(\xi) \end{pmatrix} = \begin{pmatrix} (C_1 + o(1))e^{\lambda_1 \xi} \\ (C_2 + o(1))e^{\lambda_1 \xi} \\ (C_3 + o(1))e^{\lambda_1 \xi} \end{pmatrix} \quad \text{for } c > c^*,$$

and

$$\begin{pmatrix} \varphi_1(\xi) \\ \varphi_2(\xi) \\ \varphi_3(\xi) \end{pmatrix} = \begin{pmatrix} (C_4 + C_4'\xi + o(1))e^{\lambda_1 \xi} \\ (C_4 s_1 + k_1[(3\lambda_1 d_2 - 2c)\lambda_1 - r_2]C_4'\xi + o(1))e^{\lambda_1 \xi} \\ (C_4 s_2 + k_2[(3\lambda_1 d_3 - 2c)\lambda_1 - r_3]C_4'\xi + o(1))e^{\lambda_1 \xi} \end{pmatrix} \quad \text{for } c = c^*;$$

(ii) *As* $\xi \to +\infty$,

$$\begin{pmatrix} \varphi_1(\xi) \\ \varphi_2(\xi) \\ \varphi_3(\xi) \end{pmatrix} = \begin{pmatrix} 1 - (C_5 + o(1))e^{\Lambda \xi} \\ 1 - (C_6 + o(1))e^{\Lambda_3 \xi} \\ 1 - (C_7 + o(1))e^{\Lambda_5 \xi} \end{pmatrix} \quad \text{for } c \geq c^*,$$

where $C_i > 0$, $i = 1, \ldots, 7$, $C_4' \geq 0$, $\Lambda = \Lambda(c) = \max\{\Lambda_1(c), \Lambda_3(c), \Lambda_5(c)\}$.

According to Theorem 2.1, we have the following three lemmas.

Lemma 2.2. *There are positive constants* $m_i(c)$, $l_i(c)$, $M_i(c)$ *and* $L_i(c)$ $(i = 1, 2)$*, such that*

(i) *If* $c > c^*$*, then the following results hold*

$$m_1(c)e^{\lambda_1 \xi} \leq \varphi_1(\xi), \varphi_2(\xi), \varphi_3(\xi) \leq M_1(c)e^{\lambda_1 \xi} \quad \text{for } \xi \leq 0. \tag{2.13}$$

(ii) *If* $c = c^*$*, assume that* $0 < \varepsilon < \lambda_1 := \lambda^*$*, then there exists a positive constant* K_ε *satisfying the following*

$$\max_{\xi \leq 0}\{\varphi_1(\xi), \varphi_2(\xi), \varphi_3(\xi)\} \leq K_\varepsilon e^{(\lambda^* - \varepsilon)\xi} \quad \text{for } \xi \leq 0. \tag{2.14}$$

(iii) *If* $c \geq c^*$*, then the following assertions are valid*

$$\begin{aligned} m_2(c)e^{\Lambda \xi} &\leq 1 - \varphi_1(\xi) \leq M_2(c)e^{\Lambda \xi} \quad \text{for } \xi \geq 0, \\ l_1(c)e^{\Lambda_3 \xi} &\leq 1 - \varphi_2(\xi) \leq L_1(c)e^{\Lambda_3 \xi} \quad \text{for } \xi \geq 0, \\ l_2(c)e^{\Lambda_5 \xi} &\leq 1 - \varphi_3(\xi) \leq L_2(c)e^{\Lambda_5 \xi} \quad \text{for } \xi \geq 0, \end{aligned}$$

where Λ *is given as in Theorem 2.1.*

Lemma 2.3. *There exist positive constants $\eta_i(c)$ $(i = 1, 2)$ such that*

$$\eta_1(c) \leq \frac{\varphi_1'(\xi)}{\varphi_1(\xi)}, \frac{\varphi_2'(\xi)}{\varphi_2(\xi)}, \frac{\varphi_3'(\xi)}{\varphi_3(\xi)} \leq \eta_2(c) \quad \text{for } \xi \leq 0, \tag{2.15}$$

$$\eta_1(c) \leq \frac{\varphi_1'(\xi)}{1-\varphi_1(\xi)}, \frac{\varphi_2'(\xi)}{1-\varphi_2(\xi)}, \frac{\varphi_3'(\xi)}{1-\varphi_3(\xi)} \leq \eta_2(c) \quad \text{for } \xi \geq 0. \tag{2.16}$$

Lemma 2.4. *There exist two constants $\eta_0 > 0$ and $\mu_0 > 0$ such that*

$$\varphi_2(\xi) \leq \eta_0\varphi_1(\xi) \quad \text{for } \xi \leq 0, \tag{2.17}$$

$$\varphi_3(\xi) \leq \mu_0\varphi_1(\xi) \quad \text{for } \xi \leq 0, \tag{2.18}$$

$$1-\varphi_2(\xi) \leq \eta_0(1-\varphi_1(\xi)) \quad \text{for } \xi \geq 0, \tag{2.19}$$

$$1-\varphi_3(\xi) \leq \mu_0(1-\varphi_1(\xi)) \quad \text{for } \xi \geq 0. \tag{2.20}$$

3. Existence of entire solutions

This section is devoted to the existence of the entire solutions of (1.1). As mentioned in Section 2, (1.1) is equivalent to the cooperative system (2.1). Therefore, we state the result on the system (2.1). More precisely, we have the following result.

Theorem 3.1. *Assume that* (H) *holds. Let $\Psi_i = (\varphi_{1i}, \varphi_{2i}, \varphi_{3i})$ be the solution of* (2.1) *connecting* $(0,0,0)$ *and* $(1,1,1)$ *with speed $c_i \geq c^*$, $i = 1, 2$. Then for any given constants θ_1 and θ_2,* (2.1) *has an entire solution $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ defined on $\mathbb{R}^2$ such that the following assertions are true*

(i) $(0,0,0) < u(x,t) < (1,1,1)$ *and for any $t_0 \in \mathbb{R}$,*

$$\lim_{|x|\to+\infty} \sup_{t\in(t_0,+\infty)} \|u(x,t) - 1\| = 0.$$

(ii)

$$\lim_{t\to-\infty} \left\{ \sup_{x\geq 0} \|u(x,t) - \Psi_1(x + c_1t + \theta_1)\| + \sup_{x\leq 0} \|u(x,t) - \Psi_2(-x + c_2t + \theta_2)\| \right\} = 0.$$

(iii) $\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} \|u(x,t) - 1\| = 0$, *and for any $a, b \in \mathbb{R}$ with $a < b$,*

$$\lim_{t\to-\infty} \sup_{x\in[a,b]} \|u(x,t)\| = 0.$$

The following coupled system of ordinary differential equations plays a crucial role in constructing super-solutions of (2.1) (c.f. [2, 12]):

$$\begin{aligned} &p_1'(t) = c_1 + Ne^{\alpha p_1}, \quad t < 0, \\ &p_2'(t) = c_2 + Ne^{\alpha p_1}, \quad t < 0, \\ &p_1(0) \leq 0,\ p_2(0) \leq 0, \end{aligned} \tag{3.1}$$

where c_1, c_2, N and α are positive constants, $c_2 \geq c_1 \geq c^*$ and the initial data satisfy $p_2(0) \leq p_1(0)$. A direct computation shows a solution to (3.1) as

$$p_1(t) = p_1(0) + c_1 t - \frac{1}{\alpha} \ln \left(1 + \frac{N}{c_1} e^{\alpha p_1(0)} (1 - e^{c_1 \alpha t}) \right),$$
$$p_2(t) = p_2(0) + c_2 t - \frac{1}{\alpha} \ln \left(1 + \frac{N}{c_1} e^{\alpha p_1(0)} (1 - e^{c_1 \alpha t}) \right).$$

It is easy to see that the solution $p_i(t)$ has monotone increasing property, $i = 1, 2$. Let

$$\omega_1 = p_1(0) - \frac{1}{\alpha} \ln \left(1 + \frac{N}{c_1} e^{\alpha p_1(0)} \right), \quad \omega_2 = p_2(0) - \frac{1}{\alpha} \ln \left(1 + \frac{N}{c_1} e^{\alpha p_1(0)} \right).$$

Using the fact that

$$p_i(t) - c_i t - \omega_i = -\frac{1}{\alpha} \ln \left(1 - \frac{\zeta}{1+\zeta} e^{c_1 \alpha t} \right), \quad \zeta = \frac{N}{c_1} e^{\alpha p_1(0)},$$

we can derive that there exists a constant $R_0 > 0$ such that the following relation is true

$$p_1(t) - c_1 t - \omega_1 = p_2(t) - c_2 t - \omega_2 \leq R_0 e^{c_1 \alpha t} \quad \text{for } t \leq 0.$$

Since $p_2'(t) - p_1'(t) = c_2 - c_1 \geq 0$ for all t, and $p_2(0) \leq p_1(0)$, it can be concluded that $p_2(t) \leq p_1(t)$, $t \leq 0$.

We now introduce the definition for a sub-super-solution to (2.1).

Definition 3.2. A function $\overline{u}(x,t) = (\overline{u}_1(x,t), \overline{u}_2(x,t), \overline{u}_3(x,t))$, $(x,t) \in \mathbb{R} \times (-\infty, T]$, $T \in \mathbb{R}$, is called a supper-solution of (2.1) in $(-\infty, T]$, if

$$F_i(\overline{u}(x,t)) \geq 0 \quad \text{for } (x,t) \in \mathbb{R} \times (-\infty, T], \quad i = 1, 2, 3,$$

where

$$F_1(u) = u_{1t} - d_1 u_{1xx} - r_1 u_1 (1 - a_{11} - a_{12} - u_1 + a_{11} u_2 + a_{12} u_3),$$
$$F_2(u) = u_{2t} - d_2 u_{2xx} - r_2 (1 - u_2)(a_{21} u_1 - u_2),$$
$$F_3(u) = u_{3t} - d_3 u_{3xx} - r_3 (1 - u_3)(a_{31} u_1 - u_3).$$

Similarly, the sub-solution of (2.1) is defined by reversing the above inequalities.

Let ω_1 and ω_2 be any positive constants and $\Psi_i(\xi) = (\varphi_{1i}(\xi), \varphi_{2i}(\xi), \varphi_{3i}(\xi))$ be a traveling wave solution of (2.2) connecting $(0,0,0)$ and $(1,1,1)$ with speed $c_i \geq c^*$, $i = 1, 2$. It is easy to see that the following result holds.

Lemma 3.3. *The function*

$$\underline{u}(x,t) := \max\{\Psi_1(x + c_1 t + \omega_1), \Psi_2(-x + c_2 t + \omega_2)\}$$

is a sub-solution of (2.1) *in* $(-\infty, 0]$.

Next, we construct a super-solution to (2.1).

Lemma 3.4. *Assume* (H) *holds. Take the positive constants* α *and* N *in* (3.1) *such that*

(i) *if* $c_1 = c_2 = c^*$*:* $\alpha = \lambda^* - \varepsilon$

$$N > \max_{i=1,2} K_\varepsilon \left\{ \frac{r_1 a_{11}(\eta_0+1)}{\eta_1(c^*)} + \frac{r_1 a_{12}(\mu_0+1)}{\eta_1(c^*)}, \frac{2\eta_2(c^*)}{1-\varphi_{2i}(0)} + \frac{r_2}{\eta_1(c^*)}, \frac{2\eta_2(c^*)}{1-\varphi_{3i}(0)} + \frac{r_3}{\eta_1(c^*)} \right\}$$

for some $\varepsilon \in (0, \lambda^*)$*;*

(ii) *if* $c^* = c_1 < c_2$*:* $\alpha = \lambda_1(c_2)$

$$\begin{aligned} N > \max \Bigg\{ & \frac{r_1 K_\varepsilon [a_{11}(\eta_0+1) + a_{12}(\mu_0+1)]}{\eta_1(c_2)}, \frac{r_1 M_1(c_2)[a_{11}(\eta_0+1) + a_{12}(\mu_0+1)]}{\eta_1(c^*)}, \\ & \frac{r_1 K_\varepsilon [a_{11}(\eta_0+1) + a_{12}(\mu_0+1)]}{\eta_1(c^*)}, \frac{2\eta_2(c^*)K_\varepsilon}{1-\varphi_{21}(0)} + \frac{r_2 K_\varepsilon}{\eta_1(c_2)}, \\ & \frac{2\eta_2(c_2)M_1(c_2)}{1-\varphi_{22}(0)} + \frac{r_2 M_1(c_2)}{\eta_1(c^*)}, \frac{2\eta_2(c^*)K_\varepsilon}{1-\varphi_{31}(0)} + \frac{r_3 K_\varepsilon}{\eta_1(c_2)}, \\ & \frac{2\eta_2(c_2)M_1(c_2)}{1-\varphi_{32}(0)} + \frac{r_3 M_1(c_2)}{\eta_1(c^*)} \Bigg\} \end{aligned}$$

for some $\varepsilon \in (0, \lambda^* - \lambda_1(c_2))$*, where* K_ε *was defined in Lemma* 2.2*.*

(iii) *if* $c^* < c_1 < c_2$*:* $\alpha = \lambda_1(c_2)$

$$\begin{aligned} N > \max_{i,j=1,2,i\neq j} \Bigg\{ & \frac{r_1 M_1(c_i)[a_{11}(\eta_0+1) + a_{12}(\mu_0+1)]}{\eta_1(c_j)}, \\ & \frac{2\eta_2(c_i)M_1(c_i)}{1-\varphi_{2i}(0)} + \frac{r_2 M_1(c_i)}{\eta_1(c_j)}, \frac{2\eta_2(c_i)M_1(c_i)}{1-\varphi_{3i}(0)} + \frac{r_3 M_1(c_i)}{\eta_1(c_j)} \Bigg\}. \end{aligned}$$

Then function $\overline{u}(x,t) = (\overline{u}_1(x,t), \overline{u}_2(x,t), \overline{u}_3(x,t))$ *with*

$$\begin{aligned} \overline{u}_1(x,t) &= \min\{1, \varphi_{11}(x+p_1(t)) + \varphi_{12}(-x+p_2(t))\}, \\ \overline{u}_2(x,t) &= \varphi_{21}(x+p_1(t)) + \varphi_{22}(-x+p_2(t)) - \varphi_{21}(x+p_1(t))\varphi_{22}(-x+p_2(t)), \\ \overline{u}_3(x,t) &= \varphi_{31}(x+p_1(t)) + \varphi_{32}(-x+p_2(t)) - \varphi_{31}(x+p_1(t))\varphi_{32}(-x+p_2(t)) \end{aligned}$$

is a super-solution of (2.1) *in* $(-\infty, 0]$.

Proof. The proof is divided into the following three steps.

Step 1. We prove $F_1(\overline{u}(x,t)) \geq 0$, $\forall\,(x,t) \in \mathbb{R} \times (-\infty, 0]$. Define two sets as follows:

$$\begin{aligned} S^+ &= \{(x,t) : \varphi_{11}(x+p_1(t)) + \varphi_{12}(-x+p_2(t)) \geq 1\}, \\ S^- &= \{(x,t) : \varphi_{11}(x+p_1(t)) + \varphi_{12}(-x+p_2(t)) < 1\}. \end{aligned}$$

(I) If $(x,t)\in S^+$, then $\overline{u}_1=1$ and it is obvious that

$$\begin{aligned}F_1(\overline{u})&=-r_1\overline{u}_1(1-a_{11}-a_{12}-\overline{u}_1+a_{11}\overline{u}_2+a_{12}\overline{u}_3)\\&=-r_1(-a_{11}-a_{12}+a_{11}\overline{u}_2+a_{12}\overline{u}_3)\\&=r_1(a_{11}-a_{11}\overline{u}_2+a_{12}-a_{12}\overline{u}_3)\geq 0.\end{aligned}$$

(II) If $(x,t)\in S^-$, then $\overline{u}_1=\varphi_{11}(x+p_1(t))+\varphi_{12}(-x+p_2(t))$. Consequently, by a direct computation, we can obtain

$$F_1(\overline{u})=[\varphi_{11}'(x+p_1(t))+\varphi_{12}'(-x+p_2(t))]Ne^{\alpha p_1(t)}-H_1(x,t),$$

where

$$\begin{aligned}H_1(x,t)=&\,r_1a_{11}\varphi_{11}\varphi_{22}-r_1a_{11}\varphi_{11}\varphi_{21}\varphi_{22}+r_1a_{12}\varphi_{11}\varphi_{32}\\&-r_1a_{12}\varphi_{11}\varphi_{31}\varphi_{32}+r_1a_{11}\varphi_{12}\varphi_{21}-r_1a_{11}\varphi_{12}\varphi_{21}\varphi_{22}\\&+r_1a_{12}\varphi_{12}\varphi_{31}-r_1a_{12}\varphi_{12}\varphi_{31}\varphi_{32}-2r_1\varphi_{11}\varphi_{12},\end{aligned}$$

$$\begin{aligned}&\varphi_{11}=\varphi_{11}(x+p_1(t)), &&\varphi_{12}=\varphi_{12}(-x+p_2(t)), &&\varphi_{21}=\varphi_{21}(x+p_1(t)),\\&\varphi_{22}=\varphi_{22}(-x+p_2(t)), &&\varphi_{31}=\varphi_{31}(x+p_1(t)), &&\varphi_{32}=\varphi_{32}(-x+p_2(t)).\end{aligned}$$

Let

$$U_1(x,t)=\frac{H_1(x,t)}{\varphi_{11}'(x+p_1(t))+\varphi_{12}'(-x+p_2(t))}.$$

In order to estimate the function $U_1(x,t)$, we divide $\mathbb{R}\times(-\infty,0]$ into three subsets: $A=\{p_2(t)\leq x\leq -p_1(t)\}$, $B=\{x\geq -p_1(t)\}$, $C=\{x\leq p_2(t)\}$.

Case 1. For $(x,t)\in A$, we first discuss the subcase $p_2(t)\leq x\leq 0$. If $c^*=c_1=c_2$, then it follows from (2.14), (2.15), (2.17) and (2.18) that

$$\begin{aligned}U_1(x,t)&=\frac{H_1(x,t)}{\varphi_{11}'(x+p_1(t))+\varphi_{12}'(-x+p_2(t))}\\&\leq\frac{r_1a_{11}\varphi_{11}\varphi_{22}+r_1a_{12}\varphi_{11}\varphi_{32}+r_1a_{11}\varphi_{12}\varphi_{21}+r_1a_{12}\varphi_{12}\varphi_{31}}{\varphi_{12}'(-x+p_2(t))}\\&\leq\frac{r_1a_{11}\varphi_{11}\eta_0\varphi_{12}+r_1a_{12}\varphi_{11}\mu_0\varphi_{12}+r_1a_{11}\varphi_{12}\varphi_{21}+r_1a_{12}\varphi_{12}\varphi_{31}}{\varphi_{12}'(-x+p_2(t)}\\&\leq\left(\frac{r_1a_{11}\eta_0}{\eta_1(c^*)}+\frac{r_1a_{12}\mu_0}{\eta_1(c^*)}+\frac{r_1a_{11}}{\eta_1(c^*)}+\frac{r_1a_{12}}{\eta_1(c^*)}\right)K_\varepsilon e^{(\lambda^*-\varepsilon)(x+p_1(t))}\\&\leq\left(\frac{r_1a_{11}(\eta_0+1)}{\eta_1(c^*)}+\frac{r_1a_{12}(\mu_0+1)}{\eta_1(c^*)}\right)K_\varepsilon e^{(\lambda^*-\varepsilon)p_1(t)}.\end{aligned}$$

If $c^*=c_1<c_2$, then since $\lambda^*>\lambda_1(c_2)>0$, there is a constant $\varepsilon>0$ small enough such that $\lambda^* - \varepsilon > \lambda_1(c_2)$. Accordingly, based on (2.14), (2.15), (2.17) and (2.18), we have

$$\begin{aligned} U_1(x,t) &\le \frac{r_1a_{11}\varphi_{11}\eta_0\varphi_{12} + r_1a_{12}\varphi_{11}\mu_0\varphi_{12} + r_1a_{11}\varphi_{12}\varphi_{21} + r_1a_{12}\varphi_{12}\varphi_{31}}{\varphi'_{12}(-x+p_2(t))} \\ &\le \left(\frac{r_1a_{11}\eta_0}{\eta_1(c_2)} + \frac{r_1a_{12}\mu_0}{\eta_1(c_2)} + \frac{r_1a_{11}}{\eta_1(c_2)} + \frac{r_1a_{12}}{\eta_1(c_2)}\right) K_\varepsilon e^{(\lambda^*-\varepsilon)(x+p_1(t))} \\ &\le \left(\frac{r_1a_{11}(\eta_0+1)}{\eta_1(c_2)} + \frac{r_1a_{12}(\mu_0+1)}{\eta_1(c_2)}\right) K_\varepsilon e^{\lambda_1(c_2)p_1(t)}. \end{aligned}$$

If $c^* < c_1 \le c_2$, by using (2.13), (2.15), (2.17) and (2.18), we obtain

$$\begin{aligned} U_1(x,t) &\le \frac{r_1a_{11}\varphi_{11}\eta_0\varphi_{12} + r_1a_{12}\varphi_{11}\mu_0\varphi_{12} + r_1a_{11}\varphi_{12}\varphi_{21} + r_1a_{12}\varphi_{12}\varphi_{31}}{\varphi'_{12}(-x+p_2(t))} \\ &\le \left(\frac{r_1a_{11}\eta_0}{\eta_1(c_2)} + \frac{r_1a_{12}\mu_0}{\eta_1(c_2)} + \frac{r_1a_{11}}{\eta_1(c_2)} + \frac{r_1a_{12}}{\eta_1(c_2)}\right) M_1(c_1) e^{\lambda_1(c_2)p_1(t)} \\ &\le \left(\frac{r_1a_{11}(\eta_0+1)}{\eta_1(c_2)} + \frac{r_1a_{12}(\mu_0+1)}{\eta_1(c_2)}\right) M_1(c_1) e^{\lambda_1(c_2)p_1(t)}. \end{aligned}$$

For the subcase $0 \le x \le -p_1(t)$, similar estimates can be established.

Case 2. For $(x,t) \in B$, we see that $-x+p_2(t) < 0$ and $x+p_1(t) \ge 0$. Note that

$$\begin{aligned} H_1(x,t) &\le r_1a_{11}\varphi_{11}\varphi_{22} - r_1a_{11}\varphi_{11}\varphi_{21}\varphi_{22} + r_1a_{12}\varphi_{11}\varphi_{32} - r_1a_{12}\varphi_{11}\varphi_{31}\varphi_{32} \\ &\quad + r_1a_{11}\varphi_{12}\varphi_{21} + r_1a_{12}\varphi_{12}\varphi_{31} - r_1(a_{11}+a_{12})\varphi_{11}\varphi_{12} \\ &= r_1a_{11}\varphi_{11}\varphi_{22}(1-\varphi_{21}) + r_1a_{11}\varphi_{12}(\varphi_{21}-\varphi_{11}) \\ &\quad + r_1a_{12}\varphi_{11}\varphi_{32}(1-\varphi_{31}) + r_1a_{12}\varphi_{12}(\varphi_{31}-\varphi_{11}) \\ &\le r_1a_{11}\varphi_{11}\varphi_{22}(1-\varphi_{21}) + r_1a_{11}\varphi_{12}(1-\varphi_{11}) \\ &\quad + r_1a_{12}\varphi_{11}\varphi_{32}(1-\varphi_{31}) + r_1a_{12}\varphi_{12}(1-\varphi_{11}). \end{aligned}$$

Based on (2.14), (2.16), (2.19) and (2.20), substituting $H_1(x,t)$ into $U_1(x,t)$ results in

$$\begin{aligned} &U_1(x,t) \\ &\le r_1\frac{a_{11}\varphi_{11}\varphi_{22}(1-\varphi_{21}) + a_{11}\varphi_{12}(1-\varphi_{11}) + a_{12}\varphi_{11}\varphi_{32}(1-\varphi_{31}) + a_{12}\varphi_{12}(1-\varphi_{11})}{\varphi'_{11}(x+p_1(t))} \\ &\le r_1\frac{a_{11}\varphi_{22}\eta_0(1-\varphi_{11}) + a_{11}\varphi_{12}(1-\varphi_{11}) + a_{12}\varphi_{32}\mu_0(1-\varphi_{11}) + a_{12}\varphi_{12}(1-\varphi_{11})}{\varphi'_{11}(x+p_1(t))} \\ &\le \left(\frac{r_1a_{11}\eta_0}{\eta_1(c^*)} + \frac{r_1a_{11}}{\eta_1(c^*)} + \frac{r_1a_{12}\mu_0}{\eta_1(c^*)} + \frac{r_1a_{12}}{\eta_1(c^*)}\right) K_\varepsilon e^{(\lambda^*-\varepsilon)(-x+p_2(t))} \\ &\le \left(\frac{r_1a_{11}(1+\eta_0)}{\eta_1(c^*)} + \frac{r_1a_{12}(1+\mu_0)}{\eta_1(c^*)}\right) K_\varepsilon e^{(\lambda^*-\varepsilon)p_1(t)} \end{aligned}$$

for $c^* = c_1 = c_2$. If $c^* = c_1 < c_2$, applying (2.13), (2.16), (2.19) and (2.20), one has

$$\begin{aligned} U_1(x,t) &\le r_1 \frac{a_{11}\varphi_{22}\eta_0(1-\varphi_{11}) + a_{11}\varphi_{12}(1-\varphi_{11}) + a_{12}\varphi_{32}\mu_0(1-\varphi_{11}) + a_{12}\varphi_{12}(1-\varphi_{11})}{\varphi_{11}'(x+p_1(t))} \\ &\le \left(\frac{r_1 a_{11}\eta_0}{\eta_1(c^*)} + \frac{r_1 a_{11}}{\eta_1(c^*)} + \frac{r_1 a_{12}\mu_0}{\eta_1(c^*)} + \frac{r_1 a_{12}}{\eta_1(c^*)}\right) M_1(c_2) e^{\lambda_1(c_2)(-x+p_2(t))} \\ &\le \left(\frac{r_1 a_{11}(1+\eta_0)}{\eta_1(c^*)} + \frac{r_1 a_{12}(1+\mu_0)}{\eta_1(c^*)}\right) M_1(c_2) e^{\lambda_1(c_2)p_1(t)}. \end{aligned}$$

If $c^* < c_1 \le c_2$, by (2.13), (2.16), (2.19) and (2.20), the following estimate can be obtained

$$\begin{aligned} U_1(x,t) &\le r_1 \frac{a_{11}\varphi_{22}\eta_0(1-\varphi_{11}) + a_{11}\varphi_{12}(1-\varphi_{11}) + a_{12}\varphi_{32}\mu_0(1-\varphi_{11}) + a_{12}\varphi_{12}(1-\varphi_{11})}{\varphi_{11}'(x+p_1(t))} \\ &\le \left(\frac{r_1 a_{11}\eta_0}{\eta_1(c_1)} + \frac{r_1 a_{11}}{\eta_1(c_1)} + \frac{r_1 a_{12}\mu_0}{\eta_1(c_1)} + \frac{r_1 a_{12}}{\eta_1(c_1)}\right) M_1(c_2) e^{\lambda_1(c_2)(-x+p_2(t))} \\ &\le \left(\frac{r_1 a_{11}(1+\eta_0)}{\eta_1(c_1)} + \frac{r_1 a_{12}(1+\mu_0)}{\eta_1(c_1)}\right) M_1(c_2) e^{\lambda_1(c_2)p_1(t)}. \end{aligned}$$

Case 3. $(x,t) \in C$. Note that

$$\begin{aligned} H_1(x,t) &\le r_1 a_{11}\varphi_{11}\varphi_{22} - r_1 a_{11}\varphi_{11}\varphi_{21}\varphi_{22} + r_1 a_{12}\varphi_{11}\varphi_{32} + r_1 a_{11}\varphi_{12}\varphi_{21} \\ &\quad + r_1 a_{12}\varphi_{12}\varphi_{31} - r_1 a_{12}\varphi_{12}\varphi_{31}\varphi_{32} - r_1(a_{11}+a_{12})\varphi_{11}\varphi_{12} \\ &\le r_1 a_{11}\varphi_{11}(\varphi_{22}-\varphi_{12}) + r_1 a_{11}\varphi_{21}\varphi_{12}(1-\varphi_{22}) \\ &\quad + r_1 a_{12}\varphi_{11}(\varphi_{32}-\varphi_{12}) + r_1 a_{12}\varphi_{31}\varphi_{12}(1-\varphi_{32}) \\ &\le r_1 a_{11}\varphi_{11}(1-\varphi_{12}) + r_1 a_{11}\varphi_{21}\varphi_{12}(1-\varphi_{22}) \\ &\quad + r_1 a_{12}\varphi_{11}(1-\varphi_{12}) + r_1 a_{12}\varphi_{31}\varphi_{12}(1-\varphi_{32}). \end{aligned}$$

Similar to Case 2, we can show that $U_1(x,t) \le N e^{\alpha p_1(t)}$.

From the above analysis, we conclude that

$$F_1(\overline{u}(x,t)) \ge 0, \quad \forall\, (x,t) \in \mathbb{R} \times (-\infty, 0].$$

Step 2. We now prove

$$F_2(\overline{u}(x,t)) \ge 0, \quad \forall\, (x,t) \in \mathbb{R} \times (-\infty, 0]. \tag{3.2}$$

Recall that $\overline{u}_2 = \varphi_{21} + \varphi_{22} - \varphi_{21}\varphi_{22}$, we have

$$F_2(\overline{u}(x,t)) = A_1(x,t) N e^{\alpha p_1(t)} - H(x,t),$$

where

$$\begin{aligned} A_1(x,t) &= (1-\varphi_{22})\varphi_{21}' + (1-\varphi_{21})\varphi_{22}', \\ H &= 2\varphi_{21}'\varphi_{22}' + r_2(1-\varphi_{21})(1-\varphi_{22})(a_{21}\overline{u}_1 - \varphi_{11} - \varphi_{12} + \varphi_{21}\varphi_{22}) \\ &\quad - r_2(1-\varphi_{21})(1-\varphi_{22})(2a_{21}\overline{u}_1 - \varphi_{11} - \varphi_{12}). \end{aligned}$$

We notice that the following relation is true

$$H < H_2 < H_3,$$

where

$$\begin{aligned} H_2 &= 2\varphi_{21}'\varphi_{22}' + r_2(1-\varphi_{21})(1-\varphi_{22})(\overline{u}_1 - \varphi_{11} - \varphi_{12} + \varphi_{21}\varphi_{22}) \\ &\quad - r_2(1-\varphi_{21})(1-\varphi_{22})(2\overline{u}_1 - \varphi_{11} - \varphi_{12}), \\ H_3 &= 2\varphi_{21}'\varphi_{22}' + r_2(1-\varphi_{21})(1-\varphi_{22})(\overline{u}_1 - \varphi_{11} - \varphi_{12} + \varphi_{21}\varphi_{22}). \end{aligned}$$

Hence, it suffices to show that

$$F_2(\overline{u}(x,t)) = A_1(x,t)Ne^{\alpha p_1(t)} - H_3(x,t) \geq 0.$$

Similarly to the above discussion, we divide $\mathbb{R} \times (-\infty, 0]$ into three subsets, A, B and C to estimate the function

$$U_2(x,t) := \frac{H_3(x,t)}{A_1(x,t)}.$$

(I) We first discuss the case $\overline{u}_1 = 1$, that is, $\varphi_{11}(x + p_1(t)) + \varphi_{12}(-x + p_(t)) \geq 1$. In this case, the following inequality holds

$$H_3 \leq 2\varphi_{21}'\varphi_{22}' + r_2\varphi_{21}\varphi_{22}(1-\varphi_{21})(1-\varphi_{22}).$$

Case 1. For $(x,t) \in A$, we first discuss the case $p_2(t) \leq x \leq 0$. If $c^* = c_1 = c_2$, then by using (2.14) and (2.15), we obtain

$$\begin{aligned} U_2(x,t) &\leq \frac{2\varphi_{21}'\varphi_{22}' + r_2\varphi_{21}\varphi_{22}(1-\varphi_{21})(1-\varphi_{22})}{(1-\varphi_{21})\varphi_{22}'(-x+p_2(t))} \\ &\leq \left(\frac{2\eta_2(c^*)K_\varepsilon}{1-\varphi_{21}(0)} + \frac{r_2K_\varepsilon}{\eta_1(c^*)}\right) e^{(\lambda^*-\varepsilon)p_1(t)}. \end{aligned}$$

If $c^* = c_1 < c_2$, then since $\lambda^* > \lambda_1(c_2) > 0$, it is derived that $\lambda^* - \varepsilon > \lambda_1(c_2)$ by taking $\varepsilon > 0$ sufficiently small, and hence based on (2.14) and (2.15), it follows that

$$\begin{aligned} U_2(x,t) &\leq \left(\frac{2\eta_2(c^*)K_\varepsilon}{1-\varphi_{21}(0)} + \frac{r_2K_\varepsilon}{\eta_1(c_2)}\right) e^{(\lambda^*-\varepsilon)(x+p_1(t))} \\ &\leq \left(\frac{2\eta_2(c^*)K_\varepsilon}{1-\varphi_{21}(0)} + \frac{r_2K_\varepsilon}{\eta_1(c_2)}\right) e^{\lambda_1(c_2)p_1(t)}. \end{aligned}$$

If $c^* < c_1 \leq c_2$, then applying (2.13) and (2.15), we have

$$\begin{aligned} U_2(x,t) &\leq \left(\frac{2\eta_2(c_1)M_1(c_1)}{1-\varphi_{21}(0)} + \frac{r_2M_1(c_1)}{\eta_1(c_2)}\right) e^{\lambda_1(c_1)(x+p_1(t))} \\ &\leq \left(\frac{2\eta_2(c_1)M_1(c_1)}{1-\varphi_{21}(0)} + \frac{r_2M_1(c_1)}{\eta_1(c_2)}\right) e^{\lambda_1(c_2)p_1(t)}. \end{aligned}$$

For the subcase $0 \le x \le -p_1(t)$, similar estimates can be obtained.

Case 2. For $(x,t) \in B$, in this case, $-x+p_2(t) < 0$ and $x+p_1(t) \ge 0$. If $c^* = c_1 = c_2$, thanks to (2.14), (2.15) and (2.16), then we get

$$\begin{aligned} U_2(x,t) &\le \frac{2\varphi_{21}'\varphi_{22}' + r_2\varphi_{21}\varphi_{22}(1-\varphi_{21})(1-\varphi_{22})}{(1-\varphi_{22})\varphi_{21}'(x+p_1(t))} \\ &\le \left(\frac{2\eta_2(c^*)K_\varepsilon}{1-\varphi_{22}(0)} + \frac{r_2K_\varepsilon}{\eta_1(c^*)}\right) e^{(\lambda^*-\varepsilon)(-x+p_2(t))} \\ &\le \left(\frac{2\eta_2(c^*)K_\varepsilon}{1-\varphi_{22}(0)} + \frac{r_2K_\varepsilon}{\eta_1(c^*)}\right) e^{(\lambda^*-\varepsilon)p_1(t)}. \end{aligned}$$

If $c^* = c_1 < c_2$, then by (2.13), (2.15) and (2.16), we have

$$U_2(x,t) \le \left(\frac{2\eta_2(c)M_1(c_2)}{1-\varphi_{22}(0)} + \frac{r_2M_1(c_2)}{\eta_1(c^*)}\right) e^{\lambda_1(c_2)p_1(t)}.$$

If $c^* < c_1 \le c_2$, then it follows from (2.13), (2.15) and (2.16) that

$$U_2(x,t) \le \left(\frac{2\eta_2(c_2)M_1(c_2)}{1-\varphi_{22}(0)} + \frac{r_2M_1(c_2)}{\eta_1(c_1)}\right) e^{\lambda_1(c_2)p_1(t)}.$$

Case 3. For $(x,t) \in C$, in this case, $-x+p_2(t) \ge 0$ and $x+p_1(t) < 0$. If $c^* = c_1 = c_2$, from (2.14), (2.15) and (2.16), it can be derived that

$$U_2(x,t) \le \left(\frac{2\eta_2(c^*)K_\varepsilon}{1-\varphi_{21}(0)} + \frac{r_2K_\varepsilon}{\eta_1(c^*)}\right) e^{(\lambda^*-\varepsilon)p_1(t)}.$$

If $c^* = c_1 < c_2$, then since $\lambda^* > \lambda_1(c_2) > 0$, it is concluded that $\lambda^* - \varepsilon > \lambda_1(c_2)$ by taking a positive constant ε sufficiently small, and hence we have

$$\begin{aligned} U_2(x,t) &\le \left(\frac{2\eta_2(c^*)K_\varepsilon}{1-\varphi_{21}(0)} + \frac{r_2K_\varepsilon}{\eta_1(c_2)}\right) e^{(\lambda^*-\varepsilon)p_1(t)} \\ &\le \left(\frac{2\eta_2(c^*)K_\varepsilon}{1-\varphi_{21}(0)} + \frac{r_2K_\varepsilon}{\eta_1(c_2)}\right) e^{\lambda_1(c_2)p_1(t)}. \end{aligned}$$

If $c^* < c_1 \le c_2$, by applying the decreasing of $\lambda_1(c)$ and Lemmas 2.2 and 2.3, we have

$$\begin{aligned} U_2(x,t) &\le \left(\frac{2\eta_2(c_1)M_1(c_1)}{1-\varphi_{21}(0)} + \frac{r_2M_1(c_1)}{\eta_1(c_2)}\right) e^{\lambda_1(c_1)p_1(t)} \\ &\le \left(\frac{2\eta_2(c_1)M_1(c_1)}{1-\varphi_{21}(0)} + \frac{r_2M_1(c_1)}{\eta_1(c_2)}\right) e^{\lambda_1(c_2)p_1(t)}. \end{aligned}$$

(II) Now, we study the case that $\overline{u}_1 = \varphi_{11}(x+p_1(t)) + \varphi_{12}(-x+p_2(t))$. In this case, we have

$$H_3 = 2\varphi_{21}'\varphi_{22}' + r_2\varphi_{21}\varphi_{22}(1-\varphi_{21})(1-\varphi_{22}).$$

Similar to (I), we can prove that $U_2(x,t) \le Ne^{\alpha p_1(t)}$.

Based on the above discussion, we can obtain the conclusion (3.2).

Step 3. In this step, we shall show

$$F_3(\overline{u}(x,t)) \geq 0, \quad \forall\,(x,t) \in \mathbb{R} \times (-\infty, 0]. \tag{3.3}$$

Recall that $\overline{u}_3 = \varphi_{31} + \varphi_{32} - \varphi_{31}\varphi_{32}$, we have

$$F_3(\overline{u}(x,t)) = A_*(x,t)Ne^{\alpha p_1(t)} - H_*(x,t),$$

where

$$\begin{aligned} A_*(x,t) &= (1-\varphi_{32})\varphi_{31}' + (1-\varphi_{31})\varphi_{32}', \\ H_* &= 2\varphi_{31}'\varphi_{32}' + r_3(1-\varphi_{31})(1-\varphi_{32})(a_{31}\overline{u}_1 - \varphi_{11} - \varphi_{12} + \varphi_{31}\varphi_{32}) \\ &\quad - r_3(1-\varphi_{31})(1-\varphi_{32})(2a_{31}\overline{u}_1 - \varphi_{11} - \varphi_{12}). \end{aligned}$$

We notice that the following relation

$$H_* < H_1^* < H_2^*,$$

where

$$\begin{aligned} H_1^* &= 2\varphi_{31}'\varphi_{32}' + r_2(1-\varphi_{31})(1-\varphi_{32})(\overline{u}_1 - \varphi_{11} - \varphi_{12} + \varphi_{31}\varphi_{32}) \\ &\quad - r_2(1-\varphi_{31})(1-\varphi_{32})(2\overline{u}_1 - \varphi_{11} - \varphi_{12}), \\ H_2^* &= 2\varphi_{31}'\varphi_{32}' + r_2(1-\varphi_{31})(1-\varphi_{32})(\overline{u}_1 - \varphi_{11} - \varphi_{12} + \varphi_{31}\varphi_{32}). \end{aligned}$$

Hence, it suffices to show that

$$F_3(\overline{u}(x,t)) = A_*(x,t)Ne^{\alpha p_1(t)} - H_2^*(x,t) \geq 0.$$

Let $U_3(x,t) = H_2^*(x,t)/A_*(x,t)$. Similar to the above argument, we divide $\mathbb{R} \times (-\infty, 0]$ into three subsets, A, B and C to obtain the estimate of $U_3(x,t)$.

(I) We first consider the case $\overline{u}_1 = 1$, that is, $\varphi_{11}(x+p_1(t)) + \varphi_{12}(-x+p_2(t)) \geq 1$. In this case, we have

$$H_2^* \leq 2\varphi_{31}'\varphi_{32}' + r_2\varphi_{31}\varphi_{32}(1-\varphi_{31})(1-\varphi_{32}).$$

Case 1. For $(x,t) \in A$, we first consider the subcase $p_2(t) \leq x \leq 0$. If $c^* = c_1 = c_2$, then by (2.14) and (2.15), we obtain

$$\begin{aligned} U_3(x,t) &\leq \frac{2\varphi_{31}'\varphi_{32}' + r_2\varphi_{31}\varphi_{32}(1-\varphi_{31})(1-\varphi_{32})}{(1-\varphi_{31})\varphi_{32}'(-x+p_2(t))} \\ &\leq \left(\frac{2\eta_2(c^*)K_\varepsilon}{1-\varphi_{31}(0)} + \frac{r_3K_\varepsilon}{\eta_1(c^*)}\right) e^{(\lambda^*-\varepsilon)p_1(t)}. \end{aligned}$$

If $c^* = c_1 < c_2$, then since $\lambda^* > \lambda_1(c_2) > 0$, we can obtain $\lambda^* - \varepsilon > \lambda_1(c_2)$ by choosing $\varepsilon > 0$ sufficiently small, and hence from (2.14) and (2.15), the following estimate is obtained

$$\begin{aligned} U_3(x,t) &\le \left(\frac{2\eta_2(c^*)K_\varepsilon}{1-\varphi_{31}(0)} + \frac{r_2 K_\varepsilon}{\eta_1(c_2)} \right) e^{(\lambda^*-\varepsilon)(x+p_1(t))} \\ &\le \left(\frac{2\eta_2(c^*)K_\varepsilon}{1-\varphi_{31}(0)} + \frac{r_3 K_\varepsilon}{\eta_1(c_2)} \right) e^{\lambda_1(c_2)p_1(t)}. \end{aligned}$$

If $c^* < c_1 \le c_2$, in terms of (2.13) and (2.15), we have

$$\begin{aligned} U_3(x,t) &\le \left(\frac{2\eta_2(c_1)M_1(c_1)}{1-\varphi_{31}(0)} + \frac{r_3 M_1(c_1)}{\eta_1(c_2)} \right) e^{\lambda_1(c_1)(x+p_1(t))} \\ &\le \left(\frac{2\eta_2(c_1)M_1(c_1)}{1-\varphi_{31}(0)} + \frac{r_3 M_1(c_1)}{\eta_1(c_2)} \right) e^{\lambda_1(c_2)p_1(t)}. \end{aligned}$$

For the subcase $0 \le x \le -p_1(t)$, we can obtain similar estimates.

Case 2. For $(x,t) \in B$, in this case, we can know that $-x+p_2(t) < 0$ and $x+p_1(t) \ge 0$. If $c^* = c_1 = c_2$, by (2.14), (2.15) and (2.16) one has

$$\begin{aligned} U_3(x,t) &\le \frac{2\varphi_{31}'\varphi_{32}' + r_3\varphi_{31}\varphi_{32}(1-\varphi_{31})(1-\varphi_{32})}{(1-\varphi_{32})\varphi_{31}'(x+p_1(t))} \\ &\le \left(\frac{2\eta_2(c^*)K_\varepsilon}{1-\varphi_{32}(0)} + \frac{r_3 K_\varepsilon}{\eta_1(c^*)} \right) e^{(\lambda^*-\varepsilon)(-x+p_2(t))} \\ &= \left(\frac{2\eta_2(c^*)K_\varepsilon}{1-\varphi_{32}(0)} + \frac{r_3 K_\varepsilon}{\eta_1(c^*)} \right) e^{(\lambda^*-\varepsilon)p_1(t)}. \end{aligned}$$

If $c^* = c_1 < c_2$, then

$$U_3(x,t) \le \left(\frac{2\eta_2(c_2)M_1(c_2)}{1-\varphi_{32}(0)} + \frac{r_3 M_1(c_2)}{\eta_1(c^*)} \right) e^{\lambda_1(c_2)p_1(t)}.$$

If $c^* < c_1 \le c_2$, we can derive

$$U_3(x,t) \le \left(\frac{2\eta_2(c_2)M_1(c_2)}{1-\varphi_{32}(0)} + \frac{r_2 M_1(c_2)}{\eta_1(c_1)} \right) e^{\lambda_1(c_2)p_1(t)}.$$

Case 3. For $(x,t) \in C$, in this case, $-x+p_2(t) \ge 0$ and $x+p_1(t) < 0$. If $c^* = c_1 = c_2$, from (2.14), (2.15) and (2.16), we obtain the estimate

$$U_3(x,t) \le \left(\frac{2\eta_2(c^*)K_\varepsilon}{1-\varphi_{31}(0)} + \frac{r_3 K_\varepsilon}{\eta_1(c^*)} \right) e^{(\lambda^*-\varepsilon)p_1(t)}.$$

If $c^* = c_1 < c_2$, then since $\lambda^* > \lambda_1(c_2) > 0$, it can be obtained that $\lambda^* - \varepsilon > \lambda_1(c_2)$ by taking $\varepsilon > 0$ sufficiently small, and hence we have

$$\begin{aligned} U_3(x,t) &\le \left(\frac{2\eta_2(c^*)K_\varepsilon}{1-\varphi_{31}(0)} + \frac{r_3 K_\varepsilon}{\eta_1(c_2)} \right) e^{(\lambda^*-\varepsilon)p_1(t)} \\ &\le \left(\frac{2\eta_2(c^*)K_\varepsilon}{1-\varphi_{31}(0)} + \frac{r_3 K_\varepsilon}{\eta_1(c_2)} \right) e^{\lambda_1(c_2)p_1(t)}. \end{aligned}$$

If $c^* < c_1 \le c_2$, by using the decreasing of $\lambda_1(c)$ and Lemmas 2.2 and 2.3, we have

$$\begin{aligned} U_3(x,t) &\le \left(\frac{2\eta_2(c_1)M_1(c_1)}{1-\varphi_{31}(0)} + \frac{r_3 M_1(c_1)}{\eta_1(c_2)}\right) e^{\lambda_1(c_1)p_1(t)} \\ &\le \left(\frac{2\eta_2(c_1)M_1(c_1)}{1-\varphi_{31}(0)} + \frac{r_3 M_1(c_1)}{\eta_1(c_2)}\right) e^{\lambda_1(c_2)p_1(t)}. \end{aligned}$$

(II) Now, we deal with the case that $\overline{u}_1 = \varphi_{11}(x+p_1(t)) + \varphi_{12}(-x+p_2(t))$. In this case, we have

$$H_2^* = 2\varphi_{31}'\varphi_{32}' + r_2\varphi_{31}\varphi_{32}(1-\varphi_{31})(1-\varphi_{32}).$$

Similar to Case (I), we can prove that $U_3(x,t) \le Ne^{\alpha p_1(t)}$.

From the above discussions, we see that (3.3) holds. This completes the proof. □

Based on the construction of the sub- and super-solution, we now prove Theorem 3.1.

Proof of Theorem 3.1. The proof is similar to that of [13, Theorem 1.1], see also [17, Theorem 1.1]. Here, we only sketch the outline. It is easily seen that

$$\underline{u}(x,t) \le \overline{u}(x,t), \quad \forall\,(x,t) \in \mathbb{R}\times(-\infty,0].$$

Using the method in [1, Lemma 2.1] and with the help of the comparison theorem, we can derive that there is a solution $u^* = (u_1^*, u_2^*, u_3^*)$ of (2.1) satisfying

$$\underline{u} \le u^* \le \overline{u} \quad \text{in } \mathbb{R}\times(-\infty,0].$$

Consider the the Cauchy problem of system (2.1) with the following initial data:

$$u(x,0) = u^*(x,0), \quad x \in \mathbb{R}.$$

Since $\mathbf{1} := (1,1,1)$ and $\underline{u}$ are a pair of super-solution and sub-solution of (2.1), it can be concluded that system (2.1) has a unique solution $u = (u_1, u_2, u_3)$ such that $\underline{u} \le u \le \mathbf{1}$ in $\mathbb{R}\times(-\infty,0]$. For $(x,t) \in \mathbb{R}\times(-\infty,0]$, we define $u(x,t) = u^*(x,t)$. Then $u(x,t)$ is an entire solution of system (2.1) and satisfies

$$\underline{u} \le u \le \overline{u} \quad \text{in } \mathbb{R}\times(-\infty,0] \quad \text{and} \quad \underline{u} \le u \le \mathbf{1} \quad \text{in } \mathbb{R}\times[0,\infty).$$

For any given θ_1 and θ_2, let

$$x_0 = \frac{c_2(\theta_1-\omega_1) - c_1(\theta_2-\omega_2)}{c_1+c_2}, \quad t_0 = \frac{\theta_1+\theta_2-\omega_1-\omega_2}{c_1+c_2}.$$

By a straightforward computation, we can show that

$$\begin{gathered} \varphi_{i1}(x+x_0+c_1(t+t_0)) = \varphi_{i1}(x+c_1t+\theta_1-\omega_1), \\ \varphi_{i2}(-x-x_0+c_2(t+t_0)) = \varphi_{i2}(-x+c_2t+\theta_2-\omega_2), \quad i=1,2,3. \end{gathered}$$

Set

$$\widehat{u}(x,t) = u(x + x_0, t + t_0), \quad (x,t) \in \mathbb{R}^2.$$

It is clear that $\widehat{u}(x,t)$ is an entire solution of (2.1) and satisfies the properties (i)–(iii). This completes the proof of Theorem 3.1. □

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