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Coderivatives Related to Parametric Extended Trust Region Subproblem and Their Applications

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Abstract. This paper deals with the Fréchet and Mordukhovich coderivatives of the normal cone mapping related to the parametric extended trust region subproblems (eTRS), in which the trust region intersects a ball with a single linear inequality constraint. We use the obtained results to investigate the Lipschitzian stability of parametric eTRS. We also propose a necessary condition for the local (or global) solution of the eTRS by using the coderivative tool.

1. Introduction

Quadratic program forms an important class of mathematical programming problems. Many interesting stability properties of the quadratically constrained quadratic programming (QCQP) problems were presented (see [12, 13, 18, 25]). Recently, many authors have used coderivative tools to characterize the Lipschitzian stability of linearly constrained quadratic programming (LCQP) problems and of the trust region subproblems (TRS), which are two special subclasses of the QCQP problems (see [15, 22]). We are interested in studying the stability of the parametric extend trust region subproblems (eTRS) as follows:

$$(ET(w)) \quad \min\ f(x,Q,q) := \frac{1}{2}x^TQx + q^Tx \quad \text{subject to } x \in \mathbb{R}^n : \|x\| \le r,\ a^Tx + b \le 0,$$

where the real symmetric $Q \in \mathbb{R}^{n\times n}$, vectors $a, q \in \mathbb{R}^n$ and $b, r \in \mathbb{R}$, $r > 0$, are parameters. The eTRS is a subclass of QCQP, which is a generalization of TRS and of LCQP with a linear constraint. Various aspects of eTRS were studied in the literatures (see [3, 4, 24]).

Burer and Anstreicher [3] have shown that, for the case where two parallel cuts are added to TRS, the resulting nonconvex problem has an exact representation as a semidefinite program (SDP) with additional linear and second-order-cone constraints. When the case where an additional ellipsoidal constraint is added to TRS, resulting in the two trust-region subproblem, authors have provided a new relaxation including second-order-cone

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constraints that strengthens the usual SDP relaxation. The paper conclude that if the feasible set is a ball cut by two parallel half spaces, then the problem is polynomial-time solvable. On the other hand, if the two half-spaces are not parallel and furthermore intersect within the ball, the complexity is unknown.

Bienstock and Michalka [2] have concerned the following generalized trust region subproblem

$$\min\left\{\frac{1}{2}x^TQx+q^Tx : x\in P, \|x-\mu_h\|\le r_h \text{ for } h\in S, \|x-\mu_h\|\ge r_h \text{ for } h\in K\right\},$$

with $P\subset\mathbb{R}^n$ being a polyhedral set and the $\mu_h\in\mathbb{R}^n$ and the r_h quantities being given. The authors have proved that for each fixed pair S and K, the problem can be solved in polynomial time provided that either $|K|>0$ and the number of faces of P intersecting $\bigcap_h\{x\in\mathbb{R}^n : \|x-\mu_h\|\le r_h\}$ is polynomially bounded, or $|K|=0$ and the number of inequalities defining P is bounded.

This work deals with the Fréchet and Mordukhovich coderivatives of the normal cone mapping related to the parametric eTRS. We also use the obtained results and the Mordukhovich criterion (see [16, Theorem 4.10]) for the locally Lipschitz-like property of multifunctions to investigate the Lipschitzian stability of eTRS with respect to the linear perturbations. Our results further develop some preceding works (see [15, 22]). It also agrees with the common agreement that linear and pure quadratic forms are relatively easy, but their combination is not.

Computing the Fréchet coderivative (also called the regular coderivative) and the Mordukhovich coderivative (also called the limiting coderivative or the normal coderivative) of the normal cone mapping of a system of inequalities plays an important role in sensitivity and stability analysis of parameterized optimization and equilibrium problems. This research started in the 90s with the paper [6], where the authors obtained an exact formula for the Mordukhovich normal cone in the case when the given set is a convex polyhedron, and then developed by Henrion et al. [9] and Ban et al. [1]. Recently, many authors have studied coderivatives of the normal cone mapping of polyhedral convex sets under linear and nonlinear perturbations (see, for instance, [1, 6, 9–11, 19, 20]). In [15, 22], the coderivatives of the normal cone mapping of the Euclidean ball with perturbed radius were estimated. Meanwhile, the researchers started to attack a more difficult case, when the given set is defined by many nonlinear inequalities (see [8] and references therein).

In this paper, we present the coderivatives of the normal cone mapping of the following set

$$\mathcal{F}(r,b) := \{x\in\mathbb{R}^n : \|x\|\le r, a^Tx+b\le 0\},$$

which depends on the parameter (r,b).

The organization of the paper is as follows. Section 2 recalls some concepts and facts from variational analysis. In Sections 3 and 4, the Fréchet and Mordukhovich coderivatives of normal cone mapping related to the parametric eTRS are evaluated. Section 5 estimates the Mordukhovich coderivative of the KKT point set map and characterizes the Lipschitzian stability for eTRS. Finally, Section 6 gives a necessary condition for the local (or global) solution of the extended trust region subproblem by using the coderivative tool.

2. Preliminaries

In this section, we recall the tools of variational analysis which will be used in the rest of the paper (see, [16]). The Fréchet normal cone to a set $\Omega \subset \mathbb{R}^n$ at $\overline{x} \in \Omega$ is given by

$$\widehat{N}(\overline{x};\Omega) := \left\{ x^* \in \mathbb{R}^n : \limsup_{x \xrightarrow{\Omega} \overline{x}} \frac{\langle x^*, x - \overline{x}\rangle}{\|x - \overline{x}\|} \le 0 \right\},$$

where $x \xrightarrow{\Omega} \overline{x}$ means $x \to \overline{x}$ with $x \in \Omega$. By convention, $\widehat{N}(\overline{x};\Omega) = \emptyset$ when $\overline{x} \notin \Omega$. For a multifunction $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, the *sequential Painlevé-Kuratowski upper limit* with respect to the norm topology of $\mathbb{R}^n$ is defined by

$$\operatorname*{Lim\,sup}_{x \xrightarrow{\Omega} \overline{x}} F(x) := \{x^* \in \mathbb{R}^n : \exists\, x_k \to \overline{x} \text{ and } x_k^* \to x^* \text{ with } x_k^* \in F(x_k), \text{ for } k = 1, 2, \ldots\}.$$

If Ω is locally closed around $\overline{x} \in \Omega$, the cone

$$N(\overline{x};\Omega) = \operatorname*{Lim\,sup}_{x \xrightarrow{\Omega} \overline{x}} \widehat{N}(\overline{x};\Omega)$$

is said to be the *limiting* (or *basic/Mordukhovich*) *normal cone* to Ω at $\overline{x} \in \Omega$. If $\overline{x} \notin \Omega$, $N(\overline{x};\Omega) = \emptyset$ by convention.

The graph of a multifunction $\Phi\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is defined by

$$\operatorname{gph}\Phi := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m : y \in \Phi(x)\}.$$

For every $(\overline{x},\overline{y}) \in \operatorname{gph}\Phi$, we call the multifunction $\widehat{D}^*\Phi\colon \mathbb{R}^m \rightrightarrows \mathbb{R}^n$,

$$\widehat{D}^*\Phi(\overline{x},\overline{y})(y^*) := \left\{x^* \in \mathbb{R}^n : (x^*, -y^*) \in \widehat{N}((\overline{x},\overline{y}); \operatorname{gph}\Phi)\right\}, \quad \forall\, y^* \in \mathbb{R}^m$$

the Fréchet coderivative of Φ at $(\overline{x},\overline{y})$. The multifunction $D^*\Phi(\overline{x},\overline{y})$ given by setting

$$D^*\Phi(\overline{x},\overline{y})(y^*) := \{x^* \in \mathbb{R}^n : (x^*, -y^*) \in N((\overline{x},\overline{y}); \operatorname{gph}\Phi)\}, \quad \forall\, y^* \in \mathbb{R}^m$$

is called the *Mordukhovich* (or *limiting/normal*) *coderivative* of Φ at $(\overline{x},\overline{y})$. One says that Φ is graphically regular at $(\overline{x},\overline{y}) \in \operatorname{gph}\Phi$ if

$$\widehat{D}^*\Phi(\overline{x},\overline{y})(y^*) = D^*\Phi(\overline{x},\overline{y})(y^*).$$

The last condition can be written equivalently as

$$\widehat{N}((\overline{x},\overline{y});\operatorname{gph}\Phi) = N((\overline{x},\overline{y});\operatorname{gph}\Phi).$$

The feasible region of the problem $(ET(\overline{w}))$ is rewritten as follows

$$\mathcal{F}(r,b) := \{x \in \mathbb{R}^n : \|x\| \le r, a^T x + b \le 0\},$$

which depends on the parameter (r,b).

Denote by

$$N(x;\mathcal{F}(r,b)) := \{v \in \mathbb{R}^n : \langle v, y - x\rangle \le 0, \forall\, y \in \mathcal{F}(r,b)\}$$

the normal cone to the convex set $\mathcal{F}(r,b)$ at x.

It is easy to see that

$$N(x;\mathcal{F}(r,b)) = \begin{cases} \{0\} & \text{if } \|x\| < r,\ a^T x + b < 0, \\ \{\theta x : \theta \ge 0\} & \text{if } \|x\| = r,\ a^T x + b < 0, \\ \{\gamma a : \gamma \ge 0\} & \text{if } \|x\| < r,\ a^T x + b = 0, \\ \{\theta x + \gamma a : \theta \ge 0, \gamma \ge 0\} & \text{if } \|x\| = r,\ a^T x + b = 0, \\ \emptyset & \text{if } \|x\| > r \text{ or } a^T x + b > 0. \end{cases}$$

For every $(x,r,b) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$, we put

$$\mathcal{N}(x,r,b) = N(x;\mathcal{F}(r,b)).$$

If $r \le 0$ then it is convenient to set $\mathcal{N}(x,r,b) = \emptyset$ for all $x \in \mathbb{R}^n$. Hence $\mathcal{N}\colon \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \rightrightarrows \mathbb{R}^n$ is a multifunction with closed convex values and is called *the normal cone mapping related to* $(ET(\overline{w}))$.

In the next sections, we calculate and estimate the Fréchet and Mordukhovich coderivatives of the normal cone mapping related to the parametric $(ET(\overline{w}))$.

3. Fréchet coderivative of $\mathcal{N}(\cdot)$

Fix $\overline{\omega} := (\overline{x}, \overline{r}, \overline{b}, \overline{v}) \in \operatorname{gph}\mathcal{N}$, we compute and estimate the Fréchet coderivative of the normal cone mapping. Before stating the main result, we consider the following lemmas.

Lemma 3.1. *The assertions are valid:*

(a) *If* $\|\overline{x}\| < \overline{r}$ *and* $a^T\overline{x} + \overline{b} < 0$*, then* $\overline{v} = 0$ *and*

$$\widehat{D}^*\mathcal{N}(\overline{\omega})(v^*) = \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\};$$

(b) *If* $\|\overline{x}\| = \overline{r}$, $a^T\overline{x} + \overline{b} < 0$, *and* $\overline{v} = \theta\overline{x}$ *with* $\theta > 0$ *then*

$$\widehat{D}^*\mathcal{N}(\overline{\omega})(v^*) = \begin{cases} \Omega_1(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle = 0, \\ \emptyset & \text{if } \langle v^*, \overline{x}\rangle \neq 0; \end{cases}$$

(c) *If* $\|\overline{x}\| = \overline{r}$, $a^T\overline{x} + \overline{b} < 0$, *and* $\overline{v} = 0$ *then*

$$\widehat{D}^*\mathcal{N}(\overline{\omega})(v^*) = \begin{cases} \Omega_2(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle \geq 0, \\ \emptyset & \text{if } \langle v^*, \overline{x}\rangle < 0; \end{cases}$$

(d) *If* $\|\overline{x}\| < \overline{r}$, $a^T\overline{x} + \overline{b} = 0$, *and* $\overline{v} = \gamma a$ *with* $\gamma > 0$ *then*

$$\widehat{D}^*\mathcal{N}(\overline{\omega})(v^*) = \begin{cases} \Omega_3(\overline{\omega})(v^*) & \text{if } \langle v^*, a\rangle = 0, \\ \emptyset & \text{if } \langle v^*, a\rangle \neq 0; \end{cases}$$

(e) *If* $\|\overline{x}\| < \overline{r}$, $a^T\overline{x} + \overline{b} = 0$, *and* $\overline{v} = 0$ *then*

$$\widehat{D}^*\mathcal{N}(\overline{\omega})(v^*) = \begin{cases} \Omega_4(\overline{\omega})(v^*) & \text{if } \langle v^*, a\rangle \geq 0, \\ \emptyset & \text{if } \langle v^*, a\rangle < 0; \end{cases}$$

where

$$\begin{aligned}
\Omega_1(\overline{\omega})(v^*) &:= \left\{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : b^* = 0, x^* = -\frac{r^*}{\overline{r}}\overline{x} + \theta v^*\right\}, \\
\Omega_2(\overline{\omega})(v^*) &:= \left\{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : b^* = 0, r^* \leq 0, x^* = -\frac{r^*}{\overline{r}}\overline{x}\right\}, \\
\Omega_3(\overline{\omega})(v^*) &:= \{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : r^* = 0, x^* = b^* a\}, \\
\Omega_4(\overline{\omega})(v^*) &:= \{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : r^* = 0, x^* = b^* a, b^* \geq 0\}.
\end{aligned}$$

Proof. Put

$$\mathcal{F}_1(r) := \{x \in \mathbb{R}^n : \|x\| \leq r\}, \quad \mathcal{F}_2(b) := \{x \in \mathbb{R}^n : a^T x + b \leq 0\},$$
$$\mathcal{N}_1(x, r) := N(x; \mathcal{F}_1(r)), \quad \mathcal{N}_2(x, b) := N(x; \mathcal{F}_2(b)).$$

If $a^T\overline{x} + \overline{b} < 0$, then $\mathcal{N}(\overline{\omega}) = \mathcal{N}_1(\overline{x}, \overline{r})$. Since $\mathcal{N}_1(\cdot)$ does not depend on $\overline{b}$, we have

$$\widehat{D}^*\mathcal{N}(\overline{\omega})(v^*) = \{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : b^* = 0, (x^*, r^*) \in \widehat{D}^*\mathcal{N}_1(\overline{x}, \overline{r}, \overline{v})(v^*)\}.$$

Similarly, if $\|\overline{x}\| < \overline{r}$, then $\mathcal{N}(\overline{\omega}) = \mathcal{N}_2(\overline{x}, \overline{r})$. Since $\mathcal{N}_2(\cdot)$ does not depend on r, we obtain

$$\widehat{D}^*\mathcal{N}(\overline{\omega})(v^*) = \{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : r^* = 0, (x^*, b^*) \in \widehat{D}^*\mathcal{N}_2(\overline{x}, \overline{b}, \overline{v})(v^*)\}.$$

Applying [21, Theorem 3.2] to $\mathcal{F}_1(r)$ and $\mathcal{F}_2(b)$, we deduce immediately the desired results. $\square$

Lemma 3.2. *The following assertions hold:*

(i) *If* $\|\overline{x}\| = \overline{r}$, $a^T\overline{x} + \overline{b} = 0$ *and* $\overline{v} = \theta\overline{x} + \gamma a$, $\theta > 0$, $\gamma > 0$, *then*

$$\widehat{D}^*\mathcal{N}(\overline{\omega})(v^*) \subset \begin{cases} \Omega_5(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle = 0 \text{ and } \langle v^*, a\rangle = 0, \\ \emptyset & \text{otherwise,} \end{cases}$$

where

$$\Omega_5(\overline{\omega})(v^*) := \{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : \langle x^*, \overline{x}\rangle + r^*\overline{r} + b^*\overline{b} = 0\}.$$

(ii) *If* $\|\overline{x}\| = \overline{r}$, $a^T\overline{x} + \overline{b} = 0$ *and* $\overline{v} = \theta\overline{x}$ *with* $\theta > 0$, *then*

$$\widehat{D}^*\mathcal{N}(\overline{\omega})(v^*) \subset \begin{cases} \Omega_5^1(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle = 0 \text{ and } \langle v^*, a\rangle \geq 0, \\ \emptyset & \text{otherwise,} \end{cases}$$

where

$$\Omega_5^1(\overline{\omega})(v^*) := \{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_+ : \langle x^*, \overline{x}\rangle + r^*\overline{r} + b^*\overline{b} = 0\}.$$

(iii) *If* $\|\overline{x}\| = \overline{r}$, $a^T\overline{x} + \overline{b} = 0$ *and* $\overline{v} = \gamma a$ *with* $\gamma > 0$, *then*

$$\widehat{D}^*\mathcal{N}(\overline{\omega})(v^*) \subset \begin{cases} \Omega_5^2(\overline{\omega})(v^*) & \text{if } \langle v^*, a\rangle = 0 \text{ and } \langle v^*, \overline{x}\rangle \geq 0, \\ \emptyset & \text{otherwise,} \end{cases}$$

where

$$\Omega_5^2(\overline{\omega})(v^*) := \{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R}_- \times \mathbb{R} : \langle x^*, \overline{x}\rangle + r^*\overline{r} + b^*\overline{b} = 0\}.$$

Proof. Let $(x^*, r^*, b^*) \in \widehat{D}^*\mathcal{N}(\overline{\omega})(v^*)$. This means that

$$\limsup_{(\widetilde{x},\widetilde{r},\widetilde{b},\widetilde{v})) \overset{\text{gph}\mathcal{N}}{\to} \overline{\omega}} \frac{\langle x^*, \widetilde{x} - \overline{x}\rangle + r^*(\widetilde{r} - \overline{r}) + b^*(\widetilde{b} - \overline{b}) - \langle v^*, \widetilde{v} - \overline{v}\rangle}{\|\widetilde{x} - \overline{x}\| + |\widetilde{r} - \overline{r}| + |\widetilde{b} - \overline{b}| + \|\widetilde{v} - \overline{v}\|} \leq 0. \tag{3.1}$$

Choose $\widetilde{r} \downarrow \overline{r}$, $\widetilde{x} = \frac{\widetilde{r}}{\overline{r}}\overline{x}$ and $\widetilde{b} = \frac{\widetilde{r}}{\overline{r}}\overline{b}$. Since $\|\widetilde{x}\| = \widetilde{r}$ and $a^T\widetilde{x} + \widetilde{b} = 0$, we choose $\widetilde{v} = \overline{v}$. From (3.1) it follows that

$$0 \geq \limsup_{\widetilde{r}\downarrow\overline{r}} \frac{\langle x^*, \frac{\widetilde{r}}{\overline{r}}\overline{x} - \overline{x}\rangle + r^*(\widetilde{r} - \overline{r}) + b^*\left(\frac{\widetilde{r}}{\overline{r}}\overline{b} - \overline{b}\right)}{\|\frac{\widetilde{r}}{\overline{r}}\overline{x} - \overline{x}\| + |\widetilde{r} - \overline{r}| + |\frac{\widetilde{r}}{\overline{r}}\overline{b} - \overline{b}|} = \frac{\langle x^*, \overline{x}\rangle + r^*\overline{r} + b^*\overline{b}}{\|\overline{x}\| + |\overline{r}| + |\overline{b}|},$$

which gives $\langle x^*, \overline{x}\rangle + r^*\overline{r} + b^*\overline{b} \leq 0$.

Repeating the preceding arguments for the case where $\widetilde{r} \uparrow \overline{r}$, we get

$$\langle x^*, \overline{x}\rangle + r^*\overline{r} + b^*\overline{b} \geq 0.$$

From the last two inequalities, we have

$$\langle x^*, \overline{x}\rangle + r^*\overline{r} + b^*\overline{b} = 0. \tag{3.2}$$

Choose $\widetilde{x} = \overline{x}$, $\widetilde{b} = \overline{b}$, $\widetilde{r} = \overline{r}$, and $\widetilde{v} = \overline{v} + t\overline{v}$ for $t \in \mathbb{R}$. From (3.1),

$$0 \geq \limsup_{t\uparrow 0} \frac{-\langle v^*, t\overline{v}\rangle}{\|t\overline{v}\|} = \frac{\langle v^*, \overline{v}\rangle}{\|\overline{v}\|}$$

and

$$0 \geq \limsup_{t\downarrow 0} \frac{-\langle v^*, t\overline{v}\rangle}{\|t\overline{v}\|} = -\frac{\langle v^*, \overline{v}\rangle}{\|\overline{v}\|}.$$

Hence

$$\langle v^*, \overline{v}\rangle = 0. \tag{3.3}$$

(i) Let $\widetilde{x} = \overline{x}$, $\widetilde{b} = \overline{b}$, $\widetilde{r} = \overline{r}$, and $\widetilde{v} = \overline{v} + t\overline{x}$, $t > 0$. Then, (3.1) gives

$$\langle v^*, \overline{x}\rangle \geq 0. \tag{3.4}$$

Choose $\widetilde{x} = \overline{x}$, $\widetilde{b} = \overline{b}$, $\widetilde{r} = \overline{r}$ and $\widetilde{v} = \overline{v} + ta$, $t > 0$. According to (3.1),

$$\langle v^*, a\rangle \geq 0. \tag{3.5}$$

By (3.3), (3.4) and (3.5), we obtain that $\langle v^*, \overline{x}\rangle = 0$ and $\langle v^*, a\rangle = 0$.

(ii) From (3.3) it follows $\langle v^*, \overline{x}\rangle = 0$.

Choose $\widetilde{x} = \overline{x}$, $\widetilde{b} = \overline{b}$, $\widetilde{r} = \overline{r}$, and $\widetilde{v} = \overline{v} + ta$ with $t \downarrow 0$. Then, (3.1) yields

$$0 \geq \limsup_{t\downarrow 0} \frac{-\langle v^*, ta\rangle}{\|ta\|} = -\frac{\langle v^*, a\rangle}{\|a\|}.$$

This leads to $\langle v^*, a\rangle \geq 0$.

Choose $\widetilde{x} = \overline{x}$, $\widetilde{r} = \overline{r}$, $\widetilde{b} \uparrow \overline{b}$ and $\widetilde{v} = \overline{v} = \theta\overline{x}$. From (3.1) it follows

$$0 \geq \limsup_{\widetilde{b}\uparrow\overline{b}} \frac{b^*(\widetilde{b} - \overline{b})}{|\widetilde{b} - \overline{b}|} = -b^*.$$

Hence $b^* \geq 0$.

(iii) From (3.3), we have $\langle v^*, a\rangle = 0$.

Choose $\widetilde{x} = \overline{x}$, $\widetilde{b} = \overline{b}$, $\widetilde{r} = \overline{r}$, and $\widetilde{v} = \overline{v} + t\overline{x}$ with $t \downarrow 0$. From (3.1) one has

$$0 \geq \limsup_{t\downarrow 0} \frac{-\langle v^*, t\overline{x}\rangle}{\|t\overline{x}\|} = -\frac{\langle v^*, \overline{x}\rangle}{\|\overline{x}\|},$$

which implies $\langle v^*, \overline{x}\rangle \geq 0$.

Choose $\widetilde{x} = \overline{x}$, $\widetilde{r} \downarrow \overline{r}$, $\widetilde{b} = \overline{b}$ and $\widetilde{v} = \overline{v} = \gamma a$. Then, (3.1) gives

$$0 \geq \limsup_{\widetilde{r}\downarrow\overline{r}} \frac{r^*(\widetilde{r} - \overline{r})}{|\widetilde{r} - \overline{r}|} = r^*.$$

The proof is complete. □

Denote $\mathrm{pos}\{\overline{x}, a\} := \{\theta\overline{x} + \gamma a : \theta \geq 0, \gamma \geq 0\}$.

Lemma 3.3. *If* $\|\overline{x}\| = \overline{r}$, $a^T\overline{x} + \overline{b} = 0$ *and* $\overline{v} = 0$, *then*

$$\widehat{D}^*\mathcal{N}(\overline{\omega})(v^*) \subset \begin{cases} \Omega_6(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle \geq 0 \text{ and } \langle v^*, a\rangle \geq 0, \\ \emptyset & \text{otherwise,} \end{cases}$$

where

$$\Omega_6(\overline{\omega})(v^*) := \{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : \langle x^*, \overline{x}\rangle + r^*\overline{r} + b^*\overline{b} = 0, x^* \in \mathrm{pos}\{\overline{x}, a\}, b^* \geq 0, r^* \leq 0\}.$$

Proof. Let $(x^*, r^*, b^*) \in \widehat{D}^*\mathcal{N}(\overline{\omega})(v^*)$. Then, (3.1) holds.

Choose $\widetilde{x} = \overline{x}$, $\widetilde{b} = \overline{b}$, $\widetilde{r} = \overline{r}$ and $\widetilde{v} = t\overline{x}$ with $t \downarrow 0$. According to (3.1),

$$0 \geq \limsup_{t\downarrow 0} \frac{-\langle v^*, t\overline{x}\rangle}{\|t\overline{x}\|} = -\frac{\langle v^*, \overline{x}\rangle}{\|\overline{x}\|}.$$

This implies $\langle v^*, \overline{x}\rangle \geq 0$.

We now choose $\widetilde{x} = \overline{x}$, $\widetilde{b} = \overline{b}$, $\widetilde{r} = \overline{r}$ and $\widetilde{v} = ta$ with $t \downarrow 0$. Then, (3.1) becomes

$$0 \geq \limsup_{t\downarrow 0} \frac{-\langle v^*, ta\rangle}{\|ta\|} = -\frac{\langle v^*, a\rangle}{\|a\|},$$

that is, $\langle v^*, a\rangle \geq 0$.

Choose $\widetilde{r} \downarrow \overline{r}$, $\widetilde{x} = \frac{\widetilde{r}}{\overline{r}}\overline{x}$, $\widetilde{b} = \frac{\widetilde{r}}{\overline{r}}\overline{b}$ and $\widetilde{v} = \overline{v} = 0$. Then, one has (3.2).

Next, choose $\widetilde{x} = \overline{x}$, $\widetilde{r} \downarrow \overline{r}$, $\widetilde{b} = \overline{b}$ and $\widetilde{v} = \overline{v} = 0$. From (3.1),

$$0 \geq \limsup_{\widetilde{r}\downarrow\overline{r}} \frac{r^*(\widetilde{r} - \overline{r})}{|\widetilde{r} - \overline{r}|} = r^*.$$

Choose $\widetilde{x} = \overline{x}$, $\widetilde{r} = \overline{r}$, $\widetilde{b} \uparrow \overline{b}$ and $\widetilde{v} = 0$. Then, (3.1) yields

$$0 \geq \limsup_{\widetilde{b}\uparrow\overline{b}} \frac{b^*(\widetilde{b} - \overline{b})}{|\widetilde{b} - \overline{b}|} = -b^*,$$

which means $b^* \geq 0$.

Finally, choose $\widetilde{r} = \overline{r}$, $\widetilde{b} = \overline{b}$, $\widetilde{x} \overset{\partial\mathcal{F}(\overline{r},\overline{b})}{\longrightarrow} \overline{x}$ and $\widetilde{v} = \overline{v} = 0$. By (3.1),

$$\limsup_{\widetilde{x}\overset{\partial\mathcal{F}(\overline{r},\overline{b})}{\longrightarrow}\overline{x}} \frac{\langle x^*, \widetilde{x} - \overline{x}\rangle}{\|\widetilde{x} - \overline{x}\|} \leq 0. \tag{3.6}$$

Let any $\widetilde{x}_k \overset{\partial\mathcal{F}(\overline{r},\overline{b})}{\longrightarrow} \overline{x}$ such that

$$\lim_{k\to\infty} \frac{\widetilde{x}_k - \overline{x}}{\|\widetilde{x}_k - \overline{x}\|} = u.$$

Then, $u \in T(\overline{x}, \partial\mathcal{F}(\overline{r}, \overline{b}))$. By (3.6), we have $\langle x^*, u\rangle \leq 0$ for every $u \in T(\overline{x}, \partial\mathcal{F}(\overline{r}, \overline{b}))$. This gives $x^* \in \mathrm{pos}\{\overline{x}, a\}$. The proof is complete. $\square$

By Lemmas 3.1–3.3, we get the following main theorem of this section:

Theorem 3.4. *For every $\overline{\omega} = (\overline{x}, \overline{r}, \overline{b}, \overline{v}) \in \operatorname{gph} \mathcal{N}$, the assertions are valid:*

(a) *If $\|\overline{x}\| < \overline{r}$ and $a^T\overline{x} + \overline{b} < 0$, then $\overline{v} = 0$ and*

$$\widehat{D}^*\mathcal{N}(\overline{\omega})(v^*) = \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\}.$$

(b) *If $\|\overline{x}\| = \overline{r}$, $a^T\overline{x} + \overline{b} < 0$, and $\overline{v} = \theta\overline{x}$ with $\theta > 0$ then*

$$\widehat{D}^*\mathcal{N}(\overline{\omega})(v^*) = \begin{cases} \Omega_1(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle = 0, \\ \emptyset & \text{if } \langle v^*, \overline{x}\rangle \neq 0. \end{cases}$$

(c) *If $\|\overline{x}\| = \overline{r}$, $a^T\overline{x} + \overline{b} < 0$, and $\overline{v} = 0$ then*

$$\widehat{D}^*\mathcal{N}(\overline{\omega})(v^*) = \begin{cases} \Omega_2(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle \geq 0, \\ \emptyset & \text{if } \langle v^*, \overline{x}\rangle < 0. \end{cases}$$

(d) *If $\|\overline{x}\| < \overline{r}$, $a^T\overline{x} + \overline{b} = 0$, and $\overline{v} = \gamma a$ with $\gamma > 0$ then*

$$\widehat{D}^*\mathcal{N}(\overline{\omega})(v^*) = \begin{cases} \Omega_3(\overline{\omega})(v^*) & \text{if } \langle v^*, a\rangle = 0, \\ \emptyset & \text{if } \langle v^*, a\rangle \neq 0. \end{cases}$$

(e) *If $\|\overline{x}\| < \overline{r}$, $a^T\overline{x} + \overline{b} = 0$, and $\overline{v} = 0$ then*

$$\widehat{D}^*\mathcal{N}(\overline{\omega})(v^*) = \begin{cases} \Omega_4(\overline{\omega})(v^*) & \text{if } \langle v^*, a\rangle \geq 0, \\ \emptyset & \text{if } \langle v^*, a\rangle < 0. \end{cases}$$

(f) *If $\|\overline{x}\| = \overline{r}$, $a^T\overline{x} + \overline{b} = 0$, and $\overline{v} = \theta\overline{x} + \gamma a$ with $\theta > 0$, $\gamma > 0$ then*

$$\widehat{D}^*\mathcal{N}(\overline{\omega})(v^*) \subset \begin{cases} \Omega_5(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle = 0 \text{ and } \langle v^*, a\rangle = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

(g) *If $\|\overline{x}\| = \overline{r}$, $a^T\overline{x} + \overline{b} = 0$, and $\overline{v} = \theta\overline{x}$ with $\theta > 0$, then*

$$\widehat{D}^*\mathcal{N}(\overline{\omega})(v^*) \subset \begin{cases} \Omega_5^1(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle = 0 \text{ and } \langle v^*, a\rangle \geq 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

(h) *If $\|\overline{x}\| = \overline{r}$, $a^T\overline{x} + \overline{b} = 0$, and $\overline{v} = \gamma a$ with $\gamma > 0$, then*

$$\widehat{D}^*\mathcal{N}(\overline{\omega})(v^*) \subset \begin{cases} \Omega_5^2(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle = 0 \text{ and } \langle v^*, \overline{x}\rangle \geq 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

(i) *If $\|\overline{x}\| = \overline{r}$, $a^T\overline{x} + \overline{b} = 0$, and $\overline{v} = 0$ then*

$$\widehat{D}^*\mathcal{N}(\overline{\omega})(v^*) \subset \begin{cases} \Omega_6(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle \geq 0 \text{ and } \langle v^*, a\rangle \geq 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

4. Mordukhovich coderivative of $\mathcal{N}(\cdot)$

To estimate the Mordukhovich coderivative of $\mathcal{N}(\cdot)$, we consider some lemmas.

Lemma 4.1. *For every $\overline{\omega} = (\overline{x}, \overline{r}, \overline{b}, \overline{v}) \in \operatorname{gph}\mathcal{N}$, the following assertions are valid:*

(a) *If $\|\overline{x}\| < \overline{r}$ and $a^T\overline{x} + \overline{b} < 0$, then $\overline{v} = 0$ and*

$$D^*\mathcal{N}(\overline{\omega})(v^*) = \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\}.$$

(b) *If $\|\overline{x}\| = \overline{r}$, $a^T\overline{x} + \overline{b} < 0$, and $\overline{v} = \theta\overline{x}$, with $\theta > 0$ then*

$$D^*\mathcal{N}(\overline{\omega})(v^*) = \begin{cases} \Omega_1(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle = 0, \\ \emptyset & \text{if } \langle v^*, \overline{x}\rangle \neq 0. \end{cases}$$

(c) *If $\|\overline{x}\| = \overline{r}$, $a^T\overline{x} + \overline{b} < 0$, and $\overline{v} = 0$ then*

$$D^*\mathcal{N}(\overline{\omega})(v^*) = \begin{cases} \{0_{\mathbb{R}^{n+2}}\} & \text{if } \langle v^*, \overline{x}\rangle < 0, \\ \Omega_2(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle > 0, \\ \Omega_2'(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle = 0, \end{cases}$$

where

$$\Omega_2'(\overline{\omega})(v^*) := \left\{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : b^* = 0, x^* = -\frac{r^*}{\overline{r}}\overline{x}\right\}.$$

(d) *If $\|\overline{x}\| < \overline{r}$, $a^T\overline{x} + \overline{b} = 0$, and $\overline{v} = \gamma a$ with $\gamma > 0$ then*

$$D^*\mathcal{N}(\overline{\omega})(v^*) = \begin{cases} \Omega_3(\overline{\omega})(v^*) & \text{if } \langle v^*, a\rangle = 0, \\ \emptyset & \text{if } \langle v^*, a\rangle \neq 0. \end{cases}$$

(e) *If $\|\overline{x}\| < \overline{r}$, $a^T\overline{x} + \overline{b} = 0$, and $\overline{v} = 0$ then*

$$D^*\mathcal{N}(\overline{\omega})(v^*) = \begin{cases} \{(0_{\mathbb{R}^{n+2}})\} & \text{if } \langle v^*, a\rangle < 0, \\ \Omega_4(\overline{\omega})(v^*) & \text{if } \langle v^*, a\rangle > 0, \\ \Omega_3(\overline{\omega})(v^*) & \text{if } \langle v^*, a\rangle = 0. \end{cases}$$

Proof. Repeating the arguments in the proof of Lemma 3.1 and using [21, Theorem 3.3], we get the required conclusions. □

Lemma 4.2. *Assume that $\|\overline{x}\| = \overline{r}$, $a^T\overline{x} + \overline{b} = 0$ and $\overline{v} \neq 0$. The following assertions hold:*

(i) *If* $\overline{v} = \theta \overline{x}$ *with* $\theta > 0$, *then*

$$D^*\mathcal{N}(\overline{\omega})(v^*) \subset \begin{cases} \Omega_5(\overline{\omega})(v^*) \cup \Omega_1(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x} \rangle = 0, \\ \emptyset & \text{if } \langle v^*, \overline{x} \rangle \neq 0. \end{cases}$$

(ii) *If* $\overline{v} = \gamma a$ *with* $\gamma > 0$, *then*

$$D^*\mathcal{N}(\overline{\omega})(v^*) \subset \begin{cases} \Omega_5(\overline{\omega})(v^*) \cup \Omega_3(\overline{\omega})(v^*) & \text{if } \langle v^*, a \rangle = 0, \\ \emptyset & \text{if } \langle v^*, a \rangle \neq 0. \end{cases}$$

(iii) *If* $\overline{v} = \theta \overline{x} + \gamma a$ *with* $\theta > 0$ *and* $\gamma > 0$, *then*

$$D^*\mathcal{N}(\overline{\omega})(v^*) \subset \begin{cases} \Omega_5(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{v} \rangle = 0, \\ \emptyset & \text{if } \langle v^*, \overline{v} \rangle \neq 0. \end{cases}$$

Proof. For any $(x^*, r^*, b^*) \in D^*\mathcal{N}(\overline{\omega})(v^*)$, there exist $\omega_k = (x_k, r_k, b_k, v_k)$ satisfying $\omega_k \xrightarrow{\text{gph}\,\mathcal{N}} \overline{\omega} = (\overline{x}, \overline{r}, \overline{b}, \overline{v})$ and $(x_k^*, r_k^*, b_k^*, v_k^*) \to (x^*, r^*, b^*, v^*)$ such that

$$(x_k^*, r_k^*, b_k^*, v_k^*) \in \widehat{D}^*\mathcal{N}(\omega_k)(v_k^*). \tag{4.1}$$

From $\overline{v} \neq 0$ it follows $v_k \neq 0$ for every k. We distinguish the following two cases:

Case 1: $\|x_k\| = r_k$ *for every* k *large enough.* Then, we may assume that $\|x_k\| = r_k$ for every k. We next consider the following two subcases:

Subcase 1.1: $a^T x_k + b_k = 0$ *for every* k *large enough.* Then, we can assume that $a^T x_k + b_k = 0$ for all k. By Lemma 3.2, we have $(x_k^*, r_k^*, b_k^*) \in \Omega_5(\omega_k)(v_k^*)$, that is,

$$\langle x_k^*, x_k \rangle + r_k^* r_k + b_k^* b_k = 0, \quad \langle v_k^*, x_k \rangle = 0 \quad \text{and} \quad \langle v_k^*, a \rangle = 0.$$

Letting $k \to \infty$, one has

$$\langle x^*, \overline{x} \rangle + r^* \overline{r} + b^* \overline{b} = 0, \quad \langle v^*, \overline{x} \rangle = 0 \quad \text{and} \quad \langle v^*, a \rangle = 0.$$

This leads to $(x^*, r^*, b^*) \in \Omega_5(\overline{\omega})(v^*)$. Hence

$$D^*\mathcal{N}(\overline{\omega})(v^*) \subset \Omega_5(\overline{\omega})(v^*).$$

Subcase 1.2: *there exists* $\{k_l\} \subset \{k\}$ *such that* $a^T x_{k_l} + b_{k_l} < 0$ *for all* l. Then, $v_{k_l} = \theta_{k_l} x_{k_l}$ with $0 < \theta_{k_l} \to \theta$. By Lemma 3.1, we have $(x_{k_l}^*, r_{k_l}^*, b_{k_l}^*) \in \Omega_1(p_{k_l})(v_{k_l}^*)$, that is,

$$b_{k_l}^* = 0, \quad x_{k_l}^* = -\frac{r_{k_l}^*}{r_{k_l}} x_{k_l} + \theta_{k_l} v_{k_l}^* \quad \text{and} \quad \langle v_{k_l}^*, x_{k_l} \rangle = 0.$$

Letting $l \to \infty$, one gets

$$b^* = 0, \quad x^* = -\frac{r^*}{\overline{r}}\overline{x} + \theta v^* \quad \text{and} \quad \langle v^*, \overline{x} \rangle = 0.$$

This means $(x^*, r^*, b^*) \in \Omega_1(\overline{\omega})(v^*)$. Thus

$$D^*\mathcal{N}(\overline{\omega})(v^*) \subset \Omega_1(\overline{\omega})(v^*).$$

Case 2: there exists $\{k_s\} \subset \{k\}$ *such that* $\|x_{k_s}\| < r_{k_s}$. Since $v_{k_s} \neq 0$, $a^T x_{k_s} + b_{k_s} = 0$ for every s. Then, $v_{k_s} = \gamma_{k_s} a$ with $0 < \gamma_{k_s} \to \gamma$. By Lemma 3.1, we have $(x^*_{k_s}, r^*_{k_s}, b^*_{k_s}) \in \Omega_3(p_{k_s})(v^*_{k_s})$, that is,

$$r^*_{k_s} = 0, \quad x^*_{k_s} = b^*_{k_s} a \quad \text{and} \quad \langle v^*_{k_s}, x_{k_s} \rangle = 0.$$

Passing the latter to limits as $s \to \infty$, we have

$$r^* = 0, \quad x^* = b^* a \quad \text{and} \quad \langle v^*, a \rangle = 0.$$

Hence $(x^*, r^*, b^*) \in \Omega_3(\overline{\omega})(v^*)$, and

$$D^*\mathcal{N}(\overline{\omega})(v^*) \subset \Omega_3(\overline{\omega})(v^*).$$

By the above arguments, we now prove (i), (ii) and (iii).

(i) If $\overline{v} = \theta \overline{x}$ with $\theta > 0$, then Case 2 does not occur. Hence

$$D^*\mathcal{N}(\overline{\omega})(v^*) \subset \begin{cases} \Omega_5(\overline{\omega})(v^*) \cup \Omega_1(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x} \rangle = 0, \\ \emptyset & \text{if } \langle v^*, \overline{x} \rangle \neq 0. \end{cases}$$

(ii) If $\overline{v} = \gamma a$ with $\gamma > 0$, then Subcase 1.2 does not occur. Thus

$$D^*\mathcal{N}(\overline{\omega})(v^*) \subset \begin{cases} \Omega_5(\overline{\omega})(v^*) \cup \Omega_3(\overline{\omega})(v^*) & \text{if } \langle v^*, a \rangle = 0, \\ \emptyset & \text{if } \langle v^*, a \rangle \neq 0. \end{cases}$$

(iii) If $\overline{v} = \theta \overline{x} + \gamma a$ with $\theta > 0$ and $\gamma > 0$, then both Cases 1.2 and 2 do not occur. Therefore

$$D^*\mathcal{N}(\overline{\omega})(v^*) \subset \begin{cases} \Omega_5(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{v} \rangle = 0, \\ \emptyset & \text{if } \langle v^*, \overline{v} \rangle \neq 0. \end{cases}$$

The proof is complete. □

Lemma 4.3. *If* $\|\overline{x}\| = \overline{r}$, $a^T\overline{x} + \overline{b} = 0$ *and* $\overline{v} = 0$, *then*

$$D^*\mathcal{N}(\overline{\omega})(v^*) = \begin{cases} \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\} & \text{if } \langle v^*, \overline{x}\rangle < 0 \text{ and } \langle v^*, a\rangle < 0, \\ \Omega_4(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle < 0 \text{ and } \langle v^*, a\rangle > 0, \\ \Omega_2(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle > 0 \text{ and } \langle v^*, a\rangle < 0, \\ \Omega_2'(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle = 0 \text{ and } \langle v^*, a\rangle < 0, \\ \Omega_3(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle < 0 \text{ and } \langle v^*, a\rangle = 0 \end{cases}$$

and

$$D^*\mathcal{N}(\overline{\omega})(v^*) \subset \begin{cases} \Omega_7(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle > 0 \text{ and } \langle v^*, a\rangle > 0, \\ \Omega_8(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle = 0 \text{ and } \langle v^*, a\rangle > 0, \\ \Omega_9(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle > 0 \text{ and } \langle v^*, a\rangle = 0, \\ \Omega_{10}(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle = 0 \text{ and } \langle v^*, a\rangle = 0, \end{cases}$$

where

$$\begin{aligned} \Omega_7(\overline{\omega}) &= \Omega_2(\overline{\omega})(v^*) \cup \Omega_4(\overline{\omega})(v^*) \cup \Omega_6(\overline{\omega})(v^*), \\ \Omega_8(\overline{\omega}) &= \Omega_2'(\overline{\omega})(v^*) \cup \Omega_4(\overline{\omega})(v^*) \cup \Omega_5^1(\overline{\omega})(v^*) \cup \Omega_6(\overline{\omega})(v^*), \\ \Omega_9(\overline{\omega}) &= \Omega_2(\overline{\omega})(v^*) \cup \Omega_3(\overline{\omega})(v^*) \cup \Omega_5^2(\overline{\omega})(v^*) \cup \Omega_6(\overline{\omega})(v^*), \\ \Omega_{10}(\overline{\omega}) &= \Omega_2'(\overline{\omega})(v^*) \cup \Omega_3(\overline{\omega})(v^*) \cup \Omega_5(\overline{\omega})(v^*) \cup \Omega_6(\overline{\omega})(v^*). \end{aligned}$$

Proof. We consider the following nine cases:

Case 1: $\langle v^*, \overline{x}\rangle < 0$ *and* $\langle v^*, a\rangle < 0$. Let any $(x^*, r^*, b^*) \in D^*\mathcal{N}(\overline{\omega})(v^*)$. Then, (4.1) holds. Since $\langle v^*, \overline{x}\rangle < 0$ and $\langle v^*, a\rangle < 0$, we may assume that $\langle v_k^*, x_k\rangle < 0$ and $\langle v_k^*, a\rangle < 0$ for every k. Fix any k.

If $\|x_k\| = r_k$ and $v_k \neq 0$, then $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \emptyset$, by Lemmas 3.1 and 3.2. If $\|x_k\| = r_k$ and $v_k = 0$ then, by Lemmas 3.1 and 3.3, $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \emptyset$. From Lemmas 3.1 and 3.2, $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \emptyset$ if $a^Tx_k + b_k = 0$ and $v_k \neq 0$. If $a^Tx_k + b_k = 0$ and $v_k = 0$ then, by Lemmas 3.1 and 3.3, $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \emptyset$. Hence $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) \neq \emptyset$ if $\|x_k\| < r_k$ and $a^Tx_k + b_k < 0$. From Lemma 3.1 it follows $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\}$. This gives $x_k^* = 0$, $r_k^* = 0$ and $b_k^* = 0$. Letting $k \to \infty$, one has $x^* = 0$, $r^* = 0$ and $b^* = 0$. Hence $D^*\mathcal{N}(\overline{\omega})(v^*) \subset \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\}$.

Conversely, let $r_k = \overline{r}$, $b_k = (1 - k^{-1})\overline{b} - (k^2)^{-1}$, $x_k = (1 - k^{-1})\overline{x}$ and $v_k = 0$. Then, $\|x_k\| < r_k$, $a^Tx_k + b_k = -(k^2)^{-1} < 0$ and $v_k = 0$. Let $x_k^* = 0$, $r_k^* = 0$, $b_k^* = 0$. Then, we have (4.1) by Lemma 3.1. Hence $\{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\} \subset D^*\mathcal{N}(\overline{\omega})(v^*)$. The first conclusion is proved.

Case 2: $\langle v^*, \overline{x}\rangle < 0$ *and* $\langle v^*, a\rangle > 0$. For any $(x^*, r^*, b^*) \in D^*\mathcal{N}(\overline{\omega})(v^*)$, we have (4.1). Since $\langle v^*, \overline{x}\rangle < 0$ and $\langle v^*, a\rangle > 0$, we may assume that $\langle v^*, x_k\rangle < 0$ and $\langle v_k^*, a\rangle > 0$ for every

k. Fix any k. From Lemmas 3.1 and 3.2, if $\|x_k\| = r_k$ and $v_k \neq 0$ then $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \emptyset$. If $\|x_k\| = r_k$ and $v_k = 0$ then, by Lemmas 3.1 and 3.3, $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \emptyset$.

Therefore, in order to get that $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) \neq \emptyset$, we must have $\|x_k\| < r_k$. Consider the following two subcases:

Subcase 2.1*:* $a^T x_k + b_k = 0$. By Lemma 3.1, if $v_k \neq 0$ then $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \emptyset$. If $v_k = 0$ then we have $(x_k^*, r_k^*, b_k^*) \in \Omega_4(\omega_k)(v_k^*)$, that is,

$$r_k^* = 0, \quad x_k^* = b_k^* a, \quad b_k^* \geq 0 \quad \text{and} \quad \langle v_k^*, a\rangle \geq 0.$$

Passing to the limits as $k \to \infty$, we have

$$r^* = 0, \quad x^* = b^* a, \quad b^* \geq 0 \quad \text{and} \quad \langle v^*, a\rangle \geq 0,$$

which mean $(x^*, r^*, b^*) \in \Omega_4(\overline{\omega})(v^*)$.

Subcase 2.2*:* $a^T x_k + b_k < 0$. Then $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \{0_{\mathbb{R}^{n+2}}\}$ from Lemma 3.1. This implies $x_k^* = 0$, $r_k^* = 0$ and $b_k^* = 0$. Letting $k \to \infty$, one has $x^* = 0$, $r^* = 0$ and $b^* = 0$. Hence $D^*\mathcal{N}(\overline{\omega})(v^*) \subset \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\}$.

By Subcases 2.1 and 2.2, we have $D^*\mathcal{N}(\overline{\omega})(v^*) \subset \Omega_4(\overline{\omega})(v^*)$.

Conversely, for any $(x^*, r^*, b^*) \in \Omega_4(\overline{\omega})(v^*)$, we obtain that $r^* = 0$, $b^* \geq 0$ and $\langle v^*, a\rangle \geq 0$. Choose $r_k = \overline{r}$, $b_k = (1 - k^{-1})\overline{b}$, $x_k = (1 - k^{-1})\overline{x}$. Then, $\|x_k\| < r_k$, $a^T x_k + b_k = 0$ and $v_k = 0$. We choose $r_k^* = 0$, $b_k^* = b^*$, $x_k^* = b_k^* a$ and $v_k^* = v^*$. By Lemma 3.1, we have (4.1). Hence $(x^*, r^*, b^*) \in D^*\mathcal{N}(\overline{\omega})(v^*)$. Then, we get the assertion (ii).

Case 3*:* $\langle v^*, \overline{x}\rangle > 0$ *and* $\langle v^*, a\rangle < 0$. Let $(x^*, r^*, b^*) \in D^*\mathcal{N}(\overline{\omega})(v^*)$. Then, (4.1) holds. Since $\langle v^*, \overline{x}\rangle > 0$ and $\langle v^*, a\rangle < 0$, we may assume that $\langle v^*, x_k\rangle > 0$ and $\langle v_k^*, a\rangle < 0$ for every k. Fix any k. If $a^T x_k + b_k = 0$ and $v_k \neq 0$ then $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \emptyset$ by Lemmas 3.1 and 3.2. If $a^T x_k + b_k = 0$ and $v_k = 0$ then, by Lemmas 3.1 and 3.3, $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \emptyset$. Consequently, to get $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) \neq \emptyset$, we must have $a^T x_k + b_k < 0$. We now consider the following two subcases:

Subcase 3.1*:* $\|x_k\| = r_k$. From Lemma 3.1, $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \emptyset$ if $v_k \neq 0$. If $v_k = 0$ then, by Lemma 3.1, we have $(x_k^*, r_k^*, b_k^*) \in \Omega_2(\omega_k)(v_k^*)$, that is,

$$b_k^* = 0, \quad x_k^* = -\frac{r_k^*}{r_k} x_k, \quad r_k^* \leq 0 \quad \text{and} \quad \langle v_k^*, x_k\rangle \geq 0.$$

Letting $k \to \infty$, we obtain $b^* = 0$, $x^* = -\frac{r^*}{\overline{r}}\overline{x}$, $r^* \leq 0$ and $\langle v^*, \overline{x}\rangle = 0$, which imply $(x^*, r^*, b^*) \in \Omega_2(\overline{\omega})(v^*)$.

Subcase 3.2*:* $a^T x_k + b_k < 0$. We have $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \{0_{\mathbb{R}^{n+2}}\}$ by Lemma 3.1, i.e., $x_k^* = 0$, $r_k^* = 0$ and $b_k^* = 0$. Passing the latter to limits as $k \to \infty$, one has $x^* = 0$, $r^* = 0$ and $b^* = 0$. By Subcases 3.1 and 3.2, we have $D^*\mathcal{N}(\overline{\omega})(v^*) \subset \Omega_2(\overline{\omega})(v^*)$.

Conversely, let any $(x^*, r^*, b^*) \in \Omega_2(\overline{\omega})(v^*)$, i.e., $b^* = 0$, $x^* = -\frac{r^*}{\overline{r}}\overline{x}$, $r^* \leq 0$ and $\langle v^*, \overline{x}\rangle = 0$. Choose $r_k = (1 - k^{-1})\overline{r}$, $b_k = (1 - k^{-1})\overline{b} - (k^2)^{-1}$, $x_k = (1 - k^{-1})\overline{x}$. Then,

$\|x_k\| = r_k$, $a^T x_k + b_k < 0$ and $v_k = 0$. Let $r_k^* = r^*$, $b_k^* = b^*$, $x_k^* = -\frac{r_k^*}{r_k} x_k$ and $v_k^* = v^*$. From Lemma 3.1, we get (4.1). This gives $(x^*, r^*, b^*) \in D^*\mathcal{N}(\overline{\omega})(v^*)$. The assertion (iii) is shown.

Case 4*: $\langle v^*, \overline{x}\rangle = 0$ and $\langle v^*, a\rangle < 0$.* For any $(x^*, r^*, b^*) \in D^*\mathcal{N}(\overline{\omega})(v^*)$, we get (4.1). Since $\langle v^*, a\rangle < 0$, we can assume that $\langle v_k^*, a\rangle < 0$ for every k. Fix any k. By Lemmas 3.1 and 3.2, if $a^T x_k + b_k = 0$ and $v_k \neq 0$ then $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \emptyset$. If $a^T x_k + b_k = 0$ and $v_k = 0$ then, by Lemmas 3.1 and 3.3, $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \emptyset$. Consequently, to get $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) \neq \emptyset$, we must have $a^T x_k + b_k < 0$. Consider the following three subcases:

Subcase 4.1*: $\|x_k\| = r_k$ and $v_k = 0$.* To obtain $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) \neq \emptyset$, by Lemma 3.1, we must have $\langle v_k^*, x_k\rangle \geq 0$. Then,

$$b_k^* = 0, \quad x_k^* = -\frac{r_k^*}{r_k} x_k, \quad r_k^* \leq 0 \quad \text{and} \quad \langle v_k^*, x_k\rangle \geq 0.$$

Passing the latter to limits as $k \to \infty$, we obtain

$$b^* = 0, \quad x^* = -\frac{r^*}{\overline{r}} \overline{x}, \quad r^* \leq 0 \quad \text{and} \quad \langle v^*, \overline{x}\rangle \geq 0.$$

Hence $(x^*, r^*, b^*) \in \Omega_2(\overline{\omega})(v^*) \subset \Omega_2'(\overline{\omega})(v^*)$.

Subcase 4.2*: $\|x_k\| = r_k$ and $v_k \neq 0$.* This implies $v_k = \theta_k x_k$ with $\theta_k = (\|x_k\|^{-1}\|v_k\|) \downarrow 0$. To obtain that $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) \neq \emptyset$, by Lemma 3.1, we must have $\langle v_k^*, x_k\rangle = 0$. Then,

$$b_k^* = 0, \quad x_k^* = -\frac{r_k^*}{r_k} x_k + \theta_k v_k^* \quad \text{and} \quad \langle v_k^*, x_k\rangle = 0.$$

Letting $k \to \infty$, we get $b^* = 0$, $x^* = -\frac{r^*}{\overline{r}} \overline{x}$ and $\langle v^*, \overline{x}\rangle = 0$. This leads to $(x^*, r^*, b^*) \in \Omega_2'(\overline{\omega})(v^*)$.

Subcase 4.3*: $\|x_k\| < r_k$.* By Lemma 3.1, $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \{(0_{\mathbb{R}^{n+2}})\}$, i.e., $x_k^* = 0$, $r_k^* = 0$ and $b_k^* = 0$. Letting $k \to \infty$ yields $x^* = 0$, $r^* = 0$ and $b^* = 0$. Thus $(x^*, r^*, b^*) \in \Omega_2'(\overline{\omega})(v^*)$.

Conversely, we let any $(x^*, r^*, b^*) \in \Omega_2'(\overline{\omega})(v^*)$, that is, $b^* = 0$, $x^* = -\frac{r^*}{\overline{r}} \overline{x}$ and $\langle v^*, \overline{x}\rangle = 0$. Choose $r_k = \overline{r}$, $x_k = \overline{x}$, $b_k = \overline{b} - k^{-1}$. Then, $\|x_k\| = r_k$, $a^T x_k + b_k = -k^{-1} < 0$ and $v_k = \theta_k x_k$ with $\theta_k \downarrow 0$. Let $r_k^* = r^*$, $b_k^* = b^*$, $x_k^* = -\frac{r_k^*}{r_k} x_k + \theta_k v_k^*$ and $v_k^* = v^*$. Then, we obtain (4.1) by Lemma 3.1, which follows $(x^*, r^*, b^*) \in D^*\mathcal{N}(\overline{\omega})(v^*)$. This gives the assertion (iv).

Case 5*: $\langle v^*, \overline{x}\rangle < 0$* and $\langle v^*, a\rangle = 0$. Let $(x^*, r^*, b^*) \in D^*\mathcal{N}(\overline{\omega})(v^*)$. Then, (4.1) holds. Since $\langle v^*, \overline{x}\rangle < 0$, we may assume that $\langle v_k^*, x_k\rangle < 0$ for every k. Fix any k. If $\|x_k\| = r_k$ and $v_k \neq 0$ then, by Lemmas 3.1 and 3.2, $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \emptyset$. If $\|x_k\| = r_k$ and $v_k = 0$ then, by Lemmas 3.1 and 3.3, $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \emptyset$. To get $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) \neq \emptyset$, we must have $\|x_k\| < r_k$. Consider the following three subcases:

Subcase 5.1*: $a^T x_k + b_k = 0$ and $v_k = 0$.* To get $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) \neq \emptyset$, by Lemma 3.1, we must have $\langle v_k^*, a\rangle \geq 0$. This gives

$$r_k^* = 0, \quad x_k^* = b_k^* a, \quad b_k^* \geq 0 \quad \text{and} \quad \langle v_k^*, a\rangle \geq 0.$$

Passing to limits as $k \to \infty$ yields

$$r^* = 0, \quad x^* = b^* a, \quad b^* \geq 0 \quad \text{and} \quad \langle v^*, a\rangle \geq 0,$$

which means $(x^*, r^*, b^*) \in \Omega_4(\overline{\omega})(v^*) \subset \Omega_3(\overline{\omega})(v^*)$.

Subcase 5.2*: $a^T x_k + b_k = 0$ and $v_k \neq 0$.* This implies $v_k = \theta_k a$ with $\theta_k = (\|a\|^{-1}\|v_k\|) \downarrow 0$. To get that $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) \neq \emptyset$, by Lemma 3.1, we must have $\langle v_k^*, a\rangle = 0$. Then,

$$r_k^* = 0, \quad x_k^* = b_k^* a \quad \text{and} \quad \langle v_k^*, a\rangle = 0.$$

Letting $k \to \infty$,

$$r^* = 0, \quad x^* = b^* a \quad \text{and} \quad \langle v^*, a\rangle \geq 0.$$

Hence $(x^*, r^*, b^*) \in \Omega_3(\overline{\omega})(v^*)$.

Subcase 5.3*: $a^T x_k + b_k < 0$.* By Lemma 3.1, it follows that $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\}$, that is, $x_k^* = 0$, $r_k^* = 0$ and $b_k^* = 0$. Letting $k \to \infty$, one has $x^* = 0$, $r^* = 0$ and $b^* = 0$, which gives $(x^*, r^*, b^*) \in \Omega_3(\overline{\omega})(v^*)$.

Conversely, for any $(x^*, r^*, b^*) \in \Omega_3(\overline{\omega})(v^*)$, we obtain that $r^* = 0$, $x^* = b^* a$ and $\langle v^*, a\rangle \geq 0$. Choose $r_k = \overline{r}$, $x_k = (1 - k^{-1})\overline{x}$, $b_k = (1 - k^{-1})\overline{b}$. Then, $\|x_k\| < r_k$, $a^T x_k + b_k = 0$ and $v_k = \gamma_k a$ with $\gamma_k \downarrow 0$. Let $r_k^* = r^*$, $b_k^* = b^*$, $x_k^* = b_k^* a$ and $v_k^* = v^*$. Then, we have (4.1) by Lemma 3.1. Hence $(x^*, r^*, b^*) \in D^*\mathcal{N}(\overline{\omega})(v^*)$. The assertion (v) follows.

Case 6*: $\langle v^*, \overline{x}\rangle > 0$ and $\langle v^*, a\rangle > 0$.* For $(x^*, r^*, b^*) \in D^*\mathcal{N}(\overline{\omega})(v^*)$, (4.1) follows. Since $\langle v^*, \overline{x}\rangle > 0$ and $\langle v^*, a\rangle > 0$, we may assume that $\langle v_k^*, x_k\rangle > 0$ and $\langle v_k^*, a\rangle > 0$ for every k. Fix any k. If $v_k \neq 0$ then, by Lemmas 3.1 and 3.2, $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \emptyset$. To get $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) \neq \emptyset$, we must have $v_k = 0$. Hence $(x_k^*, r_k^*, b_k^*) \in \Omega_2(\omega_k)(v_k^*) \cup \Omega_4(\omega_k)(v_k^*) \cup \Omega_6(\omega_k)(v_k^*)$ by Lemmas 3.1 and 3.3. This follows

$$(x^*, r^*, b^*) \in \Omega_2(\overline{\omega})(v^*) \cup \Omega_4(\overline{\omega})(v^*) \cup \Omega_6(\overline{\omega})(v^*),$$

which leads to the assertion (vi).

Case 7*: $\langle v^*, \overline{x}\rangle = 0$ and $\langle v^*, a\rangle > 0$.* For $(x^*, r^*, b^*) \in D^*\mathcal{N}(\overline{\omega})(v^*)$, (4.1) holds. Since $\langle v^*, a\rangle > 0$, we may assume that $\langle v_k^*, a\rangle > 0$ for every k. Fix any k. By Lemmas 3.1–3.3, we have $(x_k^*, r_k^*, b_k^*) \in \Omega_1(\omega_k)(v_k^*) \cup \Omega_2(\omega_k)(v_k^*) \cup \Omega_4(\omega_k)(v_k^*) \cup \Omega_5^1(\omega_k)(v_k^*) \cup \Omega_6(\omega_k)(v_k^*)$. This gives

$$(x^*, r^*, b^*) \in \Omega_2'(\overline{\omega})(v^*) \cup \Omega_4(\overline{\omega})(v^*) \cup \Omega_5^1(\overline{\omega})(v^*) \cup \Omega_6(\overline{\omega})(v^*).$$

The assertion (vii) is proved.

Case 8*: $\langle v^*, \overline{x}\rangle > 0$ and $\langle v^*, a\rangle = 0$.* For $(x^*, r^*, b^*) \in D^*\mathcal{N}(\overline{\omega})(v^*)$, one gets (4.1). Since $\langle v^*, \overline{x}\rangle > 0$, we may assume that $\langle v_k^*, x_k\rangle > 0$ for every $k \geq 1$. Fix any k. From Lemmas 3.1–3.3 it follows that $(x_k^*, r_k^*, b_k^*) \in \Omega_2(\omega_k)(v_k^*) \cup \Omega_3(\omega_k)(v_k^*) \cup \Omega_5^2(\omega_k)(v_k^*) \cup \Omega_6(\omega_k)(v_k^*)$.

Letting $k \to \infty$, one has

$$(x^*, r^*, b^*) \in \Omega_2(\overline{\omega})(v^*) \cup \Omega_3(\overline{\omega})(v^*) \cup \Omega_5^2(\overline{\omega})(v^*) \cup \Omega_6(\overline{\omega})(v^*).$$

The assertion (viii) is proved.

Case 9: $\langle v^*, \overline{x}\rangle = 0$ and $\langle v^*, a\rangle = 0$. For $(x^*, r^*, b^*) \in D^*\mathcal{N}(\overline{\omega})(v^*)$, (4.1) holds. Fix any k. By Lemmas 3.1–3.3, we have

$$(x_k^*, r_k^*, b_k^*) \in \big(\Omega_1(\omega_k) \cup \Omega_2(\omega_k) \cup \Omega_3(\omega_k) \cup \Omega_4(\omega_k) \cup \Omega_5(\omega_k) \cup \Omega_6(\omega_k)\big)(v_k^*).$$

Passing the latter to limits as $k \to \infty$ yields

$$(x^*, r^*, b^*) \in \Omega_2'(\overline{\omega})(v^*) \cup \Omega_3(\overline{\omega})(v^*) \cup \Omega_5(\overline{\omega})(v^*) \cup \Omega_6(\overline{\omega})(v^*).$$

The conclusion of the assertion (ix) is shown. The proof is then complete. □

By the above arguments, we get the main result in this section as follows.

Theorem 4.4. *For every $\overline{\omega} = (\overline{x}, \overline{r}, \overline{b}, \overline{v}) \in \operatorname{gph} \mathcal{N}$, the assertions are valid:*

(a) *If $\|\overline{x}\| < \overline{r}$ and $a^T\overline{x} + \overline{b} < 0$, then $\overline{v} = 0$ and*

$$D^*\mathcal{N}(\overline{\omega})(v^*) = \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\}.$$

(b) *If $\|\overline{x}\| = \overline{r}$, $a^T\overline{x} + \overline{b} < 0$, and $\overline{v} = \theta\overline{x}$ with $\theta > 0$ then*

$$D^*\mathcal{N}(\overline{\omega})(v^*) = \begin{cases} \Omega_1(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle = 0, \\ \emptyset & \text{if } \langle v^*, \overline{x}\rangle \neq 0. \end{cases}$$

(c) *If $\|\overline{x}\| = \overline{r}$, $a^T\overline{x} + \overline{b} < 0$, and $\overline{v} = 0$ then*

$$D^*\mathcal{N}(\overline{\omega})(v^*) = \begin{cases} \{0_{\mathbb{R}^{n+2}}\} & \text{if } \langle v^*, \overline{x}\rangle < 0, \\ \Omega_2(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle > 0, \\ \Omega_2'(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle = 0. \end{cases}$$

(d) *If $\|\overline{x}\| < \overline{r}$, $a^T\overline{x} + \overline{b} = 0$, and $\overline{v} = \gamma a$ with $\gamma > 0$ then*

$$D^*\mathcal{N}(\overline{\omega})(v^*) = \begin{cases} \Omega_3(\overline{\omega})(v^*) & \text{if } \langle v^*, a\rangle = 0, \\ \emptyset & \text{if } \langle v^*, a\rangle \neq 0. \end{cases}$$

(e) *If $\|\overline{x}\| < \overline{r}$, $a^T\overline{x} + \overline{b} = 0$, and $\overline{v} = 0$ then*

$$D^*\mathcal{N}(\overline{\omega})(v^*) = \begin{cases} \{0_{\mathbb{R}^{n+2}}\} & \text{if } \langle v^*, a\rangle < 0, \\ \Omega_4(\overline{\omega})(v^*) & \text{if } \langle v^*, a\rangle > 0, \\ \Omega_3(\overline{\omega})(v^*) & \text{if } \langle v^*, a\rangle = 0. \end{cases}$$

(f) *If* $\|\overline{x}\| = \overline{r}$, $a^T\overline{x} + \overline{b} = 0$, *and overlinev* $= \theta\overline{x} + \gamma a$ *with* $\theta > 0$, $\gamma > 0$ *then*

$$D^*\mathcal{N}(\overline{\omega})(v^*) \subset \begin{cases} \Omega_5(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{v}\rangle = 0, \\ \emptyset & \text{if } \langle v^*, \overline{v}\rangle \neq 0. \end{cases}$$

(g) *If* $\|\overline{x}\| = \overline{r}$, $a^T\overline{x} + \overline{b} = 0$, *and* $\overline{v} = \theta\overline{x}$ *with* $\theta > 0$ *then*

$$D^*\mathcal{N}(\overline{\omega})(v^*) \subset \begin{cases} \Omega_5(\overline{\omega})(v^*) \cup \Omega_1(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle = 0, \\ \emptyset & \text{if } \langle v^*, \overline{x}\rangle \neq 0. \end{cases}$$

(h) *If* $\|\overline{x}\| = \overline{r}$, $a^T\overline{x} + \overline{b} = 0$, *and* $\overline{v} = \gamma a$ *with* $\gamma > 0$ *then*

$$D^*\mathcal{N}(\overline{\omega})(v^*) \subset \begin{cases} \Omega_5(\overline{\omega})(v^*) \cup \Omega_3(\overline{\omega})(v^*) & \text{if } \langle v^*, a\rangle = 0, \\ \emptyset & \text{if } \langle v^*, a\rangle \neq 0. \end{cases}$$

(i) *If* $\|\overline{x}\| = \overline{r}$, $a^T\overline{x} + \overline{b} = 0$ *and* $\overline{v} = 0$ *then*

$$D^*\mathcal{N}(\overline{\omega})(v^*) = \begin{cases} \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\} & \text{if } \langle v^*, \overline{x}\rangle < 0 \text{ and } \langle v^*, a\rangle < 0, \\ \Omega_4(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle < 0 \text{ and } \langle v^*, a\rangle > 0, \\ \Omega_2(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle > 0 \text{ and } \langle v^*, a\rangle < 0, \\ \Omega_2'(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle = 0 \text{ and } \langle v^*, a\rangle < 0, \\ \Omega_3(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle < 0 \text{ and } \langle v^*, a\rangle = 0 \end{cases}$$

and

$$D^*\mathcal{N}(\overline{\omega})(v^*) \subset \begin{cases} \Omega_7(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle > 0 \text{ and } \langle v^*, a\rangle > 0, \\ \Omega_8(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle = 0 \text{ and } \langle v^*, a\rangle > 0, \\ \Omega_9(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle > 0 \text{ and } \langle v^*, a\rangle = 0, \\ \Omega_{10}(\overline{\omega})(v^*) & \text{if } \langle v^*, \overline{x}\rangle = 0 \text{ and } \langle v^*, a\rangle = 0. \end{cases}$$

5. Lipschitzian stability

In this section, we use obtained results and the Mordukhovich criterion (see [16, Theorem 4.10]) for the local Lipschitz-like property of multifunctions to investigate Lipschitzian stability of $(ET(\overline{w}))$ with respect to the linear perturbations. We always assume that $(ET(\overline{w}))$ satisfies LICQ.

The stationary solution set of $(ET(w))$ is rewritten by $S(Q, q, r, b)$. Recall that (see, for instance, [7, Proposition 1.3.4]), under LICQ, x is a stationary solution of $(ET(\overline{w}))$ if and only if

$$\langle Qx + q, y - x\rangle \geq 0, \quad \forall\, y \in \mathcal{F}(r, b),$$

i.e., x is a global optimal solution of the generalized equation

$$0 \in Qx + q + N(x; \mathcal{F}(r,b)). \tag{5.1}$$

We can rewrite (5.1) as follows

$$y \in H(x,z) + M(x,z),$$

where $y := -q$, $z := (Q,r,b)$, $H(x,z) := Qx$ and $M(x,z) := \mathcal{N}(x,r,b)$.

Denote by $\mathbb{R}^{n\times n}_s$ the linear subspace of symmetric $n \times n$ matrices in $\mathbb{R}^{n\times n}$ and put $Z := \mathbb{R}^{n\times n}_s \times \mathbb{R} \times \mathbb{R}$. Then, $S(\cdot)$ can be interpreted as the multifunction $\widetilde{S}\colon Z \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by

$$\widetilde{S}(z,y) = \{x \in \mathbb{R}^n : y \in H(x,z) + M(x,z)\}.$$

Then, we have

$$\widetilde{S}(z,y) = S(Q,q,r,b).$$

The following lemma is used to prove the main theorem.

Lemma 5.1. *The set* $\operatorname{gph}\mathcal{N}$ *is closed in* $\mathbb{P} := \mathbb{R}^n \times (0,+\infty) \times \mathbb{R} \times \mathbb{R}^n$.

Proof. Suppose that $\omega_k = (x_k, r_k, b_k, v_k) \xrightarrow{\operatorname{gph}\mathcal{N}} \overline{\omega} = (\overline{x}, \overline{r}, \overline{b}, \overline{v}) \in \mathbb{P}$. We now prove $\overline{\omega} \in \operatorname{gph}\mathcal{N}$, that is, $\overline{v} \in N(\overline{x}; \mathcal{F}(\overline{r}, \overline{b}))$. Indeed, we consider the following four cases:

Case 1*:* $\|\overline{x}\| < \overline{r}$ *and* $a^T\overline{x} + \overline{b} < 0$. For every k large enough, $\|x_k\| < r_k$, $a^T x_k + b_k < 0$ and $v_k = 0$. It follows $\overline{v} = 0$. Therefore $\overline{v} \in N(\overline{x}; \mathcal{F}(\overline{r}, \overline{b})) = \{0\}$.

Case 2*:* $\|\overline{x}\| = \overline{r}$ *and* $a^T\overline{x} + \overline{b} < 0$. Then, $N(\overline{x}; \mathcal{F}(\overline{r}, \overline{b})) = \{\theta\overline{x}, \theta \geq 0\}$. For every k large enough, we have $a^T x_k + b_k < 0$. Fix such a index k. Consider the following subcases:

Subcase 2.1*:* $\|x_k\| < r_k$. Then, $v_k = 0$, and $\overline{v} = 0 \in N(\overline{x}; \mathcal{F}(\overline{r}, \overline{b}))$.

Subcase 2.2*:* $\|x_k\| = r_k$. Then, $v_k = \theta_k x_k$ with

$$0 < \theta_k = \|x_k\|^{-1} \cdot \|v_k\| \to \overline{\theta} := \|\overline{x}\|^{-1} \cdot \|\overline{v}\|.$$

This yields $\overline{v} = \overline{\theta}\overline{x} \in N(\overline{x}; \mathcal{F}(\overline{r}, \overline{b}))$.

Case 3*:* $\|\overline{x}\| < \overline{r}$ *and* $a^T\overline{x} + \overline{b} = 0$. Then, $N(\overline{x}; \mathcal{F}(\overline{r}, \overline{b})) = \{\gamma a, \gamma \geq 0\}$. For every k large enough, we have $\|x_k\| < r_k$. Fix such a index k. Consider the following subcases:

Subcase 3.1*:* $a^T x_k + b_k < 0$. Then $v_k = 0$ and $\overline{v} = 0 \in N(\overline{x}; \mathcal{F}(\overline{r}, \overline{b}))$.

Subcase 3.2*:* $a^T x_k + b_k = 0$. In this case, we obtain that $v_k = \gamma_k a$, where $0 < \gamma_k = \|a\|^{-1} \cdot \|v_k\| \to \overline{\gamma}$ with $\overline{\gamma} := \|a\|^{-1} \cdot \|\overline{v}\|$. It follows that $\overline{v} = \overline{\gamma} a \in N(\overline{x}; \mathcal{F}(\overline{r}, \overline{b}))$.

Case 4*:* $\|\overline{x}\| = \overline{r}$ *and* $a^T\overline{x} + \overline{b} = 0$. Then, $N(\overline{x}; \mathcal{F}(\overline{r}, \overline{b})) = \operatorname{pos}\{\overline{x}, a\}$. Fix any k. Consider the following four subcases:

Subcase 4.1*:* $\|x_k\| < r_k$, $a^T x_k + b_k < 0$. Then, $v_k = 0$. This gives that $\overline{v} = 0 \in N(\overline{x}; \mathcal{F}(\overline{r}, \overline{b}))$.

Subcase 4.2*:* $\|x_k\| = r_k$*,* $a^T x_k + b_k < 0$. Then, $v_k = \theta_k x_k$ with $0 < \theta_k = \|x_k\|^{-1} \cdot \|v_k\| \to \overline{\theta} := \|\overline{x}\|^{-1} \cdot \|\overline{v}\|$ and $\overline{v} = \overline{\theta}\overline{x} \in N(\overline{x}; \mathcal{F}(\overline{r}, \overline{b}))$.

Subcase 4.3*:* $\|x_k\| < r_k$*,* $a^T x_k + b_k = 0$. Then, $v_k = \gamma_k a$ with $0 < \gamma_k = \|a\|^{-1} \cdot \|v_k\| \to \overline{\gamma} := \|a\|^{-1} \cdot \|\overline{v}\|$. Thus $\overline{v} = \overline{\gamma} a \in N(\overline{x}; \mathcal{F}(\overline{r}, \overline{b}))$.

Subcase 4.4*:* $\|x_k\| = r_k$*,* $a^T x_k + b_k = 0$. Then, $v_k = \theta_k x_k + \gamma_k a$ with $\theta_k \geq 0$ and $\gamma_k \geq 0$.

If $\|\gamma_k\| < +\infty$ then we can assume that $\gamma_k \to \overline{\gamma} \geq 0$. One has

$$\theta_k = \frac{\|v_k - \gamma_k a\|}{\|x_k\|} \to \overline{\theta} := \frac{\|\overline{v} - \overline{\gamma} a\|}{\|\overline{x}\|} \geq 0.$$

Thus $\overline{v} = \overline{\theta}\overline{x} + \overline{\gamma} a \in N(\overline{x}; \mathcal{F}(\overline{r}, \overline{b}))$.

If $\|\gamma_k\| \to +\infty$ then

$$\frac{v_k}{\gamma_k} = \frac{\theta_k}{\gamma_k} x_k + a. \tag{5.2}$$

If $\{\theta_k/\gamma_k\}$ is bounded then we can assume that $\theta_k/\gamma_k \to \mu$. From (5.2) it follows $0 = \mu\overline{x} + a$, contrary to the fact that $(ET(\overline{w}))$ satisfies (LICQ). Otherwise, if $\|\theta_k/\gamma_k\| \to +\infty$ then (5.2) gives

$$\frac{v_k}{\gamma_k} : \frac{\theta_k}{\gamma_k} = x_k + \left(\frac{\theta_k}{\gamma_k}\right)^{-1} a.$$

Letting $k \to \infty$ yields $0 = \overline{x}$. This contradicts the fact that $\|\overline{x}\| = \overline{r} > 0$. The lemma is proved. $\square$

The following theorem estimates the Mordukhovich coderivative of $\widetilde{S}(\cdot)$.

Theorem 5.2. *Consider the problem* $(ET(\overline{w}))$ *and* $(\overline{z}, \overline{y}, \overline{x}) \in \operatorname{gph} \widetilde{S}$*. For each* $x^* \in \mathbb{R}^n$*, if* $(y^*, z^*) \in D^*\widetilde{S}(\overline{z}, \overline{y}, \overline{x})(x^*)$ *then*

$$\overline{Q} y^* = 2x^*, \quad Q^*_{ij} = -y^*_i \overline{x}_j, \quad (x^*, r^*, b^*) \in D^*\mathcal{N}(\overline{x}, \overline{r}, \overline{b}, \overline{v})(-y^*),$$

where $\overline{z} = (\overline{Q}, \overline{r}, \overline{b})$*,* $\overline{v} = \overline{y} - H(\overline{x}, \overline{z}) = -\overline{q} - \overline{Q}\overline{x}$*,* $z^* = (Q^*, r^*, b^*)$ *and* Q^*_{ij} *is the* (i, j)*th element of* Q^**.*

Proof. By Lemma 5.1, we have $\mathcal{N}$ is locally closed around $(\overline{x}, \overline{r}, \overline{b}) \in \operatorname{gph} \mathcal{N}$; hence M is locally closed around $(\overline{x}, \overline{z}) \in \operatorname{gph} M$.

With similar analysis the proof of [15, Lemmas 4.1–4.3], we obtain that

$$D^*M(\overline{x}, \overline{z}, \overline{v})(v^*) = \{(x^*, 0_{\mathbb{R}^{n\times n}_s}, r^*, b^*) : (x^*, r^*, b^*) \in D^*\mathcal{N}(\overline{\omega})(v^*)\}$$

and

$$\nabla H(\overline{x}, \overline{z})^*(v^*) = \{\overline{Q} v^*\} \times (v^*_i \overline{x}_j) \times \{0_{\mathbb{R}}\},$$

where $(v^*_i \overline{x}_j)$ is the $n \times n$ matrix whose (i, j)th element is $v^*_i \overline{x}_j$.

From [14, Theorem 4.3] it follows that

$$D^*\widetilde{S}(\overline{z},\overline{y},\overline{x})(x^*) \subset \Omega_{H,\overline{y}}(x^*),$$

where

$$\Omega_{H,\overline{y}}(x^*) = \bigcup_{v^*\in\mathbb{R}^n} \big\{(z^*,y^*) \in Z^* \times \mathbb{R}^n : \\ (-x^*,z^*,y^*) \in \nabla H(\overline{x},\overline{z})^*(v^*) \times \{-v^*\} + D^*M(\overline{x},\overline{z},\overline{v})(v^*) \times \{0_{\mathbb{R}^n}\}\big\}.$$

For each $x^* \in \mathbb{R}^n$, if $(y^*,z^*) \in D^*\widetilde{S}(\overline{z},\overline{y},\overline{x})(x^*)$ then $(y^*,z^*) \in \Omega_{H,\overline{y}}(x^*)$, that is,

$$-y^* = v^*, \quad -x^* = \overline{Q}v^* + x^*, \quad Q^*_{ij} = v^*_i\overline{x}_j, \quad (x^*,r^*,b^*) \in D^*\mathcal{N}(\overline{x},\overline{r},\overline{b},\overline{v})(v^*).$$

The latter system is equivalent to

$$\overline{Q}y^* = 2x^*, \quad Q^*_{ij} = -y^*_i\overline{x}_j, \quad (x^*,r^*,b^*) \in D^*\mathcal{N}(\overline{x},\overline{r},\overline{b},\overline{v})(-y^*).$$

This establishes the desired formula. □

The Mordukhovich criterion (see [16, Theorem 4.10]) for the local Lipschitz-like property of multifunctions shows that $\widetilde{S}(\cdot)$ is locally Lipschitz-like around $(\overline{z},\overline{y},\overline{x}) \in \operatorname{gph}\widetilde{S}$ if and only if

$$D^*\widetilde{S}(\overline{z},\overline{y},\overline{x})(0) = \{0\}. \tag{5.3}$$

Since $D^*\widetilde{S}(\overline{z},\overline{y},\overline{x})(0) = \{0\}$ is equivalent to $D^*S(\overline{z},\overline{y},\overline{x})(0) = \{0\}$, we conclude that $\widetilde{S}(\cdot)$ is locally Lipschitz-like around $(\overline{z},\overline{y},\overline{x}) \in \operatorname{gph}\widetilde{S}$ if and only if S is locally Lipschitz-like around $(\overline{Q},\overline{q},\overline{r},\overline{b},\overline{x}) \in \operatorname{gph} S$.

From Theorem 5.2 it follows that (5.3) holds if the following system

$$\overline{Q}y^* = 0, \quad Q^*_{ij} = -y^*_i\overline{x}_j, \quad (0,r^*,b^*) \in D^*\mathcal{N}(\overline{\omega})(-y^*),$$

has a unique solution $(Q^*,r^*,b^*,y^*) = 0$, which is equivalent to that

$$\overline{Q}y^* = 0, \quad (0,r^*,b^*) \in D^*\mathcal{N}(\overline{\omega})(-y^*), \tag{5.4}$$

has a unique solution $(r^*,b^*,y^*) = 0$. If $\det\overline{Q} \neq 0$ then (5.4) reduces to that

$$(0,r^*,b^*) \in D^*\mathcal{N}(\overline{\omega})(0), \tag{5.5}$$

has a unique solution $(r^*,b^*) = 0$.

The following theorem shows some sufficient conditions for the local Lipschitz-like property of $S(\cdot)$.

Theorem 5.3. *The multifunction* $(\widetilde{Q},\widetilde{q},\widetilde{r},\widetilde{b}) \mapsto S(\widetilde{Q},\widetilde{q},\widetilde{r},\widetilde{b})$ *is locally Lipschitz-like around* $(\overline{Q},\overline{q},\overline{r},\overline{b},\overline{x}) \in \operatorname{gph} S$ *if at least one of the following conditions is satisfied:*

(i) $\|\overline{x}\| < \overline{r}$, $a^T\overline{x}+\overline{b} < 0$ *and* $\det \overline{Q} \neq 0$;

(ii) $\|\overline{x}\| = \overline{r}$, $a^T\overline{x}+\overline{b} < 0$ *and* $\overline{Q}\overline{x}+\overline{q} = \theta\overline{x}$, $\theta > 0$;

(iii) $\|\overline{x}\| = \overline{r}$, $a^T\overline{x}+\overline{b} < 0$, $\overline{Q}\overline{x}+\overline{q} = 0$, $\operatorname{rank}(\overline{Q};\overline{x}) = n$ *and* $\langle \overline{x}, u\rangle = 0$ *for every* $u \in \operatorname{Null}(\overline{Q})$, *where* $\operatorname{Null}(\overline{Q}) := \{x \in \mathbb{R}^n : \overline{Q}x = 0\}$;

(iv) $\|\overline{x}\| < \overline{r}$, $a^T\overline{x}+\overline{b} = 0$, $\overline{Q}\overline{x}+\overline{q} = \gamma a$, $\gamma > 0$, *and* $\operatorname{rank}(\overline{Q};a) = n$;

(v) $\|\overline{x}\| < \overline{r}$, $a^T\overline{x}+\overline{b} = 0$, $\overline{Q}\overline{x}+\overline{q} = 0$, $\operatorname{rank}(\overline{Q};a) = n$ *and* $\langle a, u\rangle = 0$ *for every* $u \in \operatorname{Null}(\overline{Q})$;

(vi) $\|\overline{x}\| = \overline{r}$, $a^T\overline{x}+b = 0$, b *is unperturbed and* $\det \overline{Q} \neq 0$.

Proof. (i) Since $\|\overline{x}\| < \overline{r}$ and $a^T\overline{x}+\overline{b} < 0$, we have $D^*\mathcal{N}(\overline{\omega})(-y^*) = \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\}$. Hence (5.4) has a unique solution $(r^*, b^*, y^*) = 0$ and $S(\cdot)$ is locally Lipschitz-like around $(\overline{Q},\overline{q},\overline{r},\overline{b},\overline{x})$.

(ii) By the assumption that $\|\overline{x}\| = \overline{r}$, $a^T\overline{x}+\overline{b} < 0$ and $\overline{Q}\overline{x}+\overline{q} = \theta\overline{x}$, $\theta > 0$, one gets $D^*\mathcal{N}(\overline{p})(v^*) = \Omega_1(\overline{\omega})(-y^*)$ if $\langle -y^*, \overline{x}\rangle = 0$. Then, (5.4) yields

$$\overline{Q}y^* = 0, \quad 0 = -\frac{r^*}{\overline{r}}\overline{x} - \theta y^*, \quad b^* = 0, \quad \langle y^*, \overline{x}\rangle = 0.$$

Combining $-\frac{r^*}{\overline{r}}\overline{x} - \theta y^* = 0$ with $\langle y^*, \overline{x}\rangle = 0$ we have $(y^*, r^*) = 0$. Hence (5.4) has only one solution $(r^*, b^*, y^*) = 0$. This leads to the desired conclusion.

(iii) By the assumption that $\|\overline{x}\| = \overline{r}$, $a^T\overline{x}+\overline{b} < 0$, and $\overline{Q}\overline{x}+\overline{q} = 0$, we obtain

$$D^*\mathcal{N}(\overline{\omega})(-y^*) = \begin{cases} \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\} & \text{if } \langle -y^*, \overline{x}\rangle < 0, \\ \Omega_2(\overline{\omega})(-y^*) & \text{if } \langle -y^*, \overline{x}\rangle > 0, \\ \Omega_2'(\overline{\omega})(-y^*) & \text{if } \langle -y^*, \overline{x}\rangle = 0. \end{cases}$$

Then, (5.4) follows that

$$\overline{Q}y^* = 0, \quad 0 = -\frac{r^*}{\overline{r}}\overline{x}, \quad b^* = 0, \quad r^* \leq 0, \quad \langle y^*, \overline{x}\rangle > 0 \tag{5.6}$$

and

$$\overline{Q}y^* = 0, \quad 0 = -\frac{r^*}{\overline{r}}\overline{x}, \quad b^* = 0, \quad \langle y^*, \overline{x}\rangle = 0. \tag{5.7}$$

Since $\langle \overline{x}, u\rangle = 0$ for every $u \in \operatorname{Null}(\overline{Q})$, (5.6) gives that $\overline{Q}y^* = 0$ and hence $\langle y^*, \overline{x}\rangle = 0$. It follows that (5.6) has no solution. Combining $\overline{Q}y^* = 0$ and $\langle y^*, \overline{x}\rangle = 0$ with the assumption $\operatorname{rank}(\overline{Q};\overline{x}) = n$, it implies $y^* = 0$. Hence (5.7) has unique solution $(r^*, b^*, y^*) = 0$. Consequently, in this case, (5.4) has only one trivial solution and the conclusion follows.

(iv) Since $\|\overline{x}\| < \overline{r}$, $a^T\overline{x} + \overline{b} = 0$ and $\overline{Q}\overline{x} + \overline{q} = \gamma a$, $\gamma > 0$, we have $D^*\mathcal{N}(\overline{p})(v^*) = \Omega_3(\overline{\omega})(-y^*)$ if $\langle -y^*, a\rangle = 0$. Then, (5.4) gives

$$\overline{Q}y^* = 0, \quad 0 = b^*a, \quad r^* = 0, \quad \langle y^*, a\rangle = 0.$$

From assumption $\operatorname{rank}(\overline{Q}; a) = n$, we get $y^* = 0$. Hence (5.4) has a unique solution $(r^*, b^*, y^*) = 0$ and $S(\cdot)$ is locally Lipschitz-like around $(\overline{Q}, \overline{q}, \overline{r}, \overline{b}, \overline{x})$.

(v) Since $\|\overline{x}\| < \overline{r}$, $a^T\overline{x} + \overline{b} = 0$, and $\overline{Q}\overline{x} + \overline{q} = 0$, we obtain

$$D^*\mathcal{N}(\overline{\omega})(v^*) = \begin{cases} \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\} & \text{if } \langle v^*, a\rangle < 0, \\ \Omega_4(\overline{\omega})(v^*) & \text{if } \langle v^*, a\rangle > 0, \\ \Omega_3(\overline{\omega})(v^*) & \text{if } \langle v^*, a\rangle = 0. \end{cases}$$

Then, (5.4) yields

$$\overline{Q}y^* = 0, \quad 0 = b^*a, \quad r^* = 0, \quad b^* \geq 0, \quad \langle y^*, a\rangle > 0 \tag{5.8}$$

and

$$\overline{Q}y^* = 0, \quad 0 = b^*a, \quad r^* = 0, \quad \langle y^*, a\rangle = 0. \tag{5.9}$$

By the assumption that $\langle \overline{x}, u\rangle = 0$ for every $u \in \operatorname{Null}(\overline{Q})$, (5.8) follows $\overline{Q}y^* = 0$. Hence $\langle y^*, a\rangle = 0$. This gives that (5.6) has no solution. Since $\operatorname{rank}(\overline{Q}; a) = n$, (5.9) has a unique solution $(r^*, b^*, y^*) = 0$. Hence in this case, (5.4) has only one solution $(r^*, b^*, y^*) = 0$ and the desired conclusion follows.

(vi) From the assumption that $\|\overline{x}\| = \overline{r}$ and $a^T\overline{x} + b = 0$ it follows that $D^*\mathcal{N}(\overline{\omega})(v^*)$ is computed and estimated as in parts (vi)–(ix) of Theorem 4.4.

Since $\det \overline{Q} \neq 0$, we now show that (5.5) has unique solution $(r^*, b^*) = 0$. Indeed, from the assumption that b is unperturbed it implies $b^* = 0$. Substituting $b^* = 0$ and $v^* = -y^* = 0$ into the formulas in parts (f)–(i) of Theorem 4.4 yields $r^* = 0$.

Consequently, in this case, (5.5) has only one trivial solution, and $S(\cdot)$ is locally Lipschitz-like around $(\overline{Q}, \overline{q}, \overline{r}, \overline{b}, \overline{x})$. The theorem is proved. □

6. Optimality conditions using the coderivative

In the recent years, the coderivative has been used as a helpful tool to characterize the optimality conditions of the mathematical programming problems. According to [17, Proposition 5.1] (applied to $(ET(w))$), if $\overline{x}$ is a local (or global) solution of the problem $(ET(\overline{w}))$ then

$$-\overline{Q}\overline{x}-\overline{q}\in N((\overline{r},\overline{b},\overline{x}):\mathcal{F}(\overline{r},\overline{b})).$$

The following result gives a necessary condition for the local (or global) solution of the extended trust region subproblem by using the coderivative tool.

Theorem 6.1. *Assume that $(\overline{x},\overline{w})$ is a local (or global) solution of the following problem*

$$\min_{x,w} f(x,Q,q) \quad \text{subject to} \quad x\in\mathcal{F}(r,b), \tag{6.1}$$

where $w=(Q,q,r,b)$ and one of the following conditions is satisfied:

(i) *(Inverse Aubin (Lipschitz-like) property): $\mathcal{F}^{-1}$ has the Aubin property at $\overline{x}$ for $(\overline{r},\overline{b})$.*

(ii) *(Metric regularity): There exist a neighborhood V of $(\overline{r},\overline{b})$, a neighborhood U of $\overline{x}$ and a non-negative real number κ such that*

$$d((r,b),\mathcal{F}^{-1}(x))\le\kappa d(x,\mathcal{F}(r,b)) \quad \text{when } (r,b)\in V,\ x\in U.$$

(iii) *(Linear openness): There exist a neighborhood V of $(\overline{r},\overline{b})$, a neighborhood U of $\overline{x}$ and a non-negative real number κ such that*

$$\mathcal{F}((r,b)+\kappa\varepsilon B)\supset[\mathcal{F}(r,b)+\varepsilon B]\cap U \quad \text{for all } w\in V,\ \varepsilon>0.$$

(iv) *(Coderivative nonsingularity): $0\in D^*\mathcal{N}(\overline{r},\overline{b},\overline{x},0)(v)=0$ only for $v=0$.*

Then, there exists $v^*\in\mathbb{R}^n$ such that

$$-\overline{Q}\overline{x}-\overline{q}\in D^*\mathcal{N}(\overline{r},\overline{b},\overline{x},0)(v^*).$$

Proof. By [23, Theorem 9.43], we get the assumptions (i), (ii), (iii) and (iv) are equivalent.

The problem (6.1) can be rewritten as follows:

$$\min_{w,x} F(w,x):=\frac{1}{2}x^TQx+q^Tx \quad \text{subject to} \quad (w,x)\in\operatorname{gph}\mathcal{S}, \tag{6.2}$$

with $\mathcal{S}$ being the global solution set of the following problem

$$\min_{x}\psi(w,x) \quad \text{subject to} \quad x\in\mathcal{F}(w):=\mathcal{F}(r,b)$$

where $\psi(w,\cdot\,)$ is constant on $\mathcal{F}(w)$.

Consider the problem (6.2) with $(\overline{w},\overline{x})$ being a local (or global) solution of (6.2) and use the assumption (iv). Applying Theorem 4.1 in [5] for the problem (6.2), we obtain that there exists $v^*\in\mathbb{R}^n$ such that

$$-\overline{Q}\overline{x}-\overline{q}\in D^*\mathcal{N}(\overline{r},\overline{b},\overline{x},0)(v^*).$$

The proof is complete. □

Remark 6.2. It is well-known that the normal cone to a convex set is the subdifferential of its indicator function. According to [16,17], coderivatives of the normal cone mapping coincide with the second-order subdifferentials of the indicator function. Hence, Theorem 6.1 can be seen as an application of second-order subdifferentials to the characterization of optimality conditions for nonlinear programming.

7. Conclusions

In this paper, the Fréchet and Mordukhovich coderivatives of the normal cone mapping related to the parametric eTRS have been computed and estimated in Theorems 3.4 and 4.4. We have used the obtained results and the Mordukhovich criterion for the locally Lipschitz-like property of multifunctions to estimate the Mordukhovich coderivative of $\widetilde{S}(\cdot)$ and to provide some sufficient conditions for the locally Lipschitz-like property of the KKT point set map of parametric eTRS with respect to the linear perturbations. We have proposed a necessary condition for the local (or global) solution of the extended trust region subproblem by using the coderivative tool.

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