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On the Numerical Quadrature of Weakly Singular Oscillatory Integral and its Fast Implementation

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Abstract. In this paper, we present a Clenshaw-Curtis-Filon-type method for the weakly singular oscillatory integral with Fourier and Hankel kernels. By interpolating the non-oscillatory and nonsingular part of the integrand at $(N+1)$ Clenshaw-Curtis points, the method can be implemented in $O(N \log N)$ operations, which requires the accurate computation of modified moments. We first give a method for the derivation of recurrence relation for the modified moments, which can be applied to the derivation of recurrence relation for the modified moments corresponding to other type oscillatory integrals. By using the recurrence relation, special functions and classic quadrature methods, the modified moments can be computed accurately and efficiently. Then, we present the corresponding error bound in inverse powers of frequencies k and ω for the proposed method. Numerical examples are provided to support the theoretical results and show the efficiency and accuracy of the method.

1. Introduction

In this work we consider the evaluation of the weakly singular oscillatory integral of the form

$$I[f] = \int_0^1 f(x) x^{\alpha} (1-x)^{\beta} \mathrm{e}^{\mathrm{i}2kx} H_{\nu}^{(1)}(\omega x) \, \mathrm{d}x \tag{1.1}$$

where $\alpha - |\nu| > -1$, $\beta > -1$, and $k \gg 1$, $\omega \gg 1$, $H_{\nu}^{(1)}(x) = J_{\nu}(x) + \mathrm{i}Y_{\nu}(x)$ is Hankel function of the first kind of order ν, and f is a sufficiently smooth function on $[0,1]$. In many areas of science and engineering, for example, in astronomy, optics, quantum mechanics, seismology image processing, electromagnetic scattering (see [2–4, 11, 19]), one will come across the computation of the integral (1.1).

The integral (1.1) has the following two characteristics:

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1. When $k+\omega \gg 1$, the integrand becomes highly oscillatory. Consequently, a prohibitively number of quadrature nodes are needed to obtain satisfied accuracy if one uses classical numerical methods like Simpson rule, Gaussian quadrature, etc. Moreover, it presents serious difficulties in obtaining numerical convergence of the integration.

2. The function $H_\nu^{(1)}(x)$ has a logarithmic singularity when $\nu = 0$, and algebraic singularity when $\nu \neq 0$ at the point $x = 0$. In addition, if $-1 < \alpha, \beta < 0$, the integrand also has algebraic singularities at two endpoints, which impacts heavily on its quadrature and error bound. For a special case that $\alpha = 0$, $\beta = 0$, and $k = 0$, the integral can be rewritten in a special form
$$I[f] = \int_0^1 f(x) H_\nu^{(1)}(\omega x)\, \mathrm{d}x.$$

In the last few years, many efficient numerical methods have been devised for the evaluation of oscillatory integrals. Here, we only mention several main methods, such as Levin method and Levin-type method [26, 27, 32], generalized quadrature rule [15, 16], Filon method and Filon-type method [12, 13, 17, 21, 36–38], Gauss-Laguerre quadrature [7–9, 19–21, 39]. In what follows, we will introduce several other papers related to the integrals considered in this paper. For the integral $\int_0^1 f(x)x^\alpha(1-x)^\beta \mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x$, as early as in 1992, Piessens [33] constructed a fast algorithm to approximate it by truncating f by its Chebyshev series and using the recurrence relation of the modified moments. Recently, the references [24, 25] developed this method by using a special Hermite interpolation at Clenshaw-Crutis points and Chebyshev expansion for $\mathrm{e}^{\mathrm{i}kx}$. If f is analytic in a sufficiently large complex region containing $[0, 1]$, a numerical steepest descent method [23] was presented by using complex integration theory. The same idea was also applied to the computation of the integral $\int_a^b (x-a)^\alpha (b-x)^\beta \ln(x-a) f(x) \mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}x$, $\alpha, \beta > -1$, based on construction of the Gauss quadrature rule with logarithmic weight function [18]. For the integral $\int_0^1 f(x)x^\alpha(1-x)^\beta J_m(\omega x)\,\mathrm{d}x$, $\alpha, \beta > -1$, a Filon-type method based on a special Hermite interpolation polynomial at Clenshaw-Curtis points was introduced in [10]. On the other hand, the reference [22] proposed a Clenshaw-Curtis-Filon method for the computation of the oscillatory Bessel integral $\int_0^1 f(x)x^\alpha \ln(x)(1-x)^\beta J_m(\omega x)\,\mathrm{d}x$, $\alpha, \beta > -1$, with algebraic or logarithmic singularities at the two endpoints.

For the evaluation of the integral (1.1), the literature [18] transformed it into two line integrals by using the analytic continuation and the construction of Gauss quadrature rules. However, this method requires that f is analytic in a enough large region. A recent work [41] presented a Clenshaw-Curtis-Filon-type method for the special case $\alpha = \beta = k = 0$ by using special functions. In addition, a composite method [12] can also be applied to the computation of this integral for this case, by absorbing the non-oscillatory part of Hankel function into f, then interpolating its product with f. However, the accuracy of this method may becomes worse as the number of Clenshaw-Curtis points increases and the fastest convergence of this method obtained is $O(\omega^{-2})$ for fixed number of Clenshaw-Curtis points [41].

In view of the advantages of Clenshaw-Curtis-Filon method, in this paper we will consider a higher order Clenshaw-Curtis-Filon-type method for the integral (1.1), which does not require that f is analytic in a enough large region. As we know, the fast implementation of Clenshaw-Curtis-Filon-type method largely depends on the accurate and efficient computation of modified moments. In addition, the key problem of the efficient computation of the modified moments is how to obtain a recurrence relation for them. Fortunately, we can give a universal method for the derivation of recurrence relation for the modified moments. Moreover, this method can be applied to the modified moments with other type kernels.

The outline of this paper is organized as follows. In Section 2, we describe a Clenshaw-Curtis-Filon-type method for the integral (1.1), and present a universal method for the derivation of recurrence relation for the modified moments, by which the modified moments can be efficiently computed with several initial values. In Section 3, we give an error bound on k and ω for the presented method. Some examples are given in Section 4 to show the efficiency and accuracy. Finally, we finish this paper in Section 5 by presenting some concluding remarks.

2. Clenshaw-Curtis-Filon-type method and its implementation

In what follows we will consider a Clenshaw-Curtis-Filon-type method for the integral (1.1) and its fast implementation. Suppose that f is a sufficiently smooth function on $[0,1]$, and let $P_{N+2s}(x)$ denote the Hermite interpolation polynomial at the Clenshaw-Curtis points

$$x_j = \frac{1+\cos(j\pi/N)}{2}, \quad j = 0, \ldots, N,$$

where s is a nonnegative integer, and for $\ell = 0, \ldots, s$, there holds

$$P^{(\ell)}_{N+2s}(0) = f^{(\ell)}(0), \quad P_{N+2s}(x_j) = f(x_j), \quad P^{(\ell)}_{N+2s}(1) = f^{(\ell)}(1), \quad j = 1, \ldots, N-1. \tag{2.1}$$

Then $P_{N+2s}(x)$ can be written in the following form

$$P_{N+2s}(x) = \sum_{n=0}^{N+2s} a_n T^*_n(x), \tag{2.2}$$

where a_n can be fast calculated by fast Fourier transform [36] with $O(N \log N)$ operations, $T^*_n(x)$ is the shifted Chebyshev polynomial of the first kind of degree n on $[0,1]$.

In view of (2.1) and (2.2), we can define a *Clenshaw-Curtis-Filon-type method* for the integral (1.1) by

$$Q_{N,s}^{CCF}[f] = \int_0^1 P_{N+2s}(x)x^\alpha(1-x)^\beta e^{i2kx} H_\nu^{(1)}(\omega x)\,dx = \sum_{n=0}^{N+2s} a_n M(n,k,\omega), \tag{2.3}$$

where the modified moments

$$M(n,k,\omega) = \int_0^1 x^\alpha(1-x)^\beta T_n^*(x) e^{i2kx} H_\nu^{(1)}(\omega x)\,dx$$

have to be computed accurately.

2.1. Recurrence relation for the modified moments

As we have stated in Section 1, the key problem of the fast computations of the modified moment $M(n,k,\omega)$ is to obtain a recurrence relation for them. In the following, we will give a universal method for the derivation of recurrence relation for the modified moments.

Theorem 2.1. *The modified moments $M(n,k,\omega)$ for $n \geq 4$, $k \geq 0$, $\omega > 0$ satisfy the following recurrence relation:*

$$\begin{aligned}
&\left(\frac{1}{16}\omega^2 - \frac{1}{4}k^2\right) M(n+4,k,\omega) + f_1(n,\alpha,\beta)M(n+3,k,\omega) \\
&+ f_2(n,\alpha,\beta)M(n+2,k,\omega) + f_3(n,\alpha,\beta)M(n+1,k,\omega) + f_4(n,\alpha,\beta)M(n,k,\omega) \\
&+ f_3(-n,\alpha,\beta)M(n-1,k,\omega) + f_2(-n,\alpha,\beta)M(n-2,k,\omega) \\
&+ f_1(-n,\alpha,\beta)M(n-3,k,\omega) + \left(\frac{1}{16}\omega^2 - \frac{1}{4}k^2\right) M(n-4,k,\omega) = 0,
\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}
f_1(n,\alpha,\beta) &= ik(\alpha+\beta+n+4) - \frac{1}{2}ik, \\
f_2(n,\alpha,\beta) &= 9 + 6(\alpha+\beta+n) + k^2 + n^2 + \alpha^2 + \beta^2 - \frac{1}{4}\omega^2 - \nu^2 \\
&\quad + 2(\alpha\beta + \alpha n + \beta n) + ik(1 - 2\alpha + 2\beta), \\
f_3(n,\alpha,\beta) &= 2n - 8\alpha + 12\beta + 4(1 - i\alpha k - i\beta k + \nu^2 + \beta n - \alpha n) \\
&\quad - \frac{31}{2}ik + 3ik(\alpha+\beta-n+4) + 4(\beta^2 - \alpha^2), \\
f_4(n,\alpha,\beta) &= 6 + 4\alpha + 12\beta - 4\alpha\beta - 2ik + 4ik(\alpha-\beta) \\
&\quad + \frac{3}{8}\omega^2 - \frac{3}{2}k^2 + 6(\alpha^2+\beta^2-\nu^2) - 2n^2.
\end{aligned}$$

Proof. First, we can rewrite the modified moments $M(n,k,\omega)$ by

$$M(n,k,\omega) = \frac{1}{2^{\alpha+\beta+1}} e^{ik} \int_{-1}^1 (1+x)^\alpha(1-x)^\beta T_n(x) e^{ikx} H_\nu^{(1)}\left(\frac{1+x}{2}\omega\right) dx,$$

where $T_n(x)$ is the Chebyshev polynomial of degree n of the first kind.

From the above equality, we can see that the modified moments $M(n,k,\omega)$ and the integral $\int_{-1}^{1}(1+x)^\alpha(1-x)^\beta T_n(x)\mathrm{e}^{\mathrm{i}kx}H_\nu^{(1)}\big(\frac{1+x}{2}\omega\big)\,\mathrm{d}x$ have the same recurrence relation.

Since the function $y=H_\nu^{(1)}(x)$ satisfies the following Bessel's differential equation [1, p. 358]

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\frac{\mathrm{d}y}{\mathrm{d}x}+(x^2-\nu^2)=0,$$

we have

$$\begin{aligned}(1+x)^2\left[H_\nu^{(1)}\left(\frac{1+x}{2}\omega\right)\right]''+(1+x)\left[H_\nu^{(1)}\left(\frac{1+x}{2}\omega\right)\right]'\\ -\left(\nu^2-\frac{(1+x)^2\omega^2}{4}\right)H_\nu^{(1)}\left(\frac{1+x}{2}\omega\right)=0.\end{aligned}\tag{2.5}$$

Let

$$\begin{aligned}K_1&=4\int_{-1}^{1}(1+x)^\alpha(1-x)^\beta(1-x)^2(1+x)^2\mathrm{e}^{\mathrm{i}kx}\left[H_\nu^{(1)}\left(\frac{1+x}{2}\omega\right)\right]''T_n(x)\,\mathrm{d}x,\\ K_2&=4\int_{-1}^{1}(1+x)^\alpha(1-x)^\beta(1-x)^2(1+x)\mathrm{e}^{\mathrm{i}kx}\left[H_\nu^{(1)}\left(\frac{1+x}{2}\omega\right)\right]'T_n(x)\,\mathrm{d}x,\end{aligned}$$

and

$$K_3=4\int_{-1}^{1}(1+x)^\alpha(1-x)^\beta(1-x)^2\left(\nu^2-\frac{(1+x)^2\omega^2}{4}\right)\mathrm{e}^{\mathrm{i}kx}H_\nu^{(1)}\left(\frac{1+x}{2}\omega\right)T_n(x)\,\mathrm{d}x.$$

It follows from (2.5) that

$$K_1+K_2-K_3=0.\tag{2.6}$$

Noting that the integrands in K_1 and K_2 have the common factor $(1-x)^2$ and using integration by parts, we can easily get

$$\begin{aligned}K_1&=4\int_{-1}^{1}\left[(1+x)^\alpha(1-x)^\beta(1-x)^2(1+x)^2\mathrm{e}^{\mathrm{i}kx}T_n(x)\right]''H_\nu^{(1)}\left(\frac{1+x}{2}\omega\right)\mathrm{d}x,\\ K_2&=4\int_{-1}^{1}\left[(1+x)^\alpha(1-x)^\beta(1-x)^2(1+x)\mathrm{e}^{\mathrm{i}kx}T_n(x)\right]'H_\nu^{(1)}\left(\frac{1+x}{2}\omega\right)\mathrm{d}x.\end{aligned}$$

According to the properties of the Chebyshev polynomial of the first kind [30]

$$x^mT_n(x)=2^{-m}\sum_{j=0}^{m}\binom{m}{j}T_{n+m-2j}(x)\quad\text{and}\quad\frac{\mathrm{d}}{\mathrm{d}x}T_n(x)=\frac{n}{2}\frac{T_{n-1}(x)-T_{n+1}(x)}{1-x^2},$$

by rewriting the integrands in K_1, K_2 and K_3 as the sum of the product of Chebyshev polynomials of different degree and $(1+x)^{\alpha}(1-x)^{\beta}\mathrm{e}^{\mathrm{i}kx}H_{\nu}^{(1)}\left(\frac{1+x}{2}\omega\right)$, we derive

$$
\begin{aligned}
K_1 = &-\frac{1}{4}k^2M(n+4,k,\omega) + f_5(n,\alpha,\beta)M(n+3,k,\omega) + f_6(n,\alpha,\beta)M(n+2,k,\omega)\\
&+ f_7(n,\alpha,\beta)M(n+1,k,\omega) + f_8(n,\alpha,\beta)M(n,k,\omega)\\
&+ f_7(-n,\alpha,\beta)M(n-1,k,\omega) + f_6(-n,\alpha,\beta)M(n-2,k,\omega)\\
&+ f_5(-n,\alpha,\beta)M(n-3,k,\omega) - \frac{1}{4}k^2M(n-4,k,\omega),
\end{aligned} \tag{2.7}
$$

where

$$
\begin{aligned}
&f_5(n,\alpha,\beta) = \mathrm{i}k(\alpha+\beta+n+4),\\
&f_6(n,\alpha,\beta) = 12 + 7(\alpha+\beta+n) + k^2 + n^2 + \alpha^2 + \beta^2 + 2(\alpha\beta + \alpha n + \beta n) + 2\mathrm{i}k(\beta-\alpha),\\
&f_7(n,\alpha,\beta) = 12(\beta-\alpha) - 4\mathrm{i}k(\alpha+\beta) + 4(\beta n - \alpha n) + 3\mathrm{i}k(\alpha+\beta-n+4) + 4(\beta^2-\alpha^2),\\
&f_8(n,\alpha,\beta) = 8 + 10(\alpha+\beta) + 4\mathrm{i}k(\alpha-\beta) - 2n^2 - \frac{3}{2}k^2 + 6(\alpha^2+\beta^2) - 4\alpha\beta,
\end{aligned}
$$

$$
\begin{aligned}
K_2 = -\Big\{&\frac{1}{2}\mathrm{i}kM(n+3,k,\omega) + (\alpha+\beta+n+3-\mathrm{i}k)M(n+2,k,\omega)\\
&- \left(\frac{1}{2}\mathrm{i}k + 4 + 4\alpha + 2n\right)M(n+1,k,\omega) + (6\alpha - 2\beta + 2\mathrm{i}k + 2)M(n,k,\omega)\\
&- \left(\frac{1}{2}\mathrm{i}k + 4 + 4\alpha + 2n\right)M(n-1,k,\omega)\\
&+ (\alpha+\beta-n+3-\mathrm{i}k)M(n-2,k,\omega) + \frac{1}{2}\mathrm{i}kM(n-3,k,\omega)\Big\},
\end{aligned} \tag{2.8}
$$

$$
\begin{aligned}
K_3 = -\frac{1}{16}\Big\{&\omega^2M(n+4,k,\omega) - (4\omega^2+16\nu^2)M(n+2,k,\omega) + 64\nu^2M(n+1,k,\omega)\\
&+ (6\omega^2 - 96\nu^2)M(n,k,\omega) + 64\nu^2M(n-1,k,\omega)\\
&- (4\omega^2+16\nu^2)M(n-2,k,\omega) + \omega^2M(n-4,k,\omega)\Big\}.
\end{aligned} \tag{2.9}
$$

A combination of (2.6), (2.7), (2.8), (2.9) leads to recurrence relation (2.4). □

In the following, let us denote by

$$
\begin{aligned}
\widetilde{M}_n^{[1]} &= \int_0^1 \ln(x)x^{\alpha}(1-x)^{\beta}T_n^*(x)\mathrm{e}^{\mathrm{i}2kx}H_{\nu}^{(1)}(\omega x)\,\mathrm{d}x,\\
\widetilde{M}_n^{[2]} &= \int_0^1 x^{\alpha}(1-x)^{\beta}\ln(1-x)T_n^*(x)\mathrm{e}^{\mathrm{i}2kx}H_{\nu}^{(1)}(\omega x)\,\mathrm{d}x,\\
\widetilde{M}_n^{[3]} &= \int_0^1 \ln(x)x^{\alpha}(1-x)^{\beta}\ln(1-x)T_n^*(x)\mathrm{e}^{\mathrm{i}2kx}H_{\nu}^{(1)}(\omega x)\,\mathrm{d}x,
\end{aligned}
$$

respectively, where $\alpha-|\nu|>-1$, $\beta>-1$. Using the fact that

$$\widetilde{M}_n^{[1]}=\frac{\partial}{\partial\alpha}M(n,k,\omega),\quad \widetilde{M}_n^{[2]}=\frac{\partial}{\partial\beta}M(n,k,\omega),\quad \widetilde{M}_n^{[3]}=\frac{\partial^2}{\partial\alpha\partial\beta}M(n,k,\omega),$$

and according to Theorem 2.1, we can readily obtain the following result.

Corollary 2.2. *The sequences $\widetilde{M}_n^{[\ell]}$, $\ell=1,2,3$ and $n\geq 4$, $k\geq 0$, $\omega>0$ satisfy the following ninth-order homogeneous recurrence relations*

$$\begin{aligned}&\left(\frac{1}{16}\omega^2-\frac{1}{4}k^2\right)\widetilde{M}_{n+4}^{[\ell]}+f_1(n,\alpha,\beta)\widetilde{M}_{n+3}^{[\ell]}+f_2(n,\alpha,\beta)\widetilde{M}_{n+2}^{[\ell]}+f_3(n,\alpha,\beta)\widetilde{M}_{n+1}^{[\ell]}\\&+f_4(n,\alpha,\beta)\widetilde{M}_{n}^{[\ell]}+f_3(-n,\alpha,\beta)\widetilde{M}_{n-1}^{[\ell]}+f_2(-n,\alpha,\beta)\widetilde{M}_{n-2}^{[\ell]}+f_1(-n,\alpha,\beta)\widetilde{M}_{n-3}^{[\ell]}\\&+\left(\frac{1}{16}\omega^2-\frac{1}{4}k^2\right)\widetilde{M}_{n-4}^{[\ell]}=r_n^{[\ell]},\end{aligned}$$

where

$$\begin{aligned}r_n^{[1]}=-\Big\{&\mathrm{i}kM(n+3,k,\omega)+(6+2\beta+2n+2\alpha+2\mathrm{i}k)M(n+2,k,\omega)\\&-(8+\mathrm{i}k+4n+8\alpha)M(n+1,k,\omega)+(4-4\beta+4\mathrm{i}k+12\alpha)M(n,k,\omega)\\&-(8+\mathrm{i}k-4n+8\alpha)M(n-1,k,\omega)+(6+2\beta-2n+2\alpha+2\mathrm{i}k)M(n-2,k,\omega)\\&+\mathrm{i}kM(n-3,k,\omega)\Big\},\\r_n^{[2]}=-\Big\{&\mathrm{i}kM(n+3,k,\omega)+(6+2\beta+2n+2\alpha-2\mathrm{i}k)M(n+2,k,\omega)\\&+(12-\mathrm{i}k+4n+8\beta)M(n+1,k,\omega)+(12-4\alpha-4\mathrm{i}k+12\beta)M(n,k,\omega)\\&+(12-\mathrm{i}k-4n+8\beta)M(n-1,k,\omega)+(6+2\beta-2n+2\alpha-2\mathrm{i}k)M(n-2,k,\omega)\\&+\mathrm{i}kM(n-3,k,\omega)\Big\},\end{aligned}$$

and

$$\begin{aligned}r_n^{[3]}=-\Big\{&\mathrm{i}k\big(\widetilde{M}_{n+3}^{[1]}+\widetilde{M}_{n+3}^{[2]}\big)+(6+2\beta+2n+2\alpha-2\mathrm{i}k)\widetilde{M}_{n+2}^{[1]}\\&+(6+2\beta+2n+2\alpha+2\mathrm{i}k)\widetilde{M}_{n+2}^{[2]}2M(n+2,k,\omega)+(12-\mathrm{i}k+4n+8\beta)\widetilde{M}_{n+1}^{[1]}\\&+(8+\mathrm{i}k+4n+8\alpha)\widetilde{M}_{n+1}^{[2]}-4M(n,k,\omega)+(12-\mathrm{i}k-4n+8\beta)\widetilde{M}_{n-1}^{[1]}\\&+(8+\mathrm{i}k-4n+8\alpha)\widetilde{M}_{n-1}^{[2]}+(6+2\beta-2n+2\alpha-2\mathrm{i}k)\widetilde{M}_{n-2}^{[1]}\\&+(6+2\beta-2n+2\alpha+2\mathrm{i}k)\widetilde{M}_{n-2}^{[2]}+2M(n-2,k,\omega)+\mathrm{i}k\big(\widetilde{M}_{n-3}^{[1]}+\widetilde{M}_{n-3}^{[2]}\big)\Big\}.\end{aligned}$$

Remark 2.3. The proof of Theorem 2.1 provides a universal method for the derivation of recurrence relation of the modified moments, which can be applied to the modified moments with other kernels that satisfy some linear differential equations. For example, for the derivation of recurrence relations of the following three kinds of modified moments

$$\int_0^1 x^\alpha (1-x)^\beta T_n^*(x) e^{i2kx} \operatorname{Ai}(-\omega x)\, dx,$$
$$\int_0^1 x^\alpha (1-x)^\beta T_n^*(x) e^{i2kx} j_\nu(\omega x)\, dx,$$
$$\int_0^1 x^\alpha (1-x)^\beta T_n^*(x) e^{i2kx} y_\nu(\omega x)\, dx,$$

the method is applicable, where $\operatorname{Ai}(x)$ is Airy function, $j_\nu(x)$, $y_\nu(x)$ are spherical Bessel functions of the first kind and second kind [1], respectively. Moreover, by differentiating the recurrence relation with respect to parameters α, β, one can also obtain recurrence relations for the modified moments with logarithmic singularities at two endpoints. As this idea is tangential to the topic of this paper, we will not study it further.

Remark 2.4. For $\omega = 2k$, the coefficients of $M(n+4,k,\omega)$ and $M(n-4,k,\omega)$ are both zero, then the recurrence relation (2.4) reduces to a seven-term recurrence relation.

2.2. Fast computations of the modified moments

In this subsection, we will be concerned with the fast computation of the modified moments by using the recurrence relation (2.4). According to the symmetry of the recurrence relation of the shifted Chebyshev polynomials $T_n^*(x)$ on $[0,1]$, it is convenient to define $T_{-n}^*(x) = T_n^*(x)$ for $n = 1,2,3,\ldots$. Consequently, $M(-n,k,\omega) = M(n,k,\omega)$, $k = 1,2,3,\ldots$. Moreover, it can be shown that (2.4) is valid, not only for $n \geq 4$, but also for all integers of n.

Unfortunately, the application of recurrence relations in the forward direction is not always numerically stable. Practical experiments show that the modified moments $M(n,k,\omega)$, $n = 0,1,2,\ldots$ can be computed accurately by using the recurrence relation (2.4) as long as $n \leq (k+\omega/2)$. However, for $n > (k+\omega/2)$, forward recursion is no longer applicable due to the loss of significant figures increases. In this case, (2.4) has to be solved as a boundary value problem. Fortunately, we can use Oliver's algorithm [31] or Lozier's algorithm [28] to solve this problem for the modified moments with five starting moments and three end moments. Particularly, for Lozier's algorithm, we can set three end moments to zero. Also, this algorithm incorporates a numerical test for determining the optimum location of the endpoint. The advantage is that a user-required accuracy is automatically obtained, without computation of the asymptotic expansion. In conclusion, several starting values for the modified moments for forward recursion and Oliver's algorithm or Lozier's algorithm are needed. In addition, three end moments can be computed by using asymptotic expansion in [14] or the method in [18].

Since the shifted Chebyshev polynomials $T_n^*(x)$ can be rewritten in terms of powers of x, the five starting modified moments can be computed by the following formulas

$$\begin{aligned}
&M(0,k,\omega) = I(0,k,\omega),\\
&M(1,k,\omega) = 2I(1,k,\omega) - I(0,k,\omega),\\
&M(2,k,\omega) = 8I(2,k,\omega) - 8I(1,k,\omega) + I(0,k,\omega),\\
&M(3,k,\omega) = 32I(3,k,\omega) - 48I(2,k,\omega) + 18I(1,k,\omega) - I(0,k,\omega),\\
&M(4,k,\omega) = 128I(4,k,\omega) - 256I(3,k,\omega) + 160I(2,k,\omega) - 32I(1,k,\omega) + I(0,k,\omega),
\end{aligned}$$

where

$$I(j,k,\omega) = \int_0^1 x^{\alpha+j}(1-x)^{\beta}\mathrm{e}^{\mathrm{i}2kx}H_\nu^{(1)}(\omega x)\,\mathrm{d}x, \tag{2.10}$$

which can be efficiently computed by the method in [18] with small number of points.

For a special case $\omega = 2k$, the computation of the integral (2.10) is reduced to the evaluation of

$$\widehat{I}(\alpha,\beta,\nu,\omega) = \int_0^1 x^{\alpha}(1-x)^{\beta}\mathrm{e}^{\mathrm{i}\omega x}H_\nu^{(1)}(\omega x)\,\mathrm{d}x, \quad \alpha > -1,\ \beta > -1,$$

which can also be accurately computed through the following theorem.

Theorem 2.5. *For all $\alpha - |\nu| > -1$, $\beta > -1$ and $\omega > 0$, it holds that*

$$\widehat{I}(\alpha,\beta,\nu,\omega) = I_1 + \mathrm{i}(I_2 + I_3) - I_4, \tag{2.11}$$

where

$$\begin{aligned}
I_1 &= CG_{4,6}^{1,4}\left(\left.\begin{matrix} -\frac{\alpha}{2}, \frac{1-\alpha}{2}, \frac{1}{4}, \frac{3}{4} \\ \frac{\nu}{2}, -\frac{\nu}{2}, \frac{1+\nu}{2}, \frac{1-\nu}{2}, -\frac{\alpha+\beta+1}{2}, -\frac{\alpha+\beta}{2} \end{matrix}\,\right|\, \omega^2\right),\\
I_2 &= \omega CG_{4,6}^{1,4}\left(\left.\begin{matrix} -\frac{\alpha+1}{2}, -\frac{\alpha}{2}, -\frac{1}{4}, \frac{1}{4} \\ \frac{\nu}{2}, -\frac{\nu}{2}, -\frac{1-\nu}{2}, -\frac{1+\nu}{2}, -\frac{\alpha+\beta+2}{2}, -\frac{\alpha+\beta+1}{2} \end{matrix}\,\right|\, \omega^2\right),\\
I_3 &= -CG_{5,7}^{2,4}\left(\left.\begin{matrix} -\frac{\alpha}{2}, \frac{1-\alpha}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1-\nu}{2} \\ -\frac{\nu}{2}, \frac{\nu}{2}, \frac{1+\nu}{2}, \frac{1-\nu}{2}, \frac{1-\nu}{2}, -\frac{\alpha+\beta+1}{2}, -\frac{\alpha+\beta}{2} \end{matrix}\,\right|\, \omega^2\right),\\
I_4 &= -\omega CG_{5,7}^{2,4}\left(\left.\begin{matrix} -\frac{\alpha+1}{2}, -\frac{\alpha}{2}, -\frac{1}{4}, \frac{1}{4}, \frac{1-\nu}{2} \\ -\frac{\nu}{2}, \frac{\nu}{2}, \frac{1-\nu}{2}, -\frac{1+\nu}{2}, \frac{\nu-1}{2}, -\frac{\alpha+\beta+2}{2}, -\frac{\alpha+\beta+1}{2} \end{matrix}\,\right|\, \omega^2\right),
\end{aligned}$$

and

$$\begin{aligned}
&G_{p,q}^{m,n}\left(\left.\begin{matrix} a_1, \ldots, a_n, a_{n+1}, \ldots, a_p \\ b_1, \ldots, b_m, b_{m+1}, \ldots, b_q \end{matrix}\,\right|\, z\right)\\
&= \frac{1}{2\pi i}\oint_L \frac{\prod_{k=1}^m \Gamma(b_k - s)\prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{k=m+1}^q \Gamma(1 - b_k + s)\prod_{j=n+1}^p \Gamma(a_j - s)} z^s\,\mathrm{d}s
\end{aligned}$$

is Meijer G-function [5], $C = 2^{-(\beta+3/2)}\Gamma(\beta+1)$.

Proof. Substituting $H_\nu^{(1)}(x) = J_\nu(x) + \mathrm{i}Y_\nu(x)$ into $\widehat{I}(\alpha,\beta,\nu,\omega)$ yields

$$\begin{aligned}
&\widehat{I}(\alpha,\beta,\nu,\omega) \\
&= \int_0^1 x^\alpha(1-x)^\beta \cos(\omega x) J_\nu(\omega x)\,\mathrm{d}x + \mathrm{i}\int_0^1 x^\alpha(1-x)^\beta \sin(\omega x) J_\nu(\omega x)\,\mathrm{d}x \\
&\quad + \mathrm{i}\int_0^1 x^\alpha(1-x)^\beta \cos(\omega x) Y_\nu(\omega x)\,\mathrm{d}x - \int_0^1 x^\alpha(1-x)^\beta \sin(\omega x) Y_\nu(\omega x)\,\mathrm{d}x.
\end{aligned} \tag{2.12}$$

Note that [42, 43]

$$ {}_0F_1(\,;b;-z^2/4)J_\nu(z) = \frac{\Gamma(b)}{\sqrt{\pi}} 2^{b-1} G_{2,4}^{1,2}\left(\left. \begin{matrix} \frac{1-b}{2}, 1-\frac{b}{2} \\ -\frac{\nu}{2}, \frac{\nu}{2}, 1-b+\frac{\nu}{2}, 1-b-\frac{\nu}{2} \end{matrix} \right| z^2 \right), \tag{2.13}$$

$$ {}_0F_1(\,;b;-z^2/4)Y_\nu(z) = \frac{\Gamma(b)}{\sqrt{\pi}} 2^{b-1} G_{2,4}^{1,2}\left(\left. \begin{matrix} \frac{1-b}{2}, 1-\frac{b}{2}, \frac{1-\nu}{2} \\ -\frac{\nu}{2}, \frac{\nu}{2}, \frac{1-\nu}{2}, 1-b+\frac{\nu}{2}, 1-b-\frac{\nu}{2} \end{matrix} \right| z^2 \right). \tag{2.14}$$

On the other hand, there holds [44]

$$\begin{aligned}
&\int_0^x t^{\alpha-1}(x-t)^{\beta-1} G_{p,q}^{m,n}\left(\left. \begin{matrix} a_1\ldots a_n, a_{n+1}\ldots a_p \\ b_1\ldots b_m, b_{m+1}\ldots b_q \end{matrix} \right| \omega t^l \right) \mathrm{d}t \\
&= \frac{l^{-\beta}\Gamma(\beta)}{x^{1-\alpha-\beta}} G_{p+l,q+l}^{m,n+l}\left(\left. \begin{matrix} \frac{1-\alpha}{l}, \ldots, \frac{l-\alpha}{l}, a_1\ldots a_n, a_{n+1}, \ldots, a_p \\ b_1, \ldots, b_m, b_{m+1}, \ldots, b_q, \frac{1-\alpha-\beta}{l}, \ldots, \frac{l-\alpha-\beta}{l} \end{matrix} \right| \omega x^l \right),
\end{aligned} \tag{2.15}$$

and [29]

$$\cos(z) = {}_0F_1(\,;1/2;-z^2/4), \quad \sin(z) = {}_0F_1(\,;3/2;-z^2/4).$$

According to (2.12), and setting $b = 1/2, 3/2$ in (2.13) and (2.14), respectively, then substituting them into (2.15), we can easily derive the result (2.11). $\square$

Remark 2.6. We choose 10 points for the Gauss-type method in [18] to evaluate $I(j,k,\omega)$, $j = 0,1,2,3,4$ for $\omega \neq 2k$. While for $\omega = 2k$, we compute them by using the formula (2.11) through Meijer G-function, which can be efficiently computed with the MATLAB code MeijerG.m [45].

3. Error estimate about k and ω for the method (2.3)

In [34, 35], Sloan and Smith presented a product-integration rule with the Clenshaw-Curtis points for approximating the integral $\int_{-1}^1 k(x)f(x)\,\mathrm{d}x$, where $k(x)$ is integrable and $f(x)$

is continuous. Moreover, the authors also considered the theoretical convergence properties of the method, and obtained the satisfactory rates of convergence for all continuous functions $f(x)$, if $k(x)$ satisfies $\int_{-1}^{1} |k(x)|^p \,\mathrm{d}x < \infty$ for some $p > 1$. Since

$$\int_0^1 \left| x^\alpha (1-x)^\beta \mathrm{e}^{\mathrm{i}2kx} H_\nu^{(1)}(\omega x) \right|^p \,\mathrm{d}x < \infty$$

for all $p > 1$ from [34, 35], we see that the Clenshaw-Curtis-Filon-type method (2.3) for integral (1.1) is uniformly convergent in N for fixed k and ω, that is

$$\lim_{N\to\infty} \int_0^1 P_{N+2s}(x) x^\alpha (1-x)^\beta \mathrm{e}^{\mathrm{i}2kx} H_\nu^{(1)}(\omega x)\,\mathrm{d}x = \int_0^1 f(x) x^\alpha (1-x)^\beta \mathrm{e}^{\mathrm{i}2kx} H_\nu^{(1)}(\omega x)\,\mathrm{d}x.$$

In what follows we will consider the error estimate on k and ω for the method (2.3). To obtain an error bound for method (2.3), we first introduce the following theorem.

Theorem 3.1. *For each $\alpha - |\nu| > -1$, $\beta > -1$, the asymptotics of the integral $\int_0^1 x^\alpha (1-x)^\beta \mathrm{e}^{\mathrm{i}2kx} H_\nu^{(1)}(\omega x)\,\mathrm{d}x$ can be estimated by the following three formulas.*

(i) *If k is fixed and $\omega \to \infty$, there holds*

$$\int_0^1 x^\alpha (1-x)^\beta \mathrm{e}^{\mathrm{i}2kx} H_\nu^{(1)}(\omega x)\,\mathrm{d}x = O\left(\frac{1}{\omega^{1+\tau_1}}\right), \tag{3.1}$$

where $\tau_1 = \min\{\alpha, \beta\}$.

(ii) *If ω is fixed and $k \to \infty$, there holds*

$$\int_0^1 x^\alpha (1-x)^\beta \mathrm{e}^{\mathrm{i}2kx} H_\nu^{(1)}(\omega x)\,\mathrm{d}x = \begin{cases} O\left(\frac{1+\ln(k)}{k^{1+\alpha}}\right) & \nu = 0,\ \alpha \le \beta, \\ O\left(\frac{1}{k^{1+\beta}}\right) & \nu = 0,\ \alpha > \beta, \\ O\left(\frac{1}{k^{1+\tau_2}}\right) & \nu \ne 0, \end{cases} \tag{3.2}$$

where $\tau_2 = \min\{\alpha - |\nu|, \beta\}$.

(iii) *If $\omega = 2k$ and $\omega \to \infty$, there holds*

$$\int_0^1 x^\alpha (1-x)^\beta \mathrm{e}^{\mathrm{i}\omega x} H_\nu^{(1)}(\omega x)\,\mathrm{d}x = O\left(\frac{1}{\omega^{1+\tau_1}}\right). \tag{3.3}$$

Proof. By using the complex integration theory and substituting the original interval of integration by the paths of steepest descent, we can rewrite the integral $\int_0^1 x^\alpha (1-x)^\beta \mathrm{e}^{\mathrm{i}2kx} H_\nu^{(1)}(\omega x)\,\mathrm{d}x$ as a sum of two line integrals (which is a special case of Eq. (20) in [18] with $f(x) = 1$, $b = 1$), that is

$$\int_0^1 x^\alpha (1-x)^\beta \mathrm{e}^{\mathrm{i}2kx} H_\nu^{(1)}(\omega x)\,\mathrm{d}x = L_0[f] - L_1[f]$$

where

$$L_0[f]=\frac{2\mathrm{i}^{\alpha}}{\mathrm{i}^{\nu}\pi(2k+\omega)^{1+\alpha}}\int_0^{\infty}\left(1-\frac{\mathrm{i}x}{2k+\omega}\right)^{\beta}K_{\nu}\left(\frac{\omega x}{2k+\omega}\right)x^{\alpha}\mathrm{e}^{-2kx/(2k+\omega)}\,\mathrm{d}x,$$
$$L_1[f]=\frac{(-\mathrm{i})^{\beta}\mathrm{i}\mathrm{e}^{\mathrm{i}\omega}}{\pi(2k+\omega)^{1+\beta}}\int_0^{\infty}\left(1+\frac{\mathrm{i}x}{2k+\omega}\right)^{\alpha}H_{\nu}^{(1)}\left(\omega+\frac{\mathrm{i}\omega x}{2k+\omega}\right)\mathrm{e}^{\omega x/(2k+\omega)}x^{\beta}\mathrm{e}^{-x}\,\mathrm{d}x,$$

here, $K_{\nu}(x)$ is the modified Bessel function of the second kind of order ν [1].

According to the Theorem in [6], when $\omega\to\infty$, for every fixed k, we have

$$L_0[f]=O\left(\frac{1}{\omega^{1+\alpha}}\right),\quad L_1[f]=O\left(\frac{1}{\omega^{1+\beta}}\right),$$

which leads to (3.1) directly.

On the other hand, when $k\to\infty$, for every fixed ω, we have

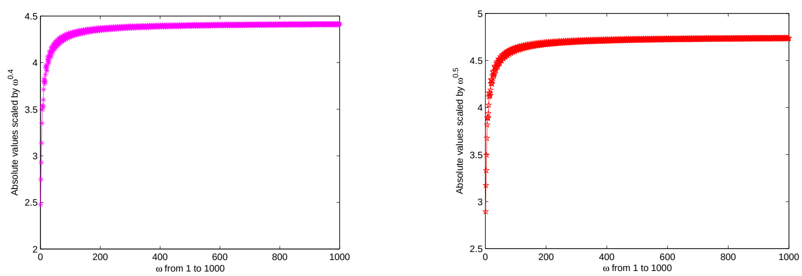
$$L_0[f]=\begin{cases}O\left(\frac{1+\ln(k)}{k^{1+\alpha}}\right) & \nu=0,\\ O\left(\frac{1}{k^{1-|\nu|+\alpha}}\right) & \nu\neq 0,\end{cases}\quad\text{and}\quad L_1[f]=O\left(\frac{1}{k^{1+\beta}}\right),$$

which derives (3.2) directly.

Equation (3.3) can be derived by a similar way to the proof of (3.1). This completes the proof. □

Example 3.2. Let us consider the asymptotics of the integral

$$\widetilde{I}_1(\alpha,\beta,\omega)=\int_0^1 x^{\alpha}(1-x)^{\beta}\mathrm{e}^{\mathrm{i}20x}H_{\nu}^{(1)}(\omega x)\,\mathrm{d}x. \tag{3.4}$$



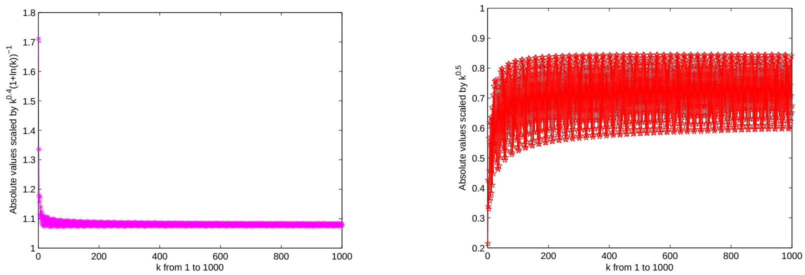
(a) scaled by $\omega^{0.4}$ with $\nu=0$, $\alpha=-0.6$ and $\beta=-0.3$

(b) scaled by $\omega^{0.5}$ with $\nu=0.3$, $\alpha=-0.5$ and $\beta=-0.2$

Figure 3.1: Absolute values of (3.4) when ω runs from 1 to 1000.

Example 3.3. Let us consider the asymptotics of the integral

$$\widetilde{I}_2(\alpha,\beta,k)=\int_0^1 x^{\alpha}(1-x)^{\beta}\mathrm{e}^{\mathrm{i}2kx}H_{\nu}^{(1)}(10x)\,\mathrm{d}x. \tag{3.5}$$



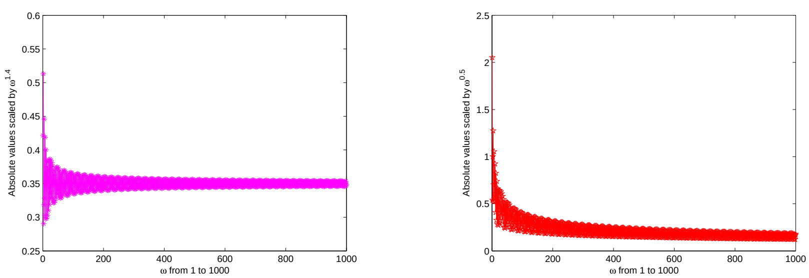
(a) scaled by $k^{0.4}(1+\ln(k))^{-1}$ with $\nu = 0$, $\alpha = -0.6$ and $\beta = -0.3$

(b) scaled by $k^{0.5}$ with $\nu = 0.3$, $\alpha = -0.2$ and $\beta = -0.4$

Figure 3.2: Absolute values of (3.5) when k runs from 1 to 1000.

Example 3.4. Let us consider the asymptotics of the integral

$$\widetilde{I}_3(\alpha, \beta, \omega) = \int_0^1 x^{\alpha}(1-x)^{\beta} \mathrm{e}^{\mathrm{i}\omega x} H_{\nu}^{(1)}(\omega x)\, \mathrm{d}x. \tag{3.6}$$



(a) scaled by $\omega^{1.4}$ with $\nu = 0$, $\alpha = 0.4$ and $\beta = 0.5$

(b) scaled by $\omega^{0.5}$ with $\nu = 0.3$, $\alpha = -0.2$ and $\beta = -0.5$

Figure 3.3: Absolute values of (3.6) when ω runs from 1 to 1000.

From Figures 3.1–3.3, we see that the asymptotic orders on k and ω stated in Theorem 3.1 are attainable.

According to Theorem 3.1, we can easily obtain the error bound for the Clenshaw-Curtis-Filon-type method (2.3), by using the technique of Theorem 3.1 in [40].

Theorem 3.5. *Suppose that $f(x)$ is a sufficiently smooth function on $[0, 1]$, then for each $\alpha - |\nu| > -1$, $\beta > -1$ and fixed N, the error bound on k and ω for the Clenshaw-Curtis-Filon-type method* (2.3) *for the integral* (1.1) *can be estimated by the following three formulas.*

(i) *For fixed k, when $\omega \to \infty$, there holds*

$$I[f] - Q_{N,s}^{CCF}[f] = O\left(\frac{1}{\omega^{s+2+\tau_1}}\right),$$

where $\tau_1 = \min\{\alpha, \beta\}$.

(ii) *For fixed ω, when $k \to \infty$, there holds*

$$I[f] - Q_{N,s}^{CCF}[f] = \begin{cases} O\left(\frac{1+\ln(k)}{k^{s+2+\alpha}}\right) & \nu = 0,\ \alpha \leq \beta, \\ O\left(\frac{1}{k^{s+2+\beta}}\right) & \nu = 0,\ \alpha > \beta, \\ O\left(\frac{1}{k^{s+2+\tau_2}}\right) & \nu \neq 0, \end{cases}$$

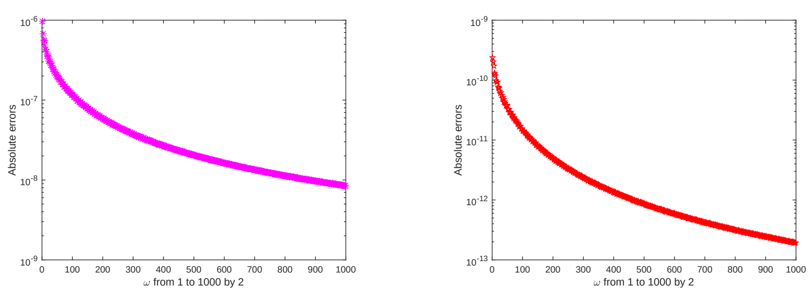
where $\tau_2 = \min\{\alpha - |\nu|, \beta\}$.

(iii) *For a special case that $\omega = 2k$, when $\omega \to \infty$, there holds*

$$I[f] - Q_{N,s}^{CCF}[f] = O\left(\frac{1}{\omega^{s+2+\tau_1}}\right).$$

4. Numerical examples

In this section, we will present several examples to illustrate the efficiency and accuracy of the proposed method. Throughout the paper, all numerical computations were implemented on the R2012a version of the MATLAB system. The experiments were performed on a computer with 3.20 GHz processor and 4 GB of RAM. In addition, the exact values of all the considered integrals $I[f]$ were computed in the MAPLE 17 using 32 decimal digits precision arithmetic.



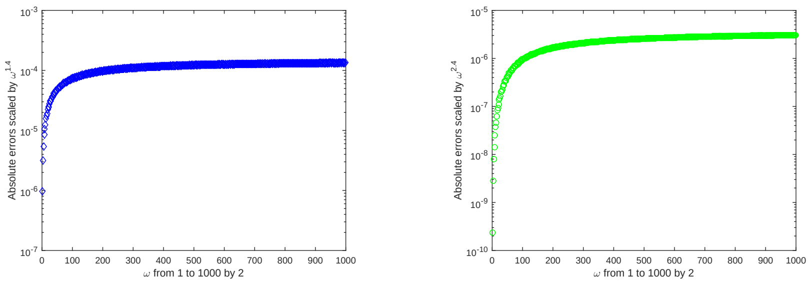
(a) $s = 0$ (b) $s = 1$

Figure 4.1: Absolute errors for the Clenshaw-Curtis-Filon-type method for the integral (4.1) when $N = 4$, $k = 50$, ω from 1 to 1000 by 2.

Example 4.1. Let us consider the computation of the integral

$$\int_0^1 x^\alpha (1-x)^\beta \cos(x) \mathrm{e}^{\mathrm{i}2kx} H_\nu^{(1)}(\omega x)\,\mathrm{d}x \tag{4.1}$$

by the Clenshaw-Curtis-Filon-type method (2.3), where $\nu = 0$, $\alpha = -0.6$ and $\beta = -0.3$. The absolute errors and scaled absolute errors are displayed in Figures 4.1 and 4.2, respectively. Also, the relative errors are displayed in Table 4.1.



(a) scaled by $\omega^{1.4}$ with $s = 0$ (b) scaled by $\omega^{2.4}$ with $s = 1$

Figure 4.2: Absolute errors for the Clenshaw-Curtis-Filon-type method for the integral (4.1) when $N = 4$, $k = 50$, ω from 1 to 1000 by 2.

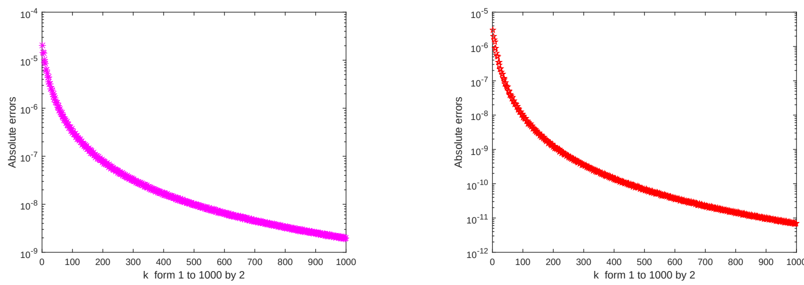
Table 4.1: Relative errors for the integral (4.1) by the Clenshaw-Curtis-Filon-type method with $k = 10$, $N = 2, 4, 6$ and $s = 0, 1, 2$.

s	N	$\omega = 10$	$\omega = 20$	$\omega = 50$
0	2	1.78×10^{-4}	1.35×10^{-4}	7.60×10^{-5}
	4	1.35×10^{-6}	8.93×10^{-7}	5.22×10^{-7}
	6	3.34×10^{-9}	1.97×10^{-9}	1.20×10^{-9}
1	2	3.94×10^{-7}	1.96×10^{-7}	5.32×10^{-8}
	4	1.04×10^{-9}	6.75×10^{-10}	1.71×10^{-10}
	6	1.72×10^{-12}	9.28×10^{-13}	2.49×10^{-13}
2	2	6.56×10^{-10}	2.20×10^{-10}	4.47×10^{-11}
	4	1.48×10^{-12}	3.74×10^{-13}	7.76×10^{-14}
	6	1.89×10^{-15}	6.79×10^{-16}	1.26×10^{-16}
Real Values		0.841824877078759 $-1.172097304662626\,\mathrm{i}$	0.708386698058846 $-0.956797421788702\,\mathrm{i}$	0.517419675175559 $-0.711685588704216\,\mathrm{i}$

Example 4.2. Let us consider the computation of the integral

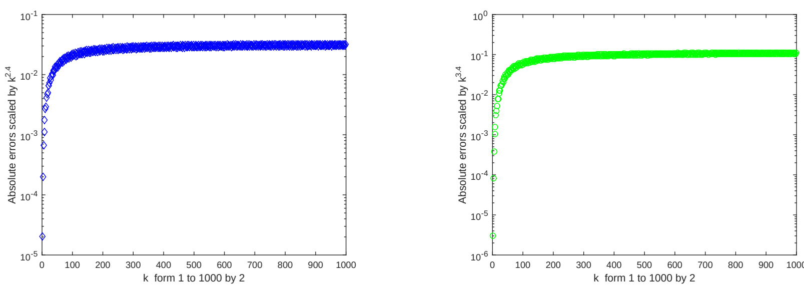
$$\int_0^1 x^\alpha (1-x)^\beta \frac{1}{1+16x^2} \mathrm{e}^{\mathrm{i}2kx} H_\nu^{(1)}(\omega x)\,\mathrm{d}x \tag{4.2}$$

by the Clenshaw-Curtis-Filon-type method (2.3), where $\nu = 0.6$, $\alpha = 0$ and $\beta = -0.3$ (see Figures 4.3, 4.4 and Table 4.2).



(a) $s = 1$ (b) $s = 2$

Figure 4.3: Absolute errors for the Clenshaw-Curtis-Filon-type method for the integral (4.2) when $N = 6$, $\omega = 50$, k from 1 to 1000 by 2.



(a) scaled by $k^{2.4}$ with $s = 1$ (b) scaled by $k^{3.4}$ with $s = 2$

Figure 4.4: Absolute errors for the Clenshaw-Curtis-Filon-type method for the integral (4.2) when $N = 6$, $\omega = 50$, k from 1 to 1000 by 2.

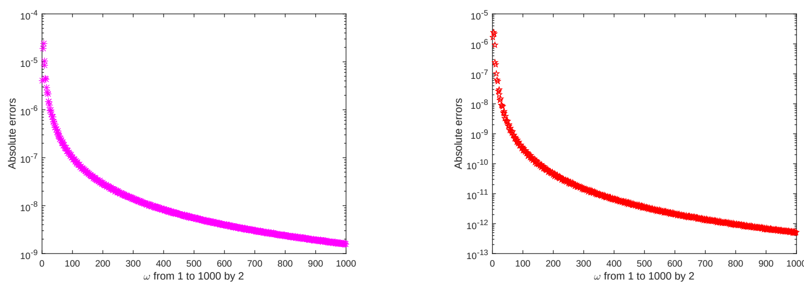
Example 4.3. Finally, we consider the computation of the integral of a special form

$$\int_0^1 x^\alpha (1-x)^\beta \frac{1}{1+(1+x)^2} \mathrm{e}^{\mathrm{i}\omega x} H_\nu^{(1)}(\omega x)\,\mathrm{d}x \tag{4.3}$$

by the Clenshaw-Curtis-Filon-type method (2.3), where $\nu = 0.3$, $\alpha = -0.2$ and $\beta = -0.3$. Figures 4.5 and 4.6 show error bound on ω for the Clenshaw-Curtis-Filon-type method for this case. Table 4.3 displays the relative errors for the proposed method with $N = 3, 6, 9$ and $s = 0, 1, 2$.

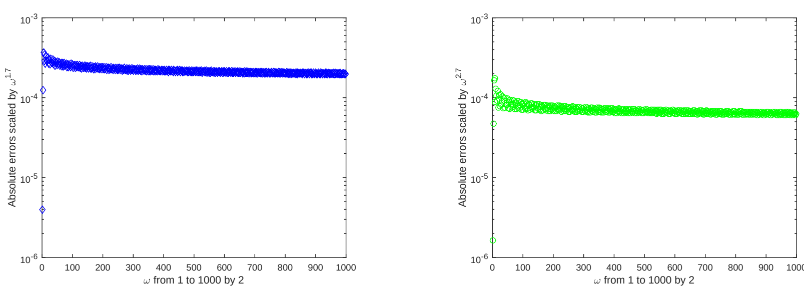
Table 4.2: Relative errors for the integral (4.2) by the Clenshaw-Curtis-Filon-type method with $\omega = 10,\ N = 8, 16, 24$ and $s = 0, 1, 2$.

s	N	$k = 80$	$k = 160$	$k = 320$
0	8	4.36×10^{-4}	2.19×10^{-4}	1.11×10^{-4}
	16	1.51×10^{-6}	8.45×10^{-7}	4.13×10^{-7}
	24	3.11×10^{-9}	1.12×10^{-9}	3.53×10^{-10}
1	8	4.80×10^{-6}	8.59×10^{-7}	2.62×10^{-7}
	16	7.80×10^{-8}	1.48×10^{-8}	3.37×10^{-9}
	24	7.96×10^{-10}	1.28×10^{-10}	2.61×10^{-11}
2	8	7.96×10^{-7}	8.89×10^{-8}	1.19×10^{-8}
	16	8.10×10^{-9}	7.17×10^{-10}	7.77×10^{-11}
	24	2.95×10^{-11}	1.81×10^{-12}	1.37×10^{-13}
Real Values		0.030083151162300 $-0.042241981991079\,\mathrm{i}$	0.023581342870858 $-0.031875514971454\,\mathrm{i}$	0.017909179561849 $-0.024353985798652\,\mathrm{i}$



(a) $s = 0$ (b) $s = 1$

Figure 4.5: Absolute errors for the Clenshaw-Curtis-Filon-type method for the integral (4.3) when $N = 4$, ω from 1 to 1000 by 2.



(a) scaled by $\omega^{1.7}$ with $s = 0$ (b) scaled by $\omega^{2.7}$ with $s = 1$

Figure 4.6: Absolute errors for the Clenshaw-Curtis-Filon-type method for the integral (4.3) when $N = 4$, ω from 1 to 1000 by 2.

Table 4.3: Relative errors for the integral (4.3) by the Clenshaw-Curtis-Filon-type method with $N = 3, 6, 9$ and $s = 0, 1, 2$.

s	N	$\omega = 25$	$\omega = 50$	$\omega = 100$
0	3	2.26×10^{-5}	9.40×10^{-6}	4.04×10^{-6}
	6	1.33×10^{-6}	5.97×10^{-7}	2.75×10^{-7}
	9	2.59×10^{-9}	1.29×10^{-9}	6.98×10^{-10}
1	3	2.03×10^{-6}	4.66×10^{-7}	1.11×10^{-7}
	6	5.78×10^{-10}	1.60×10^{-10}	2.41×10^{-11}
	9	4.42×10^{-11}	1.32×10^{-11}	2.82×10^{-12}
2	3	1.25×10^{-8}	1.98×10^{-9}	2.74×10^{-10}
	6	1.86×10^{-10}	2.26×10^{-11}	2.37×10^{-12}
	9	2.11×10^{-13}	7.48×10^{-15}	2.98×10^{-15}
Real Values		0.030229145167903 $-0.034246416918332\,\mathrm{i}$	0.017639904837672 $-0.019163197919570\,\mathrm{i}$	0.010310330002264 $-0.010688289764988\,\mathrm{i}$

Form Figures 4.2, 4.4, 4.6, we can see that the error bounds given in Theorem 3.5 for the Clenshaw-Curtis-Filon-type method are attainable. Figures 4.1, 4.3, 4.5 and Tables 4.1–4.3 show that the presented method is very efficient for the approximation of the integral (1.1). Moreover, for the well-behaved function $f(x)$, the integral (1.1) can be efficiently approximated by Clenshaw-Curtis-Filon-type method with a small number of interpolation points. In addition, the improvement of the accuracy for the integral (1.1) can be obtained by using interpolation with derivatives of higher order at two endpoints, or adding the number of the interpolation points.

5. Concluding remarks

In this paper, we consider a Clenshaw-Curtis-Filon-type method for the computation of the integral (1.1) with $(N+1)$ Clenshaw-Curtis points, which can be efficiently implemented in $O(N \log N)$ operations. Moreover, we present a universal method for the derivation of the recurrence relation for the modified moments, which can be applied to the modified moments with other type kernels. Based on this recurrence relation, the modified moments can be efficiently computed by using special functions and Gauss-type method with small number of points. Finally, an error bound on k and ω and several numerical experiments are given to show the accuracy and efficiency for the proposed method.

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