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The Spectral Method for Long-time Behavior of a Fractional Power Dissipative System

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Abstract. In this paper, we consider the fractional complex Ginzburg-Landau equation in two spatial dimensions with the dissipative effect given by a fractional Laplacian. The periodic initial value problem of the fractional complex Ginzburg-Landau equation is discretized fully by Galerkin-Fourier spectral method, and the dynamical behaviors of the discrete system are studied. The existence and convergence of global attractors of the discrete system are obtained by a priori estimates and error estimates of the discrete solution. The numerical stability and convergence of the discrete scheme are proved.

1. Introduction

Fractional differential equations have a wide range of applications in physics, biology, chemistry and other fields of science, such as kinetic theories of systems with chaotic dynamics [20, 30], pseudochaotic dynamics [32], dynamics in a complex or porous medium [18, 25], random walks with a memory and flights [24, 31], obstacle problems [2, 21]. Recently, some of the classical equations of mathematical physics have been postulated with fractional derivatives to better describe complex phenomena (e.g., [7, 10–12, 22, 26]).

The Ginzburg-Landau equation [8, 9] is one of the most-studied nonlinear equations in physics. It describes a vast variety of phenomena from nonlinear waves to second-order phase transitions, from superconductivity, superfluidity, and Bose-Einstein condensation to liquid crystals and strings in field theory. The Ginzburg-Landau equation with *fractional derivatives* was suggested in [29] and studied in [26, 27], where it is used to describe processes in media with fractal dispersion or long-range interaction.

In this work, we consider the following fractional complex Ginzburg-Landau equation [26]:

$$u_t = \rho u - (1 + \mathrm{i}\nu)(-\triangle)^\alpha u - (1 + \mathrm{i}\mu)|u|^{2\sigma} u, \quad x \in \mathbb{R}^2,\ t > 0 \tag{1.1}$$

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with the initial condition and the periodic boundary condition:

$$u(x,0) = u_0, \qquad x \in \mathbb{R}^2, \tag{1.2}$$

$$u(x + 2\pi \mathbf{e}_i, t) = u(x,t), \qquad x \in \mathbb{R}^2,\ t > 0,\ i = 1,2, \tag{1.3}$$

where $\mathbf{e}_i$ $(i = 1,2)$ is an orthonormal basis of $\mathbb{R}^2$. In (1.1), i is the imaginary unit, ν, μ, ρ are real constants, and $\rho > 0$, $\sigma > 0$, $\alpha \in (1/2, 1)$.

We would like to point out that the standard complex Ginzburg-Landau equation ($\alpha = 1$ in (1.1))

$$u_t = \rho u + (1 + \mathrm{i}\nu)\triangle u - (1 + \mathrm{i}\mu)|u|^{2\sigma} u \tag{1.4}$$

has been the object of intense study (see [1, 5, 6, 11, 14–16, 19]).

In a recent paper [13], the authors studied (1.1)–(1.3) with spatial dimension *two* and with the special pure power nonlinearity. They proved the well-posedness and studied the asymptotic behavior of the solutions, proving the existence of the global attractor. Estimates of the Hausdorff and fractal dimensions for the global attractor were also obtained.

However, these studies depended on the results of numerical experimentation to a great extent. Thus, it is worth studying whether the numerical results are reliable and the calculation schemes are suitable. In this paper, we construct a fully discrete classical Galerkin spectral scheme, which is a nonlinear scheme. We obtain the existence and convergence of global attractors of the discrete system by a priori estimates and error estimates of the discrete solution. Then we prove the numerical stability and convergence of the discrete scheme.

Let $\Omega = [0, 2\pi] \times [0, 2\pi] \subset \mathbb{R}^2$. Throughout this paper, we denote by $(\,\cdot\,,\cdot\,)$ the usual inner product of $L^2(\Omega)$, $\|\cdot\|_{H^m}$ the norm of Sobolev spaces $H^m(\Omega)$, and $\|\cdot\|_m = \|\cdot\|_{L^m(\Omega)}$ $(m = 1, 2, \ldots, \infty)$. Let $L^2_p(\Omega) = \{\varphi \in L^2(\Omega) \mid \varphi(x + 2\pi\mathbf{e}_i) = \varphi(x), i = 1,2\}$ with the norm defined just as that of $L^2(\Omega)$. Let $H^m_p(\Omega) = \{\varphi \in H^m(\Omega) \mid \varphi(x + 2\pi\mathbf{e}_i) = \varphi(x), i = 1,2\}$ with the norm defined just as that of $H^m(\Omega)$.

For any given positive integer N, let $S_N = \mathrm{Span}\{e^{\mathrm{i}k\cdot x} : |k| \leq N\}$ and denote by $P_N \colon L^2_P(\Omega) \to S_N$ the orthogonal projection operator [3].

Let τ be the mesh size in the variable t, $t_k = k\tau$, $u^k = u(x, t_k)$, $\overline{\partial}_t u^k = \frac{1}{\tau}(u^k - u^{k-1})$. We construct the Fourier spectral scheme for solving problem (1.1)–(1.3) as follows: to find $u^k_N \in S_N$ such that

$$\big(\overline{\partial}_t u^k_N - \rho u^k_N + (1 + \mathrm{i}\nu)(-\triangle)^\alpha u^k_N + (1 + \mathrm{i}\mu)|u^k_N|^{2\sigma} u^k_N, \varphi\big) = 0, \quad \forall\, \varphi \in S_N,\ k \geq 1, \tag{1.5}$$

$$u^0_N = P_N u_0. \tag{1.6}$$

It is a nonlinear iteration scheme, and by applying the fixed point theorem we can prove that there exists a unique solution u^k_N for (1.5)–(1.6).

We remark that in our work we consider periodic boundary conditions, however, we did not provide detailed justification of how to specify our boundary conditions. For those who want to learn more details, please refer to [4].

The rest of this paper is organized as follows. In Section 2, some preliminaries and notations are shown. In Section 3, the existence of discrete attractors $\mathcal{A}_N^\tau$ is obtained by a t-independent priori estimates of discrete solutions. In Section 4, the convergence of discrete attractors $\mathcal{A}_N^\tau$ is proved by the error estimates of the discrete solutions. In Section 5, the numerical stability of the discrete scheme is shown.

2. Preliminaries and notations

If u is smooth and 2π-periodic in each of the two coordinates, it can be expressed by a Fourier series $u = \sum_{k\in\mathbb{Z}^2} u_k e^{\mathrm{i}k\cdot x}$. It follows that $u_{x_i} = \sum_{k\in\mathbb{Z}^2} \mathrm{i}k_i u_k e^{\mathrm{i}k\cdot x}$ $(i = 1, 2)$, and $(-\triangle)^\alpha$ is defined by

$$(-\triangle)^\alpha u = \sum_{k\in\mathbb{Z}^2} |k|^{2\alpha} u_k e^{\mathrm{i}k\cdot x}.$$

Let $H^\beta = H^\beta(\Omega)$ denote the complete Sobolev space of order β under the norm:

$$\|u\|_{H^\beta} = \left(\sum_{k\in\mathbb{Z}^2} |k|^{2\beta}|u_k|^2 + \sum_{k\in\mathbb{Z}^2} |u_k|^2\right)^{1/2}.$$

We denote by H_p^β those functions that are 2π-periodic in all the coordinate variables and when restricted to Ω, lie in $H^\beta(\Omega)$. Throughout this paper, we denote by $(\,\cdot\,,\cdot\,)$ the usual inner product in $L^2 = L^2(\Omega;\mathbb{C})$, $\|\cdot\|_{H^m}$ the norm of Sobolev space $H^m(\Omega)$, and $\|\cdot\|_q = \|\cdot\|_{L^q(\Omega)}$, $1 \le q \le \infty$. In the forthcoming discussion, we use T to denote an arbitrary positive constant, and use c_j $(j = 1, 2, \dots)$ to denote different positive constants which depend only on the constants ρ, ν, μ, α, and σ. In addition, the following Gagliardo-Nirenberg inequality [17] is frequently used.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain having the cone property and let $u \in L^q(\Omega)$ and its derivatives of order m, $D^m u$, belong to $L^r(\Omega)$, $1 \le q, r \le \infty$. For the derivatives $D^j u$, $0 \le j < m$, the following inequalities hold*

$$\|D^j u\|_{L^p} \le c(\|D^m u\|_{L^r} + \|u\|_{L^q})^\theta \|u\|_{L^q}^{1-\theta}, \tag{2.1}$$

where

$$\frac{1}{p} = \frac{j}{n} + \theta\left(\frac{1}{r} - \frac{m}{n}\right) + (1-\theta)\frac{1}{q}$$

for all θ in the interval

$$\frac{j}{m} \le \theta \le 1,$$

(the constant c depending only on n, m, j, q, r, and θ), with the following exceptional case:

If $1 < r < \infty$, and $m - j - n/r$ is a nonnegative integer then (2.1) holds only for θ satisfying $j/m \le \theta < 1$.

For the orthogonal projection operator P_N, we have the following estimate [3].

Lemma 2.2. *If $u \in H_p^m(\Omega)$, then there exists a constant c independent of u and N such that*

$$\|u - P_N u\|_{H^s} \le c N^{s-m} \|D^m u\| \quad \text{for all } 0 \le s \le m.$$

The following lemmas are also used in this paper [23].

Lemma 2.3 (Discrete Gronwall's inequality)**.** *Let y^k, g^k, h^k be three nonnegative series satisfying*

$$\frac{y^{k+1} - y^k}{\tau} \le g^k y^k + h^k, \quad \forall\, k.$$

Then $\forall\, n > 0$, we have

$$y^n \le y^0 \exp\left(\tau \sum_{k=0}^{n} g^k\right) + \tau \sum_{k=0}^{n} h^k \exp\left(\tau \sum_{i=k}^{n} g^i\right) \quad \text{for all } k \le n+1.$$

Lemma 2.4 (Discrete uniform Gronwall's inequality)**.** *Let y^k, g^k, h^k be three nonnegative series satisfying*

$$\frac{y^{k+1} - y^k}{\tau} \le g^k y^k + h^k, \quad \forall\, k \ge k_0$$

and

$$\tau \sum_{k=k_1}^{n_0+k_1} g^k \le \alpha_1, \quad \tau \sum_{k=k_1}^{n_0+k_1} h^k \le \alpha_2, \quad \tau \sum_{k=k_1}^{n_0+k_1} y^k \le \alpha_3 \quad \text{for all } k_1 \ge k_0,$$

with $\tau n_0 = r$. Then

$$y^k \le \left(\frac{\alpha_3}{r} + \alpha_2\right) e^{\alpha_1} \quad \text{for all } k \ge n_0 + k_0.$$

In this paper, to establish the existence of the global attractor of (1.1)–(1.3), we need the following results [28].

Theorem 2.5. *Suppose that H is a Banach space, and $\{S(t)\}_{t\ge 0}$ is a semigroup of continuous operators, that map H into itself and enjoy the usual semigroup properties:*

$$S(t) \cdot S(\tau) = S(t+\tau), \quad S(0) = I,$$

where I is the identity operator. We also suppose that the operator $S(t)$ satisfies

(i) *operator $S(t)$ is bounded, i.e., for any given $R>0$, if $\|u_0\|_H \le R$, then there exists a constant $C(R)$ such that*

$$\|S(t)u_0\|_H \le C(R) \quad \textit{for } t \in [0,\infty);$$

(ii) *There is a bounded absorbing set $\mathcal{B}_1 \subset H$, i.e., for any given bounded set $\mathcal{B} \subset H$, there exists a constant $T = T(\mathcal{B})$ such that*

$$S(t)\mathcal{B} \subset \mathcal{B}_1 \quad \textit{for } t \ge T;$$

(iii) *The operator $S(t)$ is a uniformly compact for $t>0$ sufficiently large. By this mean that for every bounded set $\mathcal{B}$ there exists a constant $t_0 = t_0(\mathcal{B})$ such that*

$$\bigcup_{t \ge t_0} S(t)\mathcal{B}$$

is relatively compact in H.

Then the semigroup $\{S(t)\}_{t\ge 0}$ of operators has a compact global attractor $\mathcal{A} \subset H$. By this we mean that

(a) *$S(t)\mathcal{A} = \mathcal{A}$ for all $t \ge 0$.*

(b) *For any given bounded set $\mathcal{B} \subset H$, $\lim_{t\to\infty} \operatorname{dist}(S(t)\mathcal{B}, \mathcal{A}) = 0$, where*

$$\operatorname{dist}(X,Y) = \sup_{x\in X} \inf_{y \in Y} \|x-y\|_H.$$

3. Existence of approximation global attractors

We first obtain a priori estimates of the problem (1.5)–(1.6). In what follows, we denote $\int_\Omega f\,dx$ by the notation $\int f$.

Lemma 3.1. *Suppose that $u_0 \in L_p^2(\Omega)$, then for the solution u_N^n of (1.5)–(1.6), we have*

$$\|u_N^n\|^2 + \tau \sum_{k=1}^{n} (1+\rho\tau)^{k-1-n} \left(2\|(-\triangle)^{\alpha/2}u_N^k\|^2 + \rho\|u_N^k\|^2 + \|u_N^k\|_{2\sigma+2}^{2\sigma+2}\right) \le E_0$$

and

$$\overline{\lim_{n\to\infty}} \left(\|u_N^n\|^2 + \tau \sum_{k=1}^{n} (1+\rho\tau)^{k-1-n} \left(2\|(-\triangle)^{\alpha/2}u_N^k\|^2 + \rho\|u_N^k\|^2 + \|u_N^k\|_{2\sigma+2}^{2\sigma+2}\right)\right) \le \delta_0,$$

where the constant $\delta_0 > 0$ is independent of n, τ and $\|u_0\|$, $E_0 = E_0(\|u_0\|) > 0$ independent of n, τ.

Proof. Letting $\varphi = u_N^k$ in (1.5) and taking the real part, we obtain

$$\frac{1}{2}\overline{\partial}_t\|u_N^k\|^2 + \frac{\tau}{2}\|\overline{\partial}_t u_N^k\|^2 - \rho\|u_N^k\|^2 + \|(-\triangle)^{\alpha/2}u_N^k\|^2 + \|u_N^k\|_{2\sigma+2}^{2\sigma+2} = 0. \tag{3.1}$$

Applying Young's inequality, we see that

$$4\rho\|u_N^k\|^2 = 4\rho\int |u_N^k|^2 \le \|u_N^k\|_{2\sigma+2}^{2\sigma+2} + \rho\frac{4\sigma}{\sigma+1}\left(\frac{4\rho}{\sigma+1}\right)^{1/\sigma}|\Omega|.$$

Then (3.1) can be rewritten as

$$\begin{aligned}&\overline{\partial}_t\|u_N^k\|^2 + 2\|(-\triangle)^{\alpha/2}u_N^k\|^2 + 2\rho\|u_N^k\|^2 + \|u_N^k\|_{2\sigma+2}^{2\sigma+2}\\ &\le \rho\frac{4\sigma}{\sigma+1}\left(\frac{4\rho}{\sigma+1}\right)^{1/\sigma}|\Omega| = \rho\delta_0.\end{aligned} \tag{3.2}$$

Multiplying (3.2) by $(1+\rho\tau)^{k-1}$, and summing them for k from 1 to n, we have

$$\begin{aligned}&\|u_N^n\|^2 + \tau\sum_{k=1}^{n}(1+\rho\tau)^{k-1-n}\left(2\|(-\triangle)^{\alpha/2}u_N^k\|^2 + \rho\|u_N^k\|^2 + \|u_N^k\|_{2\sigma+2}^{2\sigma+2}\right)\\ &\le (1+\rho\tau)^{-n}\|u_0\|^2 + \delta_0.\end{aligned} \tag{3.3}$$

Let $E_0 = \|u_0\|^2 + \delta_0$, then (3.3) implies

$$\|u_N^n\|^2 + \tau\sum_{k=1}^{n}(1+\rho\tau)^{k-1-n}\left(2\|(-\triangle)^{\alpha/2}u_N^k\|^2 + \rho\|u_N^k\|^2 + \|u_N^k\|_{2\sigma+2}^{2\sigma+2}\right) \le E_0.$$

Therefore,

$$\overline{\lim_{n\to\infty}}\left(\|u_N^n\|^2 + \tau\sum_{k=1}^{n}(1+\rho\tau)^{k-1-n}\left(2\|(-\triangle)^{\alpha/2}u_N^k\|^2 + \rho\|u_N^k\|^2 + \|u_N^k\|_{2\sigma+2}^{2\sigma+2}\right)\right) \le \delta_0.$$

This completes the proof. □

Following from the above lemma, one has

Corollary 3.2. *For any $\widehat{\delta}_0 > \delta_0$ and $R > 0$, if $\|u_0\|^2 \le R$, then*

$$\|u_N^n\|^2 \le \widehat{\delta}_0 \quad \text{for all } n \ge n_0 = \frac{\ln\left(R/(\widehat{\delta}_0 - \delta_0)\right)}{\ln(1+\rho\tau)}.$$

Lemma 3.3. *Suppose that $u_0 \in H_p^1(\Omega)$, and σ satisfies the following condition*

$$\sigma \le \frac{1}{\sqrt{1+\mu^2}-1},$$

then for the solution u_N^k *of* (1.5)–(1.6)*, we have*

$$\|\nabla u_N^n\|^2 + \tau \sum_{k=1}^{n} (1+\rho\tau)^{k-1-n} \|(-\triangle)^{(\alpha+1)/2} u_N^k\|^2 \le E_1 \quad \textit{for all } n \ge 1,$$

and

$$\overline{\lim_{n\to\infty}} \left(\|\nabla u_N^n\|^2 + \tau \sum_{k=1}^{n} (1+\rho\tau)^{k-1-n} \|(-\triangle)^{(\alpha+1)/2} u_N^k\|^2 \right) \le \delta_1,$$

where the constant $\delta_1 > 0$ *is independent of* n*,* τ *and* $\|u_0\|$*,* $E_1 = E_1(\|u_0\|_{H^1}) > 0$ *independent of* n*,* τ*.*

Proof. Setting $\varphi = -\triangle u_N^k$ in (1.5) and taking the real part, we obtain

$$\begin{aligned} &\frac{1}{2}\overline{\partial}_t \|\nabla u_N^k\|^2 + \frac{\tau}{2}\|\overline{\partial}_t \nabla u_N^k\|^2 - \rho\|\nabla u_N^k\|^2 \\ &+ \|(-\triangle)^{(\alpha+1)/2} u_N^k\|^2 - \mathrm{Re}(1+\mathrm{i}\mu)(|u_N^k|^{2\sigma} u_N^k, \triangle u_N^k) = 0. \end{aligned} \tag{3.4}$$

Integrating by parts, we infer that

$$\begin{aligned} &- \mathrm{Re}(1+\mathrm{i}\mu)(|u_N^k|^{2\sigma} u_N^k, \triangle u_N^k) \\ &= \mathrm{Re}(1+\mathrm{i}\mu) \int \Big((\sigma+1)|u_N^k|^{2\sigma} |\nabla u_N^k|^2 + \sigma |u_N^k|^{2(\sigma-1)} (u_N^k \nabla \overline{u}_N^k)^2 \Big) \\ &= \frac{1}{2} \int |u_N^k|^{2(\sigma-1)} \Big(2(\sigma+1)|u_N^k|^2 |\nabla u_N^k|^2 + \sigma(1+\mathrm{i}\mu)(u_N^k \nabla \overline{u}_N^k)^2 + \sigma(1-\mathrm{i}\mu)(\overline{u}_N^k \nabla u_N^k)^2 \Big) \\ &= \frac{1}{2} \int |u_N^k|^{2(\sigma-1)} Y M Y^H, \end{aligned}$$

where

$$Y = \begin{pmatrix} \overline{u}_N^k \nabla u_N^k \\ u_N^k \nabla \overline{u}_N^k \end{pmatrix}^H, \quad M = \begin{pmatrix} \sigma+1 & \sigma(1+\mathrm{i}\mu) \\ \sigma(1-\mathrm{i}\mu) & \sigma+1 \end{pmatrix},$$

and Y^H is the conjugate transpose of the matrix Y. We observe that the condition

$$\sigma \le \frac{1}{\sqrt{1+\mu^2}-1}$$

implies that the matrix M is nonnegative definite. Then (3.4) can be rewritten as

$$\frac{1}{2}\overline{\partial}_t \|\nabla u_N^k\|^2 + \frac{\tau}{2}\|\overline{\partial}_t \nabla u_N^k\|^2 - \rho\|\nabla u_N^k\|^2 + \|(-\triangle)^{(\alpha+1)/2} u_N^k\|^2 \le 0. \tag{3.5}$$

Using Gagliardo-Nirenberg inequality, we deduce that

$$3\rho\|\nabla u_N^k\|^2 \le \|(-\triangle)^{(\alpha+1)/2} u_N^k\|^2 + c\|u_N^k\|^2. \tag{3.6}$$

Combining (3.5) and (3.6), we infer that

$$\overline{\partial}_t \|\nabla u_N^k\|^2 + \|(-\triangle)^{(\alpha+1)/2} u_N^k\|^2 + \rho\|\nabla u_N^k\|^2 \le c\|u_N^k\|^2. \tag{3.7}$$

Multiplying (3.7) by $(1+\rho\tau)^{k-1}$, summing them for k from 1 to n, and applying Lemma 3.1, we deduce that

$$\begin{aligned}
&\|\nabla u_N^n\|^2+\tau\sum_{k=1}^{n}(1+\rho\tau)^{k-1-n}\|(-\triangle)^{(\alpha+1)/2}u_N^k\|^2\\
&\le(1+\rho\tau)^{-n}\|\nabla u_0\|^2+c\tau\sum_{k=1}^{n}(1+\rho\tau)^{k-1-n}\|u_N^k\|^2\\
&\le(1+\rho\tau)^{-n}\|\nabla u_0\|^2+\frac{c}{\rho}\big((1+\rho\tau)^{-n}\|u_0\|^2+\delta_0\big)\\
&\le(1+\rho\tau)^{-n}\left(\|\nabla u_0\|^2+\frac{c}{\rho}\|u_0\|^2\right)+\frac{c}{\rho}\delta_0.
\end{aligned}$$

Let $E_1=\|\nabla u_0\|^2+\frac{c}{\rho}(\|u_0\|^2+\delta_0)$ and $\delta_1=\frac{c}{\rho}\delta_0$. Thus we obtain that

$$\|\nabla u_N^n\|^2+\tau\sum_{k=1}^{n}(1+\rho\tau)^{k-1-n}\|(-\triangle)^{(\alpha+1)/2}u_N^k\|^2\le E_1\quad\text{for all } n\ge 1$$

and

$$\varlimsup_{n\to\infty}\left(\|\nabla u_N^n\|^2+\tau\sum_{k=1}^{n}(1+\rho\tau)^{k-1-n}\|(-\triangle)^{(\alpha+1)/2}u_N^k\|^2\right)\le\delta_1,$$

which completes the proof of this lemma. □

By the above lemma, we obtain the following corollary:

Corollary 3.4. *For any $\widehat{\delta}_1>\delta_1$ and $R>0$, if $\|u_0\|_{H^1}^2\le R$, then there exists $n_1=n_1(R)>n_0$ such that*

$$\|\nabla u_N^n\|^2\le\widehat{\delta}_1\quad\textit{for all } n\ge n_1.$$

Lemma 3.5. *Suppose that $u_0\in H_p^{1+\alpha}(\Omega)$, then for the solution u_N^k of (1.5)–(1.6), we have*

$$\|(-\triangle)^{(1+\alpha)/2}u_N^n\|^2+\tau\sum_{k=1}^{n}(1+\rho\tau)^{k-1-n}\|(-\triangle)^{1/2+\alpha}u_N^k\|^2\le E_2$$

and

$$\varlimsup_{n\to\infty}\left(\|(-\triangle)^{(1+\alpha)/2}u_N^n\|^2+\tau\sum_{k=1}^{n}(1+\rho\tau)^{k-1-n}\|(-\triangle)^{1/2+\alpha}u_N^k\|^2\right)\le\delta_2,$$

where the constant $\delta_2>0$ is independent of n, τ and $\|u_0\|_{H^{1+\alpha}}$, $E_2=E_2(\|u_0\|_{H^{1+\alpha}})>0$ independent of n, τ.

Proof. Setting $\varphi=(-\triangle)^{1+\alpha}u_N^k$ in (1.5) and taking the real part, we obtain

$$\begin{aligned}
&\overline{\partial}_t\|(-\triangle)^{(1+\alpha)/2}u_N^k\|^2+\tau\|\overline{\partial}_t(-\triangle)^{(1+\alpha)/2}u_N^k\|^2\\
&\quad-2\rho\|(-\triangle)^{(1+\alpha)/2}u_N^k\|^2+2\|(-\triangle)^{\alpha+1/2}u_N^k\|^2\\
&=-2\,\mathrm{Re}(1+\mathrm{i}\mu)(|u_N^k|^{2\sigma}u_N^k,(-\triangle)^{1+\alpha}u_N^k).
\end{aligned}\tag{3.8}$$

Using Hölder inequality, Gagliardo-Nirenberg inequality and Young's inequality, we infer that, when $\sigma < 1+\alpha$, we have

$$\begin{aligned}
&-2\operatorname{Re}(1+\mathrm{i}\mu)(|u_N^k|^{2\sigma}u_N^k,(-\triangle)^{1+\alpha}u_N^k)\\
&\le 2\sqrt{1+\mu^2}\left|\left(\nabla(|u_N^k|^{2\sigma}u_N^k),(-\triangle)^{1/2+\alpha}u_N^k\right)\right|\\
&\le 2(1+2\sigma)\sqrt{1+\mu^2}\int |u_N^k|^{2\sigma}|\nabla u_N^k||(-\triangle)^{1/2+\alpha}u_N^k|\\
&\le 2(1+2\sigma)\sqrt{1+\mu^2}\|(-\triangle)^{1/2+\alpha}u_N^k\|\|\nabla u_N^k\|\|u_N^k\|_\infty^{2\sigma}\\
&\le \frac{1}{2}\|(-\triangle)^{1/2+\alpha}u_N^k\|^2+c\|\nabla u_N^k\|^2\|u_N^k\|_\infty^{4\sigma}\\
&\le \frac{1}{2}\|(-\triangle)^{1/2+\alpha}u_N^k\|^2\\
&\quad+c\|\nabla u_N^k\|^2\left(\|(-\triangle)^{(1+\alpha)/2}u_N^k\|+\|u_N^k\|\right)^{4\sigma/(\alpha+1)}\|u_N^k\|^{4\sigma\alpha/(\alpha+1)}\\
&\le \frac{1}{2}\|(-\triangle)^{1/2+\alpha}u_N^k\|^2+c\|\nabla u_N^k\|^2\|(-\triangle)^{(1+\alpha)/2}u_N^k\|^4\\
&\quad+c\|\nabla u_N^k\|^2\left(\|u_N^k\|^{4\sigma\alpha/(\alpha+1-\sigma)}+\|u_N^k\|^4\right)
\end{aligned}\tag{3.9}$$

and

$$\begin{aligned}
3\rho\|(-\triangle)^{(1+\alpha)/2}u_N^k\|^2 &\le c\|u_N^k\|_{H^{1+2\alpha}}^{2(1+\alpha)/(2\alpha+1)}\|u_N^k\|^{2\alpha/(2\alpha+1)}\\
&\le \frac{1}{2}\|(-\triangle)^{1/2+\alpha}u_N^k\|^2+c\|u_N^k\|^2.
\end{aligned}\tag{3.10}$$

Combining (3.8)–(3.10), we deduce that

$$\begin{aligned}
&\overline{\partial}_t\|(-\triangle)^{(1+\alpha)/2}u_N^k\|^2+\rho\|(-\triangle)^{(1+\alpha)/2}u_N^k\|^2+\|(-\triangle)^{1/2+\alpha}u_N^k\|^2\\
&\le g^k\|(-\triangle)^{(1+\alpha)/2}u_N^k\|^2+h^k,
\end{aligned}\tag{3.11}$$

where

$$g^k=c\|\nabla u_N^k\|^2\|(-\triangle)^{(1+\alpha)/2}u_N^k\|^2,\quad h^k=c\|\nabla u_N^k\|^2\left(\|u_N^k\|^{4\sigma\alpha/(\alpha+1-\sigma)}+\|u_N^k\|^4\right).$$

Applying (3.7) of Lemma 3.3, Corollaries 3.2 and 3.4, for any $n\ge n_1$ and any given $r>0$, k_0 satisfying $k_0\tau=r$, we obtain that

$$\begin{aligned}
\tau\sum_{k=n}^{n+k_0}g^k&=c\tau\sum_{k=n}^{n+k_0}\|\nabla u_N^k\|^2\|(-\triangle)^{(1+\alpha)/2}u_N^k\|^2\\
&\le c\widehat{\delta}_1\left(c\tau\sum_{k=n}^{n+k_0}\|u_N^k\|^2+\|\nabla u_N^{n-1}\|^2\right)\le c\widehat{\delta}_1(cr\widehat{\delta}_0+\widehat{\delta}_1)\triangleq\alpha_1,\\
\tau\sum_{k=n}^{n+k_0}h^k&=c\tau\sum_{k=n}^{n+k_0}\|\nabla u_N^k\|^2\left(\|u_N^k\|^{4\sigma\alpha/(\alpha+1-\sigma)}+\|u_N^k\|^4\right)\\
&\le c\widehat{\delta}_1 r\left(\widehat{\delta}_0^{2\sigma\alpha/(\alpha+1-\sigma)}+\widehat{\delta}_0^2\right)\triangleq\alpha_2
\end{aligned}$$

and

$$\tau\sum_{k=n}^{n+k_0} y^k = c\tau\sum_{k=n}^{n+k_0}\|(-\triangle)^{(1+\alpha)/2}u_N^k\|^2 \le c\tau\sum_{k=n}^{n+k_0}\|u_N^k\|^2 + \|\nabla u_N^{n-1}\|^2 \le cr\widehat{\delta}_0 + \widehat{\delta}_1 \triangleq \alpha_3.$$

Applying Lemma 2.4, we obtain that

$$\|(-\triangle)^{(1+\alpha)/2}u_N^k\|^2 \le \left(\frac{\alpha_3}{r} + \alpha_2\right)e^{\alpha_1} \triangleq \widehat{\delta}_2 \quad \text{for all } n \ge \widehat{n}_2 = n_1 + k_0. \tag{3.12}$$

For $n \le \widehat{n}_2$, using Lemma 2.3 for (3.11) and applying Lemmas 3.1 and 3.3, we have

$$\begin{aligned}\|(-\triangle)^{(1+\alpha)/2}u_N^n\|^2 &\le \|(-\triangle)^{(1+\alpha)/2}u_0\|^2 e^{ct_n E_1(E_0+E_1)} \\ &\quad + ct_n E_1\left(E_0^{2\sigma\alpha/(\alpha+1-\sigma)} + E_0^2\right)e^{ct_{n-k}E_1(E_0+E_1)} \\ &\triangleq \widetilde{E}_2.\end{aligned} \tag{3.13}$$

Let $E_2 = \max\left\{\widehat{\delta}_2, \widetilde{E}_2\right\}$, then from (3.12) and (3.13) we deduce that

$$\|(-\triangle)^{(1+\alpha)/2}u_N^n\|^2 \le E_2 \quad \text{for all } n \ge 1.$$

Applying (3.11), above equality and Lemma 3.3, we deduce that

$$\begin{aligned}&(1+\rho\tau)\|(-\triangle)^{(1+\alpha)/2}u_N^k\|^2 - \|(-\triangle)^{(1+\alpha)/2}u_N^{k-1}\|^2 + \tau\|(-\triangle)^{1/2+\alpha}u_N^k\|^2 \\ &\le \tau g^k\|(-\triangle)^{(1+\alpha)/2}u_N^k\|^2 + \tau h^k \le \widehat{c}\tau,\end{aligned} \tag{3.14}$$

where $\widehat{c} = cE_1(E_2^2 + E_0^{2\sigma\alpha/(\alpha+1-\sigma)} + E_0^2)$.

Multiplying (3.14) by $(1+\rho\tau)^{k-1}$, summing them for k from 1 to n, and applying Lemmas 3.1 and 3.3, we have

$$\begin{aligned}&\|(-\triangle)^{(1+\alpha)/2}u_N^n\|^2 + \tau\sum_{k=1}^{n}(1+\rho\tau)^{k-1-n}\|(-\triangle)^{1/2+\alpha}u_N^k\|^2 \\ &\le (1+\rho\tau)^{-n}\|(-\triangle)^{(1+\alpha)/2}u_0\|^2 + \widehat{c}\tau\sum_{k=1}^{n}(1+\rho\tau)^{k-1-n} \\ &\le (1+\rho\tau)^{-n}\|(-\triangle)^{(1+\alpha)/2}u_0\|^2 + \frac{\widehat{c}}{\rho}.\end{aligned}$$

It follows that

$$\overline{\lim_{n\to\infty}}\left(\|(-\triangle)^{(1+\alpha)/2}u_N^n\|^2 + \tau\sum_{k=1}^{n}(1+\rho\tau)^{k-1-n}\|(-\triangle)^{1/2+\alpha}u_N^k\|^2\right) = \frac{\widehat{c}}{\rho} := \delta_2. \tag{3.15}$$

The proof of is completed. □

Corollary 3.6. *For any $\widehat{\delta}_2 > \delta_2$ and $R > 0$, if $\|u_0\|^2_{H^{1+\alpha}} \le R$, then*

$$\|(-\triangle)^{(1+\alpha)/2}u_N^n\|^2 \le \widehat{\delta}_2 \quad \textit{for all } n \ge \widehat{n}_2.$$

Corollary 3.7. *Suppose that* $u_0 \in H_p^{1+\alpha}(\Omega)$*. One has*

$$\|u_N^k\|_\infty^2 \le C(\widehat{\delta}_0, \widehat{\delta}_2) \triangleq \widehat{\delta}_\infty \quad \text{for all } n \ge \widehat{n}_2$$

and

$$\|u_N^k\|_\infty^2 \le C(E_0, E_2) \triangleq E_\infty \quad \text{for all } n \ge 1.$$

Lemma 3.8. *Suppose that* $u_0 \in H_p^2(\Omega)$*. For the solution* u_N^k *of* (1.5)–(1.6)*, one has*

$$\|\triangle u_N^n\|^2 + \tau \sum_{k=1}^{n} (1+\rho\tau)^{k-1-n} \|(-\triangle)^{1+\alpha/2} u_N^k\|^2 \le E_3 \quad \text{for all } n \ge 1,$$

$$\overline{\lim_{n\to\infty}} \left(\|\triangle u_N^n\|^2 + \tau \sum_{k=1}^{n} (1+\rho\tau)^{k-1-n} \|(-\triangle)^{1+\alpha/2} u_N^k\|^2 \right) \le \delta_3$$

and

$$\|\overline{\partial}_t u_N^k\|^2 \le E_4 \quad \text{for all } n \ge 1,$$

where E_3 *depends on only* $\|u_0\|_{H^2}$*,* E_0*,* E_1 *and* E_2*, the constant* E_4 *depends on only* $\|u_0\|_{H^2}$*,* E_0*,* E_1 *and* E_3*, and* $\delta_3 > 0$ *depends on* δ_0*,* δ_1*,* δ_2 *and* $\|u_0\|_{H^2}$*.*

Proof. Setting $\varphi = \triangle^2 u_N^k$ in (1.5) and taking the real part, we obtain

$$\begin{aligned} &\overline{\partial}_t \|\triangle u_N^k\|^2 + \tau \|\overline{\partial}_t \triangle u_N^k\|^2 - 2\rho \|\triangle u_N^k\|^2 + 2\|(-\triangle)^{1+\alpha/2} u_N^k\|^2 \\ &= -2\,\mathrm{Re}(1+\mathrm{i}\mu)(|u_N^k|^{2\sigma} u_N^k, \triangle^2 u_N^k). \end{aligned} \tag{3.16}$$

Integrating by parts, we deduce that

$$\begin{aligned} &(|u_N^k|^{2\sigma} u_N^k, \triangle^2 u_N^k) \\ &= 2(\sigma^2+\sigma)\left(|u_N^k|^{2\sigma-2} |\nabla u_N^k|^2 u_N^k, \triangle u_N^k\right) \\ &\quad + \sigma\left((1+\sigma)|u_N^k|^{2\sigma-2}(\nabla u_N^k)^2 \overline{u}_N^k + (\sigma-1)|u_N^k|^{2\sigma-4}(\nabla \overline{u}_N^k)^2 (u_N^k)^3, \triangle u_N^k\right) \\ &\quad + \sigma\left(|u_N^k|^{2\sigma-2} \triangle \overline{u}_N^k (u_N^k)^2, \triangle u_N^k\right) + (1+\sigma)\left(|u_N^k|^{2\sigma} \triangle u_N^k, \triangle u_N^k\right). \end{aligned} \tag{3.17}$$

Similarly to (3.5), by the condition

$$\sigma \le \frac{1}{\sqrt{1+\mu^2}-1}$$

we obtain that

$$\begin{aligned} &- \mathrm{Re}(1+\mathrm{i}\mu)(1+\sigma)(|u_N^k|^{2\sigma} \triangle u_N^k, \triangle u_N^k) - \mathrm{Re}(1+\mathrm{i}\mu)\sigma(|u_N^k|^{2\sigma-2} \triangle \overline{u}_N^k (u_N^k)^2, \triangle u_N^k) \\ &= -\frac{1}{2}\int |u_N^k|^{2\sigma-2}\left(2(1+\sigma)|u_N^k|^2 |\triangle u_N^k|^2 + \sigma(1+\mathrm{i}\mu)(u_N^k \triangle \overline{u}_N^k)^2 + \sigma(1-\mathrm{i}\mu)(\overline{u}_N^k \triangle u_N^k)^2\right) \\ &= -\frac{1}{2}\int |u_N^k|^{2(\sigma-1)} Y_1 M Y_1^H \le 0, \end{aligned} \tag{3.18}$$

where

$$Y_1 = \begin{pmatrix} \overline{u}_N^k \triangle u_N^k \\ u_N^k \triangle \overline{u}_N^k \end{pmatrix}^H, \quad M = \begin{pmatrix} \sigma+1 & \sigma(1+\mathrm{i}\mu) \\ \sigma(1-\mathrm{i}\mu) & \sigma+1 \end{pmatrix},$$

and Y_1^H is the conjugate transpose of the matrix Y_1. Applying Hölder inequality, Gaglizardo-Nirenberg inequality and Young's inequality, we obtain the following estimates when $\sigma \geq 1/2$,

(3.19)

$$\begin{aligned} &(3\sigma+1)(\sigma+1)\sqrt{1+\mu^2}\int |u_N^k|^{2\sigma-1}|\nabla u_N^k|^2|\triangle u_N^k| \\ &\leq (3\sigma+1)(\sigma+1)\sqrt{1+\mu^2}\|\triangle u_N^k\|\|\nabla u_N^k\|_4^2\|u_N^k\|_\infty^{2\sigma-1} \\ &\leq c\|\nabla u_N^k\|_4^4\|u_N^k\|_\infty^{2(2\sigma-1)} + \|\triangle u_N^k\|^2 \\ &\leq c\left(\|(-\triangle)^{1+\alpha/2}u_N^k\| + \|\nabla u_N^k\|\right)^{2/(\alpha+1)}\|\nabla u_N^k\|^{2(2\alpha+1)/(\alpha+1)}\|u_N^k\|_\infty^{2(2\sigma-1)} + \|\triangle u_N^k\|^2 \\ &\leq \frac{1}{4}\|(-\triangle)^{1+\alpha/2}u_N^k\|^2 + c\left(\|\nabla u_N^k\|^2 + \|\nabla u_N^k\|^{2(2\alpha+1)/\alpha}\|u_N^k\|_\infty^{2(\alpha+1)(2\sigma-1)/\alpha}\right) + \|\triangle u_N^k\|^2. \end{aligned}$$

By (3.17)–(3.19), we deduce that

(3.20)

$$\begin{aligned} &-2\,\mathrm{Re}(1+\mathrm{i}\mu)(|u_N^k|^{2\sigma}u_N^k, \triangle^2 u_N^k) \\ &\leq \frac{1}{2}\|(-\triangle)^{1+\alpha/2}u_N^k\|^2 + c\left(\|\nabla u_N^k\|^2 + \|\nabla u_N^k\|^{2(2\alpha+1)/\alpha}\|u_N^k\|_\infty^{2(\alpha+1)(2\sigma-1)/\alpha}\right) + 2\|\triangle u_N^k\|^2. \end{aligned}$$

Plugging (3.20) into (3.16), we obtain

$$\begin{aligned} &\overline{\partial}_t\|\triangle u_N^k\|^2 + \tau\|\overline{\partial}_t\triangle u_N^k\|^2 + \frac{3}{2}\|(-\triangle)^{1+\alpha/2}u_N^k\|^2 \\ &\leq 2(\rho+1)\|\triangle u_N^k\|^2 + c\left(\|\nabla u_N^k\|^2 + \|\nabla u_N^k\|^{2(2\alpha+1)/\alpha}\|u_N^k\|_\infty^{2(\alpha+1)(2\sigma-1)/\alpha}\right). \end{aligned} \tag{3.21}$$

Applying Gaglizardo-Nirenberg inequality and Young's inequality, one has

$$(3\rho+2)\|\triangle u_N^k\|^2 \leq (3\rho+2)c\|u_N^k\|_{H^{2+\alpha}}^{4/(2+\alpha)}\|u_N^k\|^{(2\alpha)/(2+\alpha)} \leq \frac{1}{2}\|(-\triangle)^{1+\alpha/2}u_N^k\|^2 + c\|u_N^k\|^2.$$

Then (3.21) can be rewritten as

$$\begin{aligned} &\overline{\partial}_t\|\triangle u_N^k\|^2 + \|(-\triangle)^{1+\alpha/2}u_N^k\|^2 + \rho\|\triangle u_N^k\|^2 \\ &\leq c\left(\|u_N^k\|^2 + \|\nabla u_N^k\|^2 + \|\nabla u_N^k\|^{2(2\alpha+1)/\alpha}\|u_N^k\|_\infty^{2(\alpha+1)(2\sigma-1)/\alpha}\right). \end{aligned} \tag{3.22}$$

Applying Corollaries 3.2, 3.4 and 3.7, for any $n \geq \widehat{n}_2$ and any given $r > 0$, k_0 satisfying $k_0\tau = r$, we obtain that

$$\begin{aligned} \tau\sum_{k=n}^{n+k_0}\widetilde{h}^k &= c\tau\sum_{k=n}^{n+k_0}\left(\|u_N^k\|^2 + \|\nabla u_N^k\|^2 + \|\nabla u_N^k\|^{2(2\alpha+1)/\alpha}\|u_N^k\|_\infty^{2(\alpha+1)(2\sigma-1)/\alpha}\right) \\ &\leq cr\left(\widehat{\delta}_0 + \widehat{\delta}_1 + \widehat{\delta}_1^{(2\alpha+1)/\alpha}\widehat{\delta}_\infty^{(\alpha+1)(2\sigma-1)/\alpha}\right) \triangleq \widetilde{\alpha}_2 \end{aligned}$$

and

$$\tau \sum_{k=n}^{n+k_0} \widetilde{y}^k = c\tau \sum_{k=n}^{n+k_0} \|\triangle u_N^k\|^2 \le c\tau \sum_{k=n}^{n+k_0} \|u_N^k\|_{H^{1+2\alpha}}^{4/(2\alpha+1)} \|u_N^k\|^{2(2\alpha-1)/(2\alpha+1)}$$
$$\le c\tau \sum_{k=n}^{n+k_0} \left(\|(-\triangle)^{1/2+\alpha} u_N^k\|^2 + \|u_N^k\|^2 \right)$$
$$\le c\tau \sum_{k=n}^{n+k_0} \left(\|u_N^k\|^2 + \|\nabla u_N^k\|^2 \left(\|(-\triangle)^{(1+\alpha)/2} u_N^k\|^4 + \|u_N^k\|^{4\sigma\alpha/(\alpha+1-\sigma)} + \|u_N^k\|^4 \right) \right)$$
$$+ \|(-\triangle)^{(1+\alpha)/2} u_N^{n-1}\|^2$$
$$\le cr \left(\widehat{\delta}_0 + \widehat{\delta}_1^2 (\widehat{\delta}_0^2 + \widehat{\delta}_0^{2\sigma\alpha/(\alpha+1-\sigma)} + \widehat{\delta}_2^2) \right) + \widehat{\delta}_2 \triangleq \widetilde{\alpha}_3.$$

Applying Lemma 2.4, one has

$$\|\triangle u_N^k\|^2 \le \frac{\widetilde{\alpha}_3}{r} + \widetilde{\alpha}_2 \triangleq \widehat{\delta}_3 \quad \text{for all } n \ge \widehat{n}_3 = \widehat{n}_2 + k_0. \tag{3.23}$$

For $n \le \widehat{n}_3$, summing (3.22) for k from 1 to n and applying Lemmas 3.1 and 3.3, we obtain

$$\|\triangle u_N^k\|^2 \le \|\triangle u_0\|^2 + c \left(E_0 + E_1 + E_1^{(2\alpha+1)/\alpha} E_\infty^{(\alpha+1)(2\sigma-1)/\alpha} \right) t_{\widehat{n}_3} \triangleq \widetilde{E}_3. \tag{3.24}$$

Let $E_3 = \max\left\{\widehat{\delta}_3, \widetilde{E}_3\right\}$, it follows from (3.23) and (3.24) that

$$\|\triangle u_N^k\|^2 \le E_3 \quad \text{for all } n \ge 1. \tag{3.25}$$

Similarly to (3.15), one has

$$\overline{\lim_{n\to\infty}} \left(\|\triangle u_N^n\|^2 + \tau \sum_{k=1}^{n} (1+\rho\tau)^{k-1-n} \|(-\triangle)^{1+\alpha/2} u_N^k\|^2 \right) \le \delta_3.$$

Now we estimate $\|\overline{\partial}_t u_N^k\|^2$. Let $v_N^k = \overline{\partial}_t u_N^k$, applying (1.5), we deduce that $\{v_N^k\}_{k\ge1}$ satisfies

$$\left(\overline{\partial}_t v_N^k - \rho v_N^k + (1+\mathrm{i}\nu)(-\triangle)^\alpha v_N^k + \frac{1+\mathrm{i}\mu}{\tau} \left(|u_N^k|^{2\sigma} u_N^k - |u_N^{k-1}|^{2\sigma} u_N^{k-1} \right), \varphi \right) = 0. \tag{3.26}$$

Setting $\varphi = v_N^k$ in (3.26) and taking the real part, we obtain

$$\begin{aligned} &\overline{\partial}_t \|v_N^k\|^2 + 2\|(-\triangle)^{\alpha/2} v_N^k\|^2 - 2\rho \|v_N^k\|^2 \\ &+ \frac{2}{\tau} \operatorname{Re}(1+\mathrm{i}\mu) \int \left(|u_N^k|^{2\sigma} u_N^k - |u_N^{k-1}|^{2\sigma} u_N^{k-1} \right) \overline{v}_N^k \le 0. \end{aligned} \tag{3.27}$$

Now we estimate the last two terms in (3.26). First, in view of Taylor's formula, we can easily check that for $\sigma \le 1/(\sqrt{1+\mu^2}-1)$, one has

$$\operatorname{Re}(1+\mathrm{i}\mu) \int \left(|u_N^k|^{2\sigma} u_N^k - |u_N^{k-1}|^{2\sigma} u_N^{k-1} \right) \overline{v}_N^k \ge 0.$$

Applying (1.5), we have

$$2\rho\|v_N^k\|^2 = 2\rho\int\left(\rho u_N^k-(1+\mathrm{i}\nu)(-\triangle)^\alpha u_N^k-(1+\mathrm{i}\mu)|u_N^k|^{2\sigma}u_N^k\right)\overline{v}_N^k$$
$$\leq \rho\|v_N^k\|^2+6\rho^3\|u_N^k\|^2+6\rho(1+\nu^2)\|(-\triangle)^\alpha u_N^k\|^2+6\rho(1+\mu^2)\|u_N^k\|_{2(2\sigma+1)}^{2(2\sigma+1)}.$$

It follows that

$$\begin{aligned}\rho\|v_N^k\|^2 &= \rho\int\left(\rho u_N^k-(1+\mathrm{i}\nu)(-\triangle)^\alpha u_N^k-(1+\mathrm{i}\mu)|u_N^k|^{2\sigma}u_N^k\right)\overline{v}_N^k\\ &\leq 6\rho^3\|u_N^k\|^2+6\rho(1+\nu^2)\|(-\triangle)^\alpha u_N^k\|^2+6\rho(1+\mu^2)\|u_N^k\|_{2(2\sigma+1)}^{2(2\sigma+1)}\\ &\leq 6\rho^3\|u_N^k\|^2+6c\rho(1+\nu^2)\|u_N^k\|_{H^2}^{2\alpha}\|u_N^k\|^{2(1-\alpha)}+6c\rho(1+\mu^2)\|u_N^k\|_{H^2}^{2\sigma}\|u_N^k\|^{2(1+\sigma)}.\end{aligned}$$

Therefore, (3.27) can be rewritten as

$$\begin{aligned}&\overline{\partial}_t\|v_N^k\|^2+2\|(-\triangle)^{\alpha/2}v_N^k\|^2+2\rho\|v_N^k\|^2\\ &\leq 24\rho\left(\rho^2\|u_N^k\|^2+c(1+\nu^2)\|u_N^k\|_{H^2}^{2\alpha}\|u_N^k\|^{2(1-\alpha)}+c(1+\mu^2)\|u_N^k\|_{H^2}^{2\sigma}\|u_N^k\|^{2(1+\sigma)}\right).\end{aligned}\tag{3.28}$$

Applying Corollaries 3.2 and 3.4, for any $n\geq\widehat{n}_3$ and any given $r>0$, k_0 satisfying $k_0\tau=r$, we obtain that

$$\begin{aligned}&\tau\sum_{k=n}^{n+k_0}\widehat{h}^k\\ &=24\rho\tau\sum_{k=n}^{n+k_0}\left(\rho^2\|u_N^k\|^2+c(1+\nu^2)\|u_N^k\|_{H^2}^{2\alpha}\|u_N^k\|^{2(1-\alpha)}+c(1+\mu^2)\|u_N^k\|_{H^2}^{2\sigma}\|u_N^k\|^{2(1+\sigma)}\right)\\ &\leq 24\rho r\left(\rho^2\widehat{\delta}_0+c(1+\nu^2)(\widehat{\delta}_0+\widehat{\delta}_3)^\alpha\widehat{\delta}_0^{1-\alpha}+c(1+\mu^2)(\widehat{\delta}_0+\widehat{\delta}_3)^\sigma\widehat{\delta}_0^{1+\sigma}\right)\triangleq\widehat{\alpha}_2\end{aligned}$$

and

$$\begin{aligned}&\tau\sum_{k=n}^{n+k_0}\widehat{y}^k=\tau\sum_{k=n}^{n+k_0}\|v_N^k\|^2\\ &\leq 6\tau\sum_{k=n}^{n+k_0}\left(\rho^2\|u_N^k\|^2+c(1+\nu^2)\|u_N^k\|_{H^2}^{2\alpha}\|u_N^k\|^{2(1-\alpha)}+c(1+\mu^2)\|u_N^k\|_{H^2}^{2\sigma}\|u_N^k\|^{2(1+\sigma)}\right)\\ &\leq 6r\left(\rho^2\widehat{\delta}_0+c(1+\nu^2)(\widehat{\delta}_0+\widehat{\delta}_3)^\alpha\widehat{\delta}_0^{1-\alpha}+c(1+\mu^2)(\widehat{\delta}_0+\widehat{\delta}_3)^\sigma\widehat{\delta}_0^{1+\sigma}\right)\triangleq\widehat{\alpha}_3.\end{aligned}$$

Applying Lemma 2.4, we obtain that

$$\|v_N^k\|^2\leq\frac{\widetilde{\alpha}_3}{r}+\widetilde{\alpha}_2\triangleq\widehat{\delta}_4\quad\text{for all } n\geq\widehat{n}_4=\widehat{n}_3+k_0.\tag{3.29}$$

For $n\leq\widehat{n}_4$, summing (3.28) for k from 2 to n and applying Lemmas 3.1 and 3.3, one has

$$\begin{aligned}\|v_N^k\|^2&\leq\|v_N^1\|^2\\ &\quad+24\rho\left(\rho^2E_0+c(1+\nu^2)(E_0+E_3)^\alpha E_0^{1-\alpha}+c(1+\mu^2)(E_0+E_3)^\sigma E_0^{1+\sigma}\right)t_{\widehat{n}_4}.\end{aligned}\tag{3.30}$$

Here we need to estimate $\|v_N^1\|^2$. Letting $\varphi = \overline{\partial}_t u_N^k$ in (1.5), taking the real part and setting $k = 1$, one has

$$\begin{aligned}&\|\overline{\partial}_t u_N^1\|^2 - \rho\,\mathrm{Re}(u_N^1, \overline{\partial}_t u_N^1) + \mathrm{Re}(1+\mathrm{i}\nu)((-\triangle)^\alpha u_N^1, \overline{\partial}_t u_N^1)\\&+\mathrm{Re}(1+\mathrm{i}\mu)(|u_N^1|^{2\sigma}u_N^1, \overline{\partial}_t u_N^1) = 0.\end{aligned}\tag{3.31}$$

We estimate every term on the left-hand side of (3.31) below. First, applying the proof of Lemma 3.1, we have

$$\left|\rho\,\mathrm{Re}(u_N^1, \overline{\partial}_t u_N^1)\right| \le \rho\|\overline{\partial}_t u_N^1\|\|u_N^1\| \le \frac{1}{6}\|\overline{\partial}_t u_N^1\|^2 + \frac{3}{2}\rho^2(\|u_0\|^2 + \delta_0).$$

Secondly,

$$\begin{aligned}&\mathrm{Re}(1+\mathrm{i}\nu)((-\triangle)^\alpha u_N^1, \overline{\partial}_t u_N^1)\\&= \mathrm{Re}(1+\mathrm{i}\nu)\left(\tau(-\triangle)^\alpha\overline{\partial}_t u_N^1, \overline{\partial}_t u_N^1\right) + \mathrm{Re}(1+\mathrm{i}\nu)\left((-\triangle)^\alpha u_0, \overline{\partial}_t u_N^1\right)\\&= \tau\|(-\triangle)^{\alpha/2}\overline{\partial}_t u_N^1\|^2 + \mathrm{Re}(1+\mathrm{i}\nu)\left((-\triangle)^\alpha u_0, \overline{\partial}_t u_N^1\right),\end{aligned}$$

where

$$\begin{aligned}\left|\mathrm{Re}(1+\mathrm{i}\nu)\left((-\triangle)^\alpha u_0, \overline{\partial}_t u_N^1\right)\right| &\le \frac{1}{6}\|\overline{\partial}_t u_N^1\|^2 + \frac{3}{2}(1+\nu^2)\|(-\triangle)^\alpha u_0\|^2\\&\le \frac{1}{6}\|\overline{\partial}_t u_N^1\|^2 + \frac{3}{2}c(1+\nu^2)\|u_0\|_{H^2}^{2\alpha}\|u_0\|^{2(1-\alpha)}.\end{aligned}$$

Finally,

$$\begin{aligned}&\mathrm{Re}(1+\mathrm{i}\mu)(|u_N^1|^{2\sigma}u_N^1, \overline{\partial}_t u_N^1)\\&= \mathrm{Re}(1+\mathrm{i}\mu)(|u_N^1|^{2\sigma}u_N^1 - |u_0|^{2\sigma}u_0, \overline{\partial}_t u_N^1) + \mathrm{Re}(1+\mathrm{i}\mu)(|u_0|^{2\sigma}u_0, \overline{\partial}_t u_N^1).\end{aligned}$$

Applying Taylor's formula, we can easily check that for $\sigma \le 1/(\sqrt{1+\mu^2} - 1)$, one has

$$\mathrm{Re}(1+\mathrm{i}\mu)(|u_N^1|^{2\sigma}u_N^1 - |u_0|^{2\sigma}u_0, \overline{\partial}_t u_N^1) \ge 0.$$

Using Young's inequality and Gaglizardo-Nirenberg inequality, we deduce

$$\left|\mathrm{Re}(1+\mathrm{i}\mu)(|u_0|^{2\sigma}u_0, \overline{\partial}_t u_N^1)\right| \le \frac{1}{6}\|\overline{\partial}_t u_N^1\|^2 + \frac{3}{2}c(1+\mu^2)\|u_0\|_{H^2}^{2\sigma}\|u_0\|^{2(1-\sigma)}.$$

Thus substituting the above relations into (3.31) and applying Lemma 3.1 and (3.25), we obtain

$$\begin{aligned}\|v_N^1\|^2 = \|\overline{\partial}_t u_N^1\|^2 &\le 3\rho^2(\|u_0\|^2 + \delta_0)\\&+ 3c\left((1+\nu^2)\|u_0\|_{H^2}^{2\alpha}\|u_0\|^{2(1-\alpha)} + (1+\mu^2)\|u_0\|_{H^2}^{2\sigma}\|u_0\|^{2(1-\sigma)}\right)\\&\triangleq \widehat{\delta}_5.\end{aligned}\tag{3.32}$$

Thus (3.30) can be rewritten as

(3.33)
$$\begin{aligned}\|v_N^k\|^2 &\le \widehat{\delta}_5 + 24\rho\left(\rho^2 E_0 + c(1+\nu^2)(E_0+E_3)^{\alpha} E_0^{1-\alpha} + c(1+\mu^2)(E_0+E_3)^{\sigma} E_0^{1-\sigma}\right) t_{\widehat{n}_4} \\ &\triangleq \widetilde{E}_4.\end{aligned}$$

Let $E_4 = \max\left\{\widehat{\delta}_4, \widehat{\delta}_5, \widetilde{E}_4\right\}$, then from (3.29), (3.32) and (3.33), we deduce that

$$\|\overline{\partial}_t u_N^k\|^2 = \|v_N^k\|^2 \le E_4 \quad \text{for all } n \ge 1.$$

This completes the proof. □

On the basis of Theorem 2.5, we obtain our main result of this section.

Theorem 3.9. *Suppose that $u_0 \in H_p^2(\Omega)$ and σ satisfies the following condition*

$$\frac{1}{4} \le \sigma \le \min\left\{\frac{1}{\sqrt{1+\mu^2}-1}, 1+\alpha\right\}.$$

The semigroup $\{S_N^\tau(n)\}_{n\ge 0}$ of operators generated by problem (1.5)–(1.6) *has a compact global attractor $\mathcal{A}_N^\tau \subset H_p^2(\Omega) \cap S_N$.*

Proof. This theorem can be proved by checking the conditions (i)–(iii) in Theorem 2.5. Let the Banach space $H = H_p^2(\Omega) \cap S_N$ and $\{S_N^\tau\}$ be a set of the operator semigroup, which is the solution operator generated by problem (1.5)–(1.6).

First, supposing that $\mathcal{B} = \{u_N^0 \in H_p^2(\Omega) \cap S_N : \|u_N^0\|_{H^2}^2 \le R\}$, using the results of the Lemmas 3.1, 3.3, 3.5 and 3.8, we deduce that

$$\|S_N^\tau(n) u_N^0\|_{H^2}^2 \le E_0 + E_1 + E_3 \quad \text{for all } n \ge 0,$$

which means that $\{S_N^\tau(n)\}_{n\ge 0}$ are uniformly bounded in $H_p^2(\Omega)$.

Secondly, thanks to the results of the Lemmas 3.1, 3.3, 3.5 and 3.8, we infer that

$$\|S_N^\tau(n) u_N^0\|_{H^2}^2 = \|u_N^n\|_{H^2}^2 \le \widehat{\delta}_0 + \widehat{\delta}_1 + \widehat{\delta}_3 \quad \text{for all } n \ge \widehat{n}_3(R).$$

It follows that the set $\mathcal{B}_1 = \left\{u \in H_p^2(\Omega) \cap S_N : \|u\|_{H^2}^2 \le \widehat{\delta}_0 + \widehat{\delta}_1 + \widehat{\delta}_3\right\}$ is the bounded absorbing set of the semigroup of operators $\{S_N^\tau(n)\}_{n\ge 0}$.

Finally, the operators $\{S_N^\tau(n)\}_{n\ge 0}$ are uniformly compact for all $n \ge 0$, since the boundedness is equivalent to the compactness in the finite dimensional space $H_p^2(\Omega) \cap S_N$. Our result then follows from Theorem 2.5. □

4. Convergence of the global attractors $\mathcal{A}_N^\tau$

In this section, the existence of the convergence of the discrete attractor $\mathcal{A}_N^\tau$ is proved. To this end, we need the following result from [13].

Theorem 4.1. *Assume that $u_0 \in H_p^2(\Omega)$, σ satisfies the following condition*

$$\frac{1}{2} \leq \sigma \leq \min\left\{ \frac{1}{\sqrt{1+\mu^2}-1}, 1+\alpha \right\}.$$

Then there exists a unique global smooth solution $u = u(x,t)$ for the problem (1.1)–(1.3) *such that*

$$u \in L^\infty(0,T; H_p^2(\Omega)) \cap L^2(0,T; H_p^{2+\alpha}(\Omega)), \quad u_t \in L^\infty(0,T; L_p^1(\Omega)) \cap L^2(0,T; H_p^1(\Omega))$$

and

$$\int_0^t (\|u\|_{H^{2+\alpha}}^2 + \|u_t\|_{H^1}^2)\, dt \leq c(t+1), \qquad \forall\, t \geq 0,$$

$$t\|u\|_{H^{2+\alpha}}^2 \leq c(t^2+1), \quad \forall\, t \geq 0.$$

Moreover, there exists a global attractor $\mathcal{A} \subset H_p^2(\Omega)$ of the semigroup $\{S(t)\}_{t\geq 0}$ of operators generated by problem (1.1)–(1.3), *i.e., there is a set $\mathcal{A}$ such that*

(i) $S(t)\mathcal{A} = \mathcal{A}$, $t \in \mathbb{R}^+$,

(ii) $\lim_{t\to\infty} \operatorname{dist}(S(t)\mathcal{B}, \mathcal{A}) = 0$ *for any bounded $\mathcal{B} \subset H_p^2(\Omega)$, where*

$$\operatorname{dist}(X,Y) = \sup_{x\in X} \inf_{y\in Y} \|x-y\|_E.$$

Furthermore, we need the following theorem from [28].

Theorem 4.2. *Assume that*

(i) *$\{H_\eta\}_{0<\eta\leq\eta_0}$ is a family of closed subspaces of Banach space H such that $\bigcup_{0<\eta\leq\eta_0} H_\eta$ is dense in H.*

(ii) *$S_\eta(t)\colon H_\eta \to H_\eta$ and $S(t)\colon H \to H$ are the global attractors of $S_\eta(t)$ and $S(t)$, respectively.*

(iii) *For every compact interval $I \subset (0,+\infty)$,*

$$\zeta_\eta(I) = \sup_{u_0\in H_\eta} \sup_{t\in I} \operatorname{dist}(S_\eta(t)u_0, S(t)u_0) \to 0 \quad \text{as } \eta \to 0.$$

Then $\mathcal{A}_\eta$ is convergent to $\mathcal{A}$ in the sense of semi-distance:

$$\operatorname{dist}(\mathcal{A}_\eta, \mathcal{A}) \to 0 \quad \text{as } \eta \to 0.$$

Finally, similar to Lemmas 3.1, 3.3, 3.5 and 3.8, one has the following result.

Lemma 4.3. *Under the hypotheses of Theorem* 3.9*, for all* $t \in \mathbb{R}^+$*, we obtain the estimates for the smooth solution* $u(x,t)$ *of problem* (1.1)–(1.3)

$$\int_0^t \left(\|(-\triangle)^{1+\alpha/2}u\|^2 + \|(-\triangle)^{1-\alpha/2}u_t\|^2\right) dt \le c(t+1),$$

$$t\|(-\triangle)^{1-\alpha/2}u_t\|^2 + \int_0^t s\left(\|(-\triangle)^{1-\alpha}u_{tt}\|^2 + \|\triangle u_t\|^2 + \|u_{tt}\|^2 + \|\nabla u_{tt}\|^2\right) ds \le c(t^2+1),$$

$$t^2(\|(-\triangle)^{1-\alpha}u_{tt}\|^2 + \|\triangle u_t\|^2) + \int_0^t s^2\left(\|(-\triangle)^{1+\alpha/2}u_t\|^2 + \|(-\triangle)^{1-\alpha/2}u_{tt}\|^2 + \|\nabla u_{tt}\|^2\right) ds \le c(t^3+1),$$

$$t^3\|(-\triangle)^{1-\alpha/2}u_{tt}\|^2 + \int_0^t s^3\|\triangle u_{tt}\|^2\, ds \le c(t^4+1),$$

where the constant c is independent of t.

Proof. We only provide the proof for the first two inequalities. The other two can be proved similarly. First note that, by the definition of $(-\triangle)^\alpha$, one has, if $\beta \le 1/2$,

$$\begin{aligned}\|(-\triangle)^\beta u\|^2 &= \sum_{k\in Z^2} |k|^{4\beta}|u_k|^2 \le \left(\sum_{k\in Z^2}|k|^2|u_k|^2\right)^{2\beta}\left(\sum_{k\in Z^2}|u_k|^2\right)^{1-2\beta} \\ &= \|(-\triangle)^{1/2}u\|^{4\beta}\|u\|^{2-4\beta},\end{aligned}$$

if $\beta \le 1$,

$$\|(-\triangle)^\beta u\|^2 = \sum_{k\in Z^2} |k|^{4\beta}|u_k|^2 \le \left(\sum_{k\in Z^2}|k|^4|u_k|^2\right)^{\beta}\left(\sum_{k\in Z^2}|u_k|^2\right)^{1-\beta} = \|\triangle u\|^{2\beta}\|u\|^{2-2\beta}.$$

It follows that

$$\|(-\triangle)^\beta u\| \le \|(-\triangle)^{1/2}u\|^{2\beta}\|u\|^{1-2\beta} = \|\nabla u\|^{2\beta}\|u\|^{1-2\beta}, \quad \beta \le \frac{1}{2}, \tag{4.1}$$

$$\|(-\triangle)^\beta u\| \le \|\triangle u\|^{\beta}\|u\|^{1-\beta}, \quad \beta \le 1, \tag{4.2}$$

$$\|(-\triangle)^{m/2+\beta} u\| \le \|(-\triangle)^{(m+1)/2}u\|^{(m+2\beta)/(m+1)}\|u\|^{(1-2\beta)/(m+1)}, \quad 0 \le \beta < \frac{1}{2}. \tag{4.3}$$

By the inequality (4.3), we infer

$$\|\nabla u_{tt}\| \le \|(-\triangle)^{(1+\alpha)/2}u_{tt}\|^{(2\alpha-1)/(3\alpha-1)}\|(-\triangle)^{1-\alpha}u_{tt}\|^{\alpha/(3\alpha-1)}.$$

Proof of the first inequality

$$\int_0^t \left(\|(-\triangle)^{1+\alpha/2}u\|^2 + \|(-\triangle)^{1-\alpha/2}u_t\|^2\right) ds \le c(1+t).$$

Similar to Lemmas 3.1, 3.3, 3.5 and 3.8, one has

$$\|u_t\| \le c, \quad \int_0^t \|(-\triangle)^{1+\alpha/2}u\|\, ds \le c.$$

By (4.1), we obtain

$$\begin{aligned}\|(-\triangle)^{1-\alpha}(|u|^{2\sigma}u)\| &\le \|(-\triangle)^{1/2}(|u|^{2\sigma}u)\|^{2(1-\alpha)}\||u|^{2\sigma}u\|^{2\alpha-1}\\ &= \|\nabla(|u|^{2\sigma}u)\|^{2(1-\alpha)}\||u|^{2\sigma}u\|^{2\alpha-1}\\ &\le c.\end{aligned}$$

Applying equation (1.1), we infer

$$\|(-\triangle)^{1-\alpha}u_t\| \le c\left(\|\triangle u\| + \|(-\triangle)^{1-\alpha}(|u|^{2\sigma}u)\| + \|(-\triangle)^{1-\alpha}u\|\right) \le c.$$

Taking the inner product of (1.1) with $(-\triangle)^{2-\alpha}u_t$ and taking the real part, we obtain

$$\|(-\triangle)^{1-\alpha/2}u_t\|^2 \le c\left(\|(-\triangle)^{1+\alpha/2}u\|^2 + \|(-\triangle)^{1-\alpha/2}(|u|^{2\sigma}u)\|^2 + \|(-\triangle)^{1-\alpha/2}u)\|^2\right),$$

By (4.2), we have

$$\|(-\triangle)^{1-\alpha/2}(|u|^{2\sigma}u)\|^2 \le \|\triangle(|u|^{2\sigma}u)\|^{2-\alpha}\||u|^{2\sigma}u\|^{\alpha} \le c.$$

It follows that

$$\|(-\triangle)^{1-\alpha/2}u_t\|^2 \le c\left(\|(-\triangle)^{1+\alpha/2}u\|^2 + 1\right).$$

Integrating the above inequality with respect to t gives

$$\int_0^t \left(\|(-\triangle)^{1+\alpha/2}u\|^2 + \|(-\triangle)^{1-\alpha/2}u_t\|^2\right)\, ds \le c(1+t).$$

Proof of the second inequality

$$t\|(-\triangle)^{1-\alpha/2}u_t\|^2 + \int_0^t s\left(\|(-\triangle)^{1-\alpha}u_{tt}\|^2 + \|\triangle u_t\|^2 + \|(-\triangle)^{1+\alpha}u\|^2\right)\, ds \le c(1+t^2).$$

From equation (1.1), one has

$$u_{tt} = \rho u_t - (1+\mathrm{i}\nu)(-\triangle)^{\alpha}u_t - (1+\mathrm{i}\mu)(|u|^{2\sigma}u)_t. \tag{4.4}$$

Taking the inner product of (4.4) with $(-\triangle)^{2-\alpha}u_t$ and taking the real part, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(-\triangle)^{1-\alpha/2}u_t\|^2 + \|\triangle u_t\|^2 + \mathrm{Re}(1+\mathrm{i}\mu)((|u|^{2\sigma}u)_t, (-\triangle)^{2-\alpha}u_t)\\ &= \rho(u_t, (-\triangle)^{2-\alpha}u_t).\end{aligned}$$

Integrating by parts, applying Gagliardo-Nirenberg inequality and Young's inequality, one has

$$\begin{aligned}|\operatorname{Re}(1+\mathrm{i}\mu)((|u|^{2\sigma}u)_t,(-\triangle)^{2-\alpha}u_t)| &\le c\|\triangle u_t\|\|(-\triangle)^{1-\alpha}(|u|^{2\sigma}u)_t\| \\ &\le c\|\triangle u_t\|\|(-\triangle)^{1/2}(|u|^{2\sigma}u)_t\|^{2-2\alpha}\|(|u|^{2\sigma}u)_t\|^{2\alpha-1} \\ &\le \frac{1}{4}\|\triangle u_t\|^2+c\end{aligned}$$

and

$$\rho(u_t,(-\triangle)^{2-\alpha}u_t)\le \frac{1}{4}\|\triangle u_t\|^2+c.$$

Then (1.4) can be rewritten as

$$\frac{d}{dt}\|(-\triangle)^{1-\alpha/2}u_t\|^2+\|\triangle u_t\|^2\le c.$$

Multiplying both sides of above inequality by t and integrating it with respect to t, we deduce

$$t\|(-\triangle)^{1-\alpha/2}u_t\|^2+\int_0^t s\|\triangle u_t\|^2\,ds\le ct^2+\int_0^t\|(-\triangle)^{1-\alpha/2}u_t\|^2\,ds\le c(1+t^2).$$

By (4.4), we obtain

$$\int_0^t s\|u_{tt}\|^2\,ds\le c(1+t^2)$$

and

$$\int_0^t s\|\nabla u_{tt}\|^2\,ds\le ct^2+\int_0^t s\|(-\triangle)^{1-\alpha}u_{tt}\|^2\,ds\le c(1+t^2).$$ □

Next we state our main result with a detailed proof of this section.

Theorem 4.4. *Assume that $u_0\in H_p^2(\Omega)$, σ satisfies the following condition*

$$1\le\sigma\le\min\left\{\frac{1}{\sqrt{1+\mu^2}-1},1+\alpha\right\}.$$

One has

$$\operatorname{dist}(\mathcal{A}_N^\tau,\mathcal{A})\to 0\quad \textit{as } \tau\to 0,\ N\to+\infty.$$

Proof. By virtue of Theorem 4.2, we prove this theorem by the error estimates of the solution u_N^n of the discrete problem (1.5)–(1.6).

Let

$$u^k-u_N^k=(u^k-P_Nu^k)+(P_Nu^k-u_N^k)\triangleq\Psi^k+\Phi^k.$$

Then $\forall\,\varphi\in S_N$, Φ^k satisfies

$$(4.5)\quad (\overline{\partial}_t\Phi^k-\rho\Phi^k+(1+\mathrm{i}\nu)(-\triangle)^\alpha\Phi^k+(1+\mathrm{i}\mu)(|u^k|^{2\sigma}u^k-|u_N^k|^{2\sigma}u_N^k),\varphi)=(\overline{\partial}_tu^k-u_t^k,\varphi),$$

and $\Phi^0 = 0$.

Setting $\varphi = \Phi^k$ in (4.5) and taking the real part, we obtain

$$\begin{aligned}
&\frac{1}{2}\overline{\partial}_t\|\Phi^k\|^2 + \frac{\tau}{2}\|\overline{\partial}_t\Phi^k\|^2 + \|(-\triangle)^{\alpha/2}\Phi^k\|^2 - \rho\|\Phi^k\|^2 \\
&= \rho\|\Phi^k\|^2 + \mathrm{Re}(1+\mathrm{i}\mu)(|u^k|^{2\sigma}u^k - |u_N^k|^{2\sigma}u_N^k, \Phi^k) + \mathrm{Re}(\overline{\partial}_t u^k - u_t^k, \Phi^k).
\end{aligned} \tag{4.6}$$

In what follows, we estimate the three terms on the right-hand of (4.6). First, by Theorem 4.1 and Lemma 3.8, we deduce that

$$\begin{aligned}
\|\Phi^k\|^2 &= \|P_N u^k - u_N^k\|^2 \\
&= \left\|(P_N u^k - P_N u^{k-1}) + (P_N u^{k-1} - u_N^{k-1}) - (u_N^k - u_N^{k-1})\right\|^2 \\
&\le 2\left(\|P_N u^{k-1} - u_N^{k-1}\|^2 + 2\left(\|P_N u^k - P_N u^{k-1}\|^2 + \|u_N^k - u_N^{k-1}\|^2\right)\right) \\
&\le 2\left(\|\Phi^{k-1}\|^2 + 2\tau^2\left(\|\overline{\partial}_t u^k\|^2 + \|\overline{\partial}_t u_N^k\|^2\right)\right) \\
&\le 2\left(\|\Phi^{k-1}\|^2 + 2\tau^2\left(\int_{t_{k-1}}^{t_k}\|u_t\|^2\,dt + \|\overline{\partial}_t u_N^k\|^2\right)\right) \\
&\le 2\|\Phi^{k-1}\|^2 + c\tau^2.
\end{aligned} \tag{4.7}$$

Next, by Taylor's formula, Lemmas 3.1, 3.3, 3.5 and 3.8 and their corollaries, we infer that

$$\left||u^k|^{2\sigma}u^k - |u_N^k|^{2\sigma}u_N^k\right| \le (1+2\sigma)\left|\theta u^k + (1-\theta)u_N^k\right|^{2\sigma}\left(|\Psi^k| + |\Phi^k|\right) \le c\left(|\Psi^k| + |\Phi^k|\right).$$

Applying (4.7) and Lemma 2.2, we obtain that

$$\begin{aligned}
\left|\mathrm{Re}(1+\mathrm{i}\mu)\left(|u^k|^{2\sigma}u^k - |u_N^k|^{2\sigma}u_N^k, \Phi^k\right)\right| &\le c\|\Phi^k\|\left(\|\Psi^k\| + \|\Phi^k\|\right) \\
&\le c\left(\|\Phi^{k-1}\|^2 + \tau^2 + N^{-4}\right).
\end{aligned} \tag{4.8}$$

Finally, applying Young's inequality, Taylor's formula and (4.7), we deduce that

$$\begin{aligned}
\left|\mathrm{Re}(\overline{\partial}_t u^k - u_t^k, \Phi^k)\right| &\le \frac{1}{2}\|\Phi^k\|^2 + \frac{1}{2}\|\overline{\partial}_t u^k - u_t^k\|^2 \\
&\le c\left(\|\Phi^{k-1}\|^2 + \tau^2\right) + \frac{1}{2k}\int_{t_{k-1}}^{t_k} s\|u_{tt}\|^2\,ds.
\end{aligned} \tag{4.9}$$

Thus, (4.6) can be rewritten as

$$\overline{\partial}_t\|\Phi^k\|^2 + \|(-\triangle)^{\alpha/2}\Phi^k\|^2 + \|\Phi^k\|^2 \le c\left(\|\Phi^{k-1}\|^2 + N^{-4} + \tau^2 + \int_{t_{k-1}}^{t_k} s\|u_{tt}\|^2\,ds\right).$$

By applying Discrete Gronwall's inequality (Lemma 2.3) and Lemma 4.3, we deduce that

$$\|\Phi^n\|^2 \le ce^{ct_n}\left((N^{-4} + \tau^2) + \tau\int_0^{t_n} s\|u_{tt}\|^2\,ds\right) \le ce^{ct_n}(N^{-4} + \tau).$$

Letting $\varphi = (-\triangle)^{\alpha}\Phi^k$ in (4.5) and taking the real part, we obtain

$$\begin{aligned}&\frac{1}{2}\overline{\partial}_t\|(-\triangle)^{\alpha/2}\Phi^k\|^2 + \frac{\tau}{2}\|\overline{\partial}_t(-\triangle)^{\alpha/2}\Phi^k\|^2 + \|(-\triangle)^{\alpha}\Phi^k\|^2 - \rho\|(-\triangle)^{\alpha/2}\Phi^k\|^2 \\ &= -\operatorname{Re}(1+\mathrm{i}\mu)(|u^k|^{2\sigma}u^k - |u_N^k|^{2\sigma}u_N^k, (-\triangle)^{\alpha}\Phi^k) + \operatorname{Re}(\overline{\partial}_t u^k - u_t^k, (-\triangle)^{\alpha}\Phi^k).\end{aligned} \tag{4.10}$$

Similarly to (4.8) and (4.9), we obtain that

$$\begin{aligned}&\left|-2\operatorname{Re}(1+\mathrm{i}\mu)\left(|u^k|^{2\sigma}u^k - |u_N^k|^{2\sigma}u_N^k, (-\triangle)^{\alpha}\Phi^k\right)\right| \\ &\le c\|(-\triangle)^{\alpha}\Phi^k\|(\|\Psi^k\| + \|\Phi^k\|) \le \frac{1}{4}\|(-\triangle)^{\alpha}\Phi^k\|^2 + c\left(\|\Phi^{k-1}\|^2 + \tau^2 + N^{-4}\right)\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}\left|2\operatorname{Re}(\overline{\partial}_t u^k - u_t^k, (-\triangle)^{\alpha}\Phi^k)\right| &\le \frac{1}{4}\|(-\triangle)^{\alpha}\Phi^k\|^2 + 4\|\overline{\partial}_t u^k - u_t^k\|^2 \\ &\le \frac{1}{4}\|(-\triangle)^{\alpha}\Phi^k\|^2 + \frac{4}{k}\int_{t_{k-1}}^{t_k} s\|u_{tt}\|^2\,ds.\end{aligned} \tag{4.12}$$

Putting (4.11) and (4.12) into (4.10), we obtain

$$\begin{aligned}&\overline{\partial}_t\|(-\triangle)^{\alpha/2}\Phi^k\|^2 + \tau\|\overline{\partial}_t(-\triangle)^{\alpha/2}\Phi^k\|^2 + \frac{3}{2}\|(-\triangle)^{\alpha}\Phi^k\|^2 - 2\rho\|(-\triangle)^{\alpha/2}\Phi^k\|^2 \\ &\le c\left(\|\Phi^{k-1}\|^2 + \tau^2 + N^{-4} + \int_{t_{k-1}}^{t_k} s\|u_{tt}\|^2\,ds\right).\end{aligned} \tag{4.13}$$

By Gagliardo-Nirenberg inequality and Young's inequality, one has

$$3\rho\|(-\triangle)^{\alpha/2}\Phi^k\|^2 \le 3c\rho\|\Phi^k\|_{H^{2\alpha}}\|\Phi^k\| \le \frac{1}{2}\|(-\triangle)^{\alpha}\Phi^k\|^2 + c\|\Phi^k\|^2.$$

Applying (4.7) and the above inequality, (4.13) can be rewritten as

$$\begin{aligned}&\overline{\partial}_t\|(-\triangle)^{\alpha/2}\Phi^k\|^2 + \|(-\triangle)^{\alpha}\Phi^k\|^2 + \rho\|(-\triangle)^{\alpha/2}\Phi^k\|^2 \\ &\le c\left(\|\Phi^{k-1}\|^2 + \tau^2 + N^{-4} + \int_{t_{k-1}}^{t_k} s\|u_{tt}\|^2\,ds\right).\end{aligned} \tag{4.14}$$

Multiplying (4.14) by τ and taking the sum for k from 1 to n, we infer that

$$\begin{aligned}&\|(-\triangle)^{\alpha/2}\Phi^n\|^2 + \tau\sum_{k=1}^{n}\|(-\triangle)^{\alpha}\Phi^k\|^2 \\ &\le c\tau\sum_{k=1}^{n}\left(\|\Phi^{k-1}\|^2 + \tau^2 + N^{-4} + \int_{t_{k-1}}^{t_k} s\|u_{tt}\|^2\,ds\right) \\ &\le ce^{ct_n}(N^{-4} + \tau).\end{aligned}$$

Letting $\varphi = -\triangle\Phi^k$ in (4.5) and taking the real part, we obtain

$$\begin{aligned} &\frac{1}{2}\overline{\partial}_t\|\nabla\Phi^k\|^2 + \frac{\tau}{2}\|\overline{\partial}_t\nabla\Phi^k\|^2 + \|(-\triangle)^{(1+\alpha)/2}\Phi^k\|^2 - \rho\|\nabla\Phi^k\|^2 \\ &= \mathrm{Re}(1+\mathrm{i}\mu)(|u^k|^{2\sigma}u^k - |u_N^k|^{2\sigma}u_N^k, \triangle\Phi^k) - \mathrm{Re}(\overline{\partial}_t u^k - u_t^k, \triangle\Phi^k). \end{aligned} \tag{4.15}$$

By Taylor's formula, we deduce that for $\theta \in (0,1)$,

$$\begin{aligned} &\nabla(|u^k|^{2\sigma}u^k - |u_N^k|^{2\sigma}u_N^k) \\ &\le (4\sigma^2+2\sigma)|u^k - u_N^k||u_N^k + \theta(u^k - u_N^k)|^{2\sigma-1}|\nabla(u_N^k + \theta(u^k - u_N^k))| \\ &\quad + (2\sigma+1)|\nabla(u^k - u_N^k)||u_N^k + \theta(u^k - u_N^k)|^{2\sigma}. \end{aligned} \tag{4.16}$$

Integrating by parts and applying (4.16), we infer that

$$\begin{aligned} &2\,\mathrm{Re}(1+\mathrm{i}\mu)(|u^k|^{2\sigma}u^k - |u_N^k|^{2\sigma}u_N^k, \triangle\Phi^k) \\ &= -2\,\mathrm{Re}(1+\mathrm{i}\mu)(\nabla(|u^k|^{2\sigma}u^k - |u_N^k|^{2\sigma}u_N^k), \nabla\Phi^k) \\ &\le c\|\nabla\Phi^k\|\|u^k - u_N^k\|_4\|\nabla(u_N^k + \theta(u^k - u_N^k))\|_4 + c\|\nabla\Phi^k\|\|\nabla(u^k - u_N^k)\| \\ &\le c\|\nabla\Phi^k\|\|u^k - u_N^k\|_{H^1}\|u_N^k + \theta(u^k - u_N^k)\|_{H^2} + c\|\nabla\Phi^k\|\|\nabla(u^k - u_N^k)\| \\ &\le \|\nabla\Phi^k\|^2 + c(\|\Psi^k\|_{H^1}^2 + \|\Phi^k\|^2) \end{aligned} \tag{4.17}$$

and

$$-2\,\mathrm{Re}(\overline{\partial}_t u^k - u_t^k, \triangle\Phi^k) = 2\,\mathrm{Re}(\nabla(\overline{\partial}_t u^k - u_t^k), \nabla\Phi^k) \le 2\|\nabla\Phi^k\|\|\overline{\partial}_t(\nabla u^k) - (\nabla u^k)_t\|.$$

By Taylor's formula, we deduce that

$$\begin{aligned} \|\overline{\partial}_t(\nabla u^k) - (\nabla u^k)_t\|^2 &= \frac{1}{\tau^2}\left\|\int_{t_{k-1}}^{t_k}(t_{k-1}-s)\nabla u_{tt}\,ds\right\|^2 \\ &\le \frac{1}{\tau^2}\int_{t_{k-1}}^{t_k}\frac{(t_{k-1}-s)^2}{s}\,ds\int_{t_{k-1}}^{t_k}s\|\nabla u_{tt}\|^2\,ds \\ &\le \frac{1}{k}\int_{t_{k-1}}^{t_k}s\|\nabla u_{tt}\|^2\,ds. \end{aligned} \tag{4.18}$$

Thus we obtain that

$$-2\,\mathrm{Re}(\overline{\partial}_t u^k - u_t^k, \triangle\Phi^k) \le \|\nabla\Phi^k\|^2 + \frac{1}{k}\int_{t_{k-1}}^{t_k}s\|\nabla u_{tt}\|^2\,ds. \tag{4.19}$$

By (4.17) and (4.19), (4.15) can be rewritten as

$$\begin{aligned} &\overline{\partial}_t\|\nabla\Phi^k\|^2 + \tau\|\overline{\partial}_t\nabla\Phi^k\|^2 + 2\|(-\triangle)^{(1+\alpha)/2}\Phi^k\|^2 \\ &\le c\|\nabla\Phi^k\|^2 + c(\|\Psi^k\|_{H^1}^2 + \|\Phi^k\|^2) + \frac{1}{k}\int_{t_{k-1}}^{t_k}s\|\nabla u_{tt}\|^2\,ds. \end{aligned} \tag{4.20}$$

Applying Gagliardo-Nirenberg inequality and Young's inequality, we have

$$c\|\nabla\Phi^k\|^2 \le c\|\Phi^k\|_{H^{1+\alpha}}^{2/(1+\alpha)}\|\Phi^k\|^{2\alpha/(1+\alpha)} \le \|(-\triangle)^{(1+\alpha)/2}\Phi^k\|^2 + c\|\Phi^k\|^2.$$

Then, (4.20) can be rewritten as

$$\begin{aligned}&\overline{\partial}_t\|\nabla\Phi^k\|^2 + \tau\|\overline{\partial}_t\nabla\Phi^k\|^2 + \|(-\triangle)^{(1+\alpha)/2}\Phi^k\|^2\\ &\le c(\|\Psi^k\|_{H^1}^2 + \|\Phi^k\|^2) + \frac{1}{k}\int_{t_{k-1}}^{t_k} s\|\nabla u_{tt}\|^2\,ds.\end{aligned}$$

Multiplying the above formula by τ and taking the sum for k from 1 to n, we infer that

$$\begin{aligned}&\|\nabla\Phi^n\|^2 + \tau\sum_{k=1}^{n}\|(-\triangle)^{(1+\alpha)/2}\Phi^k\|^2\\ &\le c\tau\sum_{k=1}^{n}\left(\|\Psi^k\|_{H^1}^2 + \|\Phi^k\|^2 + \frac{1}{k}\int_{t_{k-1}}^{t_k} s\|\nabla u_{tt}\|^2\,ds\right)\\ &\le ce^{ct_n}(N^{-4}+\tau) + ce^{ct_n}N^{-2} + c\tau(1+t_n^2) \le ce^{ct_n}(N^{-2}+\tau).\end{aligned}$$

Letting $\varphi = (-\triangle)^{2-\alpha}\Phi^k$ in (4.5) and taking the real part, one has

$$\begin{aligned}&\frac{1}{2}\overline{\partial}_t\|(-\triangle)^{1-\alpha/2}\Phi^k\|^2 + \frac{\tau}{2}\|\overline{\partial}_t((-\triangle)^{1-\alpha/2}\Phi^k)\|^2 + \|\triangle\Phi^k\|^2 - \rho\|(-\triangle)^{1-\alpha/2}\Phi^k\|^2\\ &= -\operatorname{Re}(1+\mathrm{i}\mu)(|u^k|^{2\sigma}u^k - |u_N^k|^{2\sigma}u_N^k, (-\triangle)^{2-\alpha}\Phi^k) + \operatorname{Re}(\overline{\partial}_t u^k - u_t^k, (-\triangle)^{2-\alpha}\Phi^k).\end{aligned} \tag{4.21}$$

Integrating by parts and applying (4.16), we infer that

$$\begin{aligned}&-2\operatorname{Re}(1+\mathrm{i}\mu)(|u^k|^{2\sigma}u^k - |u_N^k|^{2\sigma}u_N^k, (-\triangle)^{2-\alpha}\Phi^k)\\ &\le 2|1+\mathrm{i}\mu||(\nabla(|u^k|^{2\sigma}u^k - |u_N^k|^{2\sigma}u_N^k), (-\triangle)^{3/2-\alpha}\Phi^k)|\\ &\le c\|(-\triangle)^{3/2-\alpha}\Phi^k\|\|u^k - u_N^k\|_4\|\nabla(u_N^k + \theta(u^k - u_N^k))\|_4 + c\|(-\triangle)^{3/2-\alpha}\Phi^k\|\|\nabla(u^k - u_N^k)\|\\ &\le c\|(-\triangle)^{3/2-\alpha}\Phi^k\|\\ &\quad\times\left(\|u^k - u_N^k\|_{H^1}\|u_N^k + \theta(u^k - u_N^k)\|_{H^2}^{3/4}\|u_N^k + \theta(u^k - u_N^k)\|^{1/4} + \|\nabla(u^k - u_N^k)\|\right)\\ &\le \frac{1}{4}\|\triangle\Phi^k\|^2 + c(\|\Psi^k\|_{H^1}^2 + \|\nabla\Phi^k\|^2 + \|\Phi^k\|^2).\end{aligned} \tag{4.22}$$

Integrating by parts and applying Gagliardo-Nirenberg inequality, Young's inequality and (4.18), one has

$$\begin{aligned}2\operatorname{Re}(\overline{\partial}_t u^k - u_t^k, (-\triangle)^{2-\alpha}\Phi^k) &\le |2(\nabla(\overline{\partial}_t u^k - u_t^k), (-\triangle)^{3/2-\alpha}\Phi^k)|\\ &\le 2\|(-\triangle)^{3/2-\alpha}\Phi^k\|\|\overline{\partial}_t(\nabla u^k) - (\nabla u^k)_t\|\\ &\le \frac{1}{4}\|\triangle\Phi^k\|^2 + c\|\Phi^k\|^2 + \frac{1}{k}\int_{t_{k-1}}^{t_k} s\|\nabla u_{tt}\|^2\,ds.\end{aligned} \tag{4.23}$$

By (4.22) and (4.23), (4.21) can be rewritten as

$$\begin{aligned}&\overline{\partial}_t\|(-\triangle)^{1-\alpha/2}\Phi^k\|^2+\tau\|\overline{\partial}_t((-\triangle)^{1-\alpha/2}\Phi^k)\|^2+\frac{3}{2}\|\triangle\Phi^k\|^2-2\rho\|(-\triangle)^{1-\alpha/2}\Phi^k\|^2\\&\le c(\|\Psi^k\|_{H^1}^2+\|\nabla\Phi^k\|^2+\|\Phi^k\|^2)+\frac{1}{k}\int_{t_{k-1}}^{t_k}s\|\nabla u_{tt}\|^2\,ds.\end{aligned}\tag{4.24}$$

Using Gagliardo-Nirenberg inequality and Young's inequality, we have

$$\begin{aligned}2\rho\|(-\triangle)^{1-\alpha/2}\Phi^k\|^2+c\|\nabla\Phi^k\|^2&\le c\|\Phi^k\|_{H^2}^{2-\alpha}\|\Phi^k\|^{\alpha}+c\|\triangle\Phi^k\|\|\Phi^k\|\\&\le\frac{1}{2}\|\triangle\Phi^k\|^2+c\|\Phi^k\|^2.\end{aligned}$$

So (4.24) can be rewritten as

$$\begin{aligned}&\overline{\partial}_t\|(-\triangle)^{1-\alpha/2}\Phi^k\|^2+\tau\|\overline{\partial}_t((-\triangle)^{1-\alpha/2}\Phi^k)\|^2+\|\triangle\Phi^k\|^2\\&\le c(\|\Psi^k\|_{H^1}^2+\|\Phi^k\|^2)+\frac{1}{k}\int_{t_{k-1}}^{t_k}s\|\nabla u_{tt}\|^2\,ds.\end{aligned}\tag{4.25}$$

Multiplying (4.25) by τ and taking the sum for k from 1 to n, we infer that

$$\begin{aligned}&\|(-\triangle)^{1-\alpha/2}\Phi^n\|^2+\tau\sum_{k=1}^{n}\|\triangle\Phi^k\|^2\\&\le c\tau\sum_{k=1}^{n}\left(\|\Psi^k\|_{H^1}^2+\|\Phi^k\|^2+\frac{1}{k}\int_{t_{k-1}}^{t_k}s\|\nabla u_{tt}\|^2\,ds\right)\\&\le ce^{ct_n}(N^{-2}+\tau)+ce^{ct_n}\cdot N^{-2}+c\tau(1+t_n^2)\le ce^{ct_n}(N^{-2}+\tau).\end{aligned}$$

Letting $\varphi=\triangle^2\Phi^k$ in (4.5) and taking the real part, we obtain

$$\begin{aligned}&\frac{1}{2}\overline{\partial}_t\|\triangle\Phi^k\|^2+\frac{\tau}{2}\|\overline{\partial}_t\triangle\Phi^k\|^2+\|(-\triangle)^{1+\alpha/2}\Phi^k\|^2-\rho\|\triangle\Phi^k\|^2\\&=-\operatorname{Re}(1+\mathrm{i}\mu)(|u^k|^{2\sigma}u^k-|u_N^k|^{2\sigma}u_N^k,\triangle^2\Phi^k)+\operatorname{Re}(\overline{\partial}_tu^k-u_t^k,\triangle^2\Phi^k).\end{aligned}\tag{4.26}$$

Applying Taylor's formula, integrating by parts and by Gagliardo-Nirenberg inequality, one has

$$\begin{aligned}&-2\operatorname{Re}(1+\mathrm{i}\mu)(|u^k|^{2\sigma}u^k-|u_N^k|^{2\sigma}u_N^k,\triangle^2\Phi^k)\\&=-2\operatorname{Re}(1+\mathrm{i}\mu)(\triangle(|u^k|^{2\sigma}u^k-|u_N^k|^{2\sigma}u_N^k),\triangle\Phi^k)\\&\le c\|\triangle\Phi^k\|\big(\|\nabla(u_N^k+\theta(u^k-u_N^k))\|_8^2\|u^k-u_N^k\|_4\\&\qquad+\|\nabla(u_N^k+\theta(u^k-u_N^k))\|_4\|\nabla(u^k-u_N^k)\|_4\big)\\&\quad+c\|\triangle\Phi^k\|\|\triangle(u^k-u_N^k)\|+c\|\triangle(u_N^k+\theta(u^k-u_N^k))\|\|\triangle\Phi^k\|_4\|u^k-u_N^k\|_4\\&\le c\|\triangle\Phi^k\|\big(\|u_N^k+\theta(u^k-u_N^k)\|_{H^2}^2\|u^k-u_N^k\|_{H^1}\\&\qquad+\|u_N^k+\theta(u^k-u_N^k)\|_{H^2}\|u^k-u_N^k\|_{H^2}+\|\triangle(u^k-u_N^k)\|\big)\\&\quad+c\|\triangle(u_N^k+\theta(u^k-u_N^k))\|\|\Phi^k\|_{H^{2+\alpha}}\|\|u^k-u_N^k\|_{H^1}\\&\le\frac{1}{4}\|(-\triangle)^{1+\alpha/2}\Phi^k\|^2+c(\|\Psi^k\|_{H^2}^2+\|\Phi^k\|_{H^1}^2)\end{aligned}\tag{4.27}$$

and

$$2\rho\|\triangle\Phi^k\|^2 \le \frac{1}{4}|(-\triangle)^{1+\alpha/2}\Phi^k\|^2 + c\|\Phi^k\|^2. \tag{4.28}$$

Substituting (4.27) and (4.28) into (4.26), we obtain

$$\begin{aligned} &\overline{\partial}_t\|\triangle\Phi^k\|^2 + \tau\|\overline{\partial}_t\triangle\Phi^k\|^2 + \frac{3}{2}\|(-\triangle)^{1+\alpha/2}\Phi^k\|^2 \\ &\le 2\,\mathrm{Re}(\overline{\partial}_t u^k - u_t^k, \triangle^2\Phi^k) + c(\|\Psi^k\|_{H^2}^2 + \|\Phi^k\|_{H^1}^2). \end{aligned} \tag{4.29}$$

Integrating by parts and using Taylor's formula, we infer that

$$\begin{aligned} &2\,\mathrm{Re}(\overline{\partial}_t u^k - u_t^k, \triangle^2\Phi^k) + c\|\Phi^k\|_{H^1}^2 \\ &= 2\,\mathrm{Re}\Big((-\triangle)^{1-\alpha/2}(\overline{\partial}_t u^k - u_t^k), (-\triangle)^{1+\alpha/2}\Phi^k\Big) + c\|\Phi^k\|_{H^1}^2 \\ &\le \frac{1}{2}\|(-\triangle)^{1+\alpha/2}\Phi^k\|^2 + c\|\Phi^k\|^2 \\ &\quad + \frac{c}{\tau^2}\int_{t_{k-1}}^{t_k}\frac{(t_{k-1}-s)^2}{s^2}\,ds\int_{t_{k-1}}^{t_k} s^2\|(-\triangle)^{1-\alpha/2}u_{tt}\|^2\,ds. \end{aligned}$$

Then, (4.29) can be written as

$$\begin{aligned} &\overline{\partial}_t\|\triangle\Phi^k\|^2 + \tau\|\overline{\partial}_t\triangle\Phi^k\|^2 + \|(-\triangle)^{1+\alpha/2}\Phi^k\|^2 \\ &\le \frac{c}{\tau^2}\int_{t_{k-1}}^{t_k}\frac{(t_{k-1}-s)^2}{s^2}\,ds\int_{t_{k-1}}^{t_k} s^2\|(-\triangle)^{1-\alpha/2}u_{tt}\|^2\,ds + c(\|\Psi^k\|_{H^2}^2 + \|\Phi^k\|^2). \end{aligned}$$

Multiplying the above formula by τt_k and taking the sum for k from 1 to n, we infer that

$$\begin{aligned} t_n\|\triangle\Phi^n\|^2 &\le \tau\sum_{k=1}^{n-1}\|\triangle\Phi^k\|^2 + c\tau\sum_{k=1}^{n} t_k(\|\Psi^k\|_{H^2}^2 + \|\Phi^k\|^2) \\ &\quad + \frac{c}{\tau}\sum_{k=1}^{n} t_k\int_{t_{k-1}}^{t_k}\frac{(t_{k-1}-s)^2}{s^2}\,ds\int_{t_{k-1}}^{t_k} s^2\|(-\triangle)^{1-\alpha/2}u_{tt}\|^2\,ds \\ &\le ce^{ct_n}(N^{-2}+\tau) + ce^{ct_n}N^{-2\alpha} + c\tau(1+t_n^3) \\ &\le ce^{ct_n}(N^{-2\alpha}+\tau). \end{aligned}$$

Namely,

$$\|\triangle\Phi^n\|^2 \le ce^{ct_n}\frac{1}{t_n}(N^{-2\alpha}+\tau), \quad \forall\, n \ge 1.$$

Using the triangle inequality, one has

$$\|u^n - u_N^n\|_{H^2}^2 \le 2(\|\Psi^n\|_{H^2}^2 + \|\Phi\|_{H^2}^2) \le ce^{ct_n}\left(1+\frac{1}{t_n}\right)(N^{-2\alpha}+\tau), \quad \forall\, n \ge 1.$$

For every interval $[t_0, T] \subset \mathbb{R}^+$, we infer that

$$\|u^n - u_N^n\|_{H^2}^2 \le ce^{cT}\left(1+\frac{1}{t_0}\right)(N^{-2\alpha}+\tau) \to 0 \quad \text{as } N \to \infty,\ \tau \to 0.$$

Our result then follows from Theorem 4.2 directly. □

5. Numerical stability of the discrete system

In this section, we focus on the numerical stability of the discrete scheme.

Theorem 5.1. *Assume that σ satisfies the following condition*

$$1 \le \sigma \le \min\left\{\frac{1}{\sqrt{1+\mu^2}-1}, 1+\alpha\right\}.$$

Let $\{u_N^n\}$, $\{v_N^n\}$ be two solutions of the discrete scheme (1.5), (1.6) *with the initial values u_N^0, v_N^0, respectively, and the initial values satisfy $\|u_0^n\|_{H^2} \le R$, $\|v_0^n\|_{H^2} \le R$. Then if the time step τ is small enough such that $\tau \le 1/(8\rho)$, we have*

$$\|u_N^n - v_N^n\|_{H^2}^2 \le ce^{ct_n}\|u_N^0 - v_N^0\|_{H^2}^2, \quad \forall\, n \ge 1.$$

Proof. Let $E_N^k = u_N^k - v_N^k$, then E_N^k satisfies

$$(\overline{\partial}_t E_N^k - \rho E_N^k + (1+\mathrm{i}\nu)(-\triangle)^\alpha E_N^k + (1+\mathrm{i}\mu)(|u_N^k|^{2\sigma}u_N^k - |v_N^k|^{2\sigma}v_N^k), \varphi) = 0 \tag{5.1}$$

for all $\varphi \in S_N$, $k \ge 1$.

Setting $\varphi = E_N^k$ in (5.1) and taking the real part, we obtain

$$\begin{aligned}&\frac{1}{2}\overline{\partial}_t\|E_N^k\|^2 + \frac{\tau}{2}\|\overline{\partial}_t E_N^k\|^2 - \rho\|E_N^k\|^2 + \|(-\triangle)^{\alpha/2}E_N^k\|^2\\&+ \mathrm{Re}(1+\mathrm{i}\mu)(|u_N^k|^{2\sigma}u_N^k - |v_N^k|^{2\sigma}v_N^k, E_N^k) = 0.\end{aligned} \tag{5.2}$$

Applying Taylor's formula, we find that if σ satisfies $\sigma \le 1/(\sqrt{1+\mu^2}-1)$, we obtain

$$\mathrm{Re}(1+\mathrm{i}\mu)(|u_N^k|^{2\sigma}u_N^k - |v_N^k|^{2\sigma}v_N^k, E_N^k) \ge 0.$$

Using Hölder's inequality and Young's inequality, we infer

$$2\rho\|E_N^k\|^2 = 2\rho(\tau\overline{\partial}_t E_N^k + E_N^{k-1}, E_N^k) \le \frac{\rho}{2}\|E_N^k\|^2 + 4\rho(\|E_N^{k-1}\|^2 + \tau^2\|\overline{\partial}_t E_N^k\|^2).$$

Hence, if $\tau \le 1/(8\rho)$, then (5.2) can be rewritten as

$$\overline{\partial}_t\|E_N^k\|^2 + \|(-\triangle)^{\alpha/2}E_N^k\|^2 + \rho\|E_N^k\|^2 \le 8\rho\|E_N^{k-1}\|^2. \tag{5.3}$$

Applying Discrete Gronwall's inequality (Lemma 2.3), we deduce

$$\|E_N^k\|^2 \le e^{8\rho t_n}\|E_N^0\|^2. \tag{5.4}$$

Taking the sum of (5.3) for k from 1 to n and using (5.4), one has

$$\|E_N^k\|^2 + \tau\sum_{k=1}^{n}(\|(-\triangle)^{\alpha/2}E_N^k\|^2 + \rho\|E_N^k\|^2) \le e^{8\rho t_n}\|E_N^0\|^2. \tag{5.5}$$

Setting $\varphi = -\triangle E_N^k$ in (5.1), taking the real part and using Gagliardo-Nirenberg inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\overline{\partial}_t\|\nabla E_N^k\|^2 + \frac{\tau}{2}\|\overline{\partial}_t\nabla E_N^k\|^2 + \|(-\triangle)^{(1+\alpha)/2}E_N^k\|^2 \\
&= \rho\|\nabla E_N^k\|^2 + \mathrm{Re}(1+\mathrm{i}\mu)(|u_N^k|^{2\sigma}u_N^k - |v_N^k|^{2\sigma}v_N^k, \triangle E_N^k) \\
&\le \frac{1}{2}\|(-\triangle)^{(1+\alpha)/2}E_N^k\|^2 + c\|E_N^k\|^2,
\end{aligned}$$

namely,

$$\overline{\partial}_t\|\nabla E_N^k\|^2 + \tau\|\overline{\partial}_t\nabla E_N^k\|^2 + \|(-\triangle)^{(1+\alpha)/2}E_N^k\|^2 \le c\|E_N^k\|^2. \tag{5.6}$$

Taking the sum of (5.6) for k from 1 to n and using (5.4), we obtain

$$\|\nabla E_N^k\|^2 + \tau\sum_{k=1}^{n}\|(-\triangle)^{(1+\alpha)/2}E_N^k\|^2 \le \|\nabla E_0^k\|^2 + ce^{8\rho t_n}\|E_N^0\|^2. \tag{5.7}$$

Setting $\varphi = \triangle^2 E_N^k$ in (5.1), taking the real part and using Gagliardo-Nirenberg inequality, one has

$$\begin{aligned}
&\frac{1}{2}\overline{\partial}_t\|\triangle E_N^k\|^2 + \frac{\tau}{2}\|\overline{\partial}_t\triangle E_N^k\|^2 + \|(-\triangle)^{1+\alpha/2}E_N^k\|^2 \\
&= \rho\|\triangle E_N^k\|^2 - \mathrm{Re}(1+\mathrm{i}\mu)(|u_N^k|^{2\sigma}u_N^k - |v_N^k|^{2\sigma}v_N^k, \triangle^2 E_N^k) \\
&\le \frac{1}{2}\|(-\triangle)^{1+\alpha/2}E_N^k\|^2 + c\|E_N^k\|_{H^1}^2,
\end{aligned}$$

namely,

$$\overline{\partial}_t\|\triangle E_N^k\|^2 + \|(-\triangle)^{1+\alpha/2}E_N^k\|^2 \le c\|E_N^k\|_{H^1}^2. \tag{5.8}$$

Taking the sum of (5.8) for k from 1 to n and using (5.4), we obtain

$$\|\triangle E_N^k\|^2 \le \|\triangle E_0^k\|^2 + c\|\nabla E_0^k\|^2 + ce^{8\rho t_n}\|E_N^0\|^2. \tag{5.9}$$

Combining (5.5), (5.7) and (5.9), we complete the proof. □

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References

[1] C. Bu, *On the Cauchy problem for the* $1+2$ *complex Ginzburg-Landau equation*, J. Austral. Math. Soc. Ser. B **36** (1994), no. 3, 313–324.

[2] L. A. Caffarelli, S. Salsa and L. Silvestre, *Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian*, Invent. Math. **171** (2008), no. 2, 425–461.

[3] C. Canuto and A. Quarteroni, *Approximation results for orthogonal polynomials in Sobolev spaces*, Math. Comp. **38** (1982), no. 157, 67–86.

[4] W. Deng, B. Li, W. Tian and P. Zhang, *Boundary problems for the fractional and tempered fractional operators*, arXiv:1702.03639.

[5] C. R. Doering, J. D. Gibbon, D. D. Holm and B. Nicolaenko, *Low-dimensional behaviour in the complex Ginzburg-Landau equation*, Nonlinearity **1** (1988), no. 2, 279–309.

[6] C. R. Doering, J. D. Gibbon and C. D. Levermore, *Weak and strong solutions of the complex Ginzburg-Landau equation*, Phys. D **71** (1994), no. 3, 285–318.

[7] J. Dong and M. Xu, *Space-time fractional Schrödinger equation with time-independent potentials*, J. Math. Anal. Appl. **344** (2008), no. 2, 1005–1017.

[8] V. L. Ginzburg, *On superconductivity and superfluidity (what I have and have not managed to do), as well as on the 'physical minimum' at the beginning of the 21st century*, Reviews of Mondern Physics **76** (2004), 981–998.

[9] V. L. Ginzburg and L. D. Landau, *On the theory of superconductivity*, Russian in Zh. Eksp. Teor. Fiz. (ZhETF) **20** (1950), 1064–1082.

[10] B. Guo, Y. Han and J. Xin, *Existence of the global smooth solution to the period boundary value problem of fractional nonlinear Schrödinger equation*, Appl. Math. Comput. **204** (2008), no. 1, 468–477.

[11] B. Guo and Z. Huo, *Global well-posedness for the fractional nonlinear Schrödinger equation*, Comm. Partial Differential Equations **36** (2011), no. 2, 247–255.

[12] B. Guo and M. Zeng, *Solutions for the fractional Landau-Lifshitz equation*, J. Math. Anal. Appl. **361** (2010), no. 1, 131–138.

[13] H. Lu, S. Lü and Z. Feng, *Asymptotic dynamics of 2D fractional complex Ginzburg-Landau equation*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **23** (2013), no. 12, 1350202, 12 pp.

[14] S. Lü, *The dynamical behavior of the Ginzburg-Landau equation and its Fourier spectral approximation*, Numer. Math. J. Chinese Univ. **22** (2000), no. 1, 1–9.

[15] S. Lü and Q. Lu, *Asymptotic behavior of three-dimensional Ginzburg-Landau type equation*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **13** (2006), no. 2, 209–220.

[16] ———, *Exponential attractor for the 3D Ginzburg-Landau type equation*, Nonlinear Anal. **67** (2007), no. 11, 3116–3135.

[17] V. G. Mazja, *Sobolev Spaces*, Springer-Verlag, 1985.

[18] R. R. Nigmatullin, *The realization of the generalized transfer equation in a medium with fractal geometry*, Phys. Stat. Solidi. B **133** (1986), no. 1, 425–430.

[19] K. Promislow, *Induced trajectories and approximate inertial manifolds for the Ginzburg-Landau partial differential equation*, Phys. D **41** (1990), no. 2, 232–252.

[20] A. I. Saichev and G. M. Zaslavsky, *Fractional kinetic equations: solutions and applications*, Chaos **7** (1997), no. 4, 753–764.

[21] S. Salsa, *Optimal regularity in lower dimensional obstacle problems*, in *Subelliptic PDE's and Applications to Geometry and Finance*, 217–226, Lect. Notes Semin. Interdiscip. Mat. **6**, Semin. Interdiscip. Mat. (S.I.M.), Potenza, 2007.

[22] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives: Theory and applications*, New York: Gordon and Breach Science, 1987.

[23] J. Shen, *Long time stability and convergence for fully discrete nonlinear Galerkin methods*, Appl. Anal. **38** (1990), no. 4, 201–209.

[24] M. F. Shlesinger, G. M. Zaslavsky and J. Klafter, *Strange kinetics*, Nature **363** (1993), no. 6424, 31–37.

[25] Y. Sire and E. Valdinoci, *Fractional Laplacian phase transitions and boundary reactions: a geometric inequality and a symmetry result*, J. Funct. Anal. **256** (2009), no. 6, 1842–1864.

[26] V. E. Tarasov and G. M. Zaslavsky, *Fractional Ginzburg-Landau equation for fractal media*, Phys. A **354** (2005), 249–261.

[27] ———, *Fractional dynamics of coupled oscillators with long-range interaction*, Chaos **16** (2006), no. 2, 023110, 13 pp.

[28] R. Temam, *Infinite-dimensional Dynamical Systems in Mechanics and Physics*, Second edition, Applied Mathematical Sciences **68**, Springer-Verlag, New York, 1997.

[29] H. Weitzner and G. M. Zaslavsky, *Some applications of fractional equations*, Commun. Nonlinear Sci. Numer. Simul. **8** (2003), no. 3-4, 273–281.

[30] G. M. Zaslavsky, *Chaos, fractional kinetics, and anomalous transport*, Phys. Rep. **371** (2002), no. 6, 461–580.

[31] ———, *Hamiltonian Chaos and Fractional Dynamics*, Oxford University Press, Oxford, 2005.

[32] G. M. Zaslavsky and M. Edelman, *Weak mixing and anomalous kinetics along filamented surfaces*, Chaos **11** (2001), no. 2, 295–305.

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