TAIWANESE JOURNAL OF MATHEMATICS
Vol. 22, No. 3, pp. 513–528, June 2018
DOI: 10.11650/tjm/170812

Distance Eigenvalues and Forwarding Indices of Circulants

Shuting Liu, Huiqiu Lin* and Jinlong Shu

Abstract. In this paper, we give the distance spectral radii of several classes of circulant graphs. We also list the elements in the first rows of their corresponding distance matrices, with which all other distance eigenvalues can be obtained. In addition, we get the relationships between the distance spectral radii and forwarding indices of circulant graphs. Finally, the exact values of the vertex-forwarding indices and some bounds of the edge-forwarding indices for these kinds of graphs are presented.

1. Introduction

An interconnection network is often modeled by a connected graph $G = (V(G), E(G))$, where the vertex set $V(G)$ corresponds to node set in a network representing communication centers or processors, and the edge set $E(G)$ represents link set used to communicate data or messages between different vertices. For notations not defined here, see [3] for references.

Let G be a connected and simple graph of order n. The distance between the vertices v_i and v_j, denoted by $d(v_i, v_j)$, is the length of a shortest path between them. The diameter of G, denoted by $\operatorname{diam}(G)$, is the maximum distance between any pair of vertices of G.

The distance matrix of G is the $n \times n$ matrix $D(G) = (d_G(v_i, v_j))_{v_i, v_j \in V(G)}$. The spectrum of distance matrix, arisen from a data communication problem studied by Graham and Pollack [6] in 1971, has been studied extensively (see [1, 8]). For $v_i \in V(G)$, the *transmission of* v_i *in* G, denoted by $\operatorname{Tr}_G(v_i)$, is defined to be the sum of distances from v_i to all other vertices of G, i.e.,

$$\operatorname{Tr}_G(v_i) = \sum_{v_j \in V(G)} d_G(v_i, v_j).$$

Note that $\operatorname{Tr}_G(v_i)$ is the row sum of $D(G)$ indexed by the vertex v_i. Let $\operatorname{Tr}(G)$ be the $n \times n$ diagonal matrix of vertex transmissions of G. That is,

$$\operatorname{Tr}(G) = \operatorname{diag}(\operatorname{Tr}_G(v_1), \operatorname{Tr}_G(v_2), \ldots, \operatorname{Tr}_G(v_n)).$$

Received June 21, 2016; Accepted August 28, 2017.
Communicated by Xuding Zhu.
2010 *Mathematics Subject Classification*. 05C50, 05C12.
Key words and phrases. circulant graph, distance eigenvalue, forwarding index.
This work is supported by National Natural Science Foundation of China (Nos. 11471121 and 11401211).
*Corresponding author.

A connected graph G is said to be *s-transmission regular* if $\mathrm{Tr}(v_i) = s$ for every vertex $v_i \in V(G)$.

The distance matrix $D(G)$ is real and symmetric, so all of its eigenvalues are real, say ρ_i, $i = 1, 2, \ldots, n$. Then we can order the distance eigenvalues as: $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$. If $\rho_1 > \rho_2 > \cdots > \rho_p$ are the distinct D-eigenvalues and $m_1, m_2, \ldots, m_p$ are their multiplicities respectively, then the D-spectrum can be represented as

$$\begin{pmatrix} \rho_1 & \rho_2 & \cdots & \rho_p \\ m_1 & m_2 & \cdots & m_p \end{pmatrix}.$$

A graph is called distance integral if its distance spectrum consists entirely of integers. The largest eigenvalue of distance matrix is called the distance spectral radius or distance index, denoted by $\rho(G)$.

Definition 1.1. [5] An $n \times n$ matrix C is circulant if it takes the following form:

$$C = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & \cdots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & & \vdots & \vdots \\ a_1 & a_2 & \cdots & a_{n-1} & a_0 \end{pmatrix}.$$

Let $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$. Suppose $S \subseteq \mathbb{Z}_n - \{0\}$ and $-S = S \pmod{n}$, namely there exist $d_1, d_2, \ldots, d_k$, such that $S = \{d_1, d_2, \ldots, d_k, n - d_1, n - d_2, \ldots, n - d_k\}$.

Definition 1.2. The graph G of order n is called circulant graph if it satisfies:

(1) $V(G) = \mathbb{Z}_n$;

(2) $E(G) = \{ij \mid j - i \in S\}$, where the operation takes module n.

The circulant graph G is denoted by $C_n(d_1, d_2, \ldots, d_k)$ or briefly $C_n(d_i)$, where $0 < d_1 < d_2 < \cdots < d_k < (n+1)/2$. The sequence (d_i) is called jump sequence and d_i is called jump.

In this paper, we give the distance spectral radii of several kinds of circulant graphs and list all elements in the first rows of their distance matrices, with which we can obtain all other distance eigenvalues of these circulant graphs. Furthermore, based on the relationships between the distance spectral radius and forwarding indices, exact values of vertex-forwarding indices and some bounds of edge-forwarding indices for these kinds of graphs are described.

2. Preliminaries

Before proceeding, we present some known results which will be useful in the proofs of our main results.

Theorem 2.1. [11] *If* $\gcd(n, a) = 1$ *or* $\gcd(n, b) = 1$*, then there exists an integer* k *satisfying* $C_n(1, k) \cong C_n(a, b)$*.*

Theorem 2.2. [2] $C_n(d_1, d_2, \dots, d_k)$ *is connected if and only if* $\gcd(d_1, d_2, \dots, d_k, n) = 1$*.*

Lemma 2.3. [5] *There is an explicit formula for the eigenvalues* λ_r $(0 \le r \le n-1)$ *of the circulant matrix* C *given by*

$$\lambda_r = \sum_{j=0}^{n-1} a_j \omega^{rj}, \tag{2.1}$$

where $\omega = e^{2\pi i/n} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ *and* i *is the imaginary unit.*

A reformulation of inequalities from the theory of nonnegative matrices (see [10, Chapter 2]) yields the following lemma.

Lemma 2.4. [10] *If* A *is a nonnegative irreducible* $n \times n$ *matrix with the largest eigenvalue* $\rho(A)$ *and row sums* $S_1, S_2, \dots, S_n$*, then*

$$\min_{1 \le i \le n}\{S_i\} \le \rho(A) \le \max_{1 \le i \le n}\{S_i\}.$$

Moreover, one of the equalities holds if and only if the row sums of A *are all equal.*

On the fact that the circulant graph is vertex-transitive, the distance matrix of the circulant graph is circulant. So the circulant graph is s-transmission regular, where $s = \mathrm{Tr}(v_i) = S_i$, for every vertex $v_i \in V(G)$. We have the following lemma.

Lemma 2.5. *Let* G *be a circulant graph and* $\rho(G)$ *be the distance spectral radius of* G*. Then*

$$\rho(G) = s.$$

The distance eigenvalues of the distance matrix of the circulant graph can be given by the formula (2.1) in Lemma 2.3. In order to obtain all other distance eigenvalues of the circulant graph, we only need to know all elements in the first row of its distance matrix. Furthermore, through the calculation by formula (2.1), we can know whether or not all the distance eigenvalues of the circulant graph are integers.

3. D-matrix and D-spectral radius

In this section, we give some results on the distance spectral radii of several classes of circulant graphs and list the elements in the first rows of some of their corresponding distance matrices.

Let the vertices of a circulant graph be labeled clockwise by $0, 1, 2, \ldots, n-1$ (corresponding to $0, -n+1, \ldots, -2, -1$ respectively), and we refer to vertex i instead of saying the vertex labeled by i. In general, we can show that a circulant graph is connected by identifying the existence of a path from 0 to t for each vertex t, i.e., we need a combination of elements of S that sum to t : $\sum_{j=1}^{\eta}(\alpha_j)(d_j) \equiv t \pmod{n}$, where (d_m) and (α_m) denote the step factor (or jump) and the step number of the $0t$-path respectively. For example, let $t = kd+i$ be one vertex of a circulant graph G of order n. Let the expression $t = (k)(d)+(i)(1)$ correspond to a $0t$-path $(0, d, 2d, \ldots, kd, kd+1, kd+2, \ldots, kd+i-1, t)$.

To obtain $\rho(G)$, by the vertex transitivity of the circulant graph G, we only need to find the distance between 0 and t for each vertex $t \in V(G)$ and calculate $\sum_{t\in V(G)} d_G(0,t)$.

3.1. $G = C_n(1,d)$, $(2 \le d = n/2)$

Let $G = C_n(1,d)$, where $2 \le d = n/2$. Then G is 3-regular.

Theorem 3.1. *Let $G = C_n(1,d)$ be a circulant graph, where $2 \le d = n/2$. Then we have the following statements.*

(1) *If d is even, then the elements in the first row of the distance matrix are* $0, 1, 2, \ldots, d/2 - 1, d/2, d/2, d/2-1, \ldots, 2, 1, 2, 3, \ldots, d/2, d/2, \ldots, 2, 1$.

(2) *If d is odd, then the elements in the first row of the distance matrix are* $0, 1, 2, \ldots, (d-1)/2, (d+1)/2, (d-1)/2, \ldots, 2, 1, 2, \ldots, (d-1)/2, (d+1)/2, (d-1)/2, \ldots, 2, 1$.

Theorem 3.2. *Let $G = C_n(1,d)$ be a circulant graph, where $2 \le d = n/2$. Then we have the following statements.*

(1) *If d is even, then $\rho(G) = d^2/2 + d - 1$;*

(2) *If d is odd, then $\rho(G) = d^2/2 + d - 1/2$.*

Proof. By Theorem 3.1, if d is even, then

$$\rho(G) = \mathrm{Tr}_G(0) = \left(1 + 2 + 3 + \cdots + \frac{d}{2}\right) \times 4 - 1 = \frac{1}{2}d^2 + d - 1.$$

If d is odd, then

$$\begin{aligned}\rho(G) = \mathrm{Tr}_G(0) &= \left(1 + 2 + 3 + \cdots + \frac{d+1}{2}\right) \times 2 + \left(1 + 2 + 3 + \cdots + \frac{d-1}{2}\right) \times 2 - 1 \\ &= \frac{1}{2}d^2 + d - \frac{1}{2}.\end{aligned}$$

□

3.2. $G = C_n(1, 2, \ldots, d)$, $(2 \le d \le n/2)$

Theorem 3.3. *Let $G = C_n(1, 2, \ldots, d)$ be a circulant graph, where $n = Kd$, $2 \le d \le n/2$ and $K \ge 2$ is even. Then*

$$\rho(G) = \frac{K}{2}\left(\frac{K+2}{2}d - 1\right).$$

Proof. On the conditions that K is even and $n = Kd$, we divide G into K equal parts of order d. For any vertex t of G, let $t = kd + i$, where $0 \le k \le K/2 - 1$ and $0 \le i < d$. When $kd + 1 \le t \le (k+1)d$, we construct a $0t$-path according to the equality $t = (k)(d) + (1)(t - kd)$. Obviously, the path given is a shortest path between vertices 0 and t, therefore $d(0, t) = k + 1$, where $0 \le k \le K/2 - 1$. Then we have

$$\begin{aligned}\rho(G) = \mathrm{Tr}_G(0) = \sum_{t \in V} d_G(0, t) &= 2\left(\sum_{k=0}^{K/2-1}(k+1)d\right) - \frac{K}{2}\\ &= \frac{K}{2}\left(\frac{K}{2} + 1\right)d - \frac{K}{2} = \frac{K}{2}\left(\frac{K+2}{2}d - 1\right).\end{aligned}$$ □

Theorem 3.4. *Let $G = C_n(1, 2, \ldots, d)$ be a circulant graph, where $n = Kd + n_0$, $0 < n_0 < d$, $2 \le d \le n/2$ and $K \ge 2$ is even. Then*

$$\rho(G) = \frac{K+2}{2}\left(\frac{K}{2}d + n_0 - 1\right).$$

Proof. The proof goes like in Theorem 3.3, but needs more care. Now we only need to consider that $(K/2)d + 1 \le t \le (K/2)d + \lfloor n_0/2 \rfloor$.

Case 1*: n_0 is odd.* It follows that

$$\begin{aligned}\rho(G) &= \mathrm{Tr}_G(0) = \sum_{t \in V} d_G(0, t)\\ &= 2\left(\sum_{k=0}^{K/2-1}(k+1)d + \left\lfloor \frac{n_0}{2} \right\rfloor \left(\frac{K}{2} + 1\right)\right) = \frac{K}{2}\left(\frac{K}{2} + 1\right)d + 2\left\lfloor \frac{n_0}{2} \right\rfloor \left(\frac{K}{2} + 1\right)\\ &= \left(\frac{K}{2} + 1\right)\left(\frac{K}{2}d + 2\left\lfloor \frac{n_0}{2} \right\rfloor\right) = \left(\frac{K}{2} + 1\right)\left(\frac{K}{2}d + n_0 - 1\right).\end{aligned}$$

Case 2*: n_0 is even.* In view of $d_G(0, n/2) = K/2 + 1$, we find that

$$\begin{aligned}\rho(G) &= \mathrm{Tr}_G(0) = \sum_{t \in V} d_G(0, t)\\ &= 2\left(\sum_{k=0}^{K/2-1}(k+1)d + \left\lfloor \frac{n_0}{2} \right\rfloor \left(\frac{K}{2} + 1\right)\right) - \left(\frac{K}{2} + 1\right)\\ &= \left(\frac{K}{2} + 1\right)\left(\frac{K}{2}d + 2\left\lfloor \frac{n_0}{2} \right\rfloor - 1\right) = \left(\frac{K}{2} + 1\right)\left(\frac{K}{2}d + n_0 - 1\right).\end{aligned}$$ □

Particularly, for the complete graph K_n, we have $K_n \cong C_n(1,2,\ldots,\lfloor n/2\rfloor)$ and $\rho(K_n) = n-1$.

Lemma 3.5. *Let $G = C_n(1,2,\ldots,d)$ be a circulant graph, where $n = Kd$, $2 \le d \le n/2$ and K is odd. Then*

$$\rho(G) = \frac{K+1}{2}\left(\frac{K+1}{2}d - 1\right).$$

Proof. On the conditions that K is odd and $n = Kd$, we divide G into K equal parts of order d. For any vertex t of G, let $t = kd + i$, where $0 \le k \le \lfloor K/2\rfloor$ and $0 \le i < d$. When $kd+1 \le t \le (k+1)d$, we construct a $0t$-path according to the equality $t = (k)(d) + (1)(t-kd)$. Obviously, the path given is a shortest path between vertices 0 and t, hence $d(0,t) = k+1$.

Now, we focus on $\lfloor K/2\rfloor d + 1 \le t \le \lfloor K/2\rfloor d + \lfloor d/2\rfloor$.

Case 1*: d is odd.* This implies that

$$\begin{aligned}
\rho(G) &= \mathrm{Tr}_G(0) = \sum_{t\in V} d_G(0,t) \\
&= 2\left(\sum_{k=0}^{\lfloor K/2\rfloor -1}(k+1)d + \left\lfloor\frac{d}{2}\right\rfloor\left(\left\lfloor\frac{K}{2}\right\rfloor + 1\right)\right) \\
&= \left\lfloor\frac{K}{2}\right\rfloor\left(\left\lfloor\frac{K}{2}\right\rfloor + 1\right)d + 2\left\lfloor\frac{d}{2}\right\rfloor\left(\left\lfloor\frac{K}{2}\right\rfloor + 1\right) \\
&= \left(\left\lfloor\frac{K}{2}\right\rfloor + 1\right)\left(\left\lfloor\frac{K}{2}\right\rfloor d + 2\left\lfloor\frac{d}{2}\right\rfloor\right) = \frac{K+1}{2}\left(\frac{K+1}{2}d - 1\right).
\end{aligned}$$

Case 2*: d is even.* Similar to the discussion in Case 1, applying with $d_G(0,n/2) = \lfloor K/2\rfloor + 1$, we obtain that

$$\begin{aligned}
\rho(G) &= \mathrm{Tr}_G(0) = \sum_{t\in V} d_G(0,t) \\
&= 2\left(\sum_{k=0}^{\lfloor K/2\rfloor -1}(k+1)d + \left\lfloor\frac{d}{2}\right\rfloor\left(\left\lfloor\frac{K}{2}\right\rfloor + 1\right)\right) - \left(\left\lfloor\frac{K}{2}\right\rfloor + 1\right) \\
&= \left(\left\lfloor\frac{K}{2}\right\rfloor + 1\right)\left(\left\lfloor\frac{K}{2}\right\rfloor d + 2\left\lfloor\frac{d}{2}\right\rfloor - 1\right) = \frac{K+1}{2}\left(\frac{K+1}{2}d - 1\right).
\end{aligned}$$

□

Lemma 3.6. *Let $G = C_n(1,2,\ldots,d)$ be a circulant graph, where $n = Kd + n_0$, $2 \le d \le n/2$, $0 < n_0 < d$ and K is odd. Then*

$$\rho(G) = \frac{K+1}{2}\left(\frac{K+1}{2}d + n_0 - 1\right).$$

Proof. Now, we mainly consider that $\lfloor K/2\rfloor d + 1 \le t \le \lfloor K/2\rfloor d + \lfloor (n_0+d)/2\rfloor$.

Case 1*: $d+n_0$ is odd.* It follows that

$$\begin{aligned}\rho(G) &= \mathrm{Tr}_G(0) = \sum_{t\in V} d_G(0,t)\\ &= 2\left(\sum_{k=0}^{\lfloor K/2\rfloor -1}(k+1)d + \left\lfloor \frac{n_0+d}{2}\right\rfloor \left(\left\lfloor \frac{K}{2}\right\rfloor +1\right)\right)\\ &= \left(\left\lfloor \frac{K}{2}\right\rfloor +1\right)\left(\left\lfloor \frac{K}{2}\right\rfloor d + 2\left\lfloor \frac{n_0+d}{2}\right\rfloor\right) = \frac{K+1}{2}\left(\frac{K+1}{2}d+n_0-1\right).\end{aligned}$$

Case 2*: $d+n_0$ is even.* It is clear that $d_G(0,n/2) = \lfloor K/2\rfloor + 1$. We see that

$$\begin{aligned}\rho(G) &= \mathrm{Tr}_G(0) = \sum_{t\in V} d_G(0,t)\\ &= 2\left(\sum_{k=0}^{\lfloor K/2\rfloor -1}(k+1)d + \left\lfloor \frac{n_0+d}{2}\right\rfloor \left(\left\lfloor \frac{K}{2}\right\rfloor +1\right)\right) - \left(\left\lfloor \frac{K}{2}\right\rfloor +1\right)\\ &= \left(\left\lfloor \frac{K}{2}\right\rfloor +1\right)\left(\left\lfloor \frac{K}{2}\right\rfloor d + 2\left\lfloor \frac{n_0+d}{2}\right\rfloor -1\right) = \frac{K+1}{2}\left(\frac{K+1}{2}d+n_0-1\right). \quad \square\end{aligned}$$

Combining Lemma 3.5 with Lemma 3.6, we have the following theorem.

Theorem 3.7. *Let $G = C_n(1,2,\ldots,d)$ be a circulant graph, where $n = Kd+n_0$, $2\le d\le n/2$, $0\le n_0 < d$ and K is odd. Then*

$$\rho(G) = \frac{K+1}{2}\left(\frac{K+1}{2}d+n_0-1\right).$$

Let $G = C_n(1,2,\ldots,d)$ $(2\le d\le n/2)$ be a circulant graph. The D-eigenvalues of G can be obtained by putting the elements in the first row of $D(G)$ into formula (2.1) in Lemma 2.3. Then we give the following theorems to describe the elements in the first rows of the distance matrices of these classes of circulants.

Theorem 3.8. *Let $G = C_n(1,2,\ldots,d)$ be a circulant graph, where $n = Kd$ and $2\le d\le n/2$.*

(1) *If K is even, then the elements in the first row of the distance matrix $D(G)$ are*

$$P = (0, \underbrace{1,\ldots,1}_{d}, \underbrace{2,\ldots,2}_{d}, \ldots, \underbrace{K/2,\ldots,K/2}_{2d-1}, \ldots, \underbrace{2,\ldots,2}_{d}, \underbrace{1,\ldots,1}_{d}).$$

(2) *If K is odd, then the elements in the first row of the distance matrix $D(G)$ are*

$$P = (0, \underbrace{1,\ldots,1}_{d}, \ldots, \underbrace{\lfloor K/2\rfloor,\ldots,\lfloor K/2\rfloor}_{d}, \underbrace{\lfloor K/2\rfloor+1,\ldots,\lfloor K/2\rfloor+1}_{d-1},$$
$$\underbrace{\lfloor K/2\rfloor,\ldots,\lfloor K/2\rfloor}_{d}, \ldots, \underbrace{1,\ldots,1}_{d}).$$

Theorem 3.9. *Let $G = C_n(1, 2, \dots, d)$ be a circulant graph, where $n = Kd + n_0$, $2 \le d \le n/2$ and $0 < n_0 \le d - 1$.*

(1) *If K is even, then the elements in the first row of the distance matrix $D(G)$ are*

$$P = (0, \underbrace{1, \dots, 1}_{d}, \dots, \underbrace{K/2, \dots, K/2}_{d}, \underbrace{K/2+1, \dots, K/2+1}_{n_0-1}, \underbrace{K/2, \dots, K/2}_{d}, \dots, \underbrace{1, \dots, 1}_{d}).$$

(2) *If K is odd, then the elements in the first row of the distance matrix $D(G)$ are*

$$P = (0, \underbrace{1, \dots, 1}_{d}, \dots, \underbrace{\lfloor K/2 \rfloor, \dots, \lfloor K/2 \rfloor}_{d}, \underbrace{\lfloor K/2 \rfloor + 1, \dots, \lfloor K/2 \rfloor + 1}_{d+n_0-1}, \underbrace{\lfloor K/2 \rfloor, \dots, \lfloor K/2 \rfloor}_{d}, \dots, \underbrace{1, \dots, 1}_{d}).$$

3.3. $G = C_n(1, d)$, $(2 \le d < n/2)$

Let $G = C_n(1, d)$ $(n \ge 6)$ be a 4-regular circulant graph. The following theorems can be obtained by Lemmas 3.1, 3.3, 3.5 and 3.7 in [9].

Theorem 3.10. *Let $G = C_n(1, d)$ be a circulant graph, where $n = Kd$ and $K \ge 4$ is even. Then*

$$\rho(G) = \begin{cases} H - K/4 & \text{if } d \text{ is even,} \\ H & \text{if } d \text{ is odd,} \end{cases}$$

where $H = \frac{K}{2}\left(\frac{K}{2}d + \frac{d^2}{2} - \frac{1}{2}\right)$.

Theorem 3.11. *Let $G = C_n(1, d)$ be a circulant graph, where $n = Kd$ and $K \ge 3$ is odd. Then*

$$\rho(G) = \begin{cases} M - (K-1)/4 & \text{if } d \text{ is even,} \\ M - 1/4 & \text{if } d \text{ is odd,} \end{cases}$$

where $M = \frac{K-1}{2}\left(\frac{K-1}{2}d + \frac{d^2}{2} + d - \frac{1}{2}\right) + \frac{d^2}{4}$.

Theorem 3.12. *Let $G = C_n(1, d)$ be a circulant graph, where $n = Kd + n_0$, $0 < n_0 < d$ and $K \ge 4$ is even.*

(1) *If n_0 is odd, then*

$$\rho(G) = \begin{cases} N - 1/4 & \text{if } d \text{ is even,} \\ N + (K-1)/4 & \text{if } d \text{ is odd,} \end{cases}$$

(2) *If n_0 is even, then*

$$\rho(G) = \begin{cases} N & \text{if } d \text{ is even,} \\ N + K/4 & \text{if } d \text{ is odd,} \end{cases}$$

where $N = \frac{K}{2}\left(\frac{K}{2}d + \frac{d^2}{2} + n_0 - 1\right) + \frac{n_0^2}{4}$.

Theorem 3.13. *Let $G = C_n(1, d)$ be a circulant graph, where $n = Kd + n_0$, $0 < n_0 < d$ and $K \geq 3$ is odd.*

(1) *If $d + n_0$ is odd, then*

$$\rho(G) = \begin{cases} S - (K+4)/4 & \text{if } d \text{ is odd,} \\ S - (2K+1)/4 & \text{if } d \text{ is even,} \end{cases}$$

(2) *If $d + n_0$ is even, then*

$$\rho(G) = \begin{cases} S - (K+1)/2 & \text{if } d \text{ is even,} \\ S - (K+1)/4 & \text{if } d \text{ is odd,} \end{cases}$$

where $S = \frac{K-1}{2}\left(\frac{K+1}{2}d + \frac{d^2}{2} + n_0\right) + \frac{{n_0}^2}{2} + \left(\frac{d-n_0}{2}\right)^2 + d$.

Elements in the first rows of the distance matrices of the 4-regular circulant graphs $G = C_n(1, d)$ $(n \geq 6)$ can be given according to the methods of constructing shortest paths provided in [9]. Now we will give a theorem to describe the elements in the first row of the distance matrix of one kind of circulant graph. And we won't detail other cases in this article.

Theorem 3.14. *Let $G = C_n(1, d)$ be a circulant graph, where $n = Kd$ and $K \geq 4$ is even. Then the elements in the first row of the distance matrix of G are:* $0, 1, \ldots, \lfloor d/2 \rfloor, d - \lfloor d/2 \rfloor, \ldots, 2, 1, 2, \ldots, \lfloor d/2 \rfloor + 1, d - \lfloor d/2 \rfloor + 1, \ldots, 3, 2, \ldots, K/2, \ldots, \lfloor d/2 \rfloor + K/2 - 1, d - \lfloor d/2 \rfloor + K/2 - 1, \ldots, K/2 + 1, K/2, K/2 + 1, \ldots, d - \lfloor d/2 \rfloor + K/2 - 1, \lfloor d/2 \rfloor + K/2 - 1, \ldots, K/2, \ldots, 2, 3, \ldots, d - \lfloor d/2 \rfloor + 1, \lfloor d/2 \rfloor + 1, \ldots, 2, 1, 2, \ldots, d - \lfloor d/2 \rfloor, \lfloor d/2 \rfloor, \ldots, 1.$

Proof. On the conditions that K is even and $n = Kd$, we divide G into K equal parts of order d. For any vertex t of G, let $t = kd + i$, where $0 \leq k \leq K/2 - 1$.

Case 1*: $kd + 1 \leq t \leq kd + \lfloor d/2 \rfloor$.* We construct a $0t$-path according to the equality $t = (k)(d) + (t - kd)(1)$. Obviously, the path given is a shortest path between vertices 0 and t, therefore $d(0, t) = k + t - kd$.

Case 1.1*: $k = 0$ and $1 \leq t \leq \lfloor d/2 \rfloor$.* Then $1 \leq d(0, t) = t \leq \lfloor d/2 \rfloor$, i.e., $d(0, t)$ are respectively $0, 1, 2, \ldots, \lfloor d/2 \rfloor - 1, \lfloor d/2 \rfloor$, when the value of t goes from 0 to $\lfloor d/2 \rfloor$.

Case 1.2*:* $k = 1$ *and* $d + 1 \leq t \leq d + \lfloor d/2 \rfloor$. Then $2 \leq d(0,t) = 1 + t - d \leq 1 + \lfloor d/2 \rfloor$, i.e., $d(0,t)$ are respectively $2, 3, \ldots, \lfloor d/2 \rfloor, \lfloor d/2 \rfloor + 1$, when the value of t goes from $d + 1$ to $d + \lfloor d/2 \rfloor$.

Case 1.3*:* $k = K/2 - 1$ *and* $(K/2 - 1)d + 1 \leq t \leq (K/2 - 1)d + \lfloor d/2 \rfloor$. Then $K/2 \leq d(0,t) \leq K/2 - 1 + \lfloor d/2 \rfloor$, i.e., $d(0,t)$ are respectively $K/2, K/2 + 1, \ldots, K/2 + \lfloor d/2 \rfloor - 2, K/2 + \lfloor d/2 \rfloor - 1$, when the value of t goes from $(K/2-1)d+1$ to $(K/2-1)d+\lfloor d/2 \rfloor$.

Case 2*:* $kd + \lfloor d/2 \rfloor + 1 \leq t \leq (k + 1)d$. Similar to Case 1, the shortest path can be constructed by the equality $t = (k + 1)(d) + ((k + 1)d - t)(-1)$, so the distance $d(0,t) = k + 1 + (k + 1)d - t$.

Case 2.1*:* $k = 0$ *and* $1 + \lfloor d/2 \rfloor \leq t \leq d$. Then $1 \leq d(0,t) = 1 + d - t \leq d - \lfloor d/2 \rfloor$, i.e., $d(0,t)$ are respectively $d - \lfloor d/2 \rfloor, d - \lfloor d/2 \rfloor - 1, \ldots, 2, 1$, when the value of t goes from $1 + \lfloor d/2 \rfloor$ to d.

Case 2.2*:* $k = 1$ *and* $d+1+\lfloor d/2 \rfloor \leq t \leq 2d$. Then $2 \leq d(0,t) = 2+2d-t \leq 1+d-\lfloor d/2 \rfloor$, i.e., $d(0,t)$ are respectively $1 + d - \lfloor d/2 \rfloor, d - \lfloor d/2 \rfloor, \ldots, 3, 2$, when the value of t goes from $d + 1 + \lfloor d/2 \rfloor$ to $2d$.

Case 2.3*:* $k = K/2 - 1$ *and* $(K/2 - 1)d + 1 + \lfloor d/2 \rfloor \leq t \leq (K/2)d$. Then $K/2 \leq d(0,t) \leq K/2 - 1 + d - \lfloor d/2 \rfloor$, i.e., $d(0,t)$ are respectively $K/2 + d - \lfloor d/2 \rfloor - 1, K/2 + d - \lfloor d/2 \rfloor - 2, \ldots, K/2 + 1, K/2$, when the value of t goes from $(K/2 - 1)d + 1 + \lfloor d/2 \rfloor$ to $(K/2)d$.

Based on the above cases, we then list the elements in the first row of the distance matrix as follows: $0, 1, \ldots, \lfloor d/2 \rfloor, d - \lfloor d/2 \rfloor, \ldots, 2, 1, 2, \ldots, \lfloor d/2 \rfloor + 1, d - \lfloor d/2 \rfloor + 1, \ldots, 3, 2, \ldots, K/2, \ldots, \lfloor d/2 \rfloor + K/2 - 1, d - \lfloor d/2 \rfloor + K/2 - 1, \ldots, K/2 + 1, K/2, K/2 + 1, \ldots, d - \lfloor d/2 \rfloor + K/2 - 1, \lfloor d/2 \rfloor + K/2 - 1, \ldots, K/2, \ldots, 2, 3, \ldots, d - \lfloor d/2 \rfloor + 1, \lfloor d/2 \rfloor + 1, \ldots, 2, 1, 2, \ldots, d - \lfloor d/2 \rfloor, \lfloor d/2 \rfloor, \ldots, 1$. □

4. Forwarding indices and D-spectral radius

To measure the load of a vertex, Chung et al. in [4] introduced the notion of forwarding index, which is one of the fault tolerance parameters of a network and has received considerable attentions owning to its importance in networks (see nice survey [16] and references therein).

A routing R of G is a set of $n(n-1)$ elementary paths $R(x,y)$ specified for all ordered pairs (x,y) of vertices of G. If each of the paths specified by R is shortest, the routing R is said to be *minimal*, denoted by R_m. If $R(x,y) = R(y,x)$ specified by R, that is to say the path $R(y,x)$ is the reverse of the path $R(x,y)$ for all x, y, then we say that the routing is *symmetric*.

Let the sets of routings and minimum routings in a graph G be denoted by $\mathcal{R}(G)$ and $\mathcal{R}_m(G)$ respectively. For a given $R \in \mathcal{R}(G)$ and $x \in V(G)$, the *load of a vertex* x in a given routing R of a graph G, denoted by $\xi_x(G, R)$, is defined as the number of paths specified by R passing through x and admitting x as an inner vertex. The *forwarding index of G with respect to R* is the maximum number of paths of R going through any vertex x in G and is denoted by

$$\xi(G, R) = \max\{\xi_x(G, R) : x \in V(G)\}.$$

The minimum forwarding index over all possible routings of a graph G, denoted by

$$\xi(G) = \min\{\xi(G, R) : R \in \mathcal{R}(G)\},$$

is called the *forwarding index of G*.

Similar concepts for the edge version of a graph were introduced by Heydemann et al. in [7]. The *load of an edge e with respect to R*, denoted by $\pi_e(G, R)$, is defined as the number of the paths specified by R going through it. The *edge forwarding index of G with respect to R* is the maximum number of paths specified by R going through any edge of G and is denoted by

$$\pi(G, R) = \max\{\pi_e(G, R) : e \in E(G)\}.$$

We call

$$\pi(G) = \min\{\pi(G, R) : R \in \mathcal{R}(G)\}$$

the *edge forwarding index of G*.

For routings of shortest paths, define

$$\xi_m(G) = \min\{\xi(G, R_m) : R_m \in \mathcal{R}_m(G)\}$$

and

$$\pi_m(G) = \min\{\pi(G, R_m) : R_m \in \mathcal{R}_m(G)\}.$$

Clearly, $\xi(G) \leq \xi_m(G)$ and $\pi(G) \leq \pi_m(G)$. The equalities, however, do not always hold and some examples can be seen in [7].

Even if the diameter of the graph is 2, the forwarding index problem is NP-complete, which was proved by Saad in [12]. Solé [13] showed that the vertex forwarding indices of graphs in a class of quasi-Cayley graphs, a new class of vertex-transitive graphs, which contain Cayley graphs, achieve the minimum. However, it is very difficult to find the exact value or a good estimate of the forwarding index of a graph, even for some special classes of graphs such as circulant graphs. In [17] Xu et al. have established upper and lower bounds of forwarding indices for circulant graphs. But these obtained bounds are difficult to compute generally. Moreover, a uniform routing of shortest paths may not exist for circulant graphs, just as the case for Cayley graphs in [14].

For the circulant graph $G = C_n(1, 3d+1, 3d^2-1)$, where $n = 3d^2+3d+1$, Thomson and Zhou [15] determined

$$\pi(G) = \frac{1}{3}d(d+1)(2d+1) \quad \text{for } d \geq 2.$$

For the circulant digraph $\overrightarrow{G}(d^k; S)$ with $S = \{1, d, \ldots, d^{k-1}\}$, $n = d^k$, $d \geq 2$ and $k \geq 2$, Xu et al. in [17], obtained

$$\xi(\overrightarrow{G}(d^k; S)) = \frac{1}{2}(d-1)d^k k - (d^k - 1) \quad \text{and} \quad \pi(\overrightarrow{G}(d^k; S)) = \frac{1}{2}(d-1)d^k.$$

Generally, it is very difficult to compute the forwarding indices of a graph. The purpose of this section is to study the forwarding indices of circulant graphs listed in the third section.

Theorem 4.1. [7] *If G is a Cayley graph of order n, then, for any vertex x in V,*

$$\xi(G) = \xi_m(G) = \sum_{y \in G} d_G(x, y) - (n-1).$$

Lemma 4.2. *If G is a connected circulant graph of order n, then*

$$\xi(G) = \xi_m(G) = \rho(G) - (n-1).$$

Some relationships between vertex-forwarding index and edge-forwarding index are presented as follows.

Theorem 4.3. [7] *For any connected graph G of order n, the maximum degree Δ and the minimum degree δ,*

(a) $2\xi(G) + 2(n-1) \leq \Delta\pi(G)$;

(b) $\pi(G) \leq \xi(G) + 2(n-1)$;

(c) $\pi_m(G) \leq \xi_m(G) + 2(n-\delta)$.

All these inequalities are also valid for symmetric routings and the inequality in (a) *is also valid for minimal routings.*

Remark 4.4. [7] In (a) the equality holds for C_n, W_n, $K_{1,n}$, the n-cube, the Petersen graph and its line graph as can be calculated with the values given in Section 6 in [7]. In (b) the equality holds for the complete graph.

Lemma 4.5. *Suppose that $G = (V, E)$ is a connected r-regular circulant graph of order n. Then*

$$\frac{2\rho(G)}{r} \leq \pi(G) \leq n + \rho(G) - (2r-1). \tag{4.1}$$

Proof. By Lemma 4.2, $\xi(G) = \xi_m(G) = \rho(G) - (n-1)$, and by (a) and (c) in Theorem 4.3, we obtain

$$\frac{2\rho(G)}{\Delta} \leq \pi(G) \leq \pi_m(G) \leq \xi_m(G) + 2(n-\delta) \leq \rho(G) + n - (2\delta - 1). \tag{4.2}$$

By Lemma 4.2 and (b) in Theorem 4.3, it follows that

$$\frac{2\rho(G)}{\Delta} \leq \pi(G) \leq \rho(G) + n - 1. \tag{4.3}$$

For a connected circulant graph G with order n and $\delta \geq 1$, combining inequality (4.2) with inequality (4.3), we have

$$\frac{2\rho(G)}{\Delta} \leq \pi(G) \leq n + \rho(G) - (2\delta - 1).$$

Also, since $r = \Delta = \delta$, it completes the proof. □

Theorem 4.6. *Let $G = C_n(1, d)$ be a 3-regular circulant graph, where $2 \leq d = n/2$.*

(1) *If d is even, then*

$$\xi(G) = \frac{d(d-2)}{2} \quad \text{and} \quad \frac{d(d+2)-2}{3} \leq \pi(G) \leq \frac{d(d+6)-12}{2}.$$

(2) *If d is odd, then*

$$\xi(G) = \frac{d(d-2)+1}{2} \quad \text{and} \quad \frac{d(d+2)-1}{3} \leq \pi(G) \leq \frac{d(d+6)-11}{2}.$$

Proof. We know that $G = C_n(1, d)$ $(n = 2d)$ is a 3-regular graph.

If d is even, by Lemma 4.2, $\xi(G) = \xi_m(G) = \rho(G) - (n-1)$. From Theorem 3.2, $\rho = d^2/2 + d - 1$ induces to $\xi(G) = d^2/2 + d - 1 - 2d + 1 = d(d-2)/2$. By inequality (4.1), i.e., $2\rho(G)/3 \leq \pi(G) \leq n + \rho(G) - 5$ in Lemma 4.5, it implies that $(d(d+2)-2)/3 \leq \pi(G) \leq (d(d+6)-12)/2$.

If d is odd, by Theorem 3.2, $\rho = d^2/2 + d - 1/2$. Taking Lemma 4.2, we see that $\xi(G) = (d(d-2)+1)/2$. By inequality (4.1) in Lemma 4.5, it holds that $(d(d+2)-1)/3 \leq \pi(G) \leq (d(d+6)-11)/2$. □

Theorem 4.7. *Let $G = C_n(1, 2, \ldots, d)$ be a circulant graph, where $n = Kd$, $2 \leq d \leq n/2$, and $K \geq 2$ is even. Then*

(1) *$\xi(G) = (n-2)(K-2)/4$;*

(2) *$((n-2)K + 2n)/(4d) \leq \pi(G) \leq (n-2)(K-2)/4 + 2(n-2d)$ when $d \neq n/2$ and $\pi(G) = 2$ when $d = n/2$.*

Proof. Suppose that the circulant graph $G = C_n(1, 2, \ldots, d)$ is a r-regular graph.

By Theorem 3.3, $\rho(G) = \frac{K}{2}\left(\frac{K+2}{2}d - 1\right)$. Using Lemma 4.2, we find that $\xi(G) = \xi_m(G) = \rho(G) - (n-1) = \frac{K}{2}\left(\frac{K+2}{2}d - 1\right) - Kd + 1 = \frac{(K-2)n}{4} - \frac{K}{2} + 1 = \frac{(K-2)(n-2)}{4}$.

By the inequality (4.1) in Lemma 4.5, we divide the proof into two cases:

Case 1: If $2 \le d < n/2$, then $r = 2d$. From the inequality $2\rho(G)/r \le \pi(G) \le n + \rho(G) - (2r-1)$, we show that $\frac{2\rho(G)}{r} = \frac{K}{2d}\left(\frac{K+2}{2}d - 1\right) = \frac{(n-2)K+2n}{4d} \le \pi(G)$ and $\pi(G) \le n + \rho(G) - (2r-1) = \frac{(n-2)(K-2)}{4} + 2(n-2d)$.

Case 2: If $d = n/2$, then $r = n-1$. $2\rho(G)/r \le \pi(G) \le n + \rho(G) - (2r-1)$ implies $\pi(G) = 2$. Indeed, in this case, $G \cong K_n$ and $\rho(K_n) = n - 1$. □

Theorem 4.8. *Let $G = C_n(1, 2, \ldots, d)$ be a circulant graph, where $n = Kd + n_0$, $2 \le d \le n/2$, $0 < n_0 < d$, and $K \ge 2$ is even. Then*

$$\xi(G) = \xi_m(G) = \rho(G) - (n-1) \quad \textit{and} \quad \frac{\rho(G)}{d} \le \pi(G) \le \rho(G) + (n - 4d + 1),$$

where $\rho(G) = \frac{K+2}{2}\left(\frac{K}{2}d + n_0 - 1\right)$.

Theorem 4.9. *Let $G = C_n(1, 2, \ldots, d)$ be a circulant graph, where $n = Kd + n_0$, $2 \le d \le n/2$, $0 \le n_0 < d$ and K is odd. Then*

$$\xi(G) = \xi_m(G) = \rho(G) - (n-1) \quad \textit{and} \quad \frac{\rho(G)}{d} \le \pi(G) \le \rho(G) + (n - 4d + 1),$$

where $\rho(G) = \frac{K+1}{2}\left(\frac{K+1}{2}d + n_0 - 1\right)$.

References

[1] M. Aouchiche and P. Hansen, *Distance spectra of graphs: a survey*, Linear Algebra Appl. **458** (2014), 301–386.

[2] F. Boesch and R. Tindell, *Circulants and their connectivities*, J. Graph Theory **8** (1984), no. 4, 487–499.

[3] J. A. Bondy and U. S. R. Murty, *Graph theory with applications*, American Elsevier Publishing Co., New York, 1976.

[4] F. R. K. Chung, E. G. Coffman, M. I. Reiman and B. Simon, *The forwarding index of communication networks*, IEEE Trans. Inform. Theory **33** (1987), no. 2, 224–232.

[5] P. J. Davis, *Circulant Matrices*, A Wiley-Interscience Publication, Pure and Applied Mathematics, John Wiley & Sons, New York, 1979.

[6] R. L. Graham and H. O. Pollak, *On the addressing problem for loop switching*, Bell System Tech. J. **50** (1971), 2495–2519.

[7] M.-C. Heydemann, J.-C. Meyer and D. Sotteau, *On forwarding index of networks*, Discrete Appl. Math. **23** (1989), no. 2, 103–123.

[8] H. Lin, Y. Hong, J. Wang and J. Shu, *On the distance spectrum of graphs*, Linear Algebra Appl. **439** (2013), no. 6, 1662–1669.

[9] S. Liu and J. Meng, *Forwarding indices of* 4*-regular circulants*, Eng. Lett. **24** (2016), no. 1, 62–68.

[10] H. Minc, *Nonnegative Matrices*, Wiley-Interscience Series in Discrete Mathematics and Optimization, A Wiley-Interscience Publication, John Wiley & Sons, New York, 1988.

[11] B. Pang, H. Shao and W. Gao, *A measurement-based admission control scheme for DiffServ network: design and application*, Chin. J. Comput. **26** (2003), no. 3, 257–265.

[12] R. Saad, *Complexity of the forwarding index problem*, SIAM J. Discrete Math. **6** (1993), no. 3, 418–427.

[13] P. Solé, *The edge-forwarding index of orbital regular graphs*, Discrete Math. **130** (1994), no. 1-3, 171–176.

[14] A. Thomson, *Frobenius Graphs as Interconnection Networks*, Ph.D Thesis, The University of Melbourne, 2011.

[15] A. Thomson and S. Zhou, *Gossiping and routing in undirected triple-loop networks*, Networks **55** (2010), no. 4, 341–349.

[16] J.-M. Xu and M. Xu, *The forwarding indices of graphs–a survey*, Opuscula Math. **33** (2013), no. 2, 345–372.

[17] M. Xu, J.-M. Xu and L. Sun, *The forwarding index of the circulant networks*, J. Math. (Wuhan) **27** (2007), no. 6, 623–629.

Shuting Liu
Department of Computer Science and Technology, East China Normal University,
Shanghai, 200062, China
E-mail address: `lst0904@163.com`

Huiqiu Lin
Department of Mathematics, East China University of Science and Technology,
Shanghai, 200237, China
E-mail address: `huiqiulin@126.com`

Jinlong Shu
Department of Computer Science and Technology, East China Normal University, Shanghai, 200062, China
E-mail address: jlshu@admin.ecnu.edu.cn