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Liouville Type Theorems for General Integral System with Negative Exponents

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Abstract. In this paper, we establish a Liouville type theorem for the following integral system with negative exponents

$$\begin{cases} u(x) = \int_{\mathbb{R}^n} |x-y|^{\nu} f(u,v)(y)\,dy, & x \in \mathbb{R}^n, \\ v(x) = \int_{\mathbb{R}^n} |x-y|^{\nu} g(u,v)(y)\,dy, & x \in \mathbb{R}^n, \end{cases}$$

where $n \geq 1$, $\nu > 0$, and f, g are continuous functions defined on $\mathbb{R}_+ \times \mathbb{R}_+$. Under nature structure conditions on f and g, we classify each pair of positive solutions for above integral system by using the method of moving sphere in integral forms. Moreover, some other Liouville theorems are established for similar integral systems.

1. Introduction

In this paper, we consider the following integral system with negative exponents

$$\begin{cases} u(x) = \int_{\mathbb{R}^n} |x-y|^{\nu} f(u,v)(y)\,dy, & x \in \mathbb{R}^n, \\ v(x) = \int_{\mathbb{R}^n} |x-y|^{\nu} g(u,v)(y)\,dy, & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $n \geq 1$, $\nu > 0$, and f, g are continuous functions defined on $\mathbb{R}_+ \times \mathbb{R}_+$. Our motivation for studying the integral system (1.1) stems from the study of the reversed Hardy-Littlewood-Sobolev inequalities and curvature problems from conformal geometry. For instance, a special case of system (1.1) is the integral system

$$\begin{cases} u(x) = \int_{\mathbb{R}^n} |x-y|^{\nu} v^{-p_1}(y)\,dy, & x \in \mathbb{R}^n, \\ v(x) = \int_{\mathbb{R}^n} |x-y|^{\nu} u^{-p_2}(y)\,dy, & x \in \mathbb{R}^n, \end{cases} \tag{1.2}$$

where $n \geq 1$, $\nu, p_1, p_2 > 0$.

Integral system (1.2) is closely related to the Euler-Lagrange equation for the extremals to the reversed Hardy-Littlewood-Sobolev inequality which was first introduced by Dou

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and Zhu in [9]. In particular, for $p_1 = p_2 = (2n+\nu)/\nu$, they employed the method of moving spheres to show that every positive measurable solution of system (1.2) has the form

$$\begin{cases} u(x) = c_1(d+|x-x_0|^2)^{\nu/2}, & x \in \mathbb{R}^n, \\ v(x) = c_2(d+|x-x_0|^2)^{\nu/2}, & x \in \mathbb{R}^n, \end{cases}$$

where $x_0 \in \mathbb{R}^n$ is some point and c_1, c_2, d are positive constants.

This classification result is a crucial step in finding the best constant in the reversed Hardy-Littlewood-Sobolev inequality. For more Hardy-Littlewood-Sobolev inequalities and its reversed versions on the half space and on compact Riemannian manifolds and their applications to curvature problems, we refer the reader to [4, 6, 9, 11, 13, 19] and the references therein. Recently, Lei [14] considered system (1.2) and obtained necessary conditions for the existence of C^1 positive entire solutions as well as necessary and sufficient conditions for the scale invariance of the system with respect to certain energy functionals. In particular, if we set $u = v$, $f(u,v) = g(u,v) = u^{-p}$, $p > 0$, integral system (1.1) will degenerate into a single equation

$$u(x) = \int_{\mathbb{R}^n} |x-y|^{\nu} u^{-p}(y)\, dy. \tag{1.3}$$

The negative exponent of integral equation (1.3) was studied by Li in [15]. Li classified the positive solutions of (1.3) for $p = (2n+\nu)/\nu$. When $n = 3$, $\nu = 1$, Xu [20] proved that if u is a C^4 entire positive solution of integral equation (1.3), then $p = 7$ and u must take the form as $u(x) = c(1+|x|^2)^{1/2}$ up to dilation and translation. Furthermore, Xu [21] proved that equation (1.3) has a C^1 positive solution if and only if $p = (2n+\nu)/\nu$. For more related results about integral system with negative exponents, we refer the reader to [5, 10, 18] and the references therein.

Similar to integral system (1.2), there is an integral system in $\mathbb{R}^n$,

$$\begin{cases} u(x) = \int_{\mathbb{R}^n} \frac{v^{p_1}(y)}{|x-y|^{n-\alpha}}\, dy, & x \in \mathbb{R}^n, \\ v(x) = \int_{\mathbb{R}^n} \frac{u^{p_2}(y)}{|x-y|^{n-\alpha}}\, dy, & x \in \mathbb{R}^n, \end{cases} \tag{1.4}$$

where $0 < \alpha < n$, $1 < p_1, p_2 < +\infty$ and $1/(p_1+1) + 1/(p_2+1) = (n-\alpha)/n$.

Chen, Li, Ou in [2] used the moving plane method in integral forms to prove that all the positive solutions are radially symmetric and monotonic decreasing about some point. Later, Hang [12] proved the same result by extending the exponent restriction to $\alpha/(n-\alpha) < p_1, p_2 < +\infty$. For $1/(p_1+1) + 1/(p_2+1) > (n-\alpha)/n$, Chen and Li in [1] proved the nonexistence of positive solutions to system (1.4) when $0 < p_1 \le \alpha/(n-\alpha)$ or $0 < p_2 \le \alpha/(n-\alpha)$. Dou, Qu and Han [7] discussed the nonexistence of positive solutions to integral system (1.4) with weighted functions. In particular, if $u = v$ and $p_1 = p_2$, system (1.4) is reduced to an integral equation, and the classification of positive solutions is studied by Chen, Li and Ou [3] using the method of moving plane and Li [15] using the method of moving sphere, respectively. In 2013, Yu [22] established the Liouville type result for more general integral system.

Inspired by the above cited papers, the purpose of this paper is to establish the Liouville type theorem for integral system (1.1). Our main result can be stated as follows.

Theorem 1.1. *For $n \geq 1$, let $(u,v) \in C^1(\mathbb{R}^n) \times C^1(\mathbb{R}^n)$ be a pair of positive solutions of* (1.1) *and $f, g\colon [0,+\infty) \times [0,+\infty) \to \mathbb{R}$ are continuous satisfying*

(i) *$f(s,t)$ and $g(s,t)$ are strictly decreasing in t for fixed s and strictly decreasing in s for fixed t;*

(ii) *there exist $\kappa_1, \theta_1 \geq 0$ satisfying $\kappa_1 + \theta_1 = (\nu + 2n)/\nu$ such that $f(s,t)s^{\kappa_1}t^{\theta_1}$ is non-decreasing in t for fixed s and non-decreasing in s for fixed t;*

(iii) *there exist $\kappa_2, \theta_2 \geq 0$ satisfying $\kappa_2 + \theta_2 = (\nu + 2n)/\nu$ such that $g(s,t)s^{\kappa_2}t^{\theta_2}$ is non-decreasing in t for fixed s and non-decreasing in s for fixed t.*

Then there exist positive constants m, l such that $f(s,t) = ms^{-\kappa_1}t^{-\theta_1}$, $g(s,t) = ls^{-\kappa_2}t^{-\theta_2}$ and $u(x)$, $v(x)$ have the form of

$$u(x) = c_1(d + |x - x_0|^2)^{\nu/2}, \quad v(x) = c_2(d + |x - x_0|^2)^{\nu/2},$$

where $x_0 \in \mathbb{R}^n$ is some point and c_1, c_2, d are positive constants.

Remark 1.2. For $f(u,v) = u^{-p_1}v^{-p_2}$ and $g(u,v) = u^{-q_1}v^{-q_2}$, $(p_1 + p_2 = q_1 + q_2 = (\nu + 2n)/\nu)$, Theorem 1.1 was established by Dou, Ren and Villavert in [8].

We also study a simper problem

$$\begin{cases} u(x) = \int_{\mathbb{R}^n} |x-y|^{\nu} f(v(y))\, dy, \\ v(x) = \int_{\mathbb{R}^n} |x-y|^{\nu} g(u(y))\, dy, \end{cases} \tag{1.5}$$

where $n \geq 1$ and $\nu > 0$.

Our second result can be stated as follows.

Theorem 1.3. *For $n \geq 1$, let $(u,v) \in C^1(\mathbb{R}^n) \times C^1(\mathbb{R}^n)$ be a pair of positive solutions of* (1.5) *and $f, g\colon [0,+\infty) \to \mathbb{R}$ are continuous satisfying*

(i) *$f(t)$ and $g(t)$ are strictly decreasing on $(0,+\infty)$ for variable t;*

(ii) *$f(t)t^{(\nu+2n)/\nu}$ and $g(t)t^{(\nu+2n)/\nu}$ are non-decreasing for variable t.*

Then there exist positive constants m, l such that $f(t) = mt^{-(\nu+2n)/\nu}$, $g(t) = lt^{-(\nu+2n)/\nu}$ and $u(x)$, $v(x)$ have the form of

$$u(x) = c_1(d + |x - x_0|^2)^{\nu/2}, \quad v(x) = c_2(d + |x - x_0|^2)^{\nu/2},$$

where $x_0 \in \mathbb{R}^n$ is some point and c_1, c_2, d are positive constants.

Using the same technique, we can consider the more general system

$$\begin{cases} u(x) = \int_{\mathbb{R}^n} |x-y|^\nu \left(\sum_{i=1}^N f_i(u,v)(y)\right) dy, \\ v(x) = \int_{\mathbb{R}^n} |x-y|^\nu \left(\sum_{i=1}^N g_i(u,v)(y)\right) dy, \end{cases} \tag{1.6}$$

where $n \geq 1$, $N \geq 1$ and $\nu > 0$.

We have the similar result as Theorem 1.1.

Theorem 1.4. *For $n \geq 1$, let $(u,v) \in C^1(\mathbb{R}^n) \times C^1(\mathbb{R}^n)$ be a pair of positive solutions of (1.6) and $f = \sum_{i=1}^N f_i$, $g = \sum_{i=1}^N g_i$, $f_i, g_i \colon [0,+\infty) \times [0,+\infty) \to \mathbb{R}$ are continuous satisfying*

(i) *$f(s,t)$ and $g(s,t)$ are strictly decreasing in t for fixed s and strictly decreasing in s for fixed t;*

(ii) *there exist $\kappa_{1i}, \theta_{1i} \geq 0$ $(i = 1,2,\ldots,N)$ satisfying $\kappa_{1i} + \theta_{1i} = (\nu+2n)/\nu$ such that $f_i(s,t)s^{\kappa_{1i}}t^{\theta_{1i}}$ is non-decreasing in t for fixed s and non-decreasing in s for fixed t;*

(iii) *there exist $\kappa_{2i}, \theta_{2i} \geq 0$ $(i = 1,2,\ldots,N)$ satisfying $\kappa_{2i} + \theta_{2i} = (\nu+2n)/\nu$ such that $g_i(s,t)s^{\kappa_{2i}}t^{\theta_{2i}}$ is non-decreasing in t for fixed s and non-decreasing in s for fixed t.*

Then there exist positive constants m_i, l_i $(i = 1,2,\ldots,N)$ such that $f_i(s,t) = m_i s^{-\kappa_{1i}} t^{-\theta_{1i}}$, $g_i(s,t) = l_i s^{-\kappa_{2i}} t^{-\theta_{2i}}$ and $u(x)$, $v(x)$ have the form of

$$u(x) = c_1(d + |x - x_0|^2)^{\nu/2}, \quad v(x) = c_2(d + |x - x_0|^2)^{\nu/2},$$

where $x_0 \in \mathbb{R}^n$ is some point and c_1, c_2, d are positive constants.

By slightly modifying our arguments in proving Theorem 1.1 for integral system (1.1), we can prove Theorems 1.3 and 1.4 and thus we omit their proofs.

Remark 1.5. In Theorems 1.1, 1.3 and 1.4, one can only assume that (u,v) is a pair of positive measurable solutions instead of C^1 positive solutions. In fact, by standard regularity lifting theory, one can verify that measurable solutions must be smooth (see [8, 15]).

2. Proof of Theorem 1.1

In this section, we use the method of moving spheres introduced by Li and Zhu in [17], which was later improved by Li and Zhang in [15,16] to prove Theorem 1.1. We first prove the following lemma.

Lemma 2.1. *For $\nu > 0$, let $(u,v) \in C^1(\mathbb{R}^n) \times C^1(\mathbb{R}^n)$ be a pair of positive solutions of* (1.1)*. Then*

$$\int_{\mathbb{R}^n} (1+|y|^\nu) f(u,v)(y)\, dy < \infty, \quad \int_{\mathbb{R}^n} (1+|y|^\nu) g(u,v)(y)\, dy < \infty,$$

and for some constant $C_1, C_2 \geq 1$,

$$\frac{1}{C_1}(1+|x|^\nu) \leq u(x) \leq C_1(1+|x|^\nu), \quad \frac{1}{C_2}(1+|x|^\nu) \leq v(x) \leq C_2(1+|x|^\nu). \tag{2.1}$$

Proof. We only prove

$$\int_{\mathbb{R}^n} (1+|y|^\nu) f(u,v)(y)\, dy < \infty, \quad \frac{1}{C_1}(1+|x|^\nu) \leq u(x) \leq C_1(1+|x|^\nu).$$

We first show

$$\int_{\mathbb{R}^n} (1+|y|^\nu) f(u,v)(y)\, dy < \infty. \tag{2.2}$$

One can write

$$\begin{aligned}\int_{\mathbb{R}^n} (1+|y|^\nu) f(u,v)(y)\, dy &= \int_{|y|<2} (1+|y|^\nu) f(u,v)(y)\, dy + \int_{|y|\geq 2} (1+|y|^\nu) f(u,v)(y)\, dy \\ &= I + II.\end{aligned}$$

For $|x| > 8$,

$$I = \int_{|y|<2} (1+|y|^\nu) f(u,v)(y)\, dy \leq C_\nu \int_{\mathbb{R}^n} |x-y|^\nu f(u,v)(y)\, dy = C_\nu u(x) < \infty. \tag{2.3}$$

For $|x| < 1/2$,

$$II = \int_{|y|\geq 2} (1+|y|^\nu) f(u,v)(y)\, dy \leq C_\nu \int_{\mathbb{R}^n} |x-y|^\nu f(u,v)(y)\, dy = C_\nu u(x) < \infty. \tag{2.4}$$

Combining (2.3) and (2.4), we obtain (2.2).

Next, we show that

$$\frac{1}{C_1}(1+|x|^\nu) \leq u(x) \leq C_1(1+|x|^\nu), \quad \forall\, x \in \mathbb{R}^n.$$

By (2.2), we obtain

$$u(x) = \int_{\mathbb{R}^n} |x-y|^\nu f(u,v)(y)\,dy \le C(1+|x|^\nu) \int_{\mathbb{R}^n} (1+|y|^\nu) f(u,v)(y)\,dy \le C(1+|x|^\nu).$$

When $|x| < 2$, we have

$$u(x) = \int_{\mathbb{R}^n} |x-y|^\nu f(u,v)(y)\,dy \ge \int_{|y|>8} |x-y|^\nu f(u,v)(y)\,dy \ge C(1+|x|^\nu).$$

When $|x| \ge 2$, we have

$$u(x) = \int_{\mathbb{R}^n} |x-y|^\nu f(u,v)(y)\,dy \ge \int_{|y|<1} |x-y|^\nu f(u,v)(y)\,dy \ge C(1+|x|^\nu).$$

Then we accomplish the proof of Lemma 2.1. □

Let u and v be positive functions in $\mathbb{R}^n$, for $x \in \mathbb{R}^n$ and $\lambda > 0$, we define

$$u_{x,\lambda}(y) = \left(\frac{|y-x|}{\lambda}\right)^\nu u(y^{x,\lambda}), \quad v_{x,\lambda}(y) = \left(\frac{|y-x|}{\lambda}\right)^\nu v(y^{x,\lambda}), \quad y \in \mathbb{R}^n,$$

where

$$y^{x,\lambda} = x + \frac{\lambda^2 (y-x)}{|y-x|^2}.$$

Lemma 2.2. *Let $(u,v) \in C^1(\mathbb{R}^n) \times C^1(\mathbb{R}^n)$ be a pair of positive solutions of (1.1), then*

(2.5)
$$\begin{aligned} u_{x,\lambda}(y) - u(y) = \int_{|z-x|\ge\lambda} & K(x,\lambda,y,z) \\ & \times \left(f(u,v) - \left(\frac{\lambda}{|z-x|}\right)^{2n+\nu} f\left[\left(\frac{\lambda}{|z-x|}\right)^\nu u_{x,\lambda}, \left(\frac{\lambda}{|z-x|}\right)^\nu v_{x,\lambda}\right]\right) dz \end{aligned}$$

and

(2.6)
$$\begin{aligned} v_{x,\lambda}(y) - v(y) = \int_{|z-x|\ge\lambda} & K(x,\lambda,y,z) \\ & \times \left(g(u,v) - \left(\frac{\lambda}{|z-x|}\right)^{2n+\nu} g\left[\left(\frac{\lambda}{|z-x|}\right)^\nu u_{x,\lambda}, \left(\frac{\lambda}{|z-x|}\right)^\nu v_{x,\lambda}\right]\right) dz, \end{aligned}$$

where

$$K(x,\lambda,y,z) = \left(\frac{|y-x|}{\lambda}\right)^\nu |y^{x,\lambda} - z|^\nu - |y-z|^\nu \tag{2.7}$$

and $K(x,\lambda,y,z) > 0$ for any $|y-x| > \lambda$, $|z-x| > \lambda$.

Proof. We first show the proof of (2.5). By (1.1), we can write

$$
\begin{aligned}
u(y) &= \int_{|z-x|\ge\lambda} |y-z|^{\nu} f(u,v)(z)\,dz + \int_{|z-x|<\lambda} |y-z|^{\nu} f(u,v)(z)\,dz \\
&= \int_{|z-x|\ge\lambda} |y-z|^{\nu} f(u,v)(z)\,dz \\
&\quad + \left(\frac{|y-x|}{\lambda}\right)^{\nu} \int_{|z-x|\ge\lambda} |y^{x,\lambda}-z|^{\nu} \left(\frac{\lambda}{|z-x|}\right)^{2n+\nu} \\
&\qquad \times f\left[\left(\frac{\lambda}{|z-x|}\right)^{\nu} u_{x,\lambda}, \left(\frac{\lambda}{|z-x|}\right)^{\nu} v_{x,\lambda}\right] dz.
\end{aligned}
\tag{2.8}
$$

Then, we have

$$
\begin{aligned}
&u_{x,\lambda}(y) \\
&= \left(\frac{|y-x|}{\lambda}\right)^{\nu} \int_{\mathbb{R}^n} |y^{x,\lambda}-z|^{\nu} f(u,v)(z)\,dz \\
&= \left(\frac{|y-x|}{\lambda}\right)^{\nu} \int_{|z-x|\ge\lambda} |y^{x,\lambda}-z|^{\nu} f(u,v)(z)\,dz \\
&\quad + \int_{|z-x|\ge\lambda} |y-z|^{\nu} \left(\frac{\lambda}{|z-x|}\right)^{2n+\nu} f\left[\left(\frac{\lambda}{|z-x|}\right)^{\nu} u_{x,\lambda}, \left(\frac{\lambda}{|z-x|}\right)^{\nu} v_{x,\lambda}\right] dz.
\end{aligned}
\tag{2.9}
$$

Combining (2.8) and (2.9), we have

$$
\begin{aligned}
&u_{x,\lambda}(y) - u(y) \\
&= \int_{|z-x|\ge\lambda} \left[\left(\frac{|y-x|}{\lambda}\right)^{\nu} |y^{x,\lambda}-z|^{\nu} - |y-z|^{\nu}\right] f(u,v)(z)\,dz \\
&\quad + \int_{|z-x|\ge\lambda} \left[|y-z|^{\nu} - \left(\frac{|y-x|}{\lambda}\right)^{\nu} |y^{x,\lambda}-z|^{\nu}\right] \left(\frac{\lambda}{|z-x|}\right)^{2n} \\
&\qquad \times f\left[\left(\frac{\lambda}{|z-x|}\right)^{\nu} u_{x,\lambda}, \left(\frac{\lambda}{|z-x|}\right)^{\nu} v_{x,\lambda}\right] dz \\
&= \int_{|z-x|\ge\lambda} K(x,\lambda,y,z) \left(f(u,v)(z) - \left(\frac{\lambda}{|z-x|}\right)^{2n+\nu} f\left[\left(\frac{\lambda}{|z-x|}\right)^{\nu} u_{x,\lambda}, \left(\frac{\lambda}{|z-x|}\right)^{\nu} v_{x,\lambda}\right]\right) dz.
\end{aligned}
$$

Similarly, we can obtain

$$
\begin{aligned}
&v_{x,\lambda}(y) - v(y) \\
&= \int_{|z-x|\ge\lambda} K(x,\lambda,y,z) \left(g(u,v)(z) - \left(\frac{\lambda}{|z-x|}\right)^{2n+\nu} g\left[\left(\frac{\lambda}{|z-x|}\right)^{\nu} u_{x,\lambda}, \left(\frac{\lambda}{|z-x|}\right)^{\nu} v_{x,\lambda}\right]\right) dz.
\end{aligned}
$$

Next, it suffices to show that $K(x,\lambda,y,z) > 0$ for $|y-x| > \lambda$ and $|z-x| > \lambda$. Without loss of generality, we may assume $x = 0$. Therefore, we can write

$$\begin{aligned} K(0, \lambda, y, z) &= \left(\frac{|y|}{\lambda}\right)^{\nu} |y^{0,\lambda} - z|^{\nu} - |y - z|^{\nu} \\ &= \left(\lambda^2 - 2y \cdot z + \frac{|y|^2|z|^2}{\lambda^2}\right)^{v/2} - (|y|^2 - 2y \cdot z + |z|^2)^{v/2} \\ &= \left(|y|^2 - 2y \cdot z + |z|^2 + (|y|^2 - \lambda^2)\left(\frac{|z|^2}{\lambda^2} - 1\right)\right)^{v/2} - (|y|^2 - 2y \cdot z + |z|^2)^{v/2} \\ &> 0. \end{aligned}$$

Thus we accomplish the proof of Lemma 2.2. □

To prove Theorem 1.1, we also need the following two key lemmas.

Lemma 2.3. *For $n \geq 1$, $\nu > 0$, let $(u, v) \in C^1(\mathbb{R}^n) \times C^1(\mathbb{R}^n)$ be a pair of positive solutions of* (1.1). *Then for any $x \in \mathbb{R}^n$, there exists $\lambda_0(x) > 0$ such that*

$$u_{x,\lambda}(y) \geq u(y), \quad v_{x,\lambda}(y) \geq v(y), \quad \forall\, 0 < \lambda < \lambda_0(x),\ |y - x| \geq \lambda. \tag{2.10}$$

Proof. Without loss of generality, we may assume $x = 0$. Since u and v are positive C^1 functions, there exists $s_0 > 0$ such that

$$\nabla_y(|y|^{-\nu/2}u(y)) \cdot y < 0, \quad \nabla_y(|y|^{-\nu/2}v(y)) \cdot y < 0, \quad \forall\, 0 < |y| < s_0.$$

Then, we have

$$u_{0,\lambda}(y) > u(y), \quad v_{0,\lambda}(y) > v(y), \quad \forall\, 0 < \lambda < |y| < s_0. \tag{2.11}$$

By (2.1), there exists C_{s_0} such that

$$u(y) \leq C_{s_0}|y|^{\nu}, \quad v(y) \leq C_{s_0}|y|^{\nu}, \quad \forall\, |y| \geq s_0.$$

Using (2.1) again, there also exists c_0 such that

$$\inf_{y \in B_{s_0}} u(y) \geq c_0 > 0, \qquad \inf_{y \in B_{s_0}} v(y) \geq c_0 > 0.$$

Then, for small $\lambda_0 \in (0, s_0)$ and $0 < \lambda < \lambda_0$, we derive

$$u_{0,\lambda}(y) = \left(\frac{|y|}{\lambda}\right)^{\nu} u\left(\frac{\lambda^2 y}{|y|^2}\right) \geq \left(\frac{|y|}{\lambda_0}\right)^{\nu} \inf_{B_{s_0}} u \geq u(y), \quad \forall\, |y| \geq s_0 \tag{2.12}$$

and

$$v_{0,\lambda}(y) = \left(\frac{|y|}{\lambda}\right)^{\nu} v\left(\frac{\lambda^2 y}{|y|^2}\right) \geq \left(\frac{|y|}{\lambda_0}\right)^{\nu} \inf_{B_{s_0}} v \geq v(y), \quad \forall\, |y| \geq s_0. \tag{2.13}$$

By (2.11), (2.12) and (2.13), we obtain (2.10). We then prove Lemma 2.3. □

For $x \in \mathbb{R}^n$, define

$$\overline{\lambda}(x) = \sup\{\mu \mid u_{x,\lambda}(y) \geq u(y), v_{x,\lambda}(y) \geq v(y), \forall\, 0 < \lambda < \mu, |y - x| \geq \lambda\}.$$

Lemma 2.4. *For $n \geq 1$, let (u, v) be a pair of positive solutions of* (1.1)*. Then for any $x \in \mathbb{R}^n$,*

$$\overline{\lambda}(x) < \infty$$

and

$$u_{x,\overline{\lambda}} \equiv u, \quad v_{x,\overline{\lambda}} \equiv v. \tag{2.14}$$

Proof. We first show that $\overline{\lambda}(x) < \infty$. By the definition of $\overline{\lambda}(x)$, for any $0 < \lambda < \overline{\lambda}(x)$, we have

$$u_{x,\lambda}(y) \geq u(y), \quad v_{x,\lambda}(y) \geq v(y), \quad \forall\, |y - x| \geq \lambda. \tag{2.15}$$

Then, we can apply (2.15) and Lemma 2.1 to obtain that

$$\begin{aligned} 0 < \int_{\mathbb{R}^n} f(u, v)(z)\, dz &= \lim_{|y| \to \infty} |y|^{-\nu} u(y) \\ &\leq \lim_{|y| \to \infty} |y|^{-\nu} \left(\frac{|y - x|}{\lambda} \right)^{\nu} u\left(x + \frac{\lambda^2 (y - x)}{|y - x|^2} \right) \\ &= \lambda^{-\nu} u(x). \end{aligned}$$

Thus $\overline{\lambda}(x) < \infty$ for all $x \in \mathbb{R}^n$.

Now we prove (2.14). Without loss of generality, we may assume $x = 0$ and we adopt the notation that $\overline{\lambda} = \overline{\lambda}(0)$, $u_\lambda = u_{0,\lambda}$ and $y^\lambda = y^{0,\lambda}$. It is easy to see that $u_{\overline{\lambda}} \geq u$ and $v_{\overline{\lambda}} \geq v$ for any $|y| \geq \overline{\lambda}$ by the definition of $\overline{\lambda}$. Then there are the following four cases:

Case (i): $u_{\overline{\lambda}}(y) = u(y)$, $v_{\overline{\lambda}}(y) = v(y)$,

Case (ii): $u_{\overline{\lambda}}(y) > u(y)$, $v_{\overline{\lambda}}(y) = v(y)$,

Case (iii): $u_{\overline{\lambda}}(y) = u(y)$, $v_{\overline{\lambda}}(y) > v(y)$,

Case (iv): $u_{\overline{\lambda}}(y) > u(y)$, $v_{\overline{\lambda}}(y) > v(y)$.

For Case (i), we are done. For Case (ii), since $g(s, t)$ is strictly decreasing in t for fixed s and strictly decreasing in s for fixed t. By (2.6) and $K(0, \overline{\lambda}, y, z) > 0$ for $|y| > \overline{\lambda}$, $|z| > \overline{\lambda}$, we have

$$0 = v_{\overline{\lambda}}(y) - v(y) \geq \int_{|z| \geq \overline{\lambda}} K(0, \overline{\lambda}, y, z) \left(g(u, v) - g(u_{\overline{\lambda}}, v_{\overline{\lambda}}) \right)\, dz > 0.$$

This is a contradiction. Similarly, one can also show that Case (iii) is impossible by (2.5). Thus it suffices to show that Case (iv) is also impossible. We carry out the argument by contradiction. Suppose that Case (iv) is valid, we will show that there exists $\varepsilon > 0$ such that

$$u_\lambda(y) \geq u(y), \quad v_\lambda(y) \geq v(y), \quad \forall\, \overline{\lambda} < \lambda < \overline{\lambda} + \varepsilon \text{ for } |y| > \lambda.$$

We only prove that

$$u_\lambda(y) \geq u(y), \quad \forall\, \overline{\lambda} < \lambda < \overline{\lambda} + \varepsilon \text{ for } |y| > \lambda.$$

By (2.5), with $x = 0$, $\lambda = \overline{\lambda}$ and the Fatou lemma, one can calculate

$$\begin{aligned}
&\lim_{|y|\to\infty} \inf |y|^{-\nu}(u_{\overline{\lambda}}(y) - u(y)) \\
&= \lim_{|y|\to\infty} \inf |y|^{-\nu} \int_{|z|\geq\overline{\lambda}} K(0,\overline{\lambda},y,z) \\
&\qquad \times \left(f(u,v) - \left(\frac{\overline{\lambda}}{|z|}\right)^{2n+\nu} f\left[\left(\frac{\overline{\lambda}}{|z-x|}\right)^{\nu} u_{x,\overline{\lambda}}, \left(\frac{\overline{\lambda}}{|z-x|}\right)^{\nu} v_{x,\overline{\lambda}} \right] \right) dz
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{|y|\to\infty} \inf |y|^{-\nu}(v_{\overline{\lambda}}(y) - v(y)) \\
&= \lim_{|y|\to\infty} \inf |y|^{-\nu} \int_{|z|\geq\overline{\lambda}} K(0,\overline{\lambda},y,z) \\
&\qquad \times \left(g(u,v) - \left(\frac{\overline{\lambda}}{|z|}\right)^{2n+\nu} f\left[\left(\frac{\overline{\lambda}}{|z-x|}\right)^{\nu} u_{x,\overline{\lambda}}, \left(\frac{\overline{\lambda}}{|z-x|}\right)^{\nu} v_{x,\overline{\lambda}} \right] \right) dz.
\end{aligned}$$

Notice the fact $\left(\frac{\overline{\lambda}}{|z|}\right)^{\nu} u_{\overline{\lambda}} \leq u_{\overline{\lambda}}$ and $\left(\frac{\overline{\lambda}}{|z|}\right)^{\nu} v_{\overline{\lambda}} \leq v_{\overline{\lambda}}$. By the monotonicity of $f(s,t)$ and $g(s,t)$, it follows that

$$\left(\left(\frac{\overline{\lambda}}{|z|}\right)^{\nu} u_{\overline{\lambda}}\right)^{\kappa_1} \left(\left(\frac{\overline{\lambda}}{|z|}\right)^{\nu} v_{\overline{\lambda}}\right)^{\theta_1} f\left(\left(\frac{\overline{\lambda}}{|z|}\right)^{\nu} u_{\overline{\lambda}}, \left(\frac{\overline{\lambda}}{|z|}\right)^{\nu} v_{\overline{\lambda}}\right) \leq f(u_{\overline{\lambda}}, v_{\overline{\lambda}}) u_{\overline{\lambda}}^{\kappa_1} v_{\overline{\lambda}}^{\theta_1}$$

and

$$\left(\left(\frac{\overline{\lambda}}{|z|}\right)^{\nu} u_{\overline{\lambda}}\right)^{\kappa_2} \left(\left(\frac{\overline{\lambda}}{|z|}\right)^{\nu} v_{\overline{\lambda}}\right)^{\theta_2} g\left(\left(\frac{\overline{\lambda}}{|z|}\right)^{\nu} u_{\overline{\lambda}}, \left(\frac{\overline{\lambda}}{|z|}\right)^{\nu} v_{\overline{\lambda}}\right) \leq g(u_{\overline{\lambda}}, v_{\overline{\lambda}}) u_{\overline{\lambda}}^{\kappa_2} v_{\overline{\lambda}}^{\theta_2}.$$

Then we have

$$\left(\frac{\overline{\lambda}}{|z|}\right)^{2n+\nu} f\left(\left(\frac{\overline{\lambda}}{|z|}\right)^{\nu} u_{\overline{\lambda}}, \left(\frac{\overline{\lambda}}{|z|}\right)^{\nu} v_{\overline{\lambda}}\right) \leq f(u_{\overline{\lambda}}, v_{\overline{\lambda}})$$

and

$$\left(\frac{\overline{\lambda}}{|z|}\right)^{2n+\nu} g\left(\left(\frac{\overline{\lambda}}{|z|}\right)^{\nu} u_{\overline{\lambda}}, \left(\frac{\overline{\lambda}}{|z|}\right)^{\nu} v_{\overline{\lambda}}\right) \leq g(u_{\overline{\lambda}}, v_{\overline{\lambda}}).$$

Then it follows that

$$\lim_{|y|\to\infty}\inf |y|^{-\nu}(u_{\overline{\lambda}}(y)-u(y)) \geq \int_{|z|\geq\overline{\lambda}} \left(\left(\frac{|z|}{\overline{\lambda}}\right)^{\nu}-1\right)(f(u,v)(z)-f(u_{\overline{\lambda}},v_{\overline{\lambda}})(z))\,dz>0$$

and

$$\lim_{|y|\to\infty}\inf |y|^{-\nu}(v_{\overline{\lambda}}(y)-v(y)) \geq \int_{|z|\geq\overline{\lambda}} \left(\left(\frac{|z|}{\overline{\lambda}}\right)^{\nu}-1\right)(g(u,v)(z)-f(u_{\overline{\lambda}},v_{\overline{\lambda}})(z))\,dz>0.$$

Therefore, for any $|y|\geq\overline{\lambda}+1$, there exists $\varepsilon_1\in(0,1)$ such that

$$u_{\overline{\lambda}}(y)-u(y)\geq\varepsilon_1|y|^{\nu},\quad v_{\overline{\lambda}}(y)-v(y)\geq\varepsilon_1|y|^{\nu}.$$

We can use the continuity of u and v with respect to variable λ to obtain that

$$u_\lambda(y)-u(y)\geq\varepsilon_1|y|^{\nu}+(u_\lambda(y)-u_{\overline{\lambda}}(y))\geq\frac{\varepsilon_1}{2}|y|^{\nu},\quad\forall\,|y|\geq\overline{\lambda}+1,\ \overline{\lambda}\leq\lambda\leq\overline{\lambda}+\varepsilon_2 \tag{2.16}$$

and

$$v_\lambda(y)-v(y)\geq\varepsilon_1|y|^{\nu}+(v_\lambda(y)-v_{\overline{\lambda}}(y))\geq\frac{\varepsilon_1}{2}|y|^{\nu},\quad\forall\,|y|\geq\overline{\lambda}+1,\ \overline{\lambda}\leq\lambda\leq\overline{\lambda}+\varepsilon_2 \tag{2.17}$$

for sufficiently small $\varepsilon_2>0$. Thus, it suffices to verify that for $\lambda<|y|<\overline{\lambda}+1$,

$$u_\lambda(y)\geq u(y),\quad\forall\,\overline{\lambda}<\lambda<\overline{\lambda}+\varepsilon.$$

By Lemma 2.2, we can write

$$\begin{aligned}
&u_\lambda(y)-u(y)\\
&=\int_{|z|\geq\lambda}K(0,\lambda,y,z)\left(f(u,v)(z)-\left(\frac{\lambda}{|z|}\right)^{2n+\nu}f\left[\left(\frac{\lambda}{|z|}\right)^{\nu}u_\lambda,\left(\frac{\lambda}{|z|}\right)^{\nu}v_\lambda\right]\right)dz\\
&\geq\int_{\lambda\leq|z|\leq\overline{\lambda}+1}K(0,\lambda,y,z)\left(f(u,v)(z)-\left(\frac{\lambda}{|z|}\right)^{2n+\nu}f\left[\left(\frac{\lambda}{|z|}\right)^{\nu}u_\lambda,\left(\frac{\lambda}{|z|}\right)^{\nu}v_\lambda\right]\right)dz\\
&\quad+\int_{\overline{\lambda}+2\leq|z|\leq\overline{\lambda}+3}K(0,\lambda,y,z)\left(f(u,v)(z)-\left(\frac{\lambda}{|z|}\right)^{2n+\nu}f\left[\left(\frac{\lambda}{|z|}\right)^{\nu}u_\lambda,\left(\frac{\lambda}{|z|}\right)^{\nu}v_\lambda\right]\right)dz.
\end{aligned}\tag{2.18}$$

By the monotonicity of $f(s,t)$ again, we obtain

$$\begin{aligned}
u_\lambda(y)-u(y)\geq&\int_{\lambda\leq|z|\leq\overline{\lambda}+1}K(0,\lambda,y,z)(f(u_{\overline{\lambda}},v_{\overline{\lambda}})-f(u_\lambda,v_\lambda))\,dz\\
&+\int_{\overline{\lambda}+2\leq|z|\leq\overline{\lambda}+3}K(0,\lambda,y,z)(f(u,v)-f(u_\lambda,v_\lambda))\,dz.
\end{aligned}$$

For $\overline{\lambda}+2 \le |z| \le \overline{\lambda}+3$, in view of (2.16) and (2.17), one can apply the monotonicity of $f(s,t)$ for each variable s and t to obtain there exists $\delta_1 > 0$ dependent of $\overline{\lambda}$ such that

$$f(u,v) - f(u_\lambda, v_\lambda) > f(u,v) - f(u + \varepsilon_2(\overline{\lambda}+2)^\nu, v + \varepsilon_2(\overline{\lambda}+2)^\nu) \ge \delta_1.$$

By (2.7), for any $\overline{\lambda} \le \lambda \le |y| \le \overline{\lambda}+1$, and $\overline{\lambda}+2 \le |z| \le \overline{\lambda}+3$, there exist $\widetilde{\delta}_2, \delta_2 > 0$ such that

$$\begin{aligned}
K(0,\lambda,y,z) &= \left(\frac{|y|}{\lambda}\right)^\nu \left|\frac{\lambda^2 y}{|y|^2} - z\right|^\nu - |y-z|^\nu \\
&= \left(\left(\frac{|y|}{\lambda}\right)^2 \left|\frac{\lambda^2 y}{|y|^2} - z\right|^2\right)^{\nu/2} - (|y-z|^2)^{\nu/2} \\
&= \widetilde{\delta}_2 \left(\left(\frac{|y|}{\lambda}\right)^2 \left|\frac{\lambda^2 y}{|y|^2} - z\right|^2 - |y-z|^2\right) \\
&\ge \delta_2(|y| - \lambda).
\end{aligned} \tag{2.19}$$

For any $\overline{\lambda} \le \lambda \le \overline{\lambda}+\varepsilon$, and $\overline{\lambda} \le \lambda \le |z| \le \overline{\lambda}+1$, we can calculate directly

$$u_\lambda(z) - u_{\overline{\lambda}}(z) = \frac{|z|^\nu u\left(\frac{\lambda^2 z}{|z|^2}\right)}{\lambda^\nu \overline{\lambda}^\nu}(\overline{\lambda}^\nu - \lambda^\nu) + \left(\frac{|z|}{\overline{\lambda}}\right)^\nu \nabla u(\xi_1)\cdot\left(\frac{\lambda^2 z}{|z|^2} - \frac{\overline{\lambda}^2 z}{|z|^2}\right) \tag{2.20}$$

and

$$v_\lambda(z) - v_{\overline{\lambda}}(z) = \frac{|z|^\nu v\left(\frac{\lambda^2 z}{|z|^2}\right)}{\lambda^\nu \overline{\lambda}^\nu}(\overline{\lambda}^\nu - \lambda^\nu) + \left(\frac{|z|}{\overline{\lambda}}\right)^\nu \nabla v(\xi_2)\cdot\left(\frac{\lambda^2 z}{|z|^2} - \frac{\overline{\lambda}^2 z}{|z|^2}\right), \tag{2.21}$$

where

$$\xi_1, \xi_2 \in \left(\frac{\overline{\lambda}^2 z}{|z|^2}, \frac{\lambda^2 z}{|z|^2}\right).$$

Then combing (2.20), (2.21) and the continuity of $f(s,t)$, we derive that

$$|f(u_{\overline{\lambda}}, v_{\overline{\lambda}})(z) - f(u_\lambda, v_\lambda)(z)| \le C\varepsilon. \tag{2.22}$$

For $\lambda \le |y| \le \overline{\lambda}+1$, we have

$$\begin{aligned}
\int_{\lambda\le|z|\le\overline{\lambda}+1} K(0,\lambda,y,z)\,dz &= \int_{\lambda\le|z|\le\overline{\lambda}+1} \left(\frac{|y|}{\lambda}\right)^\nu \left|\frac{\lambda^2 y}{|y|^2} - z\right|^\nu - |y-z|^\nu\,dz \\
&\le \int_{\lambda\le|z|\le\overline{\lambda}+1} \left(\frac{|y|}{\lambda}\right)^\nu \left|\frac{\lambda^2 y}{|y|^2} - z\right|^\nu - \left(\frac{|y|}{\lambda}\right)^\nu |y-z|^\nu\,dz \\
&\quad + \int_{\lambda\le|z|\le\overline{\lambda}+1} \left(\frac{|y|}{\lambda}\right)^\nu |y-z|^\nu - |y-z|^\nu\,dz \\
&\le C_2(|y|-\lambda).
\end{aligned} \tag{2.23}$$

By (2.18), (2.19), (2.22) and (2.23), for sufficiently small $\varepsilon > 0$, we conclude that

$$u_\lambda(y) - u(y) \geq \left(\delta_1\delta_2 \int_{\overline{\lambda}+2\leq|z|\leq\overline{\lambda}+3} dz - C_1C_2\varepsilon\right)(|y|-\lambda) \geq 0.$$

Combining with (2.16) and (2.17), one can derive a contradiction with the definition of $\overline{\lambda}$. Then this completes the proof of Lemma 2.4. $\square$

The following key calculus lemma introduced by Li in [15] is needed to carry out the final proof of Theorem 1.1.

Lemma 2.5. *For $h \in C(\mathbb{R}^n)$, $n \geq 1$, $\nu \in \mathbb{R}$, suppose that for every $x \in \mathbb{R}^n$, there exists $\lambda(x) > 0$ such that*

$$\left(\frac{\lambda(x)}{|y-x|}\right)^\nu h\left(x + \frac{\lambda(x)^2(y-x)}{|y-x|^2}\right) = h(y), \quad \forall\, y \in \mathbb{R}^n \setminus \{x\}.$$

Then for some $c \geq 0$, $d > 0$, $x_0 \in \mathbb{R}^n$, $h(x) = \pm c(|x-x_0|^2 + d)^{\nu/2}$.

Proof of Theorem 1.1. In view of Lemmas 2.4 and 2.5, one can obtain that

$$u(x) = c_1(d + |x-x_0|^2)^{\nu/2}, \quad v(x) = c_2(d + |x-x_0|^2)^{\nu/2}$$

for some $x_0 \in \mathbb{R}^n$. By $u_{x,\overline{\lambda}}(y) = u(y)$ and $v_{x,\overline{\lambda}}(y) = v(y)$, one can write that

$$\begin{aligned}
0 &= u_{x,\overline{\lambda}}(y) - u(y)\\
&= \int_{|z-x|\geq\overline{\lambda}} K(x,\overline{\lambda},y,z)\left(f(u,v) - \left(\frac{\overline{\lambda}}{|z-x|}\right)^{2n+\nu} f\left[\left(\frac{\overline{\lambda}}{|z-x|}\right)^\nu u_{x,\overline{\lambda}}, \left(\frac{\overline{\lambda}}{|z-x|}\right)^\nu v_{x,\overline{\lambda}}\right]\right) dz\\
&\geq \int_{|z-x|\geq\overline{\lambda}} K(x,\overline{\lambda},y,z)(f(u,v) - f(u_{x,\overline{\lambda}}, v_{x,\overline{\lambda}}))\, dz\\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
0 &= v_{x,\overline{\lambda}}(y) - v(y)\\
&= \int_{|z-x|\geq\overline{\lambda}} K(x,\overline{\lambda},y,z)\left(g(u,v) - \left(\frac{\overline{\lambda}}{|z-x|}\right)^{2n+\nu} g\left[\left(\frac{\overline{\lambda}}{|z-x|}\right)^\nu u_{x,\overline{\lambda}}, \left(\frac{\overline{\lambda}}{|z-x|}\right)^\nu v_{x,\overline{\lambda}}\right]\right) dz\\
&\geq \int_{|z-x|\geq\overline{\lambda}} K(x,\overline{\lambda},y,z)(g(u,v) - g(u_{x,\overline{\lambda}}, v_{x,\overline{\lambda}}))\, dz\\
&= 0.
\end{aligned}$$

Then the above inequalities must be equalities. Therefore, there exist positive constants m, l such that $f(s,t) = ms^{-\kappa_1}t^{-\theta_1}$, $g(s,t) = ls^{-\kappa_2}t^{-\theta_2}$. This accomplishes the proof of Theorem 1.1. $\square$

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